Known: n transcript

footprint counts $\underline{X} = (\underline{X}^{(n)})_{n=1}^{N}$ observed data

mRNA sequence $\underline{S} = (\underline{S}^{(n)})_{n=1}^{N}$

$E^{(n)}$ normalized factor

footprint state $\underline{Z} = (\underline{Z}^{(n)})_{n=1}^{N}$ hidden state

$M=21$ total states

$X_t^{(n)} \mid Z_t^{(n)} = z \sim Poi(\mu_{z,t}^{(n)} E^{(n)})$

$\mu_{z,t}^{(n)} \sim Gamma(\alpha_z, \beta_z)$

$x \sim Poisson(\theta)$: $f(x|\theta) = \frac{\theta^x e^{-\theta}}{x!}$

$\mu \sim gamma(\alpha, \beta)$: $g(\mu) = \frac{\beta^\alpha}{\Gamma(\alpha)} \mu^{\alpha-1} e^{-\beta\mu}$

$\mu E \sim gamma(\alpha, \frac{\beta}{E})$: $g(\mu E) = \frac{(\frac{\beta}{E})^\alpha}{\Gamma(\alpha)} (\mu E)^{\alpha-1} e^{-\frac{\beta}{E}\cdot\mu E}$

$$E(X_{z,t}) = E[E(X_{z,t}|\mu_{z,t})] = E(\mu_{z,t}E) = \mu_{z,t}E = \frac{\alpha}{\beta}E \quad ✓$$

$$\begin{aligned} Var(X_{z,t}) &= Var[E(X_{z,t}|\mu_{z,t})] + E[Var(X_{z,t}|\mu_{z,t})] \\ &= Var(\mu_{z,t}E) + E(\mu_{z,t}E) \\ &= E^2 Var(\mu_{z,t}) + E\cdot E[\mu_{z,t}] \\ &= E^2\frac{\alpha}{\beta^2} + E\frac{\alpha}{\beta} \quad ✓ \end{aligned}$$

$$X \sim NB\left(\alpha, \frac{E}{E+\beta}\right)$$

$$P(x|\alpha,\beta) = \int_0^\infty P(x, \mu E|\alpha,\beta)\, d\mu E$$

$$= \int_0^\infty P(x|\mu E, \alpha, \beta)\cdot P(\mu E|\alpha,\beta)\, d\mu E$$

$$= \int_0^\infty \frac{(\mu E)^{x}}{x!} e^{-\mu E} \cdot \frac{(\frac{\beta}{E})^\alpha}{\Gamma(\alpha)} (\mu E)^{\alpha-1} e^{-\frac{\beta}{E}\mu E}\, d\mu E$$

$$= \frac{1}{x!}\frac{(\frac{\beta}{E})^\alpha}{\Gamma(\alpha)}\frac{\Gamma(x+\alpha)}{(1+\frac{\beta}{E})^{x+\alpha}} \underbrace{\int_0^\infty \frac{(1+\frac{\beta}{E})^{x+\alpha}}{\Gamma(x+\alpha)} (\mu E)^{x+\alpha-1}\cdot e^{-(1+\frac{\beta}{E})\mu E}\, d\mu E}_{\text{gamma}(x+\alpha,\ 1+\frac{\beta}{E})\ =1}$$

$$= \frac{\Gamma(x+\alpha)}{x!\,\Gamma(\alpha)}\frac{(\frac{\beta}{E})^\alpha}{(1+\frac{\beta}{E})^x(1+\frac{\beta}{E})^\alpha}$$

$$= \frac{\Gamma(x+\alpha)}{x!\,\Gamma(\alpha)}\left(\frac{\frac{\beta}{E}}{1+\frac{\beta}{E}}\right)^\alpha\left(\frac{1}{1+\frac{\beta}{E}}\right)^x$$

$$\Gamma(n) = (n-1)! \quad , \quad C^n_r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

$$= \frac{(x+\alpha-1)!}{x!\,(\alpha-1)!}\left(\frac{\beta}{E+\beta}\right)^\alpha\left(\frac{E}{E+\beta}\right)^x$$

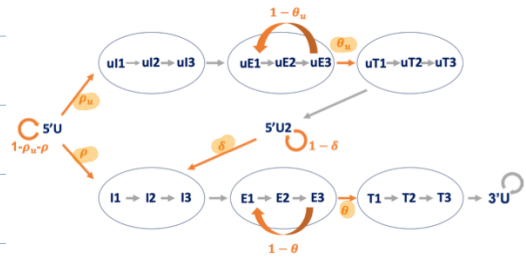
$$= \binom{x+\alpha-1}{x}\left(\frac{\beta}{E+\beta}\right)^\alpha\left(1-\frac{\beta}{E+\beta}\right)^x$$

$$\Rightarrow X_{z,t} \sim NB\left(\alpha, \frac{\beta}{E+\beta}\right) \quad ; \quad g(x|r,p) = \binom{x+r-1}{x} p^r (1-p)^x$$

check : Mean $= \frac{(1-p)r}{p} = \frac{\frac{E}{E+\beta}\alpha}{\frac{\beta}{E+\beta}} = \frac{\alpha}{\beta}E$ ✓

Variance $= \frac{(1-p)r}{p^2} = \frac{\alpha}{\beta}E\cdot\frac{1}{\frac{\beta}{E+\beta}} = \frac{\alpha E(E+\beta)}{\beta^2} = E^2\frac{\alpha}{\beta^2} + E\frac{\alpha}{\beta}$ ✓

E - step



forward algorithm: $F_m(t) = P(x_1, \cdots x_t, z_t = m \mid \theta)$

$$P(\underline{x}_{1:T} \mid \theta) = \sum_{m=1}^{M} F_m(T)$$

$$F_j(t+1) = P(x_1, \cdots, x_{t+1}, z_{t+1} = j \mid \theta)$$

$$= \sum_{m=1}^{M} P(x_1, \cdots, x_{t+1}, z_{t+1} = j, z_t = m \mid \theta)$$

$$= \sum_{m=1}^{M} P(x_{t+1} \mid x_1, \cdots, x_t, z_{t+1} = j, z_t = m, \theta) \cdot P(x_1, \cdots, x_t, z_{t+1} = j, z_t = m \mid \theta)$$

$$= \sum_{m=1}^{M} P(x_{t+1} \mid z_{t+1} = j) \cdot P(z_{t+1} = j \mid x_1, \cdots, x_t, z_t = m, \theta) \cdot P(x_1, \cdots, x_t, z_t = m \mid \theta)$$

$$= \sum_{m=1}^{M} P(x_{t+1} \mid z_{t+1} = j) \cdot P(z_{t+1} = j \mid z_t = m) \cdot F_m(t)$$

$$= \left(\sum_{m=1}^{M} F_m(t) \cdot P(z_{t+1} = j \mid z_t = m) \right) \cdot P(x_{t+1} \mid z_{t+1} = j)$$

$F_m(1) = F_1(1) = g(x_1 \mid \alpha_1, \beta_1)$ assume to start at state 1

backward algorithm: $B_m(t) = P(x_{t+1}, \cdots x_T \mid z_t = m, \theta)$

$$P(\underline{x}_{1:T} \mid \theta) = \sum_{m=1}^{M} P(\underline{x}_{1:T}, z_1 = m \mid \theta)$$

$$= \sum_{m=1}^{M} P(\underline{x}_{1:T} \mid z_1 = m, \theta) \cdot P(z_1 = m \mid \theta)$$

$$= \sum_{m=1}^{M} P(x_1 \mid \underline{x}_{2:T}, z_1 = m, \theta) \cdot (\underline{x}_{2:T} \mid z_1 = m, \theta) \cdot P(z_1 = m \mid \theta)$$

$$= \sum_{m=1}^{M} P(x_1 \mid z_1 = m, \theta) \cdot B_m(1) \cdot P(z_1 = m \mid \theta)$$

$$= P(x_1 \mid z_1 = 1, \theta) \cdot B_1(1) \cdot P(z_1 = 1 \mid \theta)$$ assume to start at state 1

$$B_m(t) = P(x_{t+1}, \cdots, x_T \mid z_t = m, \theta)$$

$$= \sum_{j=1}^{M} P(x_{t+1}, \cdots, x_T, z_{t+1} = j \mid z_t = m, \theta)$$

$$= \sum_{j=1}^{M} P(x_{t+1}, \cdots, x_T \mid z_t = m, z_{t+1} = j, \theta) \cdot P(z_{t+1} = j \mid z_t = m, \theta)$$

$$= \sum_{j=1}^{M} P(x_{t+1}, \cdots, x_T \mid z_{t+1} = j, \theta) \cdot P(z_{t+1} = j \mid z_t = m, \theta)$$

$$= \sum_{j=1}^{M} P(x_{t+1} \mid x_{t+2}, \cdots, x_T, z_{t+1} = j, \theta) \cdot P(x_{t+2}, \cdots, x_T \mid z_{t+1} = j, \theta) \cdot P(z_{t+1} = j \mid z_t = m, \theta)$$

$$= \sum_{j=1}^{M} P(x_{t+1} \mid z_{t+1} = j, \theta) \cdot B_j(t+1) \cdot P(z_{t+1} = j \mid z_t = m, \theta)$$

$B_m(T) = 1$

Viterbi algorithm: $\operatorname{argmax} P(\underline{z}_{1:T} \mid \underline{x}_{1:T}, \theta)$

$$\delta_m(t) = \max P(x_1, \cdots, x_t, z_1, \cdots, z_t = m \mid \theta)$$

$$\delta_j(t+1) = \max P(x_1, \cdots, x_{t+1}, z_1, \cdots, z_{t+1} = j \mid \theta)$$

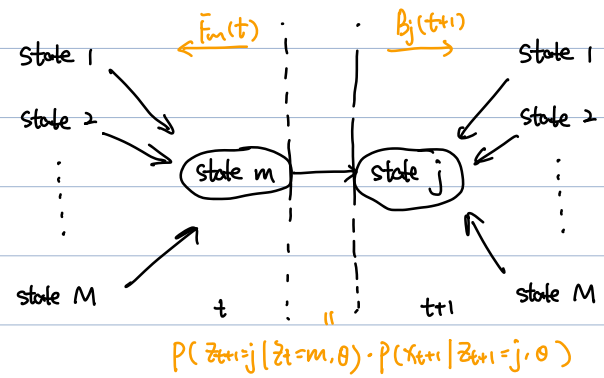
$$= \max \delta_m(t) \cdot P(z_{t+1} = j \mid z_t = m, \theta) \cdot P(x_{t+1} \mid z_{t+1} = j, \theta)$$

$$\varphi_j(t+1) = \underset{1 \le m \le M}{\operatorname{argmax}} \, \delta_m(t) \cdot P(z_{t+1} = j \mid z_t = m, \theta)$$ store index

one state probability: $L_m(t) = P(z_t = m \mid \underline{x}_{1:T}, \theta)$

$$= \frac{F_m(t) \cdot B_m(t)}{\sum_{j=1}^{M} F_j(t) \cdot B_j(t)}$$

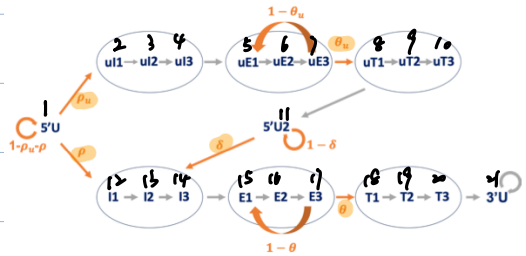
two states probability: $H_{mj}(t) = P(z_t = m, z_{t+1} = j \mid \underline{x}_{1:T}, \theta)$



$$H_{mj}(t) = \frac{F_m(t) \cdot P(z_{t+1} = j \mid z_t = m, \theta) \cdot P(x_{t+1} \mid z_{t+1} = j, \theta) \cdot B_j(t+1)}{\sum_{m=1}^{M} \sum_{j=1}^{M} F_m(t) \cdot P(z_{t+1} = j \mid z_t = m, \theta) \cdot P(x_{t+1} \mid z_{t+1} = j, \theta) \cdot B_j(t+1)}$$

$$H_{mj}(t, c_i) = \begin{cases} H_{mj}(t) & \text{if } S_{t+1:t+3} = c_i \\ 0 & \text{if } S_{t+1:t+3} \neq c_i \end{cases}$$

M - Step



$m=1,\ m'=2 : p_u$

$m=1,\ m'=12 : p$

$m=11,\ m'=12 : \delta$

$m=7,\ m'=8 : \theta_u$

$m=17,\ m'=18 : \theta$

$\begin{cases} 1 & S_{t:t+2} = c_i,\ c_i \in \text{stop codons} \\ 0 & \text{ow.} \end{cases}$

log likelihood for single transcript:

$$\log P(\underline{x}_{1:T}, \underline{z}_{1:T} \mid \theta, \underline{S}_{1:T})$$

$$= \log\left[P(\underline{x}_{1:T} \mid \underline{z}_{1:T}, \underline{S}_{1:T}, \theta)\, P(\underline{z}_{1:T} \mid \underline{S}_{1:T}, \theta)\right]$$

$$= \log\left[\prod_{t=1}^{T} P(x_t \mid z_t, \cancel{S_{t:t+2}}, \theta)\, P(z_1 \mid \cancel{S_1}, \theta) \prod_{t=2}^{T} P(z_t \mid z_{t-1}, S_{t:t+2}, \theta)\right]$$

$P(z_t = m' \mid z_{t-1} = m, S_{t:t+2}, \theta)$

$$= \log\left[\prod_{t=1}^{T}\prod_{m=1}^{M} P(x_t \mid \alpha_m, \beta_m, z_t = m)^{I(z_t=m)} \cdot \prod_{m=1}^{M} (\pi_m)^{I(z_1=m)} \cdot \underbrace{\prod_{t=2}^{T}\prod_{m=1}^{M}\prod_{m'=1}^{M} \left[a_{m,m'}\right]^{I(z_{t-1}=m,\, z_t=m')}}_{①}\right]$$

① : $\prod_{t=2}^{T}\prod_{m=1}^{M}\prod_{m'=1}^{M} \left[a_{m,m'}\right]^{I(z_{t-1}=m,\, z_t=m')} \propto \prod_{t=2}^{T} \left[a_{1,1}\right]^{I(z_{t-1}=1,\, z_t=1)} \cdot \left[a_{1,2}\right]^{I(z_{t-1}=1,\, z_t=2)} \cdot \left[a_{1,12}\right]^{I(z_{t-1}=1,\, z_t=12)} \cdot \left[a_{11,11}\right]^{I(z_{t-1}=11,\, z_t=11)}$

$$\cdot \left[a_{11,12}\right]^{I(z_{t-1}=11,\, z_t=12)}$$

$$= \prod_{t=2}^{T}\prod_{c_i \in \Omega} \left[1 - p_u^{c_i} - p^{c_i}\right]^{I(z_{t-1}=1,\, z_t=1,\, S_{t:t+2}=c_i)} \cdot \left[p_u^{c_i}\right]^{I(z_{t-1}=1,\, z_t=2,\, S_{t:t+2}=c_i)}$$

$$\cdot \left[p^{c_i}\right]^{I(z_{t-1}=1,\, z_t=12,\, S_{t:t+2}=c_i)} \cdot \left[1-\delta^{c_i}\right]^{I(z_{t-1}=11,\, z_t=11,\, S_{t:t+2}=c_i)}$$

$$\cdot \left[\delta^{c_i}\right]^{I(z_{t-1}=11,\, z_t=12,\, S_{t:t+2}=c_i)}$$

Ω: set of possible start codons

log likelihood for single transcript:

$$\log P(\underline{x}_{1:T}, \underline{z}_{1:T} \mid \theta, \underline{S}_{1:T}) \propto \sum_{t=1}^{T}\sum_{m=1}^{M} I(z_t=m) \log p(x_t \mid \alpha_m, \beta_m, z_t=m) + \sum_{m=1}^{M} I(z_1=m) \log \pi_m + \log ①$$

$$= \sum_{t=1}^{T}\sum_{m=1}^{M} I(z_t=m) \log p(x_t \mid \alpha_m, \beta_m, z_t=m) + \sum_{m=1}^{M} I(z_1=m) \log \pi_m$$

$$+ \sum_{t=2}^{T}\sum_{c_i \in \Omega} \Big\{ I(z_{t-1}=1, z_t=1, S_{t:t+2}=c_i) \log(1 - p_u^{c_i} - p^{c_i}) + I(z_{t-1}=1, z_t=2, S_{t:t+2}=c_i) \log p_u^{c_i}$$

$$+ I(z_{t-1}=1, z_t=12, S_{t:t+2}=c_i) \log p^{c_i} + I(z_{t-1}=11, z_t=11, S_{t:t+2}=c_i) \log(1-\delta^{c_i})$$

$$+ I(z_{t-1}=11, z_t=12, S_{t:t+2}=c_i) \log \delta^{c_i} \Big\}$$

log likelihood for multiple transcripts:

$$\sum_{n=1}^{N} \log p(\mathbf{x}_{1:T}^{(n)}, \mathbf{z}_{1:T}^{(n)} \mid \theta, \mathbf{s}_{1:T}^{(n)}) \propto \sum_{n=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} I(z_t^{(n)}=m) \log p(x_t^{(n)} \mid \alpha_m, \beta_m, z_t^{(n)}=m) + \sum_{n=1}^{N} \sum_{m=1}^{M} I(z_1^{(n)}=m) \log \pi_m + \sum_{n=1}^{N} \log ①$$

$$= \sum_{n=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} I(z_t^{(n)}=m) \log p(x_t^{(n)} \mid \alpha_m, \beta_m, z_t^{(n)}=m) + \sum_{n=1}^{N} \sum_{m=1}^{M} I(z_1^{(n)}=m) \log \pi_m$$

$$+ \sum_{n=1}^{N} \sum_{t=2}^{T} \sum_{c_i \in \Omega} \Big\{ I(z_{t-1}^{(n)}=1, z_t^{(n)}=1, s_{t:t+2}^{(n)}=c_i) \log(1-p_u^{c_i}-p^{c_i}) + I(z_{t-1}^{(n)}=1, z_t^{(n)}=2, s_{t:t+2}^{(n)}=c_i) \log p_u^{c_i}$$

$$+ I(z_{t-1}^{(n)}=1, z_t^{(n)}=12, s_{t:t+2}^{(n)}=c_i) \log p^{c_i} + I(z_{t-1}^{(n)}=11, z_t^{(n)}=11, s_{t:t+2}^{(n)}=c_i) \log(1-\delta^{c_i})$$

$$+ I(z_{t-1}^{(n)}=11, z_t^{(n)}=12, s_{t:t+2}^{(n)}=c_i) \log \delta^{c_i} \Big\}$$

Q function for single transcript:

$p(z_t=m \mid x_{1:T}, \theta)$ $\qquad$ $p(z_{t-1}=i, z_t=j, s_{t:t+2}=c_i \mid x_{1:T}, \theta)$

$$Q(\theta, \theta') \propto \sum_{t=1}^{T} \sum_{m=1}^{M} \boxed{p(z_t=m)} \log p(x_t \mid \alpha_m, \beta_m, z_t=m) + \sum_{m=1}^{M} p(z_1=m) \log \pi_m$$

$$+ \sum_{t=2}^{T} \sum_{c_i \in \Omega} \Big\{ p(z_{t-1}=1, z_t=1, s_{t:t+2}=c_i) \log(1-p_u^{c_i}-p^{c_i}) + p(z_{t-1}=1, z_t=2, s_{t:t+2}=c_i) \log p_u^{c_i}$$

$$+ p(z_{t-1}=1, z_t=12, s_{t:t+2}=c_i) \log p^{c_i} + p(z_{t-1}=11, z_t=11, s_{t:t+2}=c_i) \log(1-\delta^{c_i})$$

$$+ p(z_{t-1}=11, z_t=12, s_{t:t+2}=c_i) \log \delta^{c_i} \Big\}$$

Q function for multiple transcripts:

$$Q(\theta, \theta') \propto \sum_{n=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} p(z_t^{(n)}=m) \log p(x_t^{(n)} \mid \alpha_m, \beta_m, z_t^{(n)}=m) + \sum_{n=1}^{N} \sum_{m=1}^{M} p(z_1^{(n)}=m) \log \pi_m$$

$$+ \sum_{n=1}^{N} \sum_{t=2}^{T} \sum_{c_i \in \Omega} \Big\{ p(z_{t-1}^{(n)}=1, z_t^{(n)}=1, s_{t:t+2}^{(n)}=c_i) \log(1-p_u^{c_i}-p^{c_i}) + p(z_{t-1}^{(n)}=1, z_t^{(n)}=2, s_{t:t+2}^{(n)}=c_i) \log p_u^{c_i}$$

$$+ p(z_{t-1}^{(n)}=1, z_t^{(n)}=12, s_{t:t+2}^{(n)}=c_i) \log p^{c_i} + p(z_{t-1}^{(n)}=11, z_t^{(n)}=11, s_{t:t+2}^{(n)}=c_i) \log(1-\delta^{c_i})$$

$$+ p(z_{t-1}^{(n)}=11, z_t^{(n)}=12, s_{t:t+2}^{(n)}=c_i) \log \delta^{c_i} \Big\}$$

$$\frac{\partial Q(\theta, \theta')}{\partial p_u^{c_i}} = \sum_{n=1}^{N} \sum_{t=2}^{T} \left[- \frac{p(z_{t-1}^{(n)}=1, z_t^{(n)}=1, s_{t:t+2}^{(n)}=c_i)}{1-p_u^{c_i}-p^{c_i}} + \frac{p(z_{t-1}^{(n)}=1, z_t^{(n)}=2, s_{t:t+2}^{(n)}=c_i)}{p_u^{c_i}} \right] = 0$$

$$\Rightarrow \frac{(1-p_u^{c_i}-p^{c_i}) \sum_{n=1}^{N} \sum_{t=2}^{T} p(z_{t-1}^{(n)}=1, z_t^{(n)}=2, s_{t:t+2}=c_i) - p_u^{c_i} \sum_{n=1}^{N} \sum_{t=2}^{T} p(z_{t-1}^{(n)}=1, z_t^{(n)}=1, s_{t:t+2}^{(n)}=c_i)}{(p_u^{c_i})(1-p_u^{c_i}-p^{c_i})} = 0$$

$$\Rightarrow (1-p_u^{c_i}-p^{c_i}) \sum_{n=1}^{N} \sum_{t=2}^{T} p(z_{t-1}^{(n)}=1, z_t^{(n)}=2, s_{t:t+2}=c_i) = p_u^{c_i} \sum_{n=1}^{N} \sum_{t=2}^{T} p(z_{t-1}^{(n)}=1, z_t^{(n)}=1, s_{t:t+2}^{(n)}=c_i) \qquad (1)$$

$$\frac{\partial Q(\theta,\theta')}{\partial p^{ci}} = \sum_{n=1}^{N}\sum_{t=2}^{T}\left[-\frac{P(z_{t-1}^{(n)}=1, z_t^{(n)}=1, S_{t:t+2}=c_i)}{1-p_u^{ci}-p^{ci}} + \frac{P(z_{t-1}^{(n)}=1, z_t^{(n)}=12, S_{t:t+2}=c_i)}{p^{ci}}\right] = 0$$

$$\Rightarrow \frac{(1-p_u^{ci}-p^{ci})\sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=12, S_{t:t+2}=c_i) - p^{ci}\sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=1, S_{t:t+2}^{(n)}=c_i)}{(p^{ci})(1-p_u^{ci}-p^{ci})} = 0$$

$$\Rightarrow (1-p_u^{ci}-p^{ci})\sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=12, S_{t:t+2}=c_i) = p^{ci}\sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=1, S_{t:t+2}^{(n)}=c_i) \qquad (2)$$

(1) + (2) :

$$(1-p_u^{ci}-p^{ci})\left\{\sum_{n=1}^{N}\sum_{t=2}^{T}\left[P(z_{t-1}^{(n)}=1, z_t^{(n)}=2, S_{t:t+2}=c_i) + P(z_{t-1}^{(n)}=1, z_t^{(n)}=12, S_{t:t+2}=c_i)\right]\right\}$$

$$= \underbrace{(p_u^{ci}+p^{ci})}_{\substack{\| \\ 1-(1-p_u^{ci}-p^{ci})}}\sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=1, S_{t:t+2}=c_i)$$

$$\Rightarrow (1-p_u^{ci}-p^{ci})\left\{\sum_{n=1}^{N}\sum_{t=2}^{T}\boxed{\left[\underbrace{P(z_{t-1}^{(n)}=1, z_t^{(n)}=2, S_{t:t+2}=c_i)}_{=1} + \underbrace{P(z_{t-1}^{(n)}=1, z_t^{(n)}=12, S_{t:t+2}=c_i)}_{=1} + P(z_{t-1}^{(n)}=1, z_t^{(n)}=1, S_{t:t+2}=c_i)\right]}\right\} = \sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=1, S_{t:t+2}=c_i)$$

$$* \quad P(S_{t:t+2}=c_i \mid X_{1:T}) = \begin{cases} 1 & \text{if } S_{t:t+2}=c_i \\ 0 & \text{o.w.} \end{cases}$$

$$\Rightarrow (1-p_u^{ci}-p^{ci})\underbrace{\boxed{\sum_{n=1}^{N}\sum_{t=2}^{T}1}}_{N\cdot(T-1)} = \sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=1, S_{t:t+2}=c_i)$$

$$\Rightarrow 1-p_u^{ci}-p^{ci} = \frac{\sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=1, S_{t:t+2}=c_i)}{N(T-1)} = \frac{\sum_{n=1}^{N}\sum_{t=2}^{T}H_{11}(t-1, c_i)}{N(T-1)} = \frac{\sum_{n=1}^{N}\sum_{t=1}^{T-1}H_{11}(t, c_i)}{N(T-1)}$$

1 2 3 4 5 ... T

range(1, T)

from (1) :

$$p_u^{ci} = \frac{(1-p_u^{ci}-p^{ci})\sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=2, S_{t:t+2}=c_i)}{\sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=1, S_{t:t+2}=c_i)}$$

$$= \frac{\sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, \cancel{z_t^{(n)}=1}, S_{t:t+2}=c_i)}{N(T-1)} \cdot \frac{\sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=2, S_{t:t+2}=c_i)}{\sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, \cancel{z_t^{(n)}=1}, S_{t:t+2}^{(n)}=c_i)}$$

$$= \frac{\sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=2, S_{t:t+2}=c_i)}{N(T-1)} = \boxed{\frac{\sum_{n=1}^{N}\sum_{t=2}^{T}H_{12}(t-1, c_i)}{N(T-1)}}$$

$$= \sum_{n=1}^{N}\sum_{t=2}^{T}P(S_{t:t+2}=c_i \mid X_{1:T})$$

$\Rightarrow$ count how often assume c_i

from (2):

$$\rho^{c_i} = \frac{(1-\rho_u^{c_i}-\rho^{c_i})\sum_{n=1}^{N}\sum_{t=2}^{T} P(z_{t-1}^{(n)}=1, z_t^{(n)}=12, s_{t:t+2}=c_i)}{\sum_{n=1}^{N}\sum_{t=2}^{T} P(z_{t-1}^{(n)}=1, z_t^{(n)}=1, s_{t:t+2}^{(n)}=c_i)}$$

count of c_i

$$= \frac{\sum_{n=1}^{N}\sum_{t=2}^{T} P(z_{t-1}^{(n)}=1, \cancel{z_t^{(n)}=1}, s_{t:t+2}=c_i)}{N(T-1)} \cdot \frac{\sum_{n=1}^{N}\sum_{t=2}^{T} P(z_{t-1}^{(n)}=1, z_t^{(n)}=12, s_{t:t+2}=c_i)}{\sum_{n=1}^{N}\sum_{t=2}^{T} P(\cancel{z_{t-1}^{(n)}=1}, z_t^{(n)}=1, s_{t:t+2}^{(n)}=c_i)}$$

$$= \frac{\sum_{n=1}^{N}\sum_{t=2}^{T} P(z_{t-1}^{(n)}=1, z_t^{(n)}=12, s_{t:t+2}=c_i)}{N(T-1)} = \frac{\sum_{n=1}^{N}\sum_{t=2}^{T} H_{1:12}(t-1, c_i)}{N(T-1)}$$

$$\frac{\partial Q(\theta,\theta')}{\partial \delta^{c_i}} = \sum_{n=1}^{N}\sum_{t=2}^{T} \frac{P(z_{t-1}^{(n)}=11, z_t^{(n)}=12, s_{t:t+2}=c_i)}{\delta^{c_i}} - \frac{P(z_{t-1}^{(n)}=11, z_t^{(n)}=11, s_{t:t+2}^{(n)}=c_i)}{1-\delta^{c_i}}$$

$$\Rightarrow \frac{(1-\delta^{c_i})\sum_{n=1}^{N}\sum_{t=2}^{T} P(z_{t-1}^{(n)}=11, z_t^{(n)}=12, s_{t:t+2}=c_i) - \delta^{c_i}\sum_{n=1}^{N}\sum_{t=2}^{T} P(z_{t-1}^{(n)}=11, z_t^{(n)}=11, s_{t:t+2}^{(n)}=c_i)}{(\delta^{c_i})(1-\delta^{c_i})} = 0$$

$$\Rightarrow (1-\delta^{c_i})\sum_{n=1}^{N}\sum_{t=2}^{T} P(z_{t-1}^{(n)}=11, z_t^{(n)}=12, s_{t:t+2}=c_i) - \delta^{c_i}\sum_{n=1}^{N}\sum_{t=2}^{T} P(z_{t-1}^{(n)}=11, z_t^{(n)}=11, s_{t:t+2}^{(n)}=c_i) = 0$$

$$\Rightarrow \sum_{n=1}^{N}\sum_{t=2}^{T} P(z_{t-1}^{(n)}=11, z_t^{(n)}=12, s_{t:t+2}=c_i) - \delta^{c_i}\left\{\sum_{n=1}^{N}\sum_{t=2}^{T}\underbrace{\boxed{P(z_{t-1}^{(n)}=11, z_t^{(n)}=12, s_{t:t+2}=c_i) + P(z_{t-1}^{(n)}=11, z_t^{(n)}=11, s_{t:t+2}^{(n)}=c_i)}}_{*}\right\} = 0$$

$$\Rightarrow \sum_{n=1}^{N}\sum_{t=2}^{T} P(z_{t-1}^{(n)}=11, z_t^{(n)}=12, s_{t:t+2}=c_i) = \delta^{c_i}\sum_{n=1}^{N}\sum_{t=2}^{T} 1$$

$$\Rightarrow \delta^{c_i} = \frac{\sum_{n=1}^{N}\sum_{t=2}^{T} P(z_{t-1}^{(n)}=11, z_t^{(n)}=12, s_{t:t+2}=c_i)}{N(T-1)} = \frac{\sum_{n=1}^{N}\sum_{t=2}^{T} H_{11:12}(t-1, c_i)}{N(T-1)}$$

$$\Rightarrow 1-\delta^{c_i} = \frac{N(T-1) - \sum_{n=1}^{N}\sum_{t=2}^{T} P(z_{t-1}^{(n)}=11, z_t^{(n)}=12, s_{t:t+2}=c_i)}{N(T-1)} = \frac{N(T-1) - \sum_{n=1}^{N}\sum_{t=2}^{T} H_{11:12}(t-1, c_i)}{N(T-1)}$$

$$\frac{\partial Q(\theta,\theta')}{\partial \alpha_m} = \frac{\partial}{\partial \alpha_m} \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \log p(x_t^{(n)} \mid \alpha_m, \beta_m, z_t^{(n)}=m)$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \frac{\partial}{\partial \alpha_m} \log p(x_t^{(n)} \mid \alpha_m, \beta_m, z_t^{(n)}=m)$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \frac{\partial}{\partial \alpha_m} \left[\log \binom{x_t^{(n)}+\alpha_m-1}{x_t^{(n)}} \left(1-\frac{E^{(n)}}{E^{(n)}+\beta_m}\right)^{\alpha_m} \left(\frac{E^{(n)}}{E^{(n)}+\beta_m}\right)^{x_t^{(n)}} \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \frac{\partial}{\partial \alpha_m} \left[\log \binom{x_t^{(n)}+\alpha_m-1}{x_t^{(n)}} + \alpha_m \log\left(1-\frac{E^{(n)}}{E^{(n)}+\beta_m}\right) + x_t^{(n)} \log \frac{E^{(n)}}{E^{(n)}+\beta_m} \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \frac{\partial}{\partial \alpha_m} \left[\log \prod_{s=1}^{x_t^{(n)}} (\alpha_m+x_t^{(n)}-s) + \alpha_m \log\left(1-\frac{E^{(n)}}{E^{(n)}+\beta_m}\right) + x_t^{(n)} \log \frac{E^{(n)}}{E^{(n)}+\beta_m} \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \left[\sum_{s=1}^{x_t^{(n)}} \frac{\partial}{\partial \alpha_m} \log(\alpha_m+x_t^{(n)}-s) + \log\left(1-\frac{E^{(n)}}{E^{(n)}+\beta_m}\right) \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \left[\sum_{s=1}^{x_t^{(n)}} \frac{1}{\alpha_m+x_t^{(n)}-s} + \log \frac{\beta_m}{E^{(n)}+\beta_m} \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} L_m(t) \cdot \left[\sum_{s=1}^{x_t^{(n)}} \frac{1}{\alpha_m+x_t^{(n)}-s} + \log \frac{\beta_m}{E^{(n)}+\beta_m} \right]$$

$$\frac{\partial Q(\theta,\theta')}{\partial \beta_m} = \frac{\partial}{\partial \beta_m} \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \log g(x_t^{(n)} \mid \alpha_m, \beta_m, z_t^{(n)}=m)$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \frac{\partial}{\partial \beta_m} \log g(x_t^{(n)} \mid \alpha_m, \beta_m, z_t^{(n)}=m)$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \frac{\partial}{\partial \beta_m} \left[\log \left(\frac{x_t^{(n)}+\alpha_m-1}{x_t^{(n)}} \right) \left(1-\frac{E^{(n)}}{E^{(n)}+\beta_m} \right)^{\alpha_m} \left(\frac{E^{(n)}}{E^{(n)}+\beta_m} \right)^{x_t^{(n)}} \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \frac{\partial}{\partial \beta_m} \left[\log \left(\frac{x_t^{(n)}+\alpha_m-1}{x_t^{(n)}} \right) + \alpha_m \log \left(1-\frac{E^{(n)}}{E^{(n)}+\beta_m} \right) + x_t^{(n)} \log \frac{E^{(n)}}{E^{(n)}+\beta_m} \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \frac{\partial}{\partial \beta_m} \left[\alpha_m \log \frac{\beta_m}{E^{(n)}+\beta_m} + x_t^{(n)} \log \frac{E^{(n)}}{E^{(n)}+\beta_m} \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \left[\alpha_m \cdot \frac{E^{(n)}+\cancel{\beta_m}}{\beta_m} \cdot \frac{1\times(E^{(n)}+\cancel{\beta_m})-\cancel{\beta_m}\times 1}{(E^{(n)}+\beta_m)^{\cancel{2}}} + x_t^{(n)} \cdot \frac{E^{(n)}+\cancel{\beta_m}}{\cancel{E^{(n)}}} \cdot \frac{0-E^{(n)}\cancel{1}}{(E^{(n)}+\beta_m)^{\cancel{2}}} \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \left[\frac{\alpha_m \cdot E^{(n)}}{\beta_m \cdot (E^{(n)}+\beta_m)} - \frac{x_t^{(n)}}{E^{(n)}+\beta_m} \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} L_m(t) \cdot \left[\frac{\alpha_m \cdot E^{(n)}}{\beta_m \cdot (E^{(n)}+\beta_m)} - \frac{x_t^{(n)}}{E^{(n)}+\beta_m} \right]$$

$$= \cdots \left[\alpha_m \log \beta_m - \alpha_m \log(E+\beta_m) + x \log E - x \log(E+\beta_m) \right]$$

$$= \cdots \frac{\alpha_m}{\beta_m} - \frac{\alpha_m + x}{E+\beta_m}$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} L_k^{(n)}(t) \left(\frac{\alpha_k}{\beta_k} - \frac{\alpha_k + x_t^{(n)}}{E^{(n)}+\beta_k} \right)$$

logistic:

log likelihood for single transcript:

$$\log P(\underline{x}_{1:T}, \underline{z}_{1:T} \mid \theta, \underline{S}_{1:T})$$

$$= \log \left[P(\underline{x}_{1:T} \mid \underline{z}_{1:T}, \underline{S}_{1:T}, \theta)\, P(\underline{z}_{1:T} \mid \underline{S}_{1:T}, \theta) \right]$$

$$= \log \left[\prod_{t=1}^{T} P(x_t \mid z_t, S_{t:t+2}, \theta)\, \boxed{P(z_1 \mid S_1, \theta)} \prod_{t=2}^{T} P(z_t \mid z_{t-1}, S_{t:t+2}, \theta) \right]$$

$$= \log \left[\prod_{t=1}^{T} \prod_{m=1}^{M} P(x_t \mid \alpha_m, \beta_m, z_t = m)^{I(z_t = m)} \cdot \boxed{\prod_{m=1}^{M} (\pi_m)^{I(z_1 = m)}} \cdot \underbrace{\prod_{t=2}^{T} \prod_{m=1}^{M} \prod_{m'=1}^{M} \left[a_{m,m'} \right]^{I(z_{t-1}=m,\, z_t = m')}}_{①} \right]$$

$P(z_t = m' \mid z_{t-1} = m, S_{t:t+2}, \theta)$

①: $$\prod_{t=2}^{T} \prod_{m=1}^{M} \prod_{m'=1}^{M} \left[a_{m,m'} \right]^{I(z_{t-1}=m,\, z_t = m')} \propto \prod_{t=2}^{T} \prod_{c_i \in \Omega} \left[1 - \frac{1}{1+e^{-(\rho_u^0 k_t + R_u^{c_i})}} - \frac{1}{1+e^{-(\rho^0 k_t + \rho^{c_i})}} \right]^{I(z_{t-1}=1,\, z_t = 1,\, S_{t:t+2} = c_i)}$$

$$\cdot \left[\frac{1}{1+e^{-(\rho_u^0 k_t + R_u^{c_i})}} \right]^{I(z_{t-1}=1,\, z_t = 12,\, S_{t:t+2} = c_i)} \cdot \left[\frac{1}{1+e^{-(\rho^0 k_t + \rho^{c_i})}} \right]^{I(z_{t-1}=1,\, z_t = 12,\, S_{t:t+2} = c_i)}$$

$$\cdot \left[\frac{1}{1+e^{-(\delta^0 k_t + \delta^{c_i})}} \right]^{I(z_{t-1}=11,\, z_t = 12,\, S_{t:t+2} = c_i)} \cdot \left[1 - \frac{1}{1+e^{-(\delta^0 k_t + \delta^{c_i})}} \right]^{I(z_{t-1}=11,\, z_t = 11,\, S_{t:t+2} = c_i)}$$

Ω: set of possible start codons

still contains ρ_u^0

Q function for multiple transcripts:

Limits

MB

$$Q(\theta, \theta') \propto \sum_{n=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} P(z_t^{(n)} = m) \log P(x_t^{(n)} \mid \alpha_m, \beta_m, z_t^{(n)} = m) + \boxed{\sum_{n=1}^{N} \sum_{m=1}^{M} P(z_1^{(n)} = m) \log \pi_m}$$

$$+ \sum_{n=1}^{N} \sum_{t=2}^{T} \sum_{c_i \in \Omega} \Big\{ P(z_{t-1}^{(n)} = 1, z_t^{(n)} = 1, S_{t:t+2}^{(n)} = c_i) \log \left[1 - \frac{1}{1+e^{-(\rho_u^0 k_t + R_u^{c_i})}} - \frac{1}{1+e^{-(\rho^0 k_t + \rho^{c_i})}} \right]$$

$$+ P(z_{t-1}^{(n)} = 1, z_t^{(n)} = 2, S_{t:t+2}^{(n)} = c_i) \log \left[\frac{1}{1+e^{-(\rho_u^0 k_t + R_u^{c_i})}} \right]$$

$$+ P(z_{t-1}^{(n)} = 1, z_t^{(n)} = 12, S_{t:t+2}^{(n)} = c_i) \log \left[\frac{1}{1+e^{-(\rho^0 k_t + \rho^{c_i})}} \right]$$

$$+ P(z_{t-1}^{(n)} = 11, z_t^{(n)} = 11, S_{t:t+2}^{(n)} = c_i) \log \left[1 - \frac{1}{1+e^{-(\delta^0 k_t + \delta^{c_i})}} \right]$$

$$+ P(z_{t-1}^{(n)} = 11, z_t^{(n)} = 12, S_{t:t+2}^{(n)} = c_i) \log \left[\frac{1}{1+e^{-(\delta^0 k_t + \delta^{c_i})}} \right] \Big\}$$

$$\sum_{t=1}^{T-1} P(z_t = 1, z_{t+1} = 1, c_i)$$

$$1-\frac{1}{1+e^{-(\rho_u^0 K_t+\rho_u^{ci})}}-\frac{1}{1+e^{-(\rho^0 K_t+\rho^{ci})}}=\frac{\left(1+e^{-(\rho_u^0 K_t+\rho_u^{ci})}\right)\left(1+e^{-(\rho^0 K_t+\rho^{ci})}\right)-\left(1+e^{-(\rho^0 K_t+\rho^{ci})}\right)-\left(1+e^{-(\rho_u^0 K_t+\rho_u^{ci})}\right)}{\left(1+e^{-(\rho_u^0 K_t+\rho_u^{ci})}\right)\left(1+e^{-(\rho^0 K_t+\rho^{ci})}\right)}$$

$$=\frac{\cancel{1}+\cancel{e^{-(\rho^0 K_t+\rho^{ci})}}+\cancel{e^{-(\rho_u^0 K_t+\rho_u^{ci})}}+e^{-(\rho_u^0 K_t+\rho_u^{ci})-(\rho^0 K_t+\rho^{ci})}-\cancel{1}-\cancel{e^{-(\rho^0 K_t+\rho^{ci})}}-1-\cancel{e^{-(\rho_u^0 K_t+\rho_u^{ci})}}}{\left(1+e^{-(\rho_u^0 K_t+\rho_u^{ci})}\right)\left(1+e^{-(\rho^0 K_t+\rho^{ci})}\right)}$$

$$=\frac{e^{-(\rho_u^0 K_t+\rho_u^{ci})-(\rho^0 K_t+\rho^{ci})}-1}{1+e^{-(\rho^0 K_t+\rho^{ci})}+e^{-(\rho_u^0 K_t+\rho_u^{ci})}+e^{-(\rho_u^0 K_t+\rho_u^{ci})-(\rho^0 K_t+\rho^{ci})}}$$

$$=\frac{\boxed{e^{-\rho_u^0 K_t-\rho_u^{ci}-\rho^0 K_t-\rho^{ci}}-1}=①}{\boxed{1+e^{-(\rho^0 K_t+\rho^{ci})}+e^{-(\rho_u^0 K_t+\rho_u^{ci})}+e^{-\rho_u^0 K_t-\rho_u^{ci}-\rho^0 K_t-\rho^{ci}}}=②}$$

③ $=\frac{①}{②}$

$$\frac{\partial ①}{\partial \rho_u^{ci}}=-e^{-\rho_u^0 K_t-\rho_u^{ci}-\rho^0 K_t-\rho^{ci}}$$

$$\frac{\partial ②}{\partial \rho_u^{ci}}=-e^{-(\rho_u^0 K_t+\rho_u^{ci})}-e^{-\rho_u^0 K_t-\rho_u^{ci}-\rho^0 K_t-\rho^{ci}}$$

$$\frac{\partial ①}{\partial \rho_u^{ci}}\times ②=\left(-e^{-\rho_u^0 K_t-\rho_u^{ci}-\rho^0 K_t-\rho^{ci}}\right)\cdot\left(1+e^{-(\rho^0 K_t+\rho^{ci})}+e^{-(\rho_u^0 K_t+\rho_u^{ci})}+e^{-\rho_u^0 K_t-\rho_u^{ci}-\rho^0 K_t-\rho^{ci}}\right) \quad ④$$

$$①\times\frac{\partial ②}{\partial \rho_u^{ci}}=\left(e^{-\rho_u^0 K_t-\rho_u^{ci}-\rho^0 K_t-\rho^{ci}}-1\right)\cdot\left(-e^{-(\rho_u^0 K_t+\rho_u^{ci})}-e^{-\rho_u^0 K_t-\rho_u^{ci}-\rho^0 K_t-\rho^{ci}}\right) \quad ⑤$$

④ − ⑤:

$$\underline{-e^{-\rho_u^0 K_t-\rho_u^{ci}-\rho^0 K_t-\rho^{ci}}}-e^{-\rho_u^0 K_t-\rho_u^{ci}-2\rho^0 K_t-2\rho^{ci}}-e^{-2\rho_u^0 K_t-2\rho_u^{ci}-\cancel{\rho^0 K_t}-\rho^{ci}}-e^{-2\rho_u^0 K_t-\cancel{2\rho_u^{ci}}-2\rho^0 K_t-2\rho^{ci}}$$

$$+\cancel{e^{-2\rho_u^0 K_t-2\rho_u^{ci}-\rho^0 K_t-\rho^{ci}}}+\cancel{e^{-2\rho_u^0 K_t-2\rho_u^{ci}-2\rho^0 K_t-2\rho^{ci}}}-e^{-\rho_u^0 K_t-\rho_u^{ci}}-\underline{e^{-\rho_u^0 K_t-\rho_u^{ci}-\rho^0 K_t-\rho^{ci}}}$$

$$=-2e^{-\rho_u^0 K_t-\rho_u^{ci}-\rho^0 K_t-\rho^{ci}}-e^{-\rho_u^0 K_t-\rho_u^{ci}-2\rho^0 K_t-2\rho^{ci}}-e^{-\rho_u^0 K_t-\rho_u^{ci}}$$

$$\frac{\partial ③}{\partial \rho_u^{ci}}=\frac{④-⑤}{\left(1+e^{-(\rho^0 K_t+\rho^{ci})}+e^{-(\rho_u^0 K_t+\rho_u^{ci})}+e^{-\rho_u^0 K_t-\rho_u^{ci}-\rho^0 K_t-\rho^{ci}}\right)^2}$$

$$=\frac{-2e^{-\rho_u^0 K_t-\rho_u^{ci}-\rho^0 K_t-\rho^{ci}}-e^{-\rho_u^0 K_t-\rho_u^{ci}-2\rho^0 K_t-2\rho^{ci}}-e^{-\rho_u^0 K_t-\rho_u^{ci}}}{\left(1+e^{-(\rho^0 K_t+\rho^{ci})}+e^{-(\rho_u^0 K_t+\rho_u^{ci})}+e^{-\rho_u^0 K_t-\rho_u^{ci}-\rho^0 K_t-\rho^{ci}}\right)^2}$$

$$\frac{\partial Q(\theta,\theta')}{\partial \rho_u^{c_i}} = \sum_{n=1}^{N}\sum_{t=2}^{T}\Bigg\{ P(Z_{t-1}^{(n)}=1, Z_t^{(n)}=1, S_{t:t+1}=c_i)\Bigg[\Bigg(\frac{1+e^{-(\rho^0 k_t+\rho^{c_i})}+\cancel{e^{-(\rho_u^0 k_t+\rho_u^{c_i})}}+e^{-\rho_u^0 k_t-\rho_u^{c_i}-\rho^0 k_t-\rho^{c_i}}}{e^{-\rho_u^0 k_t-\rho_u^{c_i}-\rho^0 k_t-\rho^{c_i}}-1}\Bigg)$$

$$\cdot\ \frac{-2e^{-\rho_u^0 k_t-\rho_u^{c_i}-\rho^0 k_t-\rho^{c_i}}-e^{-\rho_u^0 k_t-\rho_u^{c_i}-2\rho^0 k_t-2\rho^{c_i}}-e^{-\rho_u^0 k_t-\rho_u^{c_i}}}{\left(1+e^{-(\rho^0 k_t+\rho^{c_i})}+e^{-(\rho_u^0 k_t+\rho_u^{c_i})}+e^{-\rho_u^0 k_t-\rho_u^{c_i}-\rho^0 k_t-\rho^{c_i}}\right)^{\cancel{2}}}\Bigg]$$

$$+P(Z_{t-1}^{(n)}=1, Z_t^{(n)}=2, S_{t:t+2}^{(n)}=c_i)\Bigg[\Bigg(1+\cancel{e^{-(\rho_u^0 k_t+\rho_u^{c_i})}}\Bigg)$$

$$\cdot\ \frac{0-1\times\left(-e^{-\rho_u^0 k_t-\rho_u^{c_i}}\right)}{\left[1+e^{-(\rho_u^0 k_t+\rho_u^{c_i})}\right]^{\cancel{2}}}\Bigg]\Bigg\}$$

$$= \sum_{n=1}^{N}\sum_{t=2}^{T}\Bigg\{ \underbrace{P(Z_{t-1}^{(n)}=1, Z_t^{(n)}=1, S_{t:t+2}=c_i)}_{H_{11}(t-1,c_i)}\cdot \underbrace{\frac{-2e^{-\rho_u^0 k_t-\rho_u^{c_i}-\rho^0 k_t-\rho^{c_i}}-e^{-\rho_u^0 k_t-\rho_u^{c_i}-2\rho^0 k_t-2\rho^{c_i}}-e^{-\rho_u^0 k_t-\rho_u^{c_i}}}{\left(e^{-\rho_u^0 k_t-\rho_u^{c_i}-\rho^0 k_t-\rho^{c_i}}-1\right)\cdot\left(1+e^{-(\rho^0 k_t+\rho^{c_i})}+e^{-(\rho_u^0 k_t+\rho_u^{c_i})}+e^{-\rho_u^0 k_t-\rho_u^{c_i}-\rho^0 k_t-\rho^{c_i}}\right)}}_{=(*)}$$

$$+\underbrace{P(Z_{t-1}^{(n)}=1, Z_t^{(n)}=2, S_{t:t+2}=c_i)}_{H_{12}(t-1,c_i)}\cdot \underbrace{\frac{e^{-\rho_u^0 k_t-\rho_u^{c_i}}}{\left(1+e^{-\rho_u^0 k_t+\rho_u^{c_i}}\right)}}_{=(*)}\Bigg\}$$

① ② ③ ④

$$(*) = -\boxed{e^{-\rho_u^0 k_t-\rho_u^{c_i}}}_{A}\left[2\boxed{e^{-\rho^0 k_t-\rho^{c_i}}}_{B}+e^{-2\rho^0 k_t-2\rho^{c_i}}+1\right]$$

$$\frac{\partial ①}{\partial \rho_u^0} = -k_t\, e^{-\rho_u^0 k_t-\rho_u^{c_i}-\rho^0 k_t-\rho^{c_i}}$$

$$= -A(2B+B^2+1)$$

$$= -A(B+1)^2$$

$$(*) = (AB-1)\cdot(1+A+B+AB)$$

$$\frac{\partial ②}{\partial \rho_u^0} = -k_t\, e^{-\rho_u^0 k_t-\rho_u^{c_i}} - k_t\, e^{-\rho_u^0 k_t-\rho_u^{c_i}-\rho^0 k_t-\rho^{c_i}}$$

$$\frac{\partial ①}{\partial \rho_u^0}\times ② = \left(-k_t\, e^{-\rho_u^0 k_t-\rho_u^{c_i}-\rho^0 k_t-\rho^{c_i}}\right)\times\left(1+e^{-\rho^0 k_t-\rho^{c_i}}+e^{-\rho_u^0 k_t-\rho_u^{c_i}}+e^{-\rho_u^0 k_t-\rho_u^{c_i}-\rho^0 k_t-\rho^{c_i}}\right) \quad ⑥$$

$$① \times \frac{\partial ②}{\partial \rho_u^0} = \left(e^{-\rho_u^0 k_t-\rho_u^{c_i}-\rho^0 k_t-\rho^{c_i}}-1\right)\times\left(-k_t e^{-\rho_u^0 k_t-\rho_u^{c_i}}-k_t\, e^{-\rho_u^0 k_t-\rho_u^{c_i}-\rho^0 k_t-\rho^{c_i}}\right) \quad ⑦$$

$$⑥-⑦:\ \underline{-k_t\, e^{-\rho_u^0 k_t-\rho_u^{c_i}-\rho^0 k_t-\rho^{c_i}}} - k_t\, e^{-\rho_u^0 k_t-\rho_u^{c_i}-2\rho^0 k_t-2\rho^{c_i}} - k_t\, e^{\cancel{-2\rho_u^0 k_t-2\rho_u^{c_i}-\rho^0 k_t-\rho^{c_i}}} - k_t\, e^{\cancel{-2\rho_u^0 k_t-2\rho_u^{c_i}-2\rho^0 k_t-2\rho^{c_i}}}$$

$$+k_t\, e^{\cancel{-2\rho_u^0 k_t-2\rho_u^{c_i}-\rho^0 k_t-\rho^{c_i}}} + k_t\, e^{\cancel{-2\rho_u^0 k_t-2\rho_u^{c_i}-2\rho^0 k_t-2\rho^{c_i}}} - k_t\, e^{-\rho_u^0 k_t-\rho_u^{c_i}} - \underline{k_t\, e^{-\rho_u^0 k_t-\rho_u^{c_i}-\rho^0 k_t-\rho^{c_i}}}$$

$$= -2k_t\, e^{-\rho_u^0 k_t-\rho_u^{c_i}-\rho^0 k_t-\rho^{c_i}} - k_t\, e^{-\rho_u^0 k_t-\rho_u^{c_i}-2\rho^0 k_t-2\rho^{c_i}} - k_t\, e^{-\rho_u^0 k_t-\rho_u^{c_i}}$$

$$= k_t\left(-2e^{-\rho_u^0 k_t-\rho_u^{c_i}-\rho^0 k_t-\rho^{c_i}} - e^{-\rho_u^0 k_t-\rho_u^{c_i}-2\rho^0 k_t-2\rho^{c_i}} - e^{-\rho_u^0 k_t-\rho_u^{c_i}}\right)$$

$$\frac{\partial ③}{\partial \rho_u^0} = \frac{⑥ - ⑦}{\left(1+e^{-(\rho^0 k_t+\rho^{ci})}+e^{-(\rho_u^0 k_t+\rho_u^{ci})}+e^{-\rho_u^0 k_t-\rho_u^{ci}-\rho^0 k_t-\rho^{ci}}\right)^2}$$

$$= \frac{k_t\left(-2e^{-\rho_u^0 k_t-\rho_u^{ci}-\rho^0 k_t-\rho^{ci}} - e^{-\rho_u^0 k_t-\rho_u^{ci}-2\rho^0 k_t-2\rho^{ci}} - e^{-\rho_u^0 k_t-\rho_u^{ci}}\right)}{\left(1+e^{-(\rho^0 k_t+\rho^{ci})}+e^{-(\rho_u^0 k_t+\rho_u^{ci})}+e^{-\rho_u^0 k_t-\rho_u^{ci}-\rho^0 k_t-\rho^{ci}}\right)^2}$$

$$\frac{\partial Q(\theta,\theta')}{\partial \rho_u^0} = \sum_{n=1}^{N}\sum_{t=2}^{T}\sum_{c_i=1}^{C_i}\Bigg\{ P(z_{t-1}^{(n)}=1, z_t^{(n)}=1, s_{t:t+2}^{(n)}=c_i)\left[\frac{1+e^{-(\rho^0 k_t+\rho^{ci})}+\cancel{e^{-(\rho_u^0 k_t+\rho_u^{ci})}}+e^{-\rho_u^0 k_t-\rho_u^{ci}-\rho^0 k_t-\rho^{ci}}}{e^{-\rho_u^0 k_t-\rho_u^{ci}-\rho^0 k_t-\rho^{ci}}-1}\right.$$

$$\left.\cdot\ \frac{k_t\left(-2e^{-\rho_u^0 k_t-\rho_u^{ci}-\rho^0 k_t-\rho^{ci}} - e^{-\rho_u^0 k_t-\rho_u^{ci}-2\rho^0 k_t-2\rho^{ci}} - e^{-\rho_u^0 k_t-\rho_u^{ci}}\right)}{\left(1+e^{-(\rho^0 k_t+\rho^{ci})}+e^{-(\rho_u^0 k_t+\rho_u^{ci})}+e^{-\rho_u^0 k_t-\rho_u^{ci}-\rho^0 k_t-\rho^{ci}}\right)^{\cancel{2}}}\right]$$

$$+\ P(z_{t-1}^{(n)}=1, z_t^{(n)}=2, s_{t:t+2}^{(n)}=c_i)\left[\left(1+\cancel{e^{-\rho_u^0 k_t-\rho_u^{ci}}}\right)\cdot\frac{0-1\times\left(-k_t e^{-\rho_u^0 k_t-\rho_u^{ci}}\right)}{\left(1+e^{-\rho_u^0 k_t-\rho_u^{ci}}\right)^{\cancel{2}}}\right]\Bigg\}$$

$$= k_t(*) = -k_t A(B+1)^2$$

$$= \sum_{n=1}^{N}\sum_{t=2}^{T}\sum_{c_i=1}^{C_i}\Bigg\{ \underbrace{P(z_{t-1}^{(n)}=1, z_t^{(n)}=1, s_{t:t+2}^{(n)}=c_i)}_{H_{1:}(t-1,c_i)}\cdot\frac{k_t\left(-2e^{-\rho_u^0 k_t-\rho_u^{ci}-\rho^0 k_t-\rho^{ci}} - e^{-\rho_u^0 k_t-\rho_u^{ci}-2\rho^0 k_t-2\rho^{ci}} - e^{-\rho_u^0 k_t-\rho_u^{ci}}\right)}{\left(e^{-\rho_u^0 k_t-\rho_u^{ci}-\rho^0 k_t-\rho^{ci}}-1\right)\cdot\left(1+e^{-(\rho^0 k_t+\rho^{ci})}+e^{-(\rho_u^0 k_t+\rho_u^{ci})}+e^{-\rho_u^0 k_t-\rho_u^{ci}-\rho^0 k_t-\rho^{ci}}\right)}$$

$$= (*) = (AB-1)(1+A+B+AB)$$

$$+\ \underbrace{P(z_{t-1}^{(n)}=1, z_t^{(n)}=2, s_{t:t+2}^{(n)}=c_i)}_{H_{1:2}(t-1,c_i)}\cdot\frac{k_t e^{-\rho_u^0 k_t-\rho_u^{ci}}}{1+e^{-\rho_u^0 k_t-\rho_u^{ci}}}\Bigg\}$$

$$= \frac{k_t A}{1+A}$$

$$\frac{\partial ①}{\partial \rho^{ci}} = -e^{-\rho_u^o k_t - \rho_u^{ci} - \rho^o k_t - \rho^{ci}}$$

$$\frac{\partial ②}{\partial \rho^{ci}} = -e^{-\rho^o k_t - \rho^{ci}} - e^{-\rho_u^o k_t - \rho_u^{ci} - \rho^o k_t - \rho^{ci}}$$

$$\frac{\partial ①}{\partial \rho^{ci}} \times ② = \left(-e^{-\rho_u^o k_t - \rho_u^{ci} - \rho^o k_t - \rho^{ci}}\right)\cdot\left(1 + e^{-\rho^o k_t - \rho^{ci}} + e^{-\rho_u^o k_t - \rho_u^{ci}} + e^{-\rho_u^o k_t - \rho_u^{ci} - \rho^o k_t - \rho^{ci}}\right) \quad ⑧$$

$$① \times \frac{\partial ②}{\partial \rho^{ci}} = \left(e^{-\rho_u^o k_t - \rho_u^{ci} - \rho^o k_t - \rho^{ci}} - 1\right)\cdot\left(-e^{-\rho^o k_t - \rho^{ci}} - e^{-\rho_u^o k_t - \rho_u^{ci} - \rho^o k_t - \rho^{ci}}\right) \quad ⑨$$

$$⑧ - ⑨ : \quad \underline{-e^{-\rho_u^o k_t - \rho_u^{ci} - \rho^o k_t - \rho^{ci}}} - e^{-\rho_u^o k_t - \rho_u^{ci} \cancel{- 2\rho^o k_t - 2\rho^{ci}}} - e^{-2\rho_u^o k_t - 2\rho_u^{ci} - \rho^o k_t - \rho^{ci}} - e^{-2\rho_u^o k_t - 2\rho_u^{ci} \cancel{- 2\rho^o k_t - 2\rho^{ci}}}$$

$$+ e^{-\rho_u^o k_t - \rho_u^{ci} \cancel{- 2\rho^o k_t - 2\rho^{ci}}} + e^{-2\rho_u^o k_t - 2\rho_u^{ci} \cancel{- 2\rho^o k_t - 2\rho^{ci}}} - e^{-\rho^o k_t - \rho^{ci}} \underline{- e^{-\rho_u^o k_t - \rho_u^{ci} - \rho^o k_t - \rho^{ci}}}$$

$$= -2e^{-\rho_u^o k_t - \rho_u^{ci} - \rho^o k_t - \rho^{ci}} - e^{-2\rho_u^o k_t - 2\rho_u^{ci} - \rho^o k_t - \rho^{ci}} - e^{-\rho^o k_t - \rho^{ci}}$$

$$\frac{\partial ③}{\partial \rho^{ci}} = \frac{⑧ - ⑨}{\left(1 + e^{-(\rho^o k_t + \rho^{ci})} + e^{-(\rho_u^o k_t + \rho_u^{ci})} + e^{-\rho_u^o k_t - \rho_u^{ci} - \rho^o k_t - \rho^{ci}}\right)^2}$$

$$= \frac{-2e^{-\rho_u^o k_t - \rho_u^{ci} - \rho^o k_t - \rho^{ci}} - e^{-2\rho_u^o k_t - 2\rho_u^{ci} - \rho^o k_t - \rho^{ci}} - e^{-\rho^o k_t - \rho^{ci}}}{\left(1 + e^{-(\rho^o k_t + \rho^{ci})} + e^{-(\rho_u^o k_t + \rho_u^{ci})} + e^{-\rho_u^o k_t - \rho_u^{ci} - \rho^o k_t - \rho^{ci}}\right)^2}$$

$$\frac{\partial Q(\theta, \theta')}{\partial \rho^{ci}} = \sum_{n=1}^{N}\sum_{t=2}^{T}\Bigg\{ P(z_{t-1}^{(n)} = 1, z_t^{(n)} = 1, S_{t:t+2}^{(n)} = c_i) \left[\frac{1 + e^{-(\rho^o k_t + \rho^{ci})} + \cancel{e^{-(\rho_u^o k_t + \rho_u^{ci})}} + e^{-\rho_u^o k_t - \rho_u^{ci} - \rho^o k_t - \rho^{ci}}}{e^{-\rho_u^o k_t - \rho_u^{ci} - \rho^o k_t - \rho^{ci}} - 1}\right.$$

$$\left. \cdot \frac{-2e^{-\rho_u^o k_t - \rho_u^{ci} - \rho^o k_t - \rho^{ci}} - e^{-2\rho_u^o k_t - 2\rho_u^{ci} - \rho^o k_t - \rho^{ci}} - e^{-\rho^o k_t - \rho^{ci}}}{\left(1 + e^{-(\rho^o k_t + \rho^{ci})} + e^{-(\rho_u^o k_t + \rho_u^{ci})} + e^{-\rho_u^o k_t - \rho_u^{ci} - \rho^o k_t - \rho^{ci}}\right)^{\cancel{2}}}\right]$$

$$+ P(z_{t-1}^{(n)} = 1, z_t^{(n)} = 2, S_{t:t+2}^{(n)} = c_i) \left[\left(1 + \cancel{e^{-\rho^o k_t - \rho^{ci}}}\right) \cdot \frac{0 - 1 \times \left(-e^{-\rho^o k_t - \rho^{ci}}\right)}{\left(1 + e^{-\rho^o k_t - \rho^{ci}}\right)^{\cancel{2}}}\right]\Bigg\}$$

$$= -B(A+1)^2$$

$$= \sum_{n=1}^{N}\sum_{t=2}^{T}\Bigg\{ \underbrace{P(z_{t-1}^{(n)} = 1, z_t^{(n)} = 1, S_{t:t+2}^{(n)} = c_i)}_{H_{11}(t-1, c_i)} \cdot \frac{-2e^{-\rho_u^o k_t - \rho_u^{ci} - \rho^o k_t - \rho^{ci}} - e^{-2\rho_u^o k_t - 2\rho_u^{ci} - \rho^o k_t - \rho^{ci}} - e^{-\rho^o k_t - \rho^{ci}}}{\left(e^{-\rho_u^o k_t - \rho_u^{ci} - \rho^o k_t - \rho^{ci}} - 1\right)\cdot\left(1 + e^{-(\rho^o k_t + \rho^{ci})} + e^{-(\rho_u^o k_t + \rho_u^{ci})} + e^{-\rho_u^o k_t - \rho_u^{ci} - \rho^o k_t - \rho^{ci}}\right)}$$

$$= (AB - 1)(1 + A + B + AB)$$

$$+ \underbrace{P(z_{t-1}^{(n)} = 1, z_t^{(n)} = 2, S_{t:t+2}^{(n)} = c_i)}_{H_{12}(t-1, c_i)} \cdot \frac{e^{-\rho^o k_t - \rho^{ci}}}{\left(1 + e^{-\rho^o k_t - \rho^{ci}}\right)}\Bigg\} = \frac{B}{1+B}$$

$$\frac{\partial ①}{\partial \rho^0} = -k_t\, e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}}$$

$$\frac{\partial ②}{\partial \rho^0} = -k_t\, e^{-\rho^0 k_t - \rho^{ci}} - k_t\, e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}}$$

$$\frac{\partial ①}{\partial \rho^0} \times ② = \left(-k_t\, e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}}\right) \times \left(1 + e^{-\rho^0 k_t - \rho^{ci}} + e^{-\rho_u^0 k_t - \rho_u^{ci}} + e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}}\right) \quad ⑩$$

$$① \times \frac{\partial ②}{\partial \rho^0} = \left(e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}} - 1\right) \times \left(-k_t e^{-\rho^0 k_t - \rho^{ci}} - k_t\, e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}}\right) \quad ⑪$$

⑩ − ⑪:

$$\underline{-k_t\, e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}}} - k_t\, e^{-\rho_u^0 k_t - \rho_u^{ci} \cancel{- 2\rho^0 k_t} - 2\rho^{ci}} - k_t\, e^{-2\rho_u^0 k_t - 2\rho_u^{ci} - \rho^0 k_t - \rho^{ci}} - k_t\, e^{-2\rho_u^0 k_t \cancel{- 2\rho_u^{ci}} - 2\rho^0 k_t - 2\rho^{ci}}$$

$$+ k_t\, e^{-\rho_u^0 k_t - \rho_u^{ci} \cancel{- 2\rho^0 k_t} - 2\rho^{ci}} + k_t\, e^{-2\rho_u^0 k_t \cancel{- 2\rho_u^{ci}} - 2\rho^0 k_t - 2\rho^{ci}} - k_t\, e^{-\rho^0 k_t - \rho^{ci}} \underline{- k_t\, e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}}}$$

$$= -2k_t\, e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}} - k_t\, e^{-2\rho_u^0 k_t - 2\rho_u^{ci} - \rho^0 k_t - \rho^{ci}} - k_t\, e^{-\rho^0 k_t - \rho^{ci}}$$

$$= k_t\left(-2e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}} - e^{-2\rho_u^0 k_t - 2\rho_u^{ci} - \rho^0 k_t - \rho^{ci}} - e^{-\rho_u^0 k_t - \rho_u^{ci}}\right)$$

$$\frac{\partial ③}{\partial \rho^0} = \frac{⑩ - ⑪}{\left(1 + e^{-(\rho^0 k_t + \rho^{ci})} + e^{-(\rho_u^0 k_t + \rho_u^{ci})} + e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}}\right)^2}$$

$$= \frac{k_t\left(-2e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}} - e^{-2\rho_u^0 k_t - 2\rho_u^{ci} - \rho^0 k_t - \rho^{ci}} - e^{-\rho_u^0 k_t - \rho_u^{ci}}\right)}{\left(1 + e^{-(\rho^0 k_t + \rho^{ci})} + e^{-(\rho_u^0 k_t + \rho_u^{ci})} + e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}}\right)^2}$$

$$\frac{\partial Q(\theta, \theta')}{\partial \rho^0} = \sum_{n=1}^{N} \sum_{t=2}^{T} \Bigg\{ P(Z_{t-1}^{(n)} = 1, Z_t^{(n)} = 1, S_{t:t+1}^{(n)} = c_i) \Bigg[\frac{1 + e^{-(\rho^0 k_t + \rho^{ci})} + \cancel{e^{-(\rho_u^0 k_t + \rho_u^{ci})}} + e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}}}{e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}} - 1}$$

$$\cdot \frac{k_t\left(-2e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}} - e^{-2\rho_u^0 k_t - 2\rho_u^{ci} - \rho^0 k_t - \rho^{ci}} - e^{-\rho_u^0 k_t - \rho_u^{ci}}\right)}{\left(1 + e^{-(\rho^0 k_t + \rho^{ci})} + e^{-(\rho_u^0 k_t + \rho_u^{ci})} + e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}}\right)^{\cancel{2}}} \Bigg]$$

$$+ P(Z_{t-1}^{(n)} = 1, Z_t^{(n)} = 12, S_{t:t+1}^{(n)} = c_i) \Bigg[\left(1 + \cancel{e^{-\rho^0 k_t - \rho^{ci}}}\right) \cdot \frac{0 - 1 \times \left(-k_t\, e^{-\rho^0 k_t - \rho^{ci}}\right)}{\left(1 + e^{-\rho^0 k_t - \rho^{ci}}\right)^{\cancel{2}}} \Bigg] \Bigg\}$$

$$= \sum_{n=1}^{N} \sum_{t=2}^{T} \Bigg\{ \underbrace{P(Z_{t-1}^{(n)} = 1, Z_t^{(n)} = 1, S_{t:t+1}^{(n)} = c_i)}_{H_{1,1}(t-1,\, c_i)} \cdot \frac{k_t\left(-2e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}} - e^{-2\rho_u^0 k_t - 2\rho_u^{ci} - \rho^0 k_t - \rho^{ci}} - e^{-\rho_u^0 k_t - \rho_u^{ci}}\right)}{\left(e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}} - 1\right) \cdot \left(1 + e^{-(\rho^0 k_t + \rho^{ci})} + e^{-(\rho_u^0 k_t + \rho_u^{ci})} + e^{-\rho_u^0 k_t - \rho_u^{ci} - \rho^0 k_t - \rho^{ci}}\right)}$$

$$= -k_t B (A+1)^2$$

$$= (AB - 1)(1 + A + B + AB)$$

$$+ \underbrace{P(Z_{t-1}^{(n)} = 1, Z_t^{(n)} = 12, S_{t:t+1}^{(n)} = c_i)}_{H_{1,12}(t-1,\, c_i)} \cdot \frac{k_t\, e^{-\rho^0 k_t - \rho^{ci}}}{1 + e^{-\rho^0 k_t - \rho^{ci}}} = \Bigg\} \frac{k_t B}{1 + B}$$

$$1 - \frac{1}{1+e^{-\delta^0 k_t - \delta^{c_i}}} = \frac{1+e^{-\delta^0 k_t - \delta^{c_i}} - 1}{1+e^{-\delta^0 k_t - \delta^{c_i}}} = \frac{e^{-\delta^0 k_t - \delta^{c_i}}}{1+e^{-\delta^0 k_t - \delta^{c_i}}} \quad ①2$$

$$\frac{\partial ①2}{\partial \delta^{c_i}} = \frac{\left(-e^{-\delta^0 k_t - \delta^{c_i}}\right)\left(1+e^{-\delta^0 k_t - \delta^{c_i}}\right) - \left(e^{-\delta^0 k_t - \delta^{c_i}}\right)\left(-e^{-\delta^0 k_t - \delta^{c_i}}\right)}{\left(1+e^{-\delta^0 k_t - \delta^{c_i}}\right)^2}$$

$$= \frac{-e^{-\delta^0 k_t - \delta^{c_i}} - \cancel{e^{-2\delta^0 k_t - 2\delta^{c_i}}} + \cancel{e^{-2\delta^0 k_t - 2\delta^{c_i}}}}{\left(1+e^{-\delta^0 k_t - \delta^{c_i}}\right)^2} = -\frac{e^{-\delta^0 k_t - \delta^{c_i}}}{\left(1+e^{-\delta^0 k_t - \delta^{c_i}}\right)^2}$$

$$\frac{\partial Q(\theta,\theta')}{\partial \delta^{c_i}} = \sum_{n=1}^{N}\sum_{t=2}^{T}\Big\{ P(z_{t-1}^{(n)}=11, z_t^{(n)}=11, S_{t:t+2}^{(n)}=c_i)\cdot\left[\frac{1+\cancel{e^{-\delta^0 k_t - \delta^{c_i}}}}{\cancel{e^{-\delta^0 k_t - \delta^{c_i}}}} \times \left(-\frac{\cancel{e^{-\delta^0 k_t - \delta^{c_i}}}}{(1+e^{-\delta^0 k_t - \delta^{c_i}})^{\cancel{2}}}\right)\right]$$

$$+ P(z_{t-1}^{(n)}=11, z_t^{(n)}=12, S_{t:t+2}^{(n)}=c_i)\cdot\left[(1+\cancel{e^{-\delta^0 k_t - \delta^{c_i}}}) \times \frac{0-1\times(-e^{-\delta^0 k_t - \delta^{c_i}})}{(1+e^{-\delta^0 k_t - \delta^{c_i}})^{\cancel{2}}}\right]\Big\}$$

$$= \sum_{n=1}^{N}\sum_{t=2}^{T}\Big\{ -\underbrace{P(z_{t-1}^{(n)}=11, z_t^{(n)}=11, S_{t:t+2}^{(n)}=c_i)}_{H_{11,11}(t-1,c_i)}\cdot\frac{1}{1+e^{-\delta^0 k_t - \delta^{c_i}}} + \underbrace{P(z_{t-1}^{(n)}=11, z_t^{(n)}=12, S_{t:t+2}^{(n)}=c_i)}_{H_{11,12}(t-1,c_i)}\cdot\frac{e^{-\delta^0 k_t - \delta^{c_i}}}{1+e^{-\delta^0 k_t - \delta^{c_i}}}\Big\}$$

$$\frac{\partial ①2}{\partial \delta^0} = \frac{\left(-k_t e^{-\delta^0 k_t - \delta^{c_i}}\right)\left(1+e^{-\delta^0 k_t - \delta^{c_i}}\right) - \left(e^{-\delta^0 k_t - \delta^{c_i}}\right)\left(-k_t e^{-\delta^0 k_t - \delta^{c_i}}\right)}{\left(1+e^{-\delta^0 k_t - \delta^{c_i}}\right)^2}$$

$$= \frac{-k_t e^{-\delta^0 k_t - \delta^{c_i}} - \cancel{k_t e^{-2\delta^0 k_t - 2\delta^{c_i}}} + \cancel{k_t e^{-2\delta^0 k_t - 2\delta^{c_i}}}}{\left(1+e^{-\delta^0 k_t - \delta^{c_i}}\right)^2} = -\frac{k_t e^{-\delta^0 k_t - \delta^{c_i}}}{\left(1+e^{-\delta^0 k_t - \delta^{c_i}}\right)^2}$$

$$\frac{\partial Q(\theta,\theta')}{\partial \delta^0} = \sum_{n=1}^{N}\sum_{t=2}^{T}\Big\{ P(z_{t-1}^{(n)}=11, z_t^{(n)}=11, S_{t:t+2}^{(n)}=c_i)\cdot\left[\frac{1+\cancel{e^{-\delta^0 k_t - \delta^{c_i}}}}{\cancel{e^{-\delta^0 k_t - \delta^{c_i}}}} \times \left(-\frac{k_t\cancel{e^{-\delta^0 k_t - \delta^{c_i}}}}{(1+e^{-\delta^0 k_t - \delta^{c_i}})^{\cancel{2}}}\right)\right]$$

$$+ P(z_{t-1}^{(n)}=11, z_t^{(n)}=12, S_{t:t+2}^{(n)}=c_i)\cdot\left[(1+\cancel{e^{-\delta^0 k_t - \delta^{c_i}}}) \times \frac{0-1\times(-k_t e^{-\delta^0 k_t - \delta^{c_i}})}{(1+e^{-\delta^0 k_t - \delta^{c_i}})^{\cancel{2}}}\right]\Big\}$$

$$= \sum_{n=1}^{N}\sum_{t=2}^{T}\Big\{ -\underbrace{P(z_{t-1}^{(n)}=11, z_t^{(n)}=11, S_{t:t+2}^{(n)}=c_i)}_{H_{11,11}(t-1,c_i)}\cdot\frac{k_t}{1+e^{-\delta^0 k_t - \delta^{c_i}}} + \underbrace{P(z_{t-1}^{(n)}=11, z_t^{(n)}=12, S_{t:t+2}^{(n)}=c_i)}_{H_{11,12}(t-1,c_i)}\cdot\frac{k_t e^{-\delta^0 k_t - \delta^{c_i}}}{1+e^{-\delta^0 k_t - \delta^{c_i}}}\Big\}$$