Statistics of Sharpe Ratio

Step 1: IID Setup & Objects

Data: $(R_t)_{t=1}^T$ are IID, $\mathbb{E}|R_t|^4 < \infty$.

Parameter: $\theta_0 \in \Theta \subset \mathbb{R}^p$. For Sharpe we take $\theta = (\mu, \sigma^2)^\top$.

Moment conditions (IID):

$$\mathbb{E}\big[H_t(\theta_0)\big] = \mathbf{0} \in \mathbb{R}^p, \qquad H_t : \Theta \to \mathbb{R}^p.$$

Estimator (exactly identified):

$$\overline{H}_T(\hat{\theta}) = \frac{1}{T}\sum_{t=1}^{T} H_t(\hat{\theta}) = \mathbf{0}.$$

Functional of interest: $g : \Theta \to \mathbb{R}$ (differentiable at θ_0). For Sharpe:

$$g(\theta) = \frac{\mu - R_f}{\sigma}, \qquad \sigma = \sqrt{\sigma^2}.$$

Step 2: Goal and Delta/Taylor Expansion (IID context)

Goal: Distribution of $g(\hat{\theta}) - g(\theta_0)$.

First-order Taylor at θ_0:

$$g(\hat{\theta}) - g(\theta_0) = \nabla g(\theta_0)^\top (\hat{\theta} - \theta_0) + r_T, \qquad r_T = o_p(\|\hat{\theta} - \theta_0\|).$$

Multiply by $\sqrt{T}$:

$$\sqrt{T}\big(g(\hat{\theta}) - g(\theta_0)\big) = \nabla g(\theta_0)^\top \sqrt{T}(\hat{\theta} - \theta_0) + o_p(1).$$

So we need $\sqrt{T}(\hat{\theta} - \theta_0)$ under IID.

Step 3: IID GMM Linearization (No long-run terms)

Mean-value expansion for each component of $H_t(\cdot)$ at θ_0:

$$H_t(\hat{\theta}) = H_t(\theta_0) + \nabla_\theta H_t(\theta_0)\,(\hat{\theta} - \theta_0) + r_{t,T}, \quad \frac{1}{T}\sum_{t=1}^{T} r_{t,T} = o_p(\|\hat{\theta} - \theta_0\|).$$

Average and scale:

$$\sqrt{T}\Big(\overline{H}_T(\hat{\theta}) - \overline{H}_T(\theta_0)\Big) = \sqrt{T}\left(\frac{1}{T}\sum_{t=1}^{T} \nabla_\theta H_t(\theta_0)\right)(\hat{\theta} - \theta_0) + o_p(1).$$

Since $\overline{H}_T(\hat{\theta}) = \mathbf{0}$, define $\widehat{D}_T := \frac{1}{T}\sum_{t=1}^{T} \nabla_\theta H_t(\theta_0)$ to get

$$-\sqrt{T}\,\overline{H}_T(\theta_0) = \widehat{D}_T\,\sqrt{T}(\hat{\theta} - \theta_0) + o_p(1).$$

Step 4: IID LLN/CLT Limits and V_θ

By IID LLN and CLT:

$$\widehat{D}_T \xrightarrow{p} D_0 := \mathbb{E}\big[\nabla_\theta H_t(\theta_0)\big], \qquad \sqrt{T}\,\overline{H}_T(\theta_0) \;\Rightarrow\; \mathcal{N}(\mathbf{0},\; S_{\text{IID}}),$$

with the **IID covariance**

$$\boxed{S_{\text{IID}} := \mathrm{Var}\big(H_t(\theta_0)\big) = \mathbb{E}\big[H_t(\theta_0)H_t(\theta_0)^\top\big]}.$$

Slutsky then gives

$$\sqrt{T}(\hat{\theta} - \theta_0) \Rightarrow \mathcal{N}\big(\mathbf{0},\; V_\theta\big), \qquad \boxed{V_\theta = D_0^{-1}\, S_{\text{IID}}\, D_0^{-1\top}.}$$

Step 5: Delta Method Conclusion (IID)

Combine with the Taylor step:

$$\sqrt{T}\big(g(\hat{\theta}) - g(\theta_0)\big) \Rightarrow \mathcal{N}\big(0,\ V_g\big), \quad \boxed{V_g = \nabla g(\theta_0)^\top V_\theta \, \nabla g(\theta_0).}$$

Interpretation (IID): Asymptotic variance of $g(\hat{\theta})$ equals the *gradient* of g times the *IID variance of the moments mapped through* D_0^{-1}.

Step 6: Sharpe Ratio — Choose Moments H_t

Let $\theta = (\mu, \sigma^2)^\top$, with $\mu = \mathbb{E}[R_t]$, $\sigma^2 = \mathrm{Var}(R_t) > 0$.

Define

$$H_t(\theta) = \begin{bmatrix} H_{1t} \\ H_{2t} \end{bmatrix} := \begin{bmatrix} R_t - \mu \\ (R_t - \mu)^2 - \sigma^2 \end{bmatrix}.$$

Then $\mathbb{E}[H_t(\theta_0)] = \mathbf{0}$ by construction (IID).

Step 7: Compute D_0 (IID)

Jacobian:

$$\nabla_\theta H_t(\theta) = \begin{bmatrix} -1 & 0 \\ -2(R_t - \mu) & -1 \end{bmatrix}.$$

Take expectations at θ_0 (IID, $\mathbb{E}[R_t - \mu] = 0$):

$$\boxed{D_0 = \mathbb{E}[\nabla_\theta H_t(\theta_0)] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2.}$$

Hence, for Sharpe, $V_\theta = S_{\text{IID}}$.

Step 8: Compute $S_{\text{IID}} = \text{Var}(H_t(\theta_0))$ *(1/3)*

Write central moments $\mu_k := \mathbb{E}[(R_t - \mu)^k]$; note $\mu_2 = \sigma^2$.

Entry (1,1): $\text{Var}(H_{1t}) = \text{Var}(R_t - \mu) = \sigma^2 = \mu_2$.

So far:

$$S_{\text{IID}}(1,1) = \sigma^2.$$

Step 8: Compute S_{IID} *(2/3)* — Off-diagonal

Entry (1,2)=(2,1): $\mathrm{Cov}(H_{1t}, H_{2t})$

$$\mathrm{Cov}\big(R_t - \mu,\ (R_t - \mu)^2 - \sigma^2\big) = \mathbb{E}\big[(R_t - \mu)\big((R_t - \mu)^2 - \sigma^2\big)\big]$$

$$= \ \mathbb{E}[(R_t - \mu)^3] \ - \ \sigma^2\,\mathbb{E}[R_t - \mu] = \ \mu_3 \ - \ \sigma^2 \cdot 0 \ = \ \boxed{\mu_3}.$$

Thus:

$$S_{\text{IID}}(1,2) = S_{\text{IID}}(2,1) = \mu_3.$$

Step 8: Compute S_{IID} *(3/3)* — (2,2) Entry

Entry (2,2): $\text{Var}(H_{2t}) = \text{Var}\big((R_t-\mu)^2 - \sigma^2\big)$

$$= \mathbb{E}\big[((R_t-\mu)^2-\sigma^2)^2\big] = \mathbb{E}[(R_t-\mu)^4] \; - \; 2\sigma^2\,\mathbb{E}[(R_t-\mu)^2] \; + \; \sigma^4$$

$$= \mu_4 \; - \; 2\sigma^2\cdot\sigma^2 \; + \; \sigma^4 = \boxed{\mu_4 - \sigma^4}.$$

Collecting all entries (IID):

$$\boxed{S_{\text{IID}} = \text{Var}(H_t(\theta_0)) = \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{bmatrix}.}$$

Step 9: Gradient of g (Sharpe)

Sharpe functional:

$$g(\theta) = \frac{\mu - R_f}{\sigma}, \qquad \sigma = \sqrt{\sigma^2}.$$

Gradient w.r.t (μ, σ^2):

$$\nabla g(\theta) = \begin{bmatrix} \partial g/\partial \mu \\ \partial g/\partial \sigma^2 \end{bmatrix} = \begin{bmatrix} 1/\sigma \\ -(\mu - R_f)/(2\sigma^3) \end{bmatrix} = \begin{bmatrix} 1/\sigma \\ -SR/(2\sigma^2) \end{bmatrix}, \quad SR := \frac{\mu - R_f}{\sigma}.$$

Step 10: Asymptotic Variance of $\widehat{SR}$ (IID)

With $D_0 = -I_2$ we have $V_\theta = S_{\text{IID}}$. Hence, by delta method:

$$\mathrm{AVAR}(\widehat{SR}) = \nabla g(\theta_0)^\top S_{\text{IID}} \, \nabla g(\theta_0).$$

Let $a := 1/\sigma$, $b := -SR/(2\sigma^2)$. Using $S_{\text{IID}} = \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{bmatrix}$,

$$\mathrm{AVAR}(\widehat{SR}) = a^2\sigma^2 + 2ab\,\mu_3 + b^2(\mu_4 - \sigma^4).$$

Introduce $\gamma_3 := \mu_3/\sigma^3$ and $\gamma_4 := \mu_4/\sigma^4$ to get

$$\boxed{\mathrm{AVAR}(\widehat{SR}) = 1 - SR\,\gamma_3 + \frac{SR^2}{4}(\gamma_4 - 1) = 1 + \tfrac{1}{2}SR^2 - SR\,\gamma_3 + \tfrac{SR^2}{4}(\gamma_4 - 3).}$$

Sampling variance $\approx \mathrm{AVAR}(\widehat{SR})/T$.

Step 11: Normal IID Special Case

If $R_t \sim \mathcal{N}(\mu, \sigma^2)$, then $\gamma_3 = 0$, $\gamma_4 = 3$, so

$$\mathrm{AVAR}(\widehat{SR}) = 1 + \tfrac{1}{2} SR^2.$$

Recap (IID-only)

- **IID GMM:** $\overline{H}_T(\hat{\theta}) = 0$ and $-\sqrt{T}\,\overline{H}_T(\theta_0) = \widehat{D}_T\sqrt{T}(\hat{\theta} - \theta_0) + o_p(1)$.
- **IID CLT:** $\sqrt{T}\,\overline{H}_T(\theta_0) \Rightarrow \mathcal{N}(0, S_{\text{IID}})$ with $S_{\text{IID}} = \text{Var}(H_t(\theta_0))$.
- **Parameter AVAR:** $V_\theta = D_0^{-1} S_{\text{IID}} D_0^{-1\top}$; for Sharpe, $D_0 = -I_2$, so $V_\theta = S_{\text{IID}}$.
- **Functional AVAR:** $\mathrm{AVAR}(g(\hat{\theta})) = \nabla g(\theta_0)^\top V_\theta \nabla g(\theta_0)$.
- **Sharpe S_{IID}:** $\begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{bmatrix}$, leading to the closed-form variance shown.