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HW8. MA481

1. prior distribution:

- Before education: $X_1 \ldots X_n \sim \text{Bernoulli}(\frac{13}{51})$, $p = \frac{13}{51}$, $n = 51$, $s = \sum_{i=1}^n x_i = 13$

posterior: $f(p|x^n) \propto f(p)\mathcal{L}_n(p) = p^s (1-p)^{n-s} = p^{s+1-1}(1-p)^{n-s+1-1}$

$$f(p|x^n) = \frac{\Gamma(n+2)}{\Gamma(s+1)\Gamma(n-s+1)} p^{(s+1)-1}(1-p)^{(n-s+1)-1}$$

$p|x^n \sim \text{Beta}(s+1, n-s+1) = \boxed{\text{Beta}(14, 39)}$

- After education. Use the before education Beta(14, 39) as prior

posterior: $f(p|x^n) \propto f(p)\mathcal{L}(p) = p^s(1-p)^{n-s} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}$, $\alpha = 14, \beta = 39$, $s = 50, n = 56$

$$\propto \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} p^{\alpha+s-1}(1-p)^{\beta+n-s-1}$$

$p|x^n \sim \text{Beta}(a, b) = \text{Beta}(\alpha+s, \beta+n-s) = \boxed{\text{Beta}(64, 45)}$

Compare: Besides the visual comparison, I calculate the mean of the two Beta distributions below,

$\text{Beta}(14, 39)$: $\frac{14}{14+39} = \frac{14}{53} = 0.2642$

$\text{Beta}(64, 45)$: $\frac{64}{64+45} = \frac{64}{109} = 0.5872$

There is an increase trend after the educational program, so it was effective in wearing gloves.

2. $X_1 \ldots X_n \sim \text{Poisson}(\lambda)$. $\sum_{i=1}^n x_i = 2+2+6+0+3 = 13$, $n = 5$

Prior $= P(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$ looks like gamma

$$P(x_1 \ldots x_n|\lambda) = \prod_{i=1}^n P(x_i|\lambda) = \prod_{i=1}^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \frac{\lambda^{\sum_{i=1}^n x_i} \cdot e^{-n\lambda}}{\prod_{i=1}^n x_i!}$$

likelihood. $L(\lambda|x_1 \ldots x_n) \propto p(x_1 \ldots x_n|\lambda) \propto \lambda^{\sum_{i=1}^n x_i} e^{-n\lambda}$

$$\log(L(\lambda|x_1 \ldots x_n)) = l(\lambda|x_1 \ldots x_n) \propto \sum_{i=1}^n x_i \log\lambda - \lambda n$$

$$\frac{dl}{d\lambda} \propto \sum_{i=1}^n x_i \frac{1}{\lambda} - n = 0 \Rightarrow \sum_{i=1}^n x_i \frac{1}{\lambda} - n = 0.$$

$$\hat{\lambda} = \frac{\sum_{i=1}^n x_i}{n} = \frac{2+2+6+0+3}{5} = 2.6$$

posterior $\propto$ Poisson likelihood $*$ Gamma prior

$$\propto (\lambda^{\sum x_i} e^{-n\lambda})(\frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda})$$

$$\propto \lambda^{\sum x_i + \alpha + 1} e^{-\lambda(n+\beta)}.$$

$$\Rightarrow \text{posterior} = \frac{\lambda^{\sum x_i + \alpha - 1} e^{-\lambda(n+\beta)} (n+\beta)^{\sum x_i + \alpha}}{\Gamma(\sum x_i + \alpha)}$$

$\lambda | x^n \sim \text{Gamma}(\sum x_i + \alpha,\ n+\beta)$.

$\sim \text{Gamma}(\alpha+13,\ \beta+5)$.

Compare: $\alpha=1,\ \beta=1,\ \text{Gamma}(14,6)$

$\alpha=2,\ \beta=1\ \ \text{Gamma}(15,6)$

$\alpha=3,\ \beta=1\ \ \text{Gamma}(16,6)$.

The mean of Gamma(14, 6) is $\frac{14}{6}$, mean of Gamma(15, 6) is $\frac{15}{6}$, mean of Gamma(16, 6) is $\frac{16}{6}$. The three means are not the same, so the posterior distributions of these sets of parameters for the prior gamma distribution are not the same

3. $x_1, \dots, x_n \sim \text{Exponential}(\lambda)$.

$f(x|\lambda) = \lambda e^{-\lambda x},\ x>0$. works like gamma.

$L(\lambda) = \prod_i \lambda e^{-\lambda x_i} = \lambda^n e^{-\sum \lambda x_i} = \lambda^n e^{-\lambda \sum x_i}$

Gamma Distribution: $Z \sim \text{Gamma}(\alpha, \beta)$.

$f(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\beta z},\ z>0$.

$f(\lambda | x^n) \propto L(\lambda) \cdot f(\lambda) \propto$ Exponential likelihood $\times$ Gamma prior

$\propto \lambda^n e^{-\lambda \sum x_i} \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$

$\propto \lambda^{\alpha+n-1} e^{-\lambda(\beta + \sum x_i)}$

$\Rightarrow \lambda | x^n \sim \text{Gamma}(\alpha+n,\ \beta + \sum x_i)$.