N-S equation

N-S equation in **vector** form:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla P + \frac{1}{Re}\nabla^2 \vec{u}$$

N-S equation in **component** form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Denote **convection** terms $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$ and $u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$ as $conv\_x$ and $conv\_y$, respectively.

Convection terms in **conservation** form:

$$conv\_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u\frac{\partial u}{\partial x} + u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v\frac{\partial u}{\partial y} = u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$$
$$= \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y}$$

$$conv\_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = u\frac{\partial v}{\partial x} + v\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v\frac{\partial v}{\partial y} = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} + v\frac{\partial v}{\partial y}$$
$$= \frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y}$$

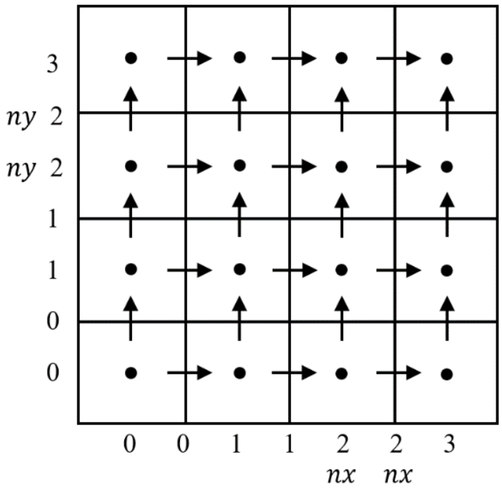
Denote **viscosity** terms $\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ and $\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$ as $visc\_x$ and $visc\_y$, respectively.

N-S equation in **discrete** form:

$$\frac{u^{n+1} - u^n}{\Delta t} + conv\_x = -\frac{\Delta P}{\Delta x} + visc\_x$$

$$\frac{v^{n+1} - v^n}{\Delta t} + conv\_y = -\frac{\Delta P}{\Delta y} + visc\_y$$

Staggered Grid



Projection Method

Take **x-direction** as an example.

Split discrete N-S equation into:

$$\frac{u^* - u^n}{\Delta t} = -conv\_x + visc\_x$$

and

$$\frac{u^{n+1} - u^*}{\Delta t} = -\frac{\Delta P}{\Delta x}$$

Step 1: predetermine

$$\frac{u^* - u^n}{\Delta t} = -conv\_x + visc\_x$$

RK2:

$$u^{n+1} = u^n + \Delta t \frac{K_1 + K_2}{2}$$

$$K_1 = f(t^n, u^n)$$

$$K_2 = f(t^n + \Delta t, u^n + \Delta t * K_1)$$

Step 2: get constriction variable P:

Taking divergence of

$$\frac{\overrightarrow{u^{n+1}} - \overrightarrow{u^*}}{\Delta t} = -\nabla P$$

yields

$$\nabla^2 P = \frac{1}{\Delta t} \nabla \cdot \overrightarrow{u^*}$$

(In following steps, P will be written in lower case, conforming to the code)

In discrete form:

$$\frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{\Delta x^2} + \frac{p_{i,j+1} - 2p_{i,j} + p_{i,j-1}}{\Delta y^2} = \frac{1}{\Delta t}\left(\frac{\Delta u^*}{\Delta x} + \frac{\Delta v^*}{\Delta y}\right)$$

Denote $\frac{1}{\Delta t}\left(\frac{\Delta u^*}{\Delta x} + \frac{\Delta v^*}{\Delta y}\right)$ as $poison\_rhs$; Denote $\left(\frac{p_{i+1,j}+p_{i-1,j}}{\Delta x^2} + \frac{p_{i,j+1}+p_{i,j-1}}{\Delta y^2}\right)$ as p_temp. Then,

$$p_{i,j}^{n+1} = \frac{poison\_rhs - p\_temp}{\left(\frac{-2}{\Delta x^2} + \frac{-2}{\Delta y^2}\right)}$$

Relaxation:

$$p_{i,j}^{n+1} = (1-\omega)p_{i,j}^{n+1} + \omega \frac{poison\_rhs - p\_temp}{\left(\frac{-2}{\Delta x^2} + \frac{-2}{\Delta y^2}\right)}$$

Step 3: correction to get final velocity

$$u^{n+1} = u^* - \Delta t \frac{\Delta P}{\Delta x}$$