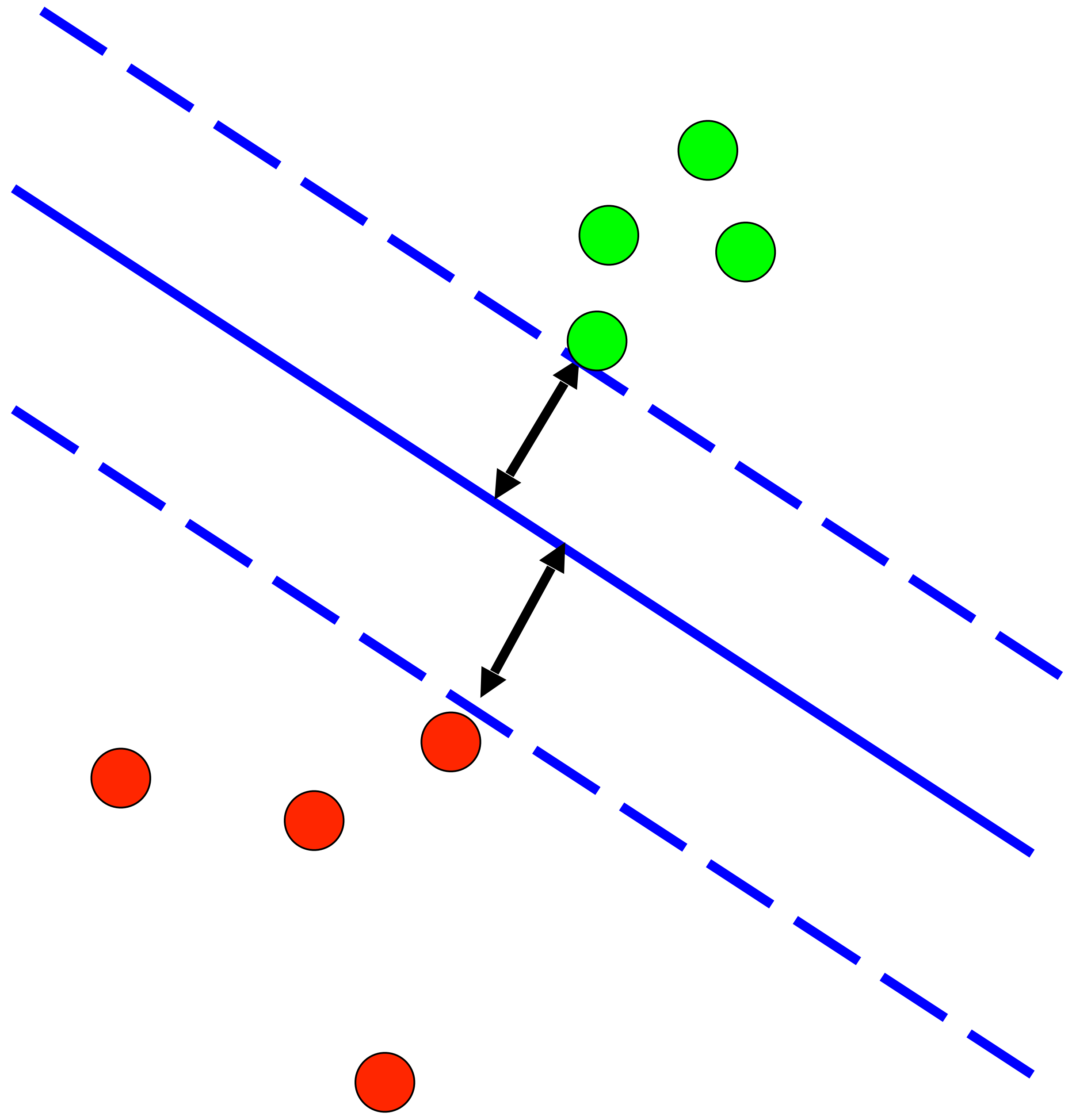
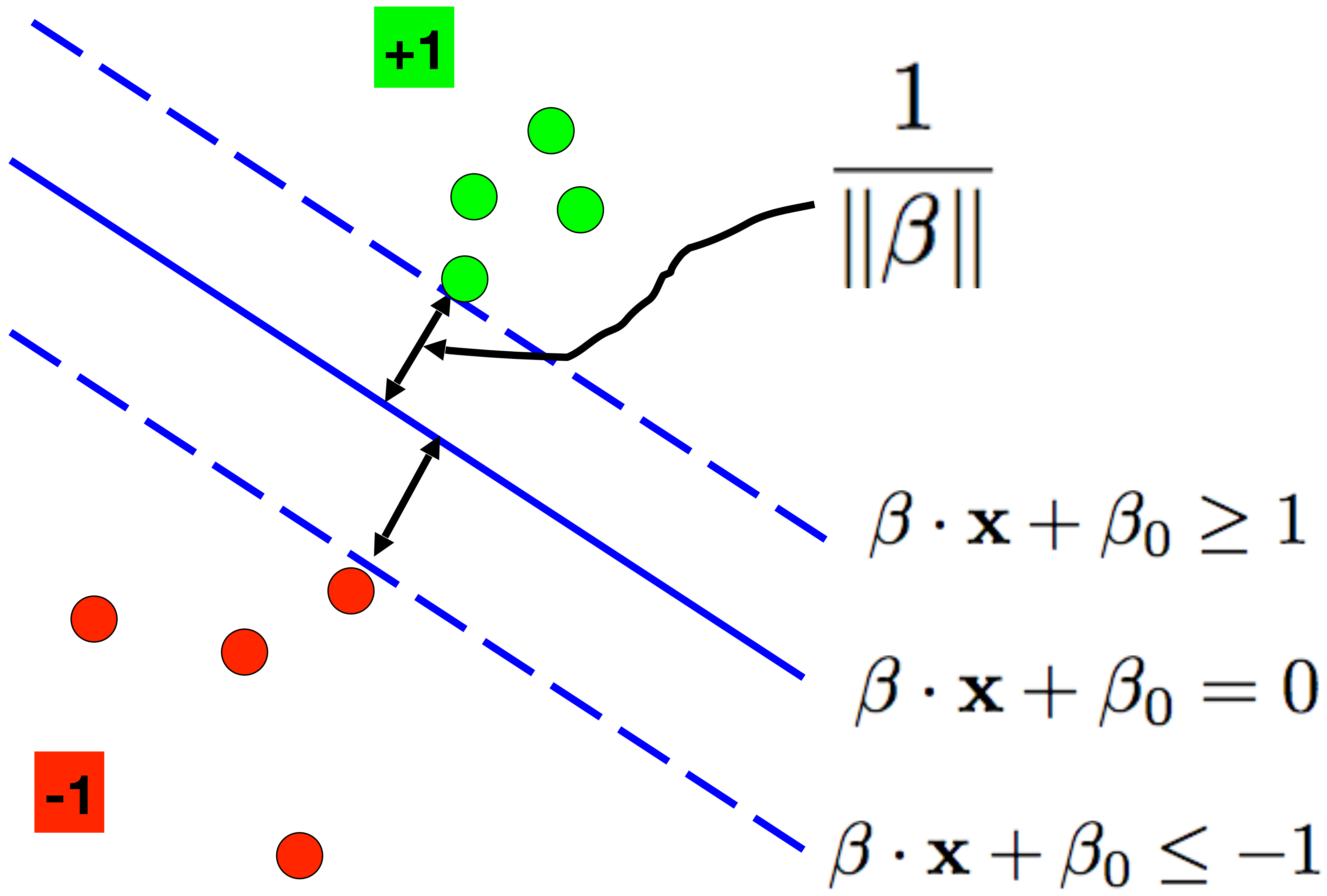


$$\beta \cdot \mathbf{x} + \beta_0 = 0$$

It is desirable to have the width (called margin) between the two lines to be large.

How to formulate this problem?

Solid Blue Line: The coefficients (b, b0) are not uniquely determined. We can scale them by any number (pos/neg), the line stays the same.



It is desirable to have the width (called margin) between the two lines to be large.

How to formulate this problem?

Solid Blue Line: The coefficients (b, b0) are not uniquely determined. We can multiple them by any number (pos/neg), the line stays the same.

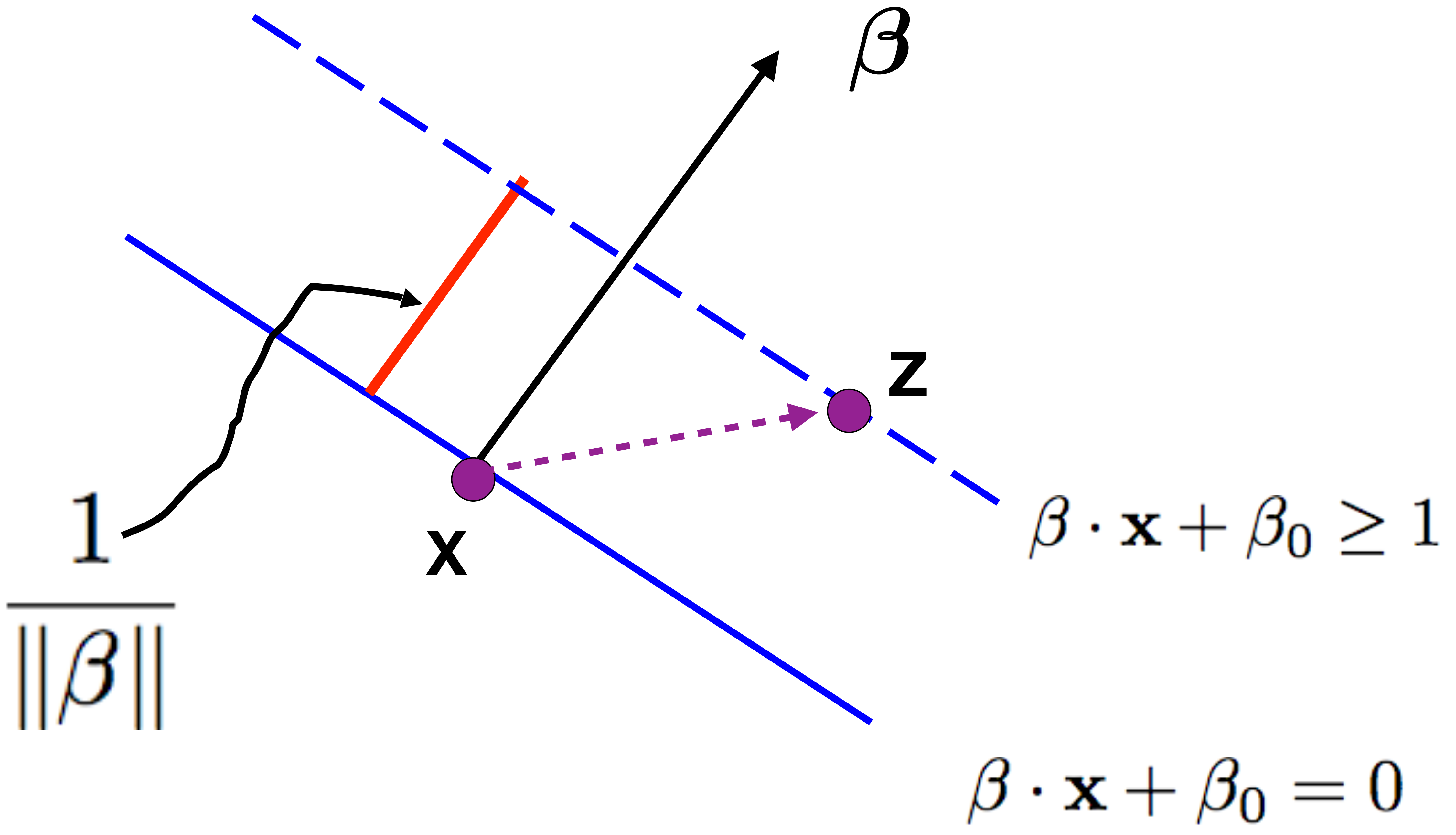
1. Fix the sign: y = +1 or -1.

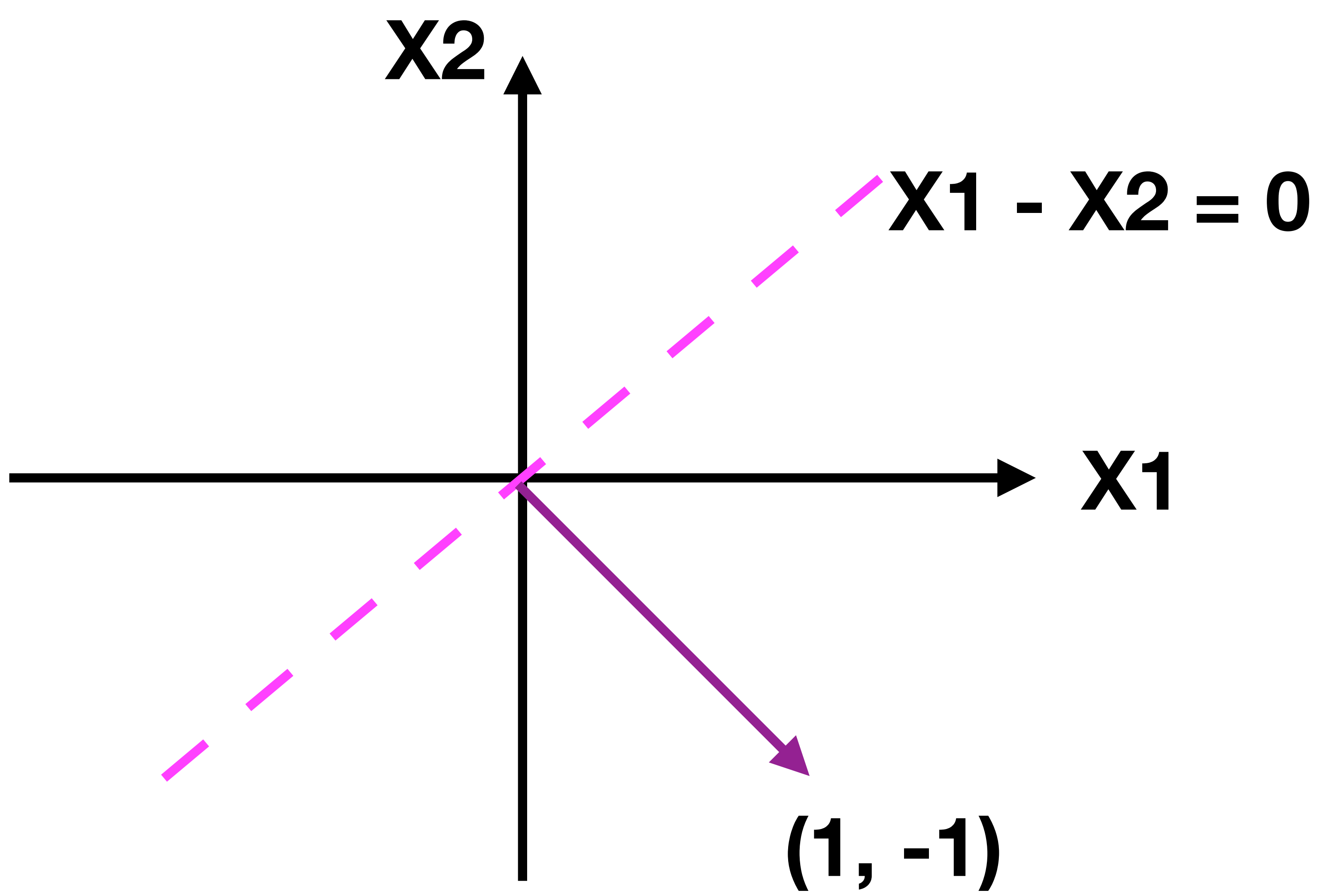
   b*x + b_0 > 0, if y = + 1
   b*x + b_0 < 0, if y = - 1

2. Fix the magnitude: parameterize the two dashed lines as

   b*x + b_0 = +1
   b*x + b_0 = -1

Two dashed lines determine this wide avenue, and the solid line is in the middle.



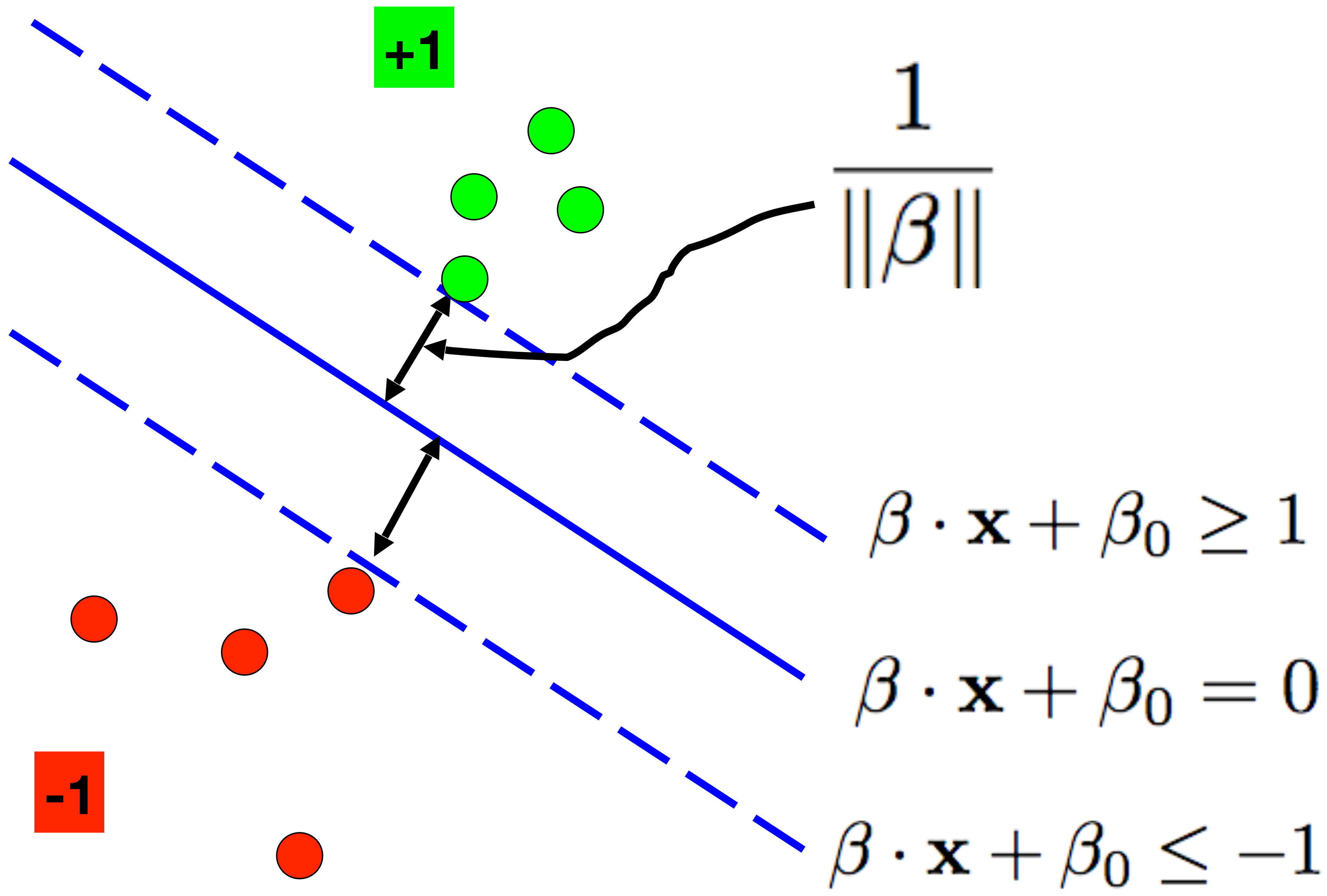


Line: b*x + b0 =0
Interpretation of b: direction that is orthogonal to the line

How to compute the distance between these two parallel lines?

$$(\mathbf{x}-\mathbf{z})^t \frac{\boldsymbol{\beta}}{\|\boldsymbol{\beta}\|} = \frac{\mathbf{x}^t\boldsymbol{\beta} - \mathbf{z}^t\boldsymbol{\beta}}{\|\boldsymbol{\beta}\|} = \frac{1}{\|\boldsymbol{\beta}\|}$$

(In my calculation, the signs may not be right, but all we care is the magnitude (i.e., we should add absolute value on each expression).

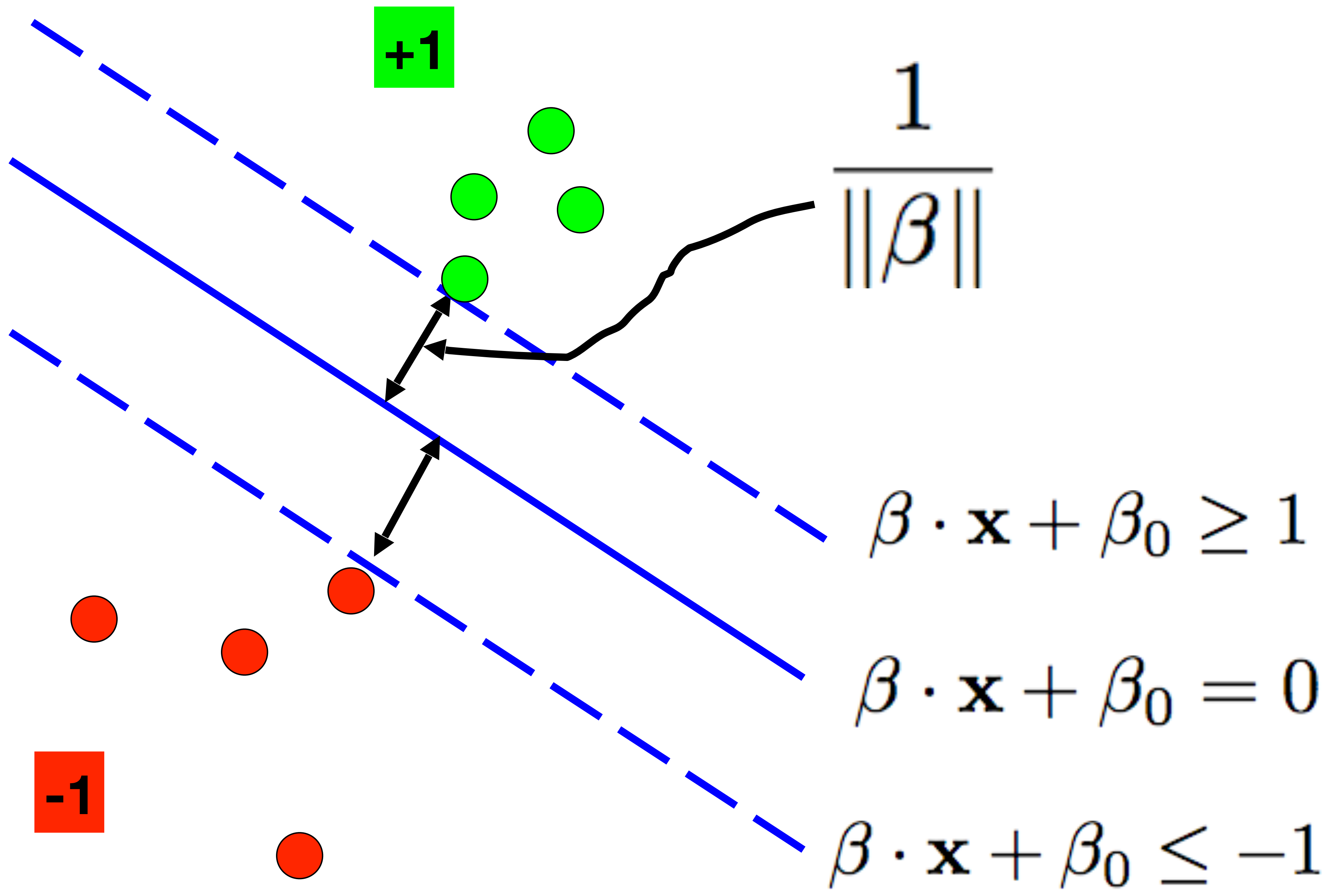


Max-Margin Problem

$$\min_{\boldsymbol{\beta},\beta_0} \quad \frac{1}{2}\|\boldsymbol{\beta}\|^2 \tag{1}$$
$$\text{subject to} \quad y_i(\boldsymbol{\beta} \cdot \mathbf{x}_i + \beta_0) - 1 \geq 0,$$

where $\boldsymbol{\beta} \cdot \mathbf{x}_i = \boldsymbol{\beta}^t \mathbf{x}_i$ denotes the (Euclidian) inner product between two vectors. The constraints are imposed to make sure that the points are on the correct side of the dashed lines, i.e.,

$$\boldsymbol{\beta} \cdot \mathbf{x}_i + \beta_0 \geq +1 \quad \text{for } y_i = +1,$$
$$\boldsymbol{\beta} \cdot \mathbf{x}_i + \beta_0 \leq -1 \quad \text{for } y_i = -1.$$



- Convex quadratic optimization problem with affine constraints.
- Any location optimum is a global optimum.
- **KKT conditions** are sufficient and necessary
- Equivalence between **the Primal** and **the Dual**.

Max-Margin Problem

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$$\min_x f(x)$$

$$\downarrow$$

First-order necessary condition

$$-\frac{\partial f(x)}{\partial x} = \mathbf{0}$$

$$\min_x f(x)$$

First-order necessary condition

$$-\frac{\partial f(x)}{\partial x} = \mathbf{0}$$

$$\min_x f(x)$$
$$\text{subj to } g(x) = b$$

$$-\frac{\partial f(x)}{\partial x} = \lambda \frac{\partial g(x)}{\partial x}$$

direction that can reduce f(x)

forbidden direction that would violate g(x)=b

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direction that can reduce f(x)

forbidden direction that would violate g(x)=b

$$\min_x f(x)$$
$$\text{subj to } g(x) \geq b$$

$$-\frac{\partial f(x)}{\partial x} = -\lambda \frac{\partial g(x)}{\partial x}$$
$$\lambda \geq 0$$
$$g(x) - b \geq 0$$
$$\lambda(g(x) - b) = 0$$

If x is a local optimum for the constrained optimization, then it must satisfy the **KKT conditions**.

- x is active (lambda nonzero)
- x is inactive (lambda = 0)

$$\min_x f(x)$$

First-order necessary condition

$$-\frac{\partial f(x)}{\partial x} = \mathbf{0}$$

$$\min_x f(x) \quad \text{subj to } g(x) = b$$

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$$-\frac{\partial f(x)}{\partial x} = -\lambda \frac{\partial g(x)}{\partial x}$$
$$\lambda \geq 0$$
$$g(x) - b \geq 0$$
$$\lambda(g(x) - b) = 0$$

$$\text{Define } L(x, \lambda) = f(x) - \lambda\big(g(x) - b\big)$$

$$\boxed{\frac{\partial}{\partial x} L = 0}$$

If x is a local optimum for the constrained optimization, then it must satisfy the **KKT conditions**.

- x is active (lambda nonzero)
- x is inactive (lambda = 0)

Primal

$$\boxed{\begin{aligned}&\min_x f(x)\\&\text{subj to } g(x) \geq b\end{aligned}}$$

$$\boxed{\min_x \max_{\lambda \geq 0} \left[\underbrace{f(x) - \lambda\big(g(x) - b\big)}_{L(x,\lambda)} \right]}$$

$$\max_{\lambda \geq 0} \left[f(x) - \lambda\big(g(x) - b\big) \right] = \begin{cases} f(x) & \text{if } g(x) \geq b \\ \infty & \text{if } g(x) < b \end{cases}$$

Primal

$$\boxed{\begin{aligned} &\min_x f(x) \\ &\text{subj to } g(x) \geq b \end{aligned}}$$

$\downarrow$

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$L(x, \lambda)$

$$\max_{\lambda \geq 0} \left[f(x) - \lambda\big(g(x) - b\big) \right] = \begin{cases} f(x) & \text{if } g(x) \geq b \\ \infty & \text{if } g(x) < b \end{cases}$$

Under some conditions that are satisfied here, we have

$$\boxed{\min_x \max_\lambda L(x, \lambda) = \max_\lambda \min_x L(x, \lambda) = L(x^*, \lambda^*)}$$

Primal

Dual

$$\min_x f(x)$$
$$\text{subj to } g(x) \geq b$$

$$\max_{\lambda \geq 0} \min_x \Big[f(x) - \lambda\big(g(x) - b\big) \Big]$$

$$\min_x \max_{\lambda \geq 0} \Big[\underbrace{f(x) - \lambda\big(g(x) - b\big)}_{L(x, \lambda)} \Big]$$

Equivalent and KKT conditions can link the two sets of solutions: x* and lambda*

$$\max_{\lambda \geq 0} \Big[f(x) - \lambda\big(g(x) - b\big) \Big] = \begin{cases} f(x) & \text{if } g(x) \geq b \\ \infty & \text{if } g(x) < b \end{cases}$$

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