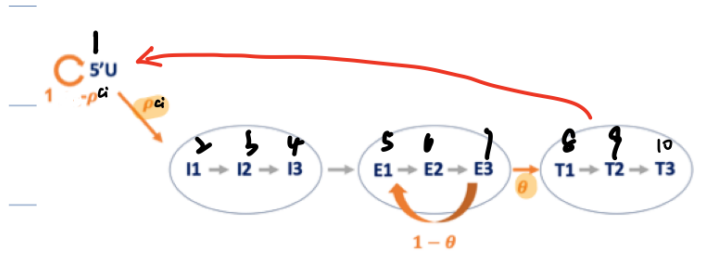


Known: n transcript

footprint counts $\underline{X} = (\underline{X}^{(n)})_{n=1}^{N}$ observed data

mRNA sequence $\underline{S} = (\underline{S}^{(n)})_{n=1}^{N}$

$E^{(n)}$ normalized factor

footprint state $\underline{Z} = (\underline{Z}^{(n)})_{n=1}^{N}$ hidden state

$M = 10$ total states

$X_t^{(n)} \mid Z_t^{(n)} = z \sim Poi(\mu_{z,t}^{(n)} E^{(n)})$

$\mu_{z,t}^{(n)} \sim Gamma(\alpha_z, \beta_z)$

$x \sim Poisson(\theta) : f(x|\theta) = \frac{\theta^x e^{-\theta}}{x!}$

$\mu \sim gamma(\alpha, \beta) : g(\mu) = \frac{\beta^\alpha}{\Gamma(\alpha)} \mu^{\alpha-1} e^{-\beta\mu}$

$\mu E \sim gamma(\alpha, \frac{\beta}{E}) : g(\mu E) = \frac{(\frac{\beta}{E})^\alpha}{\Gamma(\alpha)} (\mu E)^{\alpha-1} e^{-\frac{\beta}{E}\cdot\mu E}$

$$E(X_{z,t}) = E[E(X_{z,t}|\mu_{z,t})] = E(\mu_{z,t}E) = \mu_{z,t}E = \frac{\alpha}{\beta}E \quad ✓$$

$$\begin{aligned} Var(X_{z,t}) &= Var[E(X_{z,t}|\mu_{z,t})] + E[Var(X_{z,t}|\mu_{z,t})] \\ &= Var(\mu_{z,t}E) + E(\mu_{z,t}E) \\ &= E^2 Var(\mu_{z,t}) + E\cdot E[\mu_{z,t}] \\ &= E^2\frac{\alpha}{\beta^2} + E\frac{\alpha}{\beta} \quad ✓ \end{aligned}$$

$$X \sim NB\left(\alpha, \frac{E}{E+\beta}\right)$$

$$P(x|\alpha,\beta) = \int_0^\infty P(x, \mu E|\alpha,\beta)\, d\mu E$$

$$= \int_0^\infty P(x|\mu E, \alpha, \beta)\cdot P(\mu E|\alpha,\beta)\, d\mu E$$

$$= \int_0^\infty \frac{(\mu E)^{x}}{x!} e^{-\mu E} \cdot \frac{(\frac{\beta}{E})^{\alpha}}{\Gamma(\alpha)} (\mu E)^{\alpha-1} e^{-\frac{\beta}{E}\mu E}\, d\mu E$$

$$= \frac{1}{x!}\frac{(\frac{\beta}{E})^{\alpha}}{\Gamma(\alpha)}\frac{\Gamma(x+\alpha)}{(1+\frac{\beta}{E})^{x+\alpha}} \underbrace{\int_0^\infty \frac{(1+\frac{\beta}{E})^{x+\alpha}}{\Gamma(x+\alpha)} (\mu E)^{x+\alpha-1}\cdot e^{-(1+\frac{\beta}{E})\mu E}\, d\mu E}_{\text{gamma}(x+\alpha,\ 1+\frac{\beta}{E})\ =1}$$

$$= \frac{\Gamma(x+\alpha)}{x!\,\Gamma(\alpha)}\frac{(\frac{\beta}{E})^{\alpha}}{(1+\frac{\beta}{E})^{x}(1+\frac{\beta}{E})^{\alpha}}$$

$$= \frac{\Gamma(x+\alpha)}{x!\,\Gamma(\alpha)}\left(\frac{\frac{\beta}{E}}{1+\frac{\beta}{E}}\right)^{\alpha}\left(\frac{1}{1+\frac{\beta}{E}}\right)^{x}$$

$$\Gamma(n) = (n-1)! \quad , \quad C^n_r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

$$= \frac{(x+\alpha-1)!}{x!\,(\alpha-1)!}\left(\frac{\beta}{E+\beta}\right)^{\alpha}\left(\frac{E}{E+\beta}\right)^{x}$$

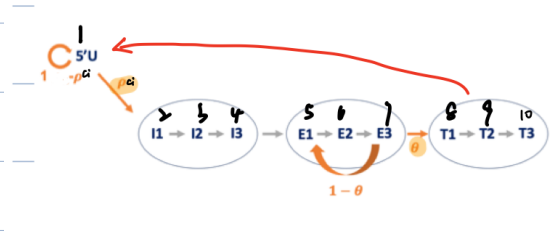
$$= \binom{x+\alpha-1}{x}\left(\frac{\beta}{E+\beta}\right)^{\alpha}\left(1-\frac{\beta}{E+\beta}\right)^{x}$$

$$\Rightarrow X_{z,t} \sim NB\left(\alpha, \frac{\beta}{E+\beta}\right) \quad ; \quad g(x|r,p) = \binom{x+r-1}{x} p^{r}(1-p)^{x}$$

check : Mean $= \frac{(1-p)r}{p} = \frac{\frac{E}{E+\beta}\alpha}{\frac{\beta}{E+\beta}} = \frac{\alpha}{\beta}E$ ✓

Variance $= \frac{(1-p)r}{p^2} = \frac{\alpha}{\beta}E\cdot\frac{1}{\frac{\beta}{E+\beta}} = \frac{\alpha E(E+\beta)}{\beta^2} = E^2\frac{\alpha}{\beta^2} + E\frac{\alpha}{\beta}$ ✓

E - step



forward algorithm: $F_m(t) = P(x_1, \cdots x_t, z_t = m \mid \theta)$

$$P(\underline{x}_{1:T} \mid \theta) = \sum_{m=1}^{M} F_m(T)$$

$$F_j(t+1) = P(x_1, \cdots, x_{t+1}, z_{t+1} = j \mid \theta)$$

$$= \sum_{m=1}^{M} P(x_1, \cdots, x_{t+1}, z_{t+1} = j, z_t = m \mid \theta)$$

$$= \sum_{m=1}^{M} P(x_{t+1} \mid x_1, \cdots, x_t, z_{t+1} = j, z_t = m, \theta) \cdot P(x_1, \cdots, x_t, z_{t+1} = j, z_t = m \mid \theta)$$

$$= \sum_{m=1}^{M} P(x_{t+1} \mid z_{t+1} = j) \cdot P(z_{t+1} = j \mid x_1, \cdots, x_t, z_t = m, \theta) \cdot P(x_1, \cdots, x_t, z_t = m \mid \theta)$$

$$= \sum_{m=1}^{M} P(x_{t+1} \mid z_{t+1} = j) \cdot P(z_{t+1} = j \mid z_t = m) \cdot F_m(t)$$

$$= \left(\sum_{m=1}^{M} F_m(t) \cdot P(z_{t+1} = j \mid z_t = m) \right) \cdot P(x_{t+1} \mid z_{t+1} = j)$$

$F_m(1) = F_1(1) = g(x_1 \mid \alpha_1, \beta_1)$ assume to start at state 1

backward algorithm: $B_m(t) = P(x_{t+1}, \cdots x_T \mid z_t = m, \theta)$

$$P(\underline{x}_{1:T} \mid \theta) = \sum_{m=1}^{M} P(\underline{x}_{1:T}, z_1 = m \mid \theta)$$

$$= \sum_{m=1}^{M} P(\underline{x}_{1:T} \mid z_1 = m, \theta) \cdot P(z_1 = m \mid \theta)$$

$$= \sum_{m=1}^{M} P(x_1 \mid \underline{x}_{2:T}, z_1 = m, \theta) \cdot (\underline{x}_{2:T} \mid z_1 = m, \theta) \cdot P(z_1 = m \mid \theta)$$

$$= \sum_{m=1}^{M} P(x_1 \mid z_1 = m, \theta) \cdot B_m(1) \cdot P(z_1 = m \mid \theta)$$

$$= P(x_1 \mid z_1 = 1, \theta) \cdot B_1(1) \cdot P(z_1 = 1 \mid \theta)$$ assume to start at state 1

$$B_m(t) = P(x_{t+1}, \cdots, x_T \mid z_t = m, \theta)$$

$$= \sum_{j=1}^{M} P(x_{t+1}, \cdots, x_T, z_{t+1} = j \mid z_t = m, \theta)$$

$$= \sum_{j=1}^{M} P(x_{t+1}, \cdots, x_T \mid z_t = m, z_{t+1} = j, \theta) \cdot P(z_{t+1} = j \mid z_t = m, \theta)$$

$$= \sum_{j=1}^{M} P(x_{t+1}, \cdots, x_T \mid z_{t+1} = j, \theta) \cdot P(z_{t+1} = j \mid z_t = m, \theta)$$

$$= \sum_{j=1}^{M} P(x_{t+1} \mid x_{t+2}, \cdots, x_T, z_{t+1} = j, \theta) \cdot P(x_{t+2}, \cdots, x_T \mid z_{t+1} = j, \theta) \cdot P(z_{t+1} = j \mid z_t = m, \theta)$$

$$= \sum_{j=1}^{M} P(x_{t+1} \mid z_{t+1} = j, \theta) \cdot B_j(t+1) \cdot P(z_{t+1} = j \mid z_t = m, \theta)$$

$B_m(T) = 1$

$$\sum_{n=1}^{N} \log P(\underline{x}_{1:T}^{(n)}, \underline{z}_{1:T}^{(n)} \mid \theta, \underline{s}_{1:T}^{(n)}) \propto \sum_{n=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} I(z_{t}^{(n)} = m) \log P(x_t^{(n)} \mid z_t^{(n)} = m, \theta)$$

$$+ \sum_{n=1}^{N} \sum_{t=2}^{T} \sum_{c_i \in \Omega} \left\{ I(s_{t+1:t+3}^{(n)} = c_i) \log P(z_{t+1}^{(n)} = j \mid z_t^{(n)} = m, \theta, s_{t+1:t+3}^{(n)} = c_i) \right\}$$

Viterbi algorithm:

$$\underset{\underline{z}_{1:T}}{\operatorname{argmax}}\ P(\underline{z}_{1:T} \mid \underline{x}_{1:T}, \theta)$$

$$\delta_m(t) = \max P(x_1, \cdots, x_t$$

$$\delta_j(t+1) = \max P(x_1, \cdots, x_{t+1}$$

$$= \max \delta_m(t) \cdot P(z_{t+1}$$

$$\varphi_j(t+1) = \underset{1 \le m \le M}{\operatorname{argmax}}\ \delta_m(t) \cdot P(z_{t+1} = j \mid z_t = m, \theta)$$

store index

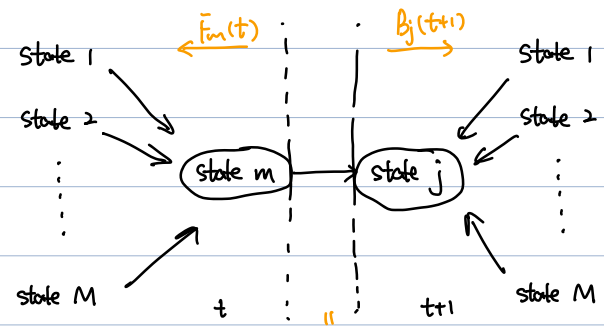
one state probability:

$$L_m(t) = P(z_t = m \mid \underline{x}_{1:T}, \theta)$$

$$= \frac{F_m(t) \cdot B_m(t)}{\sum_{j=1}^{M} F_j(t) \cdot B_j(t)}$$

two states probability:

$$H_{mj}(t) = P(z_t = m, z_{t+1} = j \mid \underline{x}_{1:T}, \theta)$$

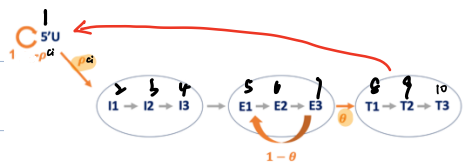


$$= P(z_{t+1} = j \mid z_t = m, \theta) \cdot P(x_{t+1} \mid z_{t+1} = j, \theta)$$

$$H_{mj}(t) = \frac{F_m(t) \cdot P(z_{t+1} = j \mid z_t = m, \theta) \cdot P(x_{t+1} \mid z_{t+1} = j, \theta) \cdot B_j(t+1)}{\sum_{m=1}^{M} \sum_{j=1}^{M} F_m(t) \cdot P(z_{t+1} = j \mid z_t = m, \theta) \cdot P(x_{t+1} \mid z_{t+1} = j, \theta) \cdot B_j(t+1)}$$

$$H_{mj}(t, c_i) = \begin{cases} H_{mj}(t) & \text{if } s_{t+1:t+3} = c_i \\ 0 & \text{if } s_{t+1:t+3} \neq c_i \end{cases}$$

M - Step



$m=1,\ m'=2 : p^{ci}$

$m=7,\ m'=8 : \theta$

$\begin{cases} 1 & S_{t:t+2} = ci,\ ci \in \text{stop codons} \\ 0 & \text{ow.} \end{cases}$

log likelihood for single transcript:

$$\log P(\underline{x}_{1:T}, \underline{z}_{1:T} \mid \theta, \underline{S}_{1:T})$$

$$= \log \left[P(\underline{x}_{1:T} \mid \underline{z}_{1:T}, \underline{S}_{1:T}, \theta)\, P(\underline{z}_{1:T} \mid \underline{S}_{1:T}, \theta) \right]$$

$$= \log \left[\prod_{t=1}^{T} P(x_t \mid z_t, \cancel{S_{t:t+2}}, \theta)\, P(z_1 \mid \cancel{S_1}, \theta) \prod_{t=2}^{T} P(z_t \mid z_{t-1}, S_{t:t+2}, \theta) \right]$$

$$= \log \left[\prod_{t=1}^{T} \prod_{m=1}^{M} P(x_t \mid \alpha_m, \beta_m, z_t = m)^{I(z_t = m)} \cdot \prod_{m=1}^{M} (\pi_m)^{I(z_1 = m)} \cdot \underbrace{\prod_{t=2}^{T} \prod_{m=1}^{M} \prod_{m'=1}^{M} \left[P(z_t = m' \mid z_{t-1} = m, S_{t:t+2}, \theta) \right]^{I(z_{t-1} = m,\, z_t = m')}}_{①} \right]$$

$$① : \prod_{t=2}^{T} \prod_{m=1}^{M} \prod_{m'=1}^{M} \left[a_{m,m'} \right]^{I(z_{t-1} = m,\, z_t = m')} \propto \prod_{t=2}^{T} \left[a_{1,1} \right]^{I(z_{t-1} = 1,\, z_t = 1)} \cdot \left[a_{1,2} \right]^{I(z_{t-1} = 1,\, z_t = 2)}$$

$$= \prod_{t=2}^{T} \prod_{ci \in \Omega} \left[1 - p^{ci} \right]^{I(z_{t-1} = 1,\, z_t = 1,\, S_{t:t+2} = ci)} \cdot \left[p^{ci} \right]^{I(z_{t-1} = 1,\, z_t = 2,\, S_{t:t+2} = ci)}$$

Ω: set of possible start codons

log likelihood for single transcript:

$$\log P(\underline{x}_{1:T}, \underline{z}_{1:T} \mid \theta, \underline{S}_{1:T}) \propto \sum_{t=1}^{T} \sum_{m=1}^{M} I(z_t = m) \log p(x_t \mid \alpha_m, \beta_m, z_t = m) + \sum_{m=1}^{M} I(z_1 = m) \log \pi_m + \log ①$$

$$= \sum_{t=1}^{T} \sum_{m=1}^{M} I(z_t = m) \log p(x_t \mid \alpha_m, \beta_m, z_t = m) + \sum_{m=1}^{M} I(z_1 = m) \log \pi_m$$

$$+ \sum_{t=2}^{T} \sum_{ci \in \Omega} \left\{ I(z_{t-1} = 1, z_t = 1, S_{t:t+2} = ci) \log(1 - p^{ci}) + I(z_{t-1} = 1, z_t = 2, S_{t:t+2} = ci) \log p^{ci} \right\}$$

log likelihood for multiple transcripts:

$$\sum_{n=1}^{N} \log p(\underline{x}_{1:T}^{(n)}, \underline{z}_{1:T}^{(n)} \mid \theta, \underline{S}_{1:T}^{(n)}) \propto \sum_{n=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} I(z_t^{(n)}=m) \log p(x_t^{(n)} \mid \alpha_m, \beta_m, z_t^{(n)}=m) + \sum_{n=1}^{N} \sum_{m=1}^{M} I(z_1^{(n)}=m) \log \pi_m + \sum_{n=1}^{N} \log ①$$

$$= \sum_{n=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} I(z_t^{(n)}=m) \log p(x_t^{(n)} \mid \alpha_m, \beta_m, z_t^{(n)}=m) + \sum_{n=1}^{N} \sum_{m=1}^{M} I(z_1^{(n)}=m) \log \pi_m$$

$$+ \sum_{n=1}^{N} \sum_{t=2}^{T} \sum_{c_i \in \Omega} \left\{ I(z_{t-1}^{(n)}=1, z_t^{(n)}=1, S_{t:t+2}^{(n)}=c_i) \log (\quad 1 - p^{c_i}) + I(z_{t-1}^{(n)}=1, z_t^{(n)}=2, S_{t:t+2}^{(n)}=c_i) \log p^{c_i} \right\}$$

Q function for single transcript:

$p(z_t=m \mid x_{1:T}, \theta)$ $\qquad$ $p(z_{t-1}=i, z_t=j, S_{t:t+2}=c_i \mid x_{1:T}, \theta)$

$$Q(\theta, \theta') \propto \sum_{t=1}^{T} \sum_{m=1}^{M} \boxed{p(z_t=m)} \log p(x_t \mid \alpha_m, \beta_m, z_t=m) + \sum_{m=1}^{M} p(z_1=m) \log \pi_m$$

$$+ \sum_{t=2}^{T} \sum_{c_i \in \Omega} \left\{ \underline{p(z_{t-1}=1, z_t=1, S_{t:t+2}=c_i)} \log (\quad 1 - p^{c_i}) + \underline{p(z_{t-1}=1, z_t=2, S_{t:t+2}=c_i)} \log p^{c_i} \right\}$$

Q function for multiple transcripts:

$$Q(\theta, \theta') \propto \sum_{n=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} p(z_t^{(n)}=m) \log p(x_t^{(n)} \mid \alpha_m, \beta_m, z_t^{(n)}=m) + \sum_{n=1}^{N} \sum_{m=1}^{M} p(z_1^{(n)}=m) \log \pi_m$$

$$+ \sum_{n=1}^{N} \sum_{t=2}^{T} \sum_{c_i \in \Omega} \left\{ \underline{p(z_{t-1}^{(n)}=1, z_t^{(n)}=1, S_{t:t+2}^{(n)}=c_i)} \log (\quad 1 - p^{c_i}) + \underline{p(z_{t-1}^{(n)}=1, z_t^{(n)}=2, S_{t:t+2}^{(n)}=c_i)} \log p^{c_i} \right\}$$

$$\frac{\partial Q(\theta,\theta')}{\partial p^{c_i}} = \sum_{n=1}^{N}\sum_{t=2}^{T}\left[-\frac{P(z_{t-1}^{(n)}=1, z_t^{(n)}=1, S_{t:t+2}^{(n)}=c_i)}{1-p^{c_i}} + \frac{P(z_{t-1}^{(n)}=1, z_t^{(n)}=2, S_{t:t+2}=c_i)}{p^{c_i}}\right] = 0$$

$$\Rightarrow \frac{(1-p^{c_i})\sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=2, S_{t:t+2}=c_i) - p^{c_i}\sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=1, S_{t:t+2}^{(n)}=c_i)}{(p^{c_i})(1-p^{c_i})} = 0$$

$$\Rightarrow (1-p^{c_i})\sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=2, S_{t:t+2}=c_i) = p^{c_i}\sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=1, S_{t:t+2}^{(n)}=c_i) \qquad (2)$$

$$p^{c_i}\left\{\sum_{n=1}^{N}\sum_{t=2}^{T}\left[P(z_{t-1}^{(n)}=1, z_t^{(n)}=1, S_{t:t+2}=c_i) + P(z_{t-1}^{(n)}=1, z_t^{(n)}=2, S_{t:t+2}=c_i)\right]\right\}$$
$$= \sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=2, S_{t:t+2}=c_i)$$

$$\Rightarrow p^{c_i}\left\{\sum_{n=1}^{N}\sum_{t=2}^{T}\underbrace{\left[P(z_{t-1}^{(n)}=1, z_t^{(n)}=1, S_{t:t+2}=c_i) + P(z_{t-1}^{(n)}=1, z_t^{(n)}=2, S_{t:t+2}=c_i)\right]}_{\parallel}\right\}$$
$$= \sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=2, S_{t:t+2}=c_i)$$

$$* \quad P(S_{t:t+2}=c_i \mid X_{1:T}) = \begin{cases} 1 & \text{if } S_{t:t+2}=c_i \\ 0 & \text{o.w.} \end{cases}$$

$$\Rightarrow p^{c_i}\sum_{n=1}^{N}\sum_{t=2}^{T}P(S_{t:t+2}=c_i \mid X_{1:T}) = \sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=2, S_{t:t+2}=c_i)$$

$$\Rightarrow p^{c_i} = \frac{\sum_{n=1}^{N}\sum_{t=2}^{T}P(z_{t-1}^{(n)}=1, z_t^{(n)}=2, S_{t:t+2}=c_i)}{\underbrace{\sum_{n=1}^{N}\sum_{t=2}^{T}P(S_{t:t+2}=c_i \mid X_{1:T})}}$$

$$\Rightarrow p^{c_i} = \frac{\sum_{n=1}^{N}\sum_{t=2}^{T}H_{1,2}(t-1, c_i)}{\underbrace{\sum_{n=1}^{N}\sum_{t=2}^{T}P(S_{t:t+2}=c_i \mid X_{1:T})}}$$

count of c_i

$$\Rightarrow p^{c_i} = \frac{\sum_{n=1}^{N}\sum_{t=1}^{T-1}H_{1,2}(t, c_i)}{\underbrace{\sum_{n=1}^{N}\sum_{t=2}^{T}P(S_{t:t+2}=c_i \mid X_{1:T})}}$$

$$\frac{\partial Q(\theta,\theta')}{\partial \alpha_m} = \frac{\partial}{\partial \alpha_m} \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \log p(x_t^{(n)} \mid \alpha_m, \beta_m, z_t^{(n)}=m)$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \frac{\partial}{\partial \alpha_m} \log p(x_t^{(n)} \mid \alpha_m, \beta_m, z_t^{(n)}=m)$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \frac{\partial}{\partial \alpha_m} \left[\log \binom{x_t^{(n)}+\alpha_m-1}{x_t^{(n)}} \left(1-\frac{E^{(n)}}{E^{(n)}+\beta_m}\right)^{\alpha_m} \left(\frac{E^{(n)}}{E^{(n)}+\beta_m}\right)^{x_t^{(n)}} \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \frac{\partial}{\partial \alpha_m} \left[\log \binom{x_t^{(n)}+\alpha_m-1}{x_t^{(n)}} + \alpha_m \log\left(1-\frac{E^{(n)}}{E^{(n)}+\beta_m}\right) + x_t^{(n)} \log \frac{E^{(n)}}{E^{(n)}+\beta_m} \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \frac{\partial}{\partial \alpha_m} \left[\log \prod_{s=1}^{x_t^{(n)}} (\alpha_m+x_t^{(n)}-s) + \alpha_m \log\left(1-\frac{E^{(n)}}{E^{(n)}+\beta_m}\right) + x_t^{(n)} \log \frac{E^{(n)}}{E^{(n)}+\beta_m} \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \left[\sum_{s=1}^{x_t^{(n)}} \frac{\partial}{\partial \alpha_m} \log(\alpha_m+x_t^{(n)}-s) + \log\left(1-\frac{E^{(n)}}{E^{(n)}+\beta_m}\right) \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \left[\sum_{s=1}^{x_t^{(n)}} \frac{1}{\alpha_m+x_t^{(n)}-s} + \log \frac{\beta_m}{E^{(n)}+\beta_m} \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} L_m(t) \cdot \left[\sum_{s=1}^{x_t^{(n)}} \frac{1}{\alpha_m+x_t^{(n)}-s} + \log \frac{\beta_m}{E^{(n)}+\beta_m} \right]$$

$$\frac{\partial Q(\theta,\theta')}{\partial \beta_m} = \frac{\partial}{\partial \beta_m} \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \log g(x_t^{(n)} \mid \alpha_m, \beta_m, z_t^{(n)}=m)$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \frac{\partial}{\partial \beta_m} \log g(x_t^{(n)} \mid \alpha_m, \beta_m, z_t^{(n)}=m)$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \frac{\partial}{\partial \beta_m} \left[\log \left(\frac{x_t^{(n)}+\alpha_m-1}{x_t^{(n)}} \right) \left(1-\frac{E^{(n)}}{E^{(n)}+\beta_m} \right)^{\alpha_m} \left(\frac{E^{(n)}}{E^{(n)}+\beta_m} \right)^{x_t^{(n)}} \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \frac{\partial}{\partial \beta_m} \left[\log \left(\frac{x_t^{(n)}+\alpha_m-1}{x_t^{(n)}} \right) + \alpha_m \log \left(1-\frac{E^{(n)}}{E^{(n)}+\beta_m} \right) + x_t^{(n)} \log \frac{E^{(n)}}{E^{(n)}+\beta_m} \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \frac{\partial}{\partial \beta_m} \left[\alpha_m \log \frac{\beta_m}{E^{(n)}+\beta_m} + x_t^{(n)} \log \frac{E^{(n)}}{E^{(n)}+\beta_m} \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \left[\alpha_m \cdot \frac{E^{(n)}+\cancel{\beta_m}}{\beta_m} \cdot \frac{1\times(E^{(n)}+\cancel{\beta_m})-\cancel{\beta_m}\times 1}{(E^{(n)}+\beta_m)^{\cancel{2}}} + x_t^{(n)} \cdot \frac{E^{(n)}+\cancel{\beta_m}}{\cancel{E^{(n)}}} \cdot \frac{0-E^{(n)}\cancel{1}}{(E^{(n)}+\beta_m)^{\cancel{2}}} \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} P(z_t^{(n)}=m) \left[\frac{\alpha_m \cdot E^{(n)}}{\beta_m \cdot (E^{(n)}+\beta_m)} - \frac{x_t^{(n)}}{E^{(n)}+\beta_m} \right]$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} L_m(t) \cdot \left[\frac{\alpha_m \cdot E^{(n)}}{\beta_m \cdot (E^{(n)}+\beta_m)} - \frac{x_t^{(n)}}{E^{(n)}+\beta_m} \right]$$

$$= \cdots \left[\alpha_m \log \beta_m - \alpha_m \log(E+\beta_m) + x \log E - x \log(E+\beta_m) \right]$$

$$= \cdots \frac{\alpha_m}{\beta_m} - \frac{\alpha_m + x}{E+\beta_m}$$

$$= \sum_{n=1}^{N}\sum_{t=1}^{T} L_k^{(n)}(t) \left(\frac{\alpha_k}{\beta_k} - \frac{\alpha_k + x_t^{(n)}}{E^{(n)}+\beta_k} \right)$$