Team Control Number

12736

Problem Chosen

A

2022

HiMCM/MidMCM
Summary Sheet

Modelling of Honeybee Population Dynamics

Honeybee is widely distributed in the Earth and plays an important role of ecosystem and agriculture worldwide. However, the bee colonies decline dramatically in the recent years. Since 2006, the decline rate of bee colonies has been more than 30%. From 2021 to 2022, American beekeepers lost about 39.0% of their bee colonies. The phenomena of colony collapse disorder have caused the beekeepers to incur huge economic losses. Thus, it is essential to find an applicable method to predict the development of bee colonies and to quantify the benefit from bee.

There are 2 objectives of this report. The first objective is to estimate the population dynamic of bee colony and analyze the influence of factors on the development of bee colony. The second objective to quantify and maximize the net profit of utilizing bees in crop parcel. The 2 objectives correspond to the forementioned significance of bees and its contribution to agriculture.

In the first part, we discussed the eusociality of beehive, process of bee colony's development and the factors influencing bee colony and current methodology simulating bee population dynamic according to literature review. With the discussion, we established the ordinary differential equations including season and food to simulate the population change of bee colony. With the model, we simulated the population after 22 days of a bee colony and compared the results with observed data in real world. The comparison shows that the average error of our simulation is lower than 15% of the observed data.

Then, the sensitivity analysis was conducted. The sensitivity to the initial number of worker bees, the egg laying rate of queen, the food collection rate and the death rate of bees were investigated. The investigation on initial number of worker bees shows that within the interval of 22 days, the larger the initial number of worker bees is, the more broods will be at the end of the interval. When the initial number of worker bees increases, the ratio of the number of broods to the number of worker bees increases initially and then decreases. For the spawning rate, the change in broods with laying rate in the steady state has a greater slope than that of worker bees. The investigation on death rate shows that the bee colony will collapse if the death rate is greater than 0.55. The effects of different food collection rates shows that broods, hive bees and foragers start to increase rapidly when the rate is higher than 0.092. After the food collection rate reaches 0.102, further increases in food collection rate cannot prompt the growth of the honeybee population anymore. The analysis of joint influence from death and food collection indicates that it is possible to maintain a larger bee population with a higher death rate if the food collection rate is also higher.

In the second part, we discussed the function of quantifying the cost and benefit of using bees and obtained the profitable interval of the number of hives under different conditions. The cost includes the cost for purchasing beehives and hiring labor to maintain the beehives. The benefit includes the benefit from honey and the benefit from incremental crops because of bee pollination. Firstly, we discussed the benefit and cost of benefit in single hive condition. The function for benefit and cost are set based on the bee density estimation, bloom season estimation and marginal utility of benefit. Then, we extend the function by considering multiple beehives. The modified function for multiple beehives considered not only the marginal utility but also the incremental cost. With the parameters of unit benefit and cost, we optimized the optimal number of beehives and discussed the influences of flower blooming season and unit cost of beehive. The result shows that the maximum net profit is $0.275Million, in which the unit cost of beehive is moderate, and the bloom season is May. When crops bloom too early or too late, the optimal number of hives is lower, and the net profit is also lower. The influence of unit cost of beehive was also discussed based on similar methodology.

One of the features of this report is that we considered both food and season in the population dynamic model of bee colony and the simulation by this model is validated by real-world data, which proves the credibility of our model. Besides, the sensitivity analysis reveals the joint influence of multiple factors according to the proposed population dynamic model. Moreover, the report proposes a method to optimize the beehives distribution considering bee population dynamics, flower blooming season and marginal utility of benefit and cost. The proposed models are supposed to be applicable in wide range of applications in environmental engineering, entomology and agriculture.

Keywords: ordinary differential equations, sensitivity analysis, optimization, bee population

NOV. 2022 | 11.14

"BEE" FRIEND

Written by Team number 12736

HTTPS://WWW.EFSA.EUROPA.EU/EN/NEWS/BEE-HEALTH-NEW-DATA-HUB-GOES-LIVE

INTRODUCTION

Honeybees are our ecological friends. Through pollinating various plants in diligence, they provide us with the food essential to our survival. However, as human society develops, honeybees have been fading out from our view in the shadow of extinction: In the US, beekeepers have reported an average bee colony decline rate of 30% each winter since 2006; In 2007, the term **Colony Collapse Disorder** (CCD) was coined to describe this population plummet; more recently, from 2021 to 2022, American beekeepers lost about 39.0% of their bee colonies……What causes the worsening situation for honeybees, and what should we do?

HTTPS://WWW.FAMILYHANDYMAN.COM/ARTICLE/10-CRAZY-THINGS-YOU-DIDNT-KNOW-ABOUT-BEES/

FACTORS AFFECTING THE NUMBER OF BEES

In the natural environment, there are many factors affecting the changes in bee population. A range of factors including light, temperature, precipitation, bee activity, pollen species, infectious diseases, vegetation coverage of habitats, were all carefully investigated and discussed in various previous simulation models. In this blog, we summarize the two most important factors.

Food: In the brood stage of bees, only enough food and stable environment can make them grow healthily. If the food collected is insufficient due to various factors, the number of bees will decrease.

Season: In the bee population, the most important food source is nectar, which is collected by forager (a kind of worker bee) working outside. Every spring, the nectar and pollen source plants become abundant, and the workload of foragers starts to increase. At this time, some overloaded foragers will die, and the mortality of the whole colony will increase

HOW TO KEEP BEES ?

What about rearing your own bees and protecting them? Great idea. Here are somethings for existing or future beekeepers concerning their possible costs and benefits of raising honeybees:

- **Cost**
  - **Costs of beehives**

Since the equipment fees are mostly steady, simply consider about the best-fit amount of beehives according to your lands available

  - **Human costs of management/manual operation**

These include monitoring bee colonies, preventing outbreaks of virus, repelling alien colonies, and collecting honey. Beware that human costs are cumulative and may rocket!

- **Benefit**
  - Revenue from selling honey

Pay attention to market trends of bees and honey sales, try to get the best sale of your honeybee!

  - Increased crop yield

The activities of your honeybees bring rise to adjacent crop yields—yours or probably your neighbors'—by pollination.

The above is our summary of the possible factors contributing to honeybee colony collapses and corresponding instructions for beekeeping. Did we inspire you?

Don't hesitate to take care of your hives. Also, we at **www.beefriend.com** sincerely hope that you can revisit, repost, and donate to this site to contribute further to protecting the friends of mankind-- bees.

.

Contents

1 Introduction . . . 1
  1.1 Background . . . 1
  1.2 Requirements . . . 1
  1.3 Our Approach . . . 1
2 General Assumptions and Model Overview . . . 2
3 Notations . . . 2
  3.1 Notations . . . 2
4 Population Dynamic of Bee Colony . . . 3
  4.1 Key issues in Analysis of Bee Colony . . . 3
  4.2 Model for Population Dynamic of Bee Colony . . . 3
    4.2.1 The Selection of Model . . . 3
    4.2.2 The Eusociality and Influencing Factors of Honeybee Colony . . . 4
  4.3 Transformation between the types of bees . . . 5
  4.4 Model Solution . . . 6
    4.4.1 Parameter Estimation . . . 6
    4.4.2 Algorithm . . . 7
    4.4.3 Results and Discussions . . . 8
5 Sensitivity Analysis of Simulation Model . . . 9
  5.1 Critical Factors Analyzed in the Model . . . 9
  5.2 Range of the Critical Parameters in Sensitivity Analysis . . . 9
  5.3 Results and Discussion . . . 10
    5.3.1 Waterlevel Forecast based on ARIMA model . . . 10
6 Distribution of Hive Among Crop Parcel . . . 13
  6.1 Key Issues in Arranging Bee Hives . . . 13
  6.2 Benefit and Cost Model Optimizing Hives for Crops . . . 13
    6.2.1 Benefit and Cost Analysis of Single Hive . . . 13
    6.2.2 Benefit and Cost Analysis of Multiple Hives . . . 15
    6.2.3 Constraint Conditions . . . 16
  6.3 Model solution . . . 16
    6.3.1 Algorithms . . . 16
    6.3.2 Results and Discussions . . . 16
7 Model Evaluation . . . 19
  7.1 Advantages . . . 19
  7.2 Possible Improvements . . . 19
References . . . 20
Appendices . . . 20
Appendix A Tools and software . . . 20
Appendix B The MATLAB Code . . . 21

1 Introduction

1.1 Background

Honeybees are our ecological friends in almost every ecosystem on the earth that supports insect-pollinated blooming plants, despite the only exception of Antarctica. While they provide us with the food essential to our survival through pollinating various plants diligently, humans have been keeping bee colonies for millennia.

However, as human society develops, honeybees have been fading from our view in the shadow of extinction: In the US, beekeepers have reported an average bee colony decline rate of 30% each winter since 2006; In 2007, the term Colony Collapse Disorder (CCD) was coined to describe this population plummet; more recently, from 2021 to 2022, American beekeepers lost about 39.0% of their bee colonies. Since many agricultural commodities throughout the world rely on pollination by honeybees, that the Food and Agriculture Organization of the United Nations (FAO) reported that the value of all crops pollinated by honeybees worldwide was projected to be close to US$200 billion in 2005—the declining bee population, especially the phenomenon referred as CCD, might cause the beekeepers to incur huge economic losses. Thus, it is essential to find an applicable method to predict the development of bee colonies and to quantify the benefit from bee.

1.2 Requirements

- Develop a model to determine the population of a honeybee colony over time.
- Conduct sensitivity analysis on your model to determine which factors (e.g., lifespans, egg laying rates, fertilized/unfertilized egg ratios, or other factors) have the greatest impact on honeybee colony size.
- Model and predict how many honeybee hives you will need to support pollination of a 20-acre (81,000 square meters) parcel of land containing crops that benefit from pollination.
- Create a non-technical, one-page blog or infographic for a website that provides the information you developed.

1.3 Our Approach

- To solve the first requirement, we reviewed relevant literature to investigate what factors affect the number of bee populations. Secondly, we built relevant mathematical models to simulate and calculate the trend of bee population change.
- To solve the second requirement, based on the model established in the first requirement, we conducted a sensitivity analysis on the critical parameters that affect the bee population and then identified the joint effects of the critical factors to the bee population.
- To solve requirement 3, we reviewed literatures on this topic and investigated the specific relationship between the number of hives and bees and the economic benefits generated by bee pollination. Then, we found the number of hives that could maximize the economic benefits.

- After satisfying the above three requirements, we integrated the conclusions from the above three questions to write our blog.

2 General Assumptions and Model Overview

To simplify the problem, we make the following basic assumptions, each of which is properly justified.

- **Assumption 1**: The death rate of hive bees is ignored since it is significantly less than the death rate of forager bees and broods.
- **Assumption 2**: The influence of infectious disease is ignored. It can lead to rapid colony extinction and hard to be reversed so it is unnecessary to consider the diseases in a healthy and sustainable bee colony.
- **Assumption 3**: The differences between food resources are ignored since different food resources lead to similar death rate of bees and are not influence the population significantly.
- **Assumption 4**: The bee hives are assumed to be at the same location of the crop parcel since the area of the crop parcel is strikingly smaller than the bees' flight area.
- **Assumption 5**: The spatial distribution of bees from a hive and the time distribution of flower blooming follow normal distribution since it is the most common distribution for natural phenomena.
- **Assumption 6**: Several values of parameters for benefit and cost analysis were assumed in a reasonable range.

In this report, we developed 2 models for the population dynamic of bee colony and bee hive optimization in crop field. The ordinary differential equations including eusociality of bees and enviornmental conditions were established to simulate the population dynamic of bee colony. The sensitivity analysis was conducted on this model to evaluate the influence of egg laying rate, death rate and food collection rate. For arranging bee hives in the crop field, an optimization model considering bee and crop density in time and spatial dimension was established.

3 Notations

3.1 Notations

Important notations used in this article are listed in Table 1.

Table 1: Notations

Symbol	Description
$c_b c$	The initial cost of a single hive (dollars)
c_m	The cost of manual operation when having a single hive
b_h	The income derived from harvesting honey from a single hive
B_h	benefit from honey
$A_c f$	The area of crop field
d_1	the distance between hive and crop field in the first dimension
d_2	the distance between hive and crop field in the second dimension
$n_F(t)$	The number of forager in one bee colony at the tth day
$\rho_F(t, d_1, d_2,$	the density of the bees per square meters
$\rho_c(t)$	Within a square meter, the number of crops' flowers available for pollination at the tth day
$R_F c(t)$	the cumulative density of the bees at tt day
$n_b c$	Number of beehives
n_m	The maximum number of hives that can be cared for by one person
P_N	Net profit
B	Total benefit
N	the optimal number of hives by maximizing the net benefit

4 Population Dynamic of Bee Colony

4.1 Key issues in Analysis of Bee Colony

equirement 1 asks us to model the changes in the number of honeybees in bee colonies over time.

For the first requirement, we need to investigate the following three aspects

- The eusociality of bee colony
- Reasons and influencing factors of the increase and decrease of bee population
- Transformation between different jobs of honeybees

For the second and third aspects, we need to further focus on the following two points.

- Select appropriate function to simulate the change of bee population and the transformation between different jobs of bees within the bee colony.
- Quantify critical factors and introduce them into the constructed function

Based on the above discussion, we will model the population dynamic in different types of bees under multiple influencing factors in section 4.2.

4.2 Model for Population Dynamic of Bee Colony

4.2.1 The Selection of Model

In previous studies on the population of honeybees in bee colonies were mainly based on two methods. The first method is to count the total population of bees in the hive after the bee colonies have been kept for a certain period of time under different breeding or environmental conditions, after that, the scientists conducted regression analysis between the population of bees and the breeding or environmental variables.

This method helps we obtain the data about the correlation between honeybee population and breeding conditions in the real world, and the data can be utilized to predict the trend of honeybees through regression analysis. However, this method requires a lot of time on breeding bees. Meanwhile, because honeybees have larger flight range, which can be more than 6 kilometers and are easily affected by natural environmental factors, the variables in this method cannot be controlled accurately. Thus, larger errors would be introduced into the research and affect the following quantitative analysis.

In addition, it is difficult for us to count the same bee colony in high frequency. Although this method can be utilized to discuss the correlation between different influencing factors and the change of honeybee population, the above disadvantages of this method indicate the difficulty of utilizing this method to obtain an accurate data-driven model of honeybee population.

The second method is to use ODE model to simulate the population dynamic of bees. The advantage of this method is that it can simulate the change of bee population and control different breeding conditions accurately by adjusting parameters in the model. The impacts of the parameters in the bee colony model also can be analyzed, and we can define appropriate time intervals to slice the process and analyze the trend of population change.

Using this method to analyze the bee population needs to build a model meets the real world about the operation mechanism and influencing factors of bee population related and select appropriate parameters for simulation. Now, there have already been a lot of studies on the operation mechanism and influencing factors of bee population, and a large number of real-world studies have also provided reference for parameter selection. Therefore, we choose the ODE method to model and analyze the bee population. The operating mechanism and influencing factors of bee population will be discussed in the next part.

4.2.2 The Eusociality and Influencing Factors of Honeybee Colony

In a bee colony, the active bees within a bee colony can be generally divided into four types, including queen bee, worker bee, brood and drone bee.

- Queen bee: The queen bee is the main reproducer in the colony, whose task is to lay eggs. The egg can be either fertilized or unfertilized.

- Brood: The brood are from eggs, and do not undertake work at this stage. The broods born in fertilized eggs will become worker bees. The broods born in unfertilized eggs will become drone bees.

- Worker bee: The worker bees are developed from the brood through the pupation, and all works in the hive are basically completed by the worker bees. Worker bees can be further classified as hive bees and foragers. Hive bees work in the hive, whose works are taking care of broods or maintaining the hive. Foragers work outside the hive to collect honey and pollen and provide food for the hive. Foragers die after a period of work.

- Drone bee: Drone bees are born in unfertilized eggs. The drone does not participate in collecting food or maintaining hive, and is not responsible for feeding young bees. Its only responsibility is to mate with the queen bee, and die after mating.

There is only one queen bee in each colony. The drone bees pass away soon after fertilizing and do not contribute to collecting nectar and pollen. Thus, the queen bee and drone bees are not considered in this population dynamic model. There are 3 types of bees, brood, hive bee and forager, that are considered in this model. The number of brood, The rates of change of these types is discussed in the following parts. The simplified relations are shown in the following figure.

Previous studies have discussed a variety of factors affecting honeybee populations, including light, temperature, precipitation, bee activity, types of pollen foraging, infectious diseases, and vegetation cover of the habitat. Sunlight, temperature, and precipitation are highly season-related and these factors are coupled with other seasonal factors, such as bloom cycles of plants. (Burrill and Dietz [1981]) Based on past studies of the effect of seasons on honeybee populations, a mixed season function representing the overall effect of light, temperature, precipitation, and plant growth cycles can be introduced in the analysis. (Fukuda and SEKIGUCHI [1966])

Since previous studies show that bees foraging for different pollen species demonstrate similar death rates and bees tend to mix different kinds of pollen in real-world foraging, food can be utilized as a combined factor effecting the development of bee colonies. (Di Pasquale et al. [2013]) For infectious diseases, because they can lead to rapid colony extinction and the trend is often irreversible in the same hive after occurrence, it is unneccessary to generalize their effects in the model for a single hive in a natural environment. (Yuan et al. [2021])

The vegetation conditions of habitats mainly affect honeybees' food sources and it can be reflected by adjusting the parameters regarding food. (Ballare et al. [2019]) In summary, two factors, season and food, were introduced in the model building process to analyze honeybee colony population dynamics.

4.3 Transformation between the types of bees

As mentioned above, bees are all incubated from the eggs laid by the queen bee and the survival rate of the brood is related to the food sufficiency and the rearing ability of hive bee. The surviving brood transforms into hive bee after a specific period of growing and pupation, the death of the hive bee mainly occurred in the forager stage. Forager's lifespan is mainly affected by work intensity. In spring and summer, forager's work intensity is greater, so their life is shorter than at other seasons. Therefore, the following differential equations are modeled to describe the changes in the number of bees in the hive.

The rate of change of brood number (n_B) can be expressed as follow.

$$\frac{dn_B}{dt} = s(t) \cdot r_L \cdot r_S(n_H, w_f) - \phi n_B \tag{1}$$

In this equation, r_L represents the laying rate of the queen bee. $s(t)$ is the season factor, which will be discussed in the following parts. $r_S(n_H, w_f)$ represents the survival rate of eggs which is related to the number of hive bees (n_H) and food (w_f). The $r_S(n_H, n_F)$ is modeled in the following form. The $\phi n_B(t, \tau)$ are the number of broods who enter pupation at the time of t,

$$r_S(n_H, n_F) = \frac{w_f^2}{w_f^2 + \beta^2} \frac{n_H}{n_H + \nu} \tag{2}$$

The β and ν are the parameters to adjust the slope of this function of survival rate corresponding to food (w_f) and number of hive bees (n_H).

After several days (τ_p), the broods become hive bees. The rate of change of hive bees (n_H) is shown in the following equation.

$$\frac{dn_H}{dt} = \phi n_B(t, \tau_p) - n_H \cdot r_T(n_H, n_F, w_f) \tag{3}$$

The $\phi n_B(t, \tau_p)$ are the broods who entered pupation at $t - \tau_p$, the n_H is the current number of hive bees. The $r_T(n_H, n_F, w_f)$ is the function of transfer rate from hive bees to forager bees, which is related to the current number of hive bees (n_H), the number of forager bees (n_F), and food (w_f).

The transfer function $r_T(n_H, n_F, w_f)$ is listed as follow.

$$r_T(n_H, n_F, w_f) = r_{T,\min} + r_{T,\max}\frac{\beta^2}{\beta^2 + w_f^2} - r_{T,re}\frac{n_F}{n_F + n_H} \tag{4}$$

The $r_{T,\min}$, $r_{T,\max}$ and $r_{T,re}$ are 3 basic rate of transfer.

All forager bees are from hive bees. The rate of change of forager bees (n_F) is shown in the following equation.

$$\frac{dn_F}{dt} = n_H \cdot r_T(n_H, n_F, w_f) - s(t) \cdot m_0 \cdot n_F \tag{5}$$

In this equation, the $n_H \cdot r_T(n_H, n_F, w_f)$ represents the forager bees transferred from hive bees. The m_0 represents the maximum death rate of forager bees. The $s(t)$ is the season factor, which is a function of time. (Schmickl & Crailsheim, 2007)

The $s(t)$ is expressed as follow.

$$s(t) = \max(1 - \frac{1}{1 + \alpha_1 e^{-2t/\alpha_2}}, \frac{1}{1 + \alpha_3 e^{-2(t-\alpha_4)/\alpha_5}}) \tag{6}$$

In this equation, the $s(t)$ is between 0 and 1. x_1 to x_5 are the parameters to adjust the shape of season factor function.

The change rate of food is shown in the following formula.

$$\frac{dw_f}{dt} = \omega_c \cdot n_F - \omega_W(n_F + n_H) - \omega_B \cdot n_B \tag{7}$$

In this formula, the ω_c is the rate of food collection by forager bees. ω_W is the rate of food consumption by worker bees, including hive bees and forager bees. The ω_B is the rate of food consumption by broods.

4.4 Model Solution

4.4.1 Parameter Estimation

To solve the model, $r_L = 2000$ was defined as the basic egg production rate. Past practices have shown that the maximum egg production rate of honeybees can range from 1500 to 3000 eggs per day. In this case, a basic rate of 2000 eggs per day was chosen (Cramp, 2008). The required number of worker bees for the hive varies from season to season due to the different workload; this variation in the required worker bee population is also reflected in the egg production rate and shows clear seasonal period. Therefore, the basic rate of egg production as the maximum rate was defined to be reached in spring, and then decreased from summer and reached the minimum in winter. This seasonal variation was reflected by the seasonal coefficient $s(t)$.

We defined $v = 5000$ since we assumed that half of the brood would die due to overpopulation and the worker bees' overload when there were 5000 broods.

$m_0 = 0.02$ was defined since previous studies have shown that the daily death rate of worker bees with different pollen feeding conditions is close to 2

$r_{T,\min} = 0.25$ $r_{T,re} = 0.75$ $r_{T,\max} = 0.25$. $r_{T,\min} = 0.25$ controlled hive bees takes at least 4 days to convert into a forager bee. $r_{T,re} = 0.75$ indicated that if the number of foragers was greater than 1/3 of the total number of worker bees and food was abundant, foragers would return to the hive to work as hive bees; $r_{T,\max} = 0.25$ meant that when food was empty, double the number of hive bees would leave the hive to collect food as foragers.

$\phi = 1/9$ meant that brood needed 9 days to enter pupation.

$\tau = 12$ meant that pupation needed 12 days.

$\omega_c = 0.1$ was because forager can collect 0.1g of food per day on average; $\omega_B = 0.018$ was because each brood consumes 0.018g of food per day; $\omega_W = 0.007$ meant that each worker bee consumed 0.007g of food per day; $b = 500$ was because of the hive capacity limit.

α_1 to α_5 were defined as 385 25 36 125 600. The season factors in 365 days are shown in the Figure 1, where the horizontal axis is the number of day and the vertical axis is the $s(t)$.

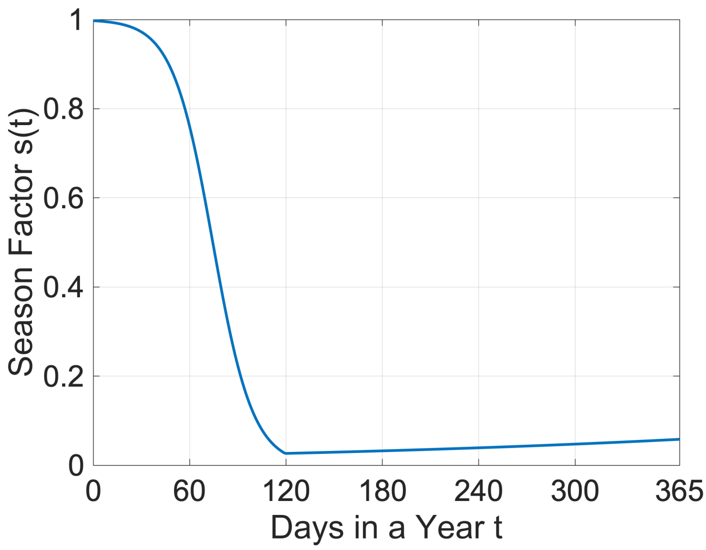

Figure 1: The season factor $s(t)$ vs. no. of the days in a year(t)

4.4.2 Algorithm

The adaptive Euler method was utilized to solve the differential equations. The steps are as follows:

- Step 1: Set the time t, step length t_s, and end time t_{end}. Input previously set the initial parameters and initial value n_F honeybees populations $n_F(t)$, $n_H(t)$, $n_B(t)$, set the initial food 0.05 ($w_f(t) = 0.05$).
- Step 2: Calculate dn_F/dt, dn_H/dt, dn_B/dt, dw_f/dt;
- Step 3: Calculate the value of

$$n_F(t+t_s) = n_F(t) + \frac{t_s}{2}(n_F(t)\frac{dn_F}{dt} + (n_F(t) + t_s n_F(t)\frac{dn_F}{dt})\frac{dn_F}{dt}),$$
$$n_H(t+t_s) = n_H(t) + \frac{t_s}{2}(n_H(t)\frac{dn_H}{dt} + (n_H(t) + t_s n_H(t)\frac{dn_H}{dt})\frac{dn_H}{dt}),$$
$$n_B(t+t_s) = n_B(t) + \frac{t_s}{2}(n_B(t)\frac{dn_B}{dt} + (n_B(t) + t_s n_B(t)\frac{dn_B}{dt})\frac{dn_B}{dt}),$$
$$w_f(t+t_s) = w_f(t) + \frac{t_s}{2}(w_f(t)\frac{dw_f}{dt} + (w_f(t) + t_s w_f(t)\frac{dw_f}{dt})\frac{dw_f}{dt}).$$

- Step 4: $t = t + t_s$
- Step 5: returns Step 2 until $t = t_{end}$

4.4.3 Results and Discussions

We set the initial values of the ODE model according to the initial values of real-world experimental measurements (Harbo, 1986), and simulated the honeybee colony's population changes with the step length of 0.01 day. The total duration of the simulation was 40 days. We summarized the number of honeybees taking up different jobs in the colony on each simulated day and compared the simulated values with the observed values. The comparison is shown in the table 2 and figure 2 below. In figure 2, the blue circles are the observation data by Harbo in 1986, and the orange circle are the simulation by our model.

Table 2: The comparison bewteen our simulation and experimental data from Harbo in 1986

Initial adult bee		Final brood cells		Brood cells per A		Percentage survival	
observed	model	observed	model	observed	model	observed	model
2316	2400	4325	4005	2.41	2.13	56	56
4515	4500	11162	9154	3.04	2.55	64	60
9352	9000	16275	17542	2.21	2.40	58	62
17099	18000	22875	26493	1.67	1.79	63	64
37061	36000	27878	33599	0.97	1.13	55	65

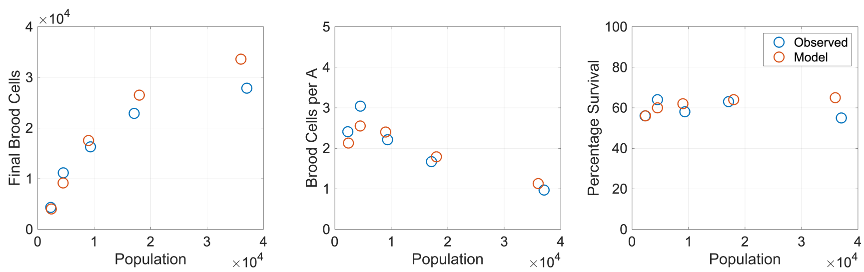

Figure 2: The scatter of observations and simulations

In Figure 2-a, the horizontal axis represents the initial number of adult honeybees while the vertical axis represents the number of broods after 22 days. In Figure 2-b, the vertical axis represents the ratio of the number of brood to the number of adult bees after 22 days. In Figure 2-c, the vertical axis is the survival rate of bees after 22 days. Figures above show that the trend of simulation and the trend of experimental results are consistent and numerically similar.

Furthermore, we measured the proportion of error in the simulated results to the experimental data. The error between the total number of simulated broods and the total number of broods from the actual data ranged from 7.3% to 20.5% of the actual data, with an average error of 13.9%; the error in the predicted proportion of broods and adult bees compared to the actual data ranged from 7.1% to 16.5%, with an average error of 12.0%; the error in the survival rate of bees after 22 days compared to the actual data ranged from 0% to 18%, with an average error of 6.5%. This comparison validates the accuracy of the result of our simulation model.

The largest error occurred in the simulation of brood numbers, which may be due to the fact that queen bee's spawning rates were affected by random variables such as weather in the actual experiment while our model utilized a smooth seasonal function to evaluate the effects of environmental variables. The bias generated by the random fluctuations in spawning rate is directly reflected in the brood number. Nevertheless, the errors above are within an acceptable range and the model still reflects the magnitude and trend of the honeybee population in the hive generally. Therefore, the model can be utilized to simulate the changes in honeybee population in a single hive.

5 Sensitivity Analysis of Simulation Model

5.1 Critical Factors Analyzed in the Model

In our simulation model, the simulation results will be affected by multiple parameter. The main factors are:

- The number of worker bees in initial state ($n_F + n_H$)
- The egg laying rate (or spawning rate) of queen bee (r_L)
- The food collection rate (ω_c)
- The death rate (m_0)

Since the above factors will have direct or indirect effects on food collection and consumption, the proportion of bee colony types of work, quantity change rate and other variables, we will analyze the single and combined effects of the above factors. Although time for entering population (ϕ) and time for population (τ) will also affect the population composition and change rate of honeybees, it is mainly related to bee species. For the same species of bees, the two indicators do not have a large range of change in different environments, so we will not discuss them here.

5.2 Range of the Critical Parameters in Sensitivity Analysis

The number of worker bees in the initial state stays between 2000 and 40000 based on previous real world research data. To compare with previous studies, we selected $n_F + n_H$ between 2400 and 36,000 at the beginning of the simulation.

As mentioned in the previous chapter, the maximum egg laying rate of queen bee under artificial culture is generally between 1500 and 3000 eggs per day. Therefore, we choose r_L between 1500 and 3000.

The food collection rate (ω_c) depends on honeybees' mobility, local vegetation density and species, local bee density, bee collection preference, etc. The average collection rate in different environments is 0.1 gram per day. To study the influence of different acquisition rates, we choose 0.08 to 0.12 as the research interval.

The death rate of honeybee (m_0) is affected by many factors, such as activity, food type, geographical location, species competition, infectious diseases, etc. When the food is sufficient and the survival pressure is low, its value is around 0.02. When there are predators, bee colony competition or infectious diseases, the death rate can be as high as 50%. Therefore, we selected the death rate of 0 to 0.7 to analyze the impact of bee mortality on the population.

5.3 Results and Discussion

5.3.1 Waterlevel Forecast based on ARIMA model

In the figure 1, it is shown that within the interval of 22 days, the larger the initial number of worker bees is, the more broods will be after 22 days. A possible reason is that a higher initial number of worker bees provides more food and could takes care of more broods.

When the initial number of worker bees increases, the ratio of the number of broods to the number of worker bees increases initially and then decreases. The reason for this initial increase may be that the initial number of worker bees is too limited. The limited number of worker bees results in low survival rate at a given spawning rate and thus a smaller number of worker bees after 22 days. When the number of worker bees increases further, the ratio of brood to adult bee decreases. It is probably because the initial brood survival rate increases, which lead to more worker bees after pupation. Thus, the ratio of brood to adult bee decreases when the spawning rate is constant and the survival rate of reaches a high level.

The sensitivity analysis for different queen bees' spawning rates is shown in Figure 3. The horizontal axis of the figure is the spawning rate and the vertical axis is the number of bees. The figure shows that higher spawning rate leads to higher growth rate of bee population in the initial stage and higher total number of bees in the steady state. The reason is that a constant high spawning rate produces more broods and consequently more worker bees.

The change in broods with laying rate in the steady state has a greater slope than that of worker bees, suggesting that the growth of spawning rate has the greatest effect on brood numbers, while worker bee numbers are less affected by egg laying rate due to the limitations of job conversion, food distribution and death.

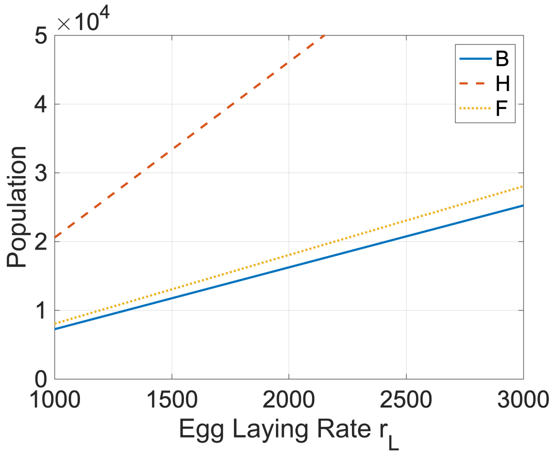

Figure 3: Bees with different egg laying rate

The effect of different death rates on steady state is shown in Figure 4. The horizontal axis of the figure is the death rate and the vertical axis is the number of bees. When the death rate increases, broods, hive bees and foragers in the steady state decreases. When the death rate is lower than 0.1, the number of foragers is higher than broods. When the death rate is higher than 0.1, the number of broods become higher than the number of foragers. The reason is the rapid loss of foragers caused by high death rate. When the death rate is higher than 0.4, the number of hive bees and broods decreases rapidly as the death rate increases because the number of foragers is no longer sufficient to collect enough food to keep the number of hive bees and broods at a high level. When the death rate is greater than 0.55, this bee colony collapses.

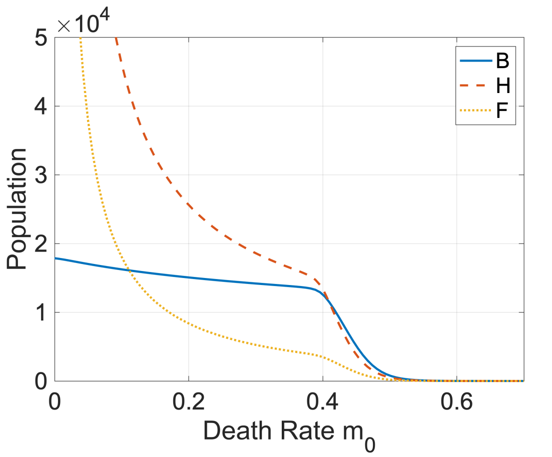

Figure 4: Bees with different death rate

The effects of different food collection rates on the steady state is shown in Figure 5. The horizontal axis of the figure represents the food collection rate and the vertical axis is the number of bees. When the food collection rate is lower than 0.092, it is difficult to maintain the bee population in the hive; when the rate is higher than 0.092, broods, hive bees and foragers starts to increase rapidly.

When food collection rate increases, the slope of increasing hive bees and broods are similar, while foragers increases more slowly. With increasing food collection rate, the growth of honeybee population gradually slows down, showing the diminishing marginal utility of the increase in food collection rate. After the food collection rate reaches 0.102, further increases in food collection rate could no longer drive the growth of the honeybee population.

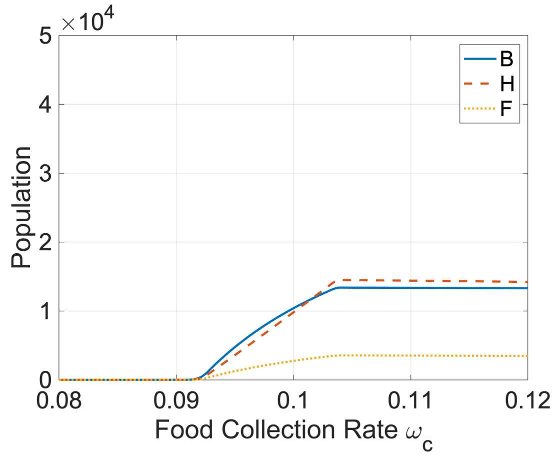

Figure 5: Bees with different food collection rate

The joint influence of foraging and mortality are shown in Figure 6. The horizontal axis of the figure represents death rate and the vertical axis represents the food collection rate. The more the color is skewed towards yellow represents the higher number of bees of that worker species.

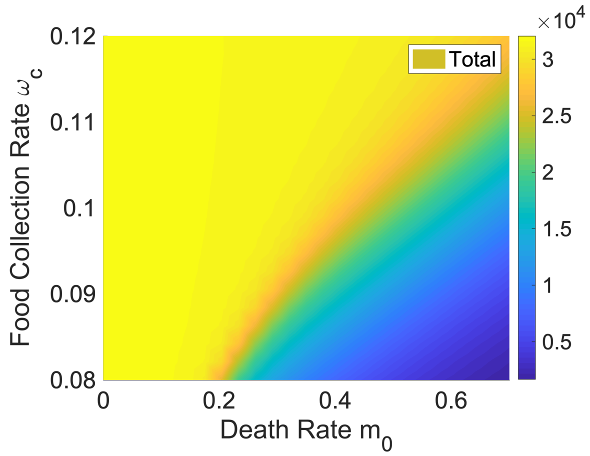

Figure 6: Joint influence of food collection rate and death rate on total number of bees

Figure 7-a shows the number of broods in the steady state for different food collection rates and death rates. Figure 7-b shows the number of hive bees in the steady state. Figure 7-c demonstrates the number of forager bees in the steady state. For the 7-a and 7-c, more the color is skewed towards blue represents the higher number of broods or forager bees. For the 7-b, more the color is skewed towards yellow represents the higher number of hive bees.

The sensitivity analysis of worker bees and broods suggests that if the food collection rate is also higher, it is possible to maintain a larger bee population with a higher death rate. In addition, the boundary between the yellow and blue areas reveals the dividing line where the rapidest change occurs in the sustainable honeybee population at different death rates. For example, at a food collection rate of 0.08, if the death rate is greater than 0.2, the colony size will decrease rapidly. The boundary also shows that the critical death rate will increase when there is a higher food collection rate.

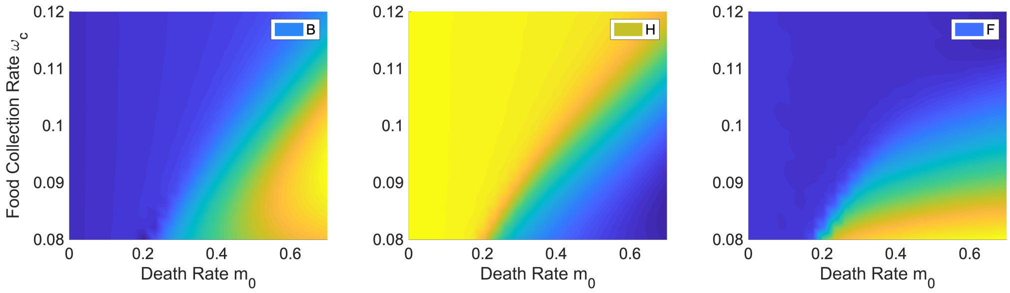

Figure 7: Joint influence of food collection rate and death rate on broods, hive bees and forager bees.

6 Distribution of Hive Among Crop Parcel

6.1 Key Issues in Arranging Bee Hives

We are required to establish a model predicting the number of hives needed to pollinate 81000 square meters of crops and benefit from it. For this issue, the cost and benefit need to be discussed separately. In order to benefit from it, we should find a scheme in which the benefits of bee pollination outweigh the costs.

The cost of utilizing bees to pollinate crops can be divided into two parts:

- The cost of installing hives
- The labor cost of maintaining bee colony manually

The benefits of utilizing bees to pollinate crops can be divided into two parts:

- Honey benefit
- Benefit of increasing crops yield

Based on the above discussions, in 6.2, we will discuss the function of quantifying the cost and benefit of using bees, and obtain the profitable interval of the number of hives under different conditions.

6.2 Benefit and Cost Model Optimizing Hives for Crops

In this part, we start to establish its cost and benefit model from a single hive, and then extend it to the scenario of multiple hives to obtain the cost and benefit model for multiple hives, as well as establishing the net profit model.

6.2.1 Benefit and Cost Analysis of Single Hive

We define the initial cost of a single hive as c_{bc}, in million USD. The initial cost includes the cost of building beehives, purchasing or attracting bee seeds, which is necessary for pollination by bees. c_m is the cost required for manual operation when having a single beehive, and the unit is million USD. In order to maintain the bee colony in the hive and increase the income from the hive, manual operation includes monitoring the state of bee colony, preventing the outbreak of infectious diseases, driving away other bees, collecting honey, etc. Considering the above two parts of costs, we obtain the cost of a single hive as follows.

$$C = c_{bc} + c_m \tag{8}$$

As described in 6.1, the benefit of a single hive can be divided into two parts: honey benefit and pollination benefit.

We define b_h as the benefit from harvesting honey from a single hive. At the end of the flowering season, the honey reserves in the honeycomb will generally reach the maximum, so that it can supply the consumption in rest of the year. In order to maintain the sustainable development of bee colonies, bee farmers will not collect and sell all honey, but only collect honey of fixed quality or proportion. Therefore, considering that the manually set beehives generally have the same storage volume and fixed collection mode, we assume b_h as a same value, In the case of a single hive, all honey benefit B_h equals to b_h.

$$B_h = b_h \tag{9}$$

The benefit of honeybee pollination are reflected in the incremental crop yield. Previous studies have shown that crop yield increment is mainly related to the relative density of bees and crops. (Morris et al. [2010])

The area of crop field is A_{cf}. Assume that the crop field is square. The distance between the bee colony and the crop field is $\sqrt{d_1^2 + d_2^2}$. The number of forager in one bee colony at the tth day is $n_F(t)$, which can be generated by our model proposed in previous sections. Assume that the bees follow normal distribution around the bee colony. At the area of interest A_{cf}, the density of the bees per square meters is $\rho_F(t, d_1, d_2, A_{cf})$.

$$f_N(x_1, x_2, t) = \frac{n_F(t)}{(2\pi)^{\frac{p}{2}}|\boldsymbol{\Sigma}|^{-\frac{1}{2}}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})} \tag{10}$$

$$\rho_F(t, d_1, d_2, A_{cf}) = \frac{\int_{d_2-\sqrt{A_{cf}}/2}^{d_2+\sqrt{A_{cf}}/2} \int_{d_1-\sqrt{A_{cf}}/2}^{d_1+\sqrt{A_{cf}}/2} f_N(x_1, x_2) dx_1 dx_2}{A_{cf}} \tag{11}$$

Since the area of a single crop field is often much smaller than the active area of bees, we can approximate that $d_1 = d_2 = 0$ when the hives are arranged in the crop field. At this point:

$$\rho_F(t, A_{cf}) = \frac{\int_{-\sqrt{A_{cf}}/2}^{\sqrt{A_{cf}}/2} \int_{-\sqrt{A_{cf}}/2}^{\sqrt{A_{cf}}/2} f_N(x_1, x_2) dx_1 dx_2}{A_{cf}} \tag{12}$$

Assume bloom season is presented in normal distribution, so we utilized a model of normal distribution on time to represent the density of pollenable flowers per unit area. Within a square meter, the number of crops' flowers available for pollination at the t th day is $\rho_c(t)$.

$$\rho_c(t) = \rho_{c0} e^{-\frac{(t-t_0)^2}{2\sigma^2}} \tag{13}$$

Where ρ_{c0} is the highest number of crops' flowers per square meters. At the t_0th day, the number of crops' flowers per square meters reaches the maximum. If we define the duration of bloom season as τ_{fs} and the season starts from $t = 0$, then $t_0 = \frac{\tau_{fs}}{2}$.

In the crop field A_{cf}, the ratio between foraging bees and crops is defined as $\rho_F(t, A_{cf})/\rho_c(t)$. The higher value of $\rho_F(t, A_{cf})/\rho_c(t)$ indicates the higher intensity of pollination. Because the forager bees collects food and participate in pollination everyday during the bloom season, the daily ratio of active bees over flowers can be cumulated. We define the cumulative density of the bees at t day as $R_{Fc}(t)$.

$$R_{Fc}(t) = \sum_{\tau=1}^{t} \frac{\rho_F(\tau, A_{fc})}{\rho_c(\tau)} \tag{14}$$

When the $R_{Fc}(t)$ increases, the cumulated pollination by bees increases, and the benefit increases. However, the increasing benefit will reach to saturation when the d_F/d_c is too high. (Morris et cl., 2010) During the whole bloom season τ_{fs} The hyperbolic tangent function is utilized to illustrate this relation, where B_c is the increased benefit because of bee pollination and b_{c0} is the maximum benefit that can be achieved when the number of bees is much larger than the number of plants.

$$B_c = b_{c0} \cdot \frac{e^{R_{Fc}(\tau_{fs})} - e^{-R_{Fc}(\tau_{fs})}}{e^{R_{Fc}(\tau_{fs})} + e^{-R_{Fc}(\tau_{fs})}} \tag{15}$$

Thus, the overall benefit (B) is

$$B = B_h + B_c \tag{16}$$

6.2.2 Benefit and Cost Analysis of Multiple Hives

For the case of setting multiple hives, we need to consider a generalization to the case of a single hive and make some modifications to the previous cost-benefit functions.

When there are multiple hives, we define the number of hives as n_{bc}.

When arranging multiple hives, we still need to spend c_{bc} for each hive. For the manual operation, the increase of marginal cost on increasing number of hives is non-linear because a single person can take care of multiple hives at the same time. Based on this, we use step function to describe the increasing cost of the manual operation.

$$C = n_{bc} \cdot c_{bc} + \lceil n_{bc}/n_m \rceil c_m \tag{17}$$

n_m is the maximum number of hives that a single person can take care of.

As described in 6.2.1, we expect to get the benefit

$$b_h$$

from a single hive. This is also true when there are multiple hives. Therefore, the benefit in the case of multiple hives is given by

$$B_h = n_{bc} \cdot b_h \tag{18}$$

Similarly, since the increasing benefit will reach to saturation as the relative density of bees to the plant increases, we continue to use the hyperbolic tangent function to describe the crop yield benefit from bee pollination.

$$B_c = b_{c0} \cdot \frac{e^{R_{Fc}(\tau_{fs})} - e^{-R_{Fc}(\tau_{fs})}}{e^{R_{Fc}(\tau_{fs})} + e^{-R_{Fc}(\tau_{fs})}} \tag{19}$$

In the equation, the R_{Fc} is a function of n_{bc} because of the overlapping between the cover areas of bee colony. When there are multiple hives with bee activity covering the same area, the density of bees in this area is the superposition of multiple colonies:

$$R_{Fc}(\tau_{fs}) = n_{bc} \cdot \sum_{t=1}^{\tau_{fs}} \frac{\rho_{F,n}(t, A_{fc})}{\rho_c(t)} \tag{20}$$

As such, the function of net profit, P_N, is

$$P_N = B - C \tag{21}$$

As a result, we get the net income that can describe the number of hives arranged in this area. An optimal number of hives N can be found by maximizing the net income.

$$N = \arg\max P_N = \arg\max(B - C). \tag{22}$$

6.2.3 Constraint Conditions

For the function presented in 6.2.2, we have following constraints.

First, the number of beehives should not be less than 0.

$$n_{bc} \geq 0 \tag{23}$$

Also, the distance from the hives to the crop field should not be less than 0.

$$d_F \geq 0 \tag{24}$$

Moreover, we only discuss the bloom season of agricultural crops.

$$0 < t \leq \tau_{fs} \tag{25}$$

6.3 Model solution

6.3.1 Algorithms

Considering the actual value range of the decision variable N in objective function is a positive integer, we use the fixed step size search algorithm to find the optimal number of hives. The steps are shown as follows:

6.3.2 Results and Discussions

The crop field of 81,000 square meters given in the question is much smaller than the activity range of bees with a radius of 6km, so we can assume that $d_1 = d_2 = 0$ and the hives provide the same density of bee population when they are at different spots within the field. The standard deviation in the two-dimensional distribution density function is $\sigma = 6$km. We assume the benefit of a single hive to \$0.02 million during the bloom season and the equipment cost of a single hive to \$0.05 million.

Assume a single beekeeper can be fully responsible for taking care of 4 hives at a cost of \$0.05M per beekeeper hired. Since the crop type is not given, we set the length of the bloom season of the crops grown in the field to 45 days, and the bloom season is from April 16 to May 30. The initial parameters of beehives are the same as in Section 4. We obtain the relationship between benefit and net profit with respect to the number of beehives by the algorithm described above. The variation of benefit, cost and net profit with the number of hives for a given flowering time and length is shown in Figure 8.

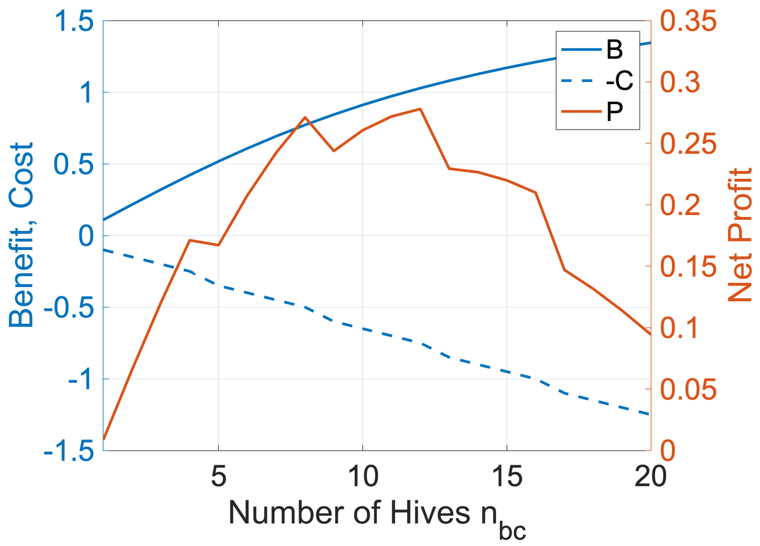

Figure 8: The benefit, cost and net profit with different number of hives

The horizontal axis is the number of hives, the left vertical axis is the benefit, and the right vertical axis is the net profit. In the figure, the benefit increases as the number of hives increases, but the marginal utility from each additional hive is decreasing. When the number of hives increases, the overall trend of net profit also increases and then decreases. However, whenever new beekeepers are hired, net profit decreases because of the rising cost, so the curve shows a jagged fluctuation locally. From the figure, the maximum net profit is reached at $n_{bc} = 8$, and the maximum net profit is \$0.275Million.

To further discuss the effects of using bees to pollinate crops at different times, sensitivity analysis was conducted for different starting times of flowering. In Figure 9, the horizontal axis is the date t_0, which is the midpoint of the bloom season, and the vertical axis is the number of hives laid. In the figure, a more yellow color represents a higher net profit, and a bluer color represents a lower net profit.

As shown in the figure, when t_0 is too large or too small, that is, when crops bloom too early or too late, the optimal number of hives is lower, and the net profit is also lower than when the midpoint of flowering is in May. The reason is that the number of bees is insufficient when flowering too early or too late, resulting in low net profits.

Since the cost of a single hive is fixed at different times, the maximum point of net profit corresponds to a smaller number of hives to save costs and improve the net profit under this scenario. This phenomenon also indicates that when using bees for pollination, we should select the appropriate bee species corresponding to the bloom season to match the occurance of maximum number of forager bees. In addition, this phenomenon also supports the nature of honeybees and plants to co-evolve in mutually-beneficial symbiosis.

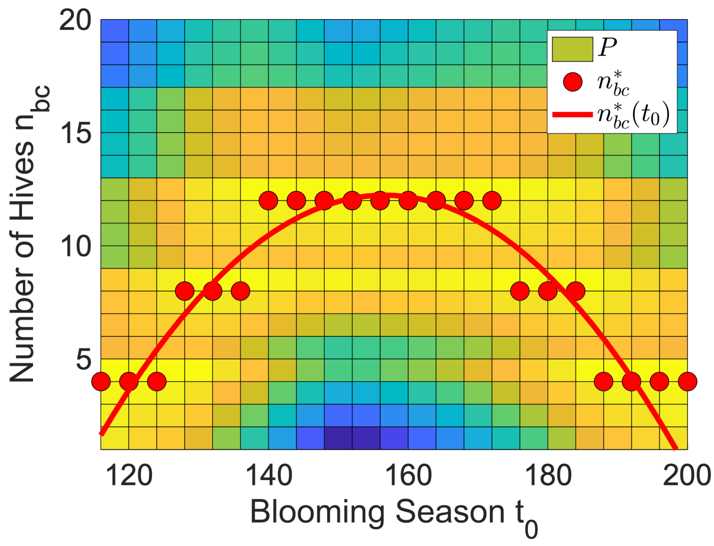

Figure 9: The heatmap of net profit with different blooming season and number of moderate-cost hives

To discuss the effect of hive costs on the optimal scheme, we further discuss the effect of different hive costs on the optimal net profit.

Figures 10, 9, and 11 respectively show the optimal net profit of pollination of crops by bees at different times when the hive cost is high, moderate, and low.

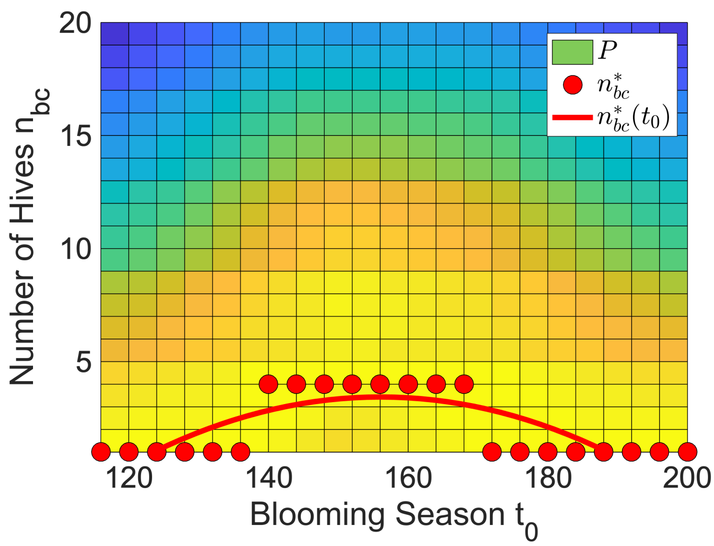

Figure 10: The heatmap of net profit with low hive cost

As shown in the figure 10, when the cost of honeybee hive equipment is too high, the optimal plan needs to reduce the number of hives in all bloom seasons.

When the maximum number of bees does not match the midpoint of the bloom season, the model suggests that the number of hives should be reduced to save costs because of high hive cost.

When the equipment cost of honeybee hives was moderate, the optimal scheme is also sensitive to the matching degree of the honeybee maximum and bloom season, and the optimal number of honeybee higher than the condition when hive cost is higher.

When the cost installing honeybee hive is low, the optimal number of hives is higher.

When the maximum number of bees does not match the midpoint of the bloom season, we can choose to increase the number of beehives to obtain higher net profits and improve net profit during optimization since the cost of hive is low.

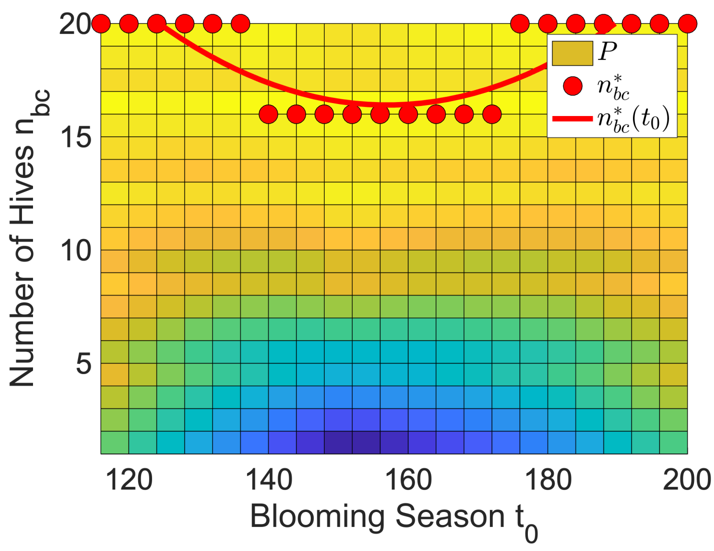

Figure 11: The heatmap of net profit with high hive cost

7 Model Evaluation

7.1 Advantages

- First, we developed our model for simulation from real-world trend of bee population and various related factors extracted from previous studies and experiments; also, we validated and thus confirmed our simulation results by comparing them with the real-world statistics. It ensures that our research is factual and applicable.
- We simulated the population dynamics of bees by ODE model, which allowed us to control various parameters affecting the bee population precisely and analyze each of them.
- We conducted sensitivity analysis investigating the critical factors affecting the bee population. It allowed us to identify the most important contributors to bee population changes.
- We proposed a reasonable optimization model for distributing bee hives in crop parcels. It can be applicable in various bees, plants and area of interest because of its adjustable parameters.

7.2 Possible Improvements

- The real-world data in the research are still not perfectly adequate and since it was discrete in time due to the limits of experimental conditions. For improvement, we can further improve our data by either collecting more existing studies or conducting an experiment to ensure the consistency of data.

- Regarding the accuracy of our simulation results, there are still relatively larger errors like the simulation of brood numbers in 4.3.3. They're usually due to inadequacy in controlling the variables. We may further consider more variables and their effects in every stage of bee population dynamics for improvement.

- The two factors that we considered most important are mostly based on the existing research, so the parameters we controlled in the ODE model might not be enough either. For improvement, if we have extended time, the more in-depth investigation in either literature reviews or real-world experiments could be conducted to involve more possible factors.

References

Kimberly M Ballare, John L Neff, Rebecca Ruppel, and Shalene Jha. Multi-scalar drivers of biodiversity: Local management mediates wild bee community response to regional urbanization. *Ecological Applications*, 29(3):e01869, 2019.

Robert M Burrill and Alfred Dietz. The response of honey bees to variations in solar radiation and temperature. *Apidologie*, 12(4):319–328, 1981.

Garance Di Pasquale, Marion Salignon, Yves Le Conte, Luc P Belzunces, Axel Decourtye, André Kretzschmar, Séverine Suchail, Jean-Luc Brunet, and Cédric Alaux. Influence of pollen nutrition on honey bee health: do pollen quality and diversity matter? *PloS one*, 8 (8):e72016, 2013.

Hiromi Fukuda and Kiichi SEKIGUCHI. Seasonal change of the honeybee worker longevity in sapporo, north japan, with notes on some factors affecting the life-span. *Japanese Journal of Ecology*, 16(5):206–212, 1966.

William F Morris, Diego P Vázquez, and Natacha P Chacoff. Benefit and cost curves for typical pollination mutualisms. *Ecology*, 91(5):1276–1285, 2010.

Chunying Yuan, Xuejian Jiang, Man Liu, Sa Yang, Shuai Deng, and Chunsheng Hou. An investigation of honey bee viruses prevalence in managed honey bees (and) undergone colony decline. *The Open Microbiology Journal*, 15(1), 2021.

Appendices

Appendix A Tools and software

Article written and generated via LaTeX, free distribution.
Graph generated via MATLAB R2020a and Xmind Zen, free trial license.
Calculation using MATLAB R2020a.

Appendix B The MATLAB Code

```
clc
clear
close all
cc.L    = 2000;
cc.phi  = 1/9;
cc.v    = 5000;
cc.sig  = 0.75;
cc.amin = 0.25;
cc.amax = 0.25;
cc.b    = 500;
cc.c    = 0.1;
cc.yA   = 0.007;
cc.yB   = 0.018;
lag = 12;
cc.m    = 0.1;
sol = dde23(@(~,X,Z)odefun(X,Z,cc),lag,[0;16000;8000;0],[0,500]);
p = plot(sol.x.',sol.y(1:3,:).','LineWidth',2);
[p.LineStyle] = deal('-', '--', ':');
ylim([0,50000]);
xlim([0,500]);
grid on
legend B H F
set(gca,'FontSize',16,'FontName','Arial');
function cc = patch(cc,t)
   if t >= 50
      cc.m = 0.42;
   end
end
function dX = odefun(X,Z,cc)
    [L,phi,b,v,amin,amax,sig,yA,yB,c,m] = unpack(cc);
    B = X(1);
    H = X(2);
    F = X(3);
    f = X(4);
    zB = Z(1);
    S = f^2/(b^2+f^2)*H/(H+v);
    R = amin+amax*b^2/(b^2+f^2)-sig*(F/(F+H));
    dB = L*S-phi*B;
    dH = phi*zB-H*R;
    dF = H*R-m*F;
    df = c*F-yA*(F+H)-yB*B;
    dX = [dB;dH;dF;df];
end
function [L,phi,b,v,amin,amax,sig,yA,yB,c,m] = unpack(cc)
   L    = cc.L;
   phi  = cc.phi;
   b    = cc.b;
   v    = cc.v;
   amin = cc.amin;
   amax = cc.amax;
   sig  = cc.sig;
   yA   = cc.yA;
   yB   = cc.yB;
   c    = cc.c;
   m    = cc.m;
end
```