

Solutions to Quiz 1

Problem 1 (15 points). Let G be an undirected graph. Let $P(x, y)$ mean that there is a path from vertex x to vertex y . Express each of the following sentences in terms of P , quantifiers, logical connectives, and equality, using variables that range over the vertices of G . (Reminder: there is a zero-length path from any vertex to itself.)

(a) (3 points) Vertices x and y are in the same connected component.

Solution. $P(x, y)$ ■

(b) (3 points) G has a vertex of degree zero. (Reminder: undirected graphs only have edges between distinct vertices, that is, no self-loops.)

Solution. $\exists x \forall y. P(x, y) \longrightarrow x = y.$ ■

(c) (4 points) G has at least three connected components.

Solution.

$$\exists x, y, z. \neg P(x, y) \wedge \neg P(x, z) \wedge \neg P(y, z).$$
 ■

(d) (5 points) There is a positive-length *simple* path from x to y .

Solution. There is a positive length simple path between two vertices iff the vertices are distinct and there is a path between them, so we can express this as

$$x \neq y \wedge P(x, y).$$
 ■

Problem 2 (10 points). Classify each of the following binary relations as

E: An equivalence relation.

T: A Total order,

P: A Partial order that is not total.

S: A Symmetric relation that is not transitive.

N: None of the above.

(a) (2 points) The relation between times during a single day: x and y are at most twenty minutes apart. ____

Solution. S: This relation is obviously symmetric. It is not transitive: $1:00R1:15$, and $1:15R1:22$, but $\neg(1:00R1:22)$. ■

(b) (2 points) The relation between times during a single day: x is more than twenty minutes later than y . ____

Solution. P: This relation is asymmetric and transitive. It is not a total ordering because some times are incomparable to each other. For example, 1:15 is incomparable to 1:22. ■

(c) (2 points) The relation between vertices in an arbitrary digraph: there is a path from v to w . ____

Solution. N: This relation is transitive but won't be a partial order unless the graph is acyclic. ■

(d) (2 points) The relation between vertices in an undirected graph: there is a path from v to w . ____

Solution. E: In fact, the connected components of the graph are the equivalence classes of this relation. ■

(e) (2 points) The relation between Fall '05 6.042 students: student s is older but also shorter than t . ____

Solution. P: This relation is the product of the "older than" and the "shorter than" orders, and so is a partial order. It is not total because there is a younger student who is taller than an older student. ■

Problem 3 (20 points). Let $G_0 = 1$, $G_1 = 2$, $G_2 = 4$, and define

$$G_n ::= G_{n-1} + 2G_{n-2} + G_{n-3} \quad (1)$$

for $n \geq 3$. Show by induction that $G_n \leq (2.2)^n$ for all $n \geq 0$.

Solution. The proof is by strong induction with hypothesis $P(n) := G_n \leq (2.2)^n$.

Proof. **Base Cases**

$$n = 0: G_0 ::= 1 = (2.2)^0.$$

$$n = 1: G_1 ::= 2 < 2.2 = (2.2)^1.$$

$$n = 2: G_2 ::= 4 < 4.84 = (2.2)^2.$$

Inductive Step: Assume $n \geq 2$ and $P(k)$ for all k such that $0 \leq k \leq n$.

$$\begin{aligned} G_{n+1} &::= G_n + 2G_{n-1} + G_{n-2} && \text{by (1)} \\ &\leq (2.2)^n + 2(2.2)^{n-1} + (2.2)^{n-2} && \text{by induction hypothesis} \\ &= (2.2)^{n-2}[(2.2)^2 + 2(2.2) + 1] \\ &= (2.2)^{n-2}[(2.2 + 1)^2] \\ &= (2.2)^{n-2}(3.2)^2 \\ &= (2.2)^{n-2}(10.24) \\ &< (2.2)^{n-2}(10.648) \\ &= (2.2)^{n-2}(2.2)^3 \\ &= (2.2)^{n+1} \end{aligned}$$

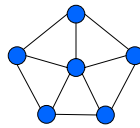
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■

Problem 4 (20 points). An *intersection graph* is an undirected graph whose vertices are sets and whose edges are specified by the rule that there is an edge between vertices A and B iff $A \neq B$ and $A \cap B \neq \emptyset$.

(a) (1 point) Draw the intersection graph whose vertices are the sets

$$\{1, 2, 3\}, \{1, 9, 10\}, \{2, 4, 6, 8, 10\}, \{3, 4, 5\}, \{5, 6, 7\}, \{7, 8, 9\}$$



Solution. ■

(b) (3 points) What is the chromatic number of the graph in part (a)? ____

Solution. 4. The graph is isomorphic to the wheel, W_5 . All wheels with an odd-length rim have chromatic number 4, as explained in lecture. ■

(c) (3 points) What is the largest k such that the graph in part (a) is k -connected? ____

Solution. 3. ■

We now consider an arbitrary undirected graph, G . For any vertex, v , of G , let $I(v)$ be the set of edges incident to v .

(d) (3 points) Explain how to uniquely determine the vertex v given any two edges in $I(v)$.

Solution. Just find a vertex that the two edges have in common. This must be v . ■

(e) (10 points) An *incidence-set* is the set of edges incident to some vertex, that is, a set equal to $I(v)$ for some vertex v of G . Prove that if G is a graph whose vertices all have degree greater than 1, then the function, I , is an isomorphism between G and the intersection graph whose vertices are the incidence-sets of G .

Solution. Let H be the intersection graph whose vertices are the incidence-sets of G . By definition of H , the function, I , is a surjection onto the vertices of H . Moreover, since each incidence-set contains at least two edges, part (d) implies that given $I(v)$, we can uniquely determine v . This means I is also an injection into the vertices of H , and therefore is a bijection from the vertices of G to the vertices of H . Also

$$\begin{aligned}
 v-w \text{ is an edge of } G & \text{ iff } [v-w \in I(v) \text{ and } v-w \in I(w)] \\
 & \text{ iff } v \neq w \text{ and } I(v) \cap I(w) \neq \emptyset \\
 & \text{ iff } I(v)-I(w) \text{ is an edge of } H,
 \end{aligned}$$

which proves that I is an isomorphism.

Incidentally, if G happens to have any vertices of degree < 2 , then it's easy to assign special sets to these vertices in such a way that these special sets, along with the incidence sets of vertices of degree ≥ 2 , are the vertices of an intersection graph that is isomorphic to G . This shows that *every* undirected graph is isomorphic to an intersection graph. ■

Problem 5 (10 points). Two banks only allow transactions that are multiples of 3^9 dollars or 5^7 dollars. Is there a series of transactions whose net result is a payment of 1 dollar from the first bank to the second bank? Briefly explain why or why not.

Solution. Yes. Since 3^9 and 5^7 have no common prime factors, their gcd is 1, and so there are positive integers s, t such that $s(3^9) - t(5^7) = 1$. So the first bank should pay the second bank 3^9 dollars s times, and accept payment of 5^7 dollars from the second bank t times for a net transfer of 1 dollar to the second bank. ■

Problem 6 (15 points). Each year, Santa's reindeer hold "Reindeer Games", from which Rudolph is pointedly excluded. The Games consist of a sequence of matches, where one reindeer competes against another. Draws are not possible.

On Christmas Eve, Santa produces a rank list of all his reindeer. If reindeer p lost a match to reindeer q , then p appears below q in Santa's ranking, but if he has any choice because of unplayed matches, Santa can give higher rank to the reindeer he likes better. To prevent confusion, two reindeer may not play a match if either outcome would lead to a cycle of reindeer, where each lost to the next.

Though it is only October, the 2004 Reindeer Games have already begun. We can describe the results so far with a binary relation, L , on the set of reindeer, where pLq means that reindeer p lost a match to reindeer q . Let L^+ be the corresponding positive-length path relation¹. Note that L^+ is a partial order, so we can regard a match loser as "smaller" than the winner.

On the following page you'll find a list of terms and a sequence of statements. Add the appropriate term to each statement.

¹Thus, reindeer p is related to reindeer q by L^+ if p lost to q or if p lost to a reindeer who lost to q or if p lost to a reindeer who lost to a reindeer who lost to q , etc.

Terms

a strict partial order	a weak partial order	a total order
comparable elements	incomparable elements	a chain
an antichain	a maximal antichain	a topological sort
a minimum element	a minimal element	
a maximum element	a maximal element	

Statements

(a) (1 point) An unbeaten reindeer is

_____ of the partial order L^+ .

Solution. a maximal element



(b) (1 point) A reindeer who has lost every match so far is

_____ of the partial order L^+ .

Solution. a minimal element



(c) (1 point) Two reindeer can *not* play a match if they are

_____ of L^+ .

Solution. comparable elements



(d) (1 point) A reindeer assured of first place in Santa's ranking is

_____ of L^+ .

Solution. a maximum element



(e) (1 point) A sequence of reindeer which *must* appear in the same order in Santa's rank list is

_____.

Solution. a chain



(f) (2 points) A set of reindeer such that any two could still play a match is

_____.

Solution. an antichain ■

(g) (2 points) The fact that no reindeer loses a match to himself implies that L^+ is _____.

Solution. a strict partial order ■

(h) (2 points) Santa's final ranking of his reindeer on Christmas Eve must be _____ of L^+ .

Solution. a topological sort ■

(i) (2 points) No more matches are possible if and only if L^+ is _____.

Solution. a total order ■

(j) (2 points) Suppose that Santa has 11 reindeer. If no more matches can be played, what is the smallest possible number of matches already played? ____

Solution. 10.

Call the reindeer r_1, \dots, r_{11} . Then no more matches can be played if r_1 lost to r_2 , r_2 lost to r_3 , etc. If only ten matches have been played, then the corresponding digraph will not even be connected if we regard the edges as undirected. So there would be two disjoint sets of reindeer, S and T , such that no reindeer in S has played a match against a reindeer in T . Therefore, more matches can be played; in particular, any reindeer in S can play against any reindeer in T . ■

Problem 7 (10 points). A *map* is a connected planar graph with a planar drawing whose face boundaries are simple cycles.

(a) (7 points) Prove that if a map has no simple cycle of length 3, then

$$e \leq 2v - 4, \quad (2)$$

where v is the number of vertices and e is the number of edges in the graph.

Solution. Since there is no simple cycle of length 3 and face boundaries are simple cycles, every face must be of length 4 or more. We know that in a planar drawing, each edge appears twice on the face boundaries, so

$$2e = \sum_{b \in \text{boundaries}} \text{length}(b) \geq \left(\sum_{b \in \text{boundaries}} 4 \right) = 4f, \quad (3)$$

where f is the number of faces. By Euler's formula, $f = e - v + 2$, so substituting for f in (3), yields

$$2e \geq 4(e - v + 2),$$

which simplifies to (2). ■

(b) (3 points) Prove that $K_{3,3}$ is not a map. ($K_{3,3}$ is the graph with six vertices and an edge from each of the first three vertices to each of the last three.)

Solution. $K_{3,3}$ is by definition bipartite, and hence has no odd-length simple cycle, and in particular, no length 3 cycle. (This could also be checked exhaustively.)

However, it has 9 edges and 6 vertices, and since $9 > 8 = 2 \cdot 6 - 4$, it does not satisfy (2), and so cannot be a map.

Incidentally, this implies that $K_{3,3}$ is not planar. The reason is that connected planar graphs which are not maps are the ones with “dongles” or “crossbars,” as explained in lecture. But all edges in dongles and crossbars are cut edges. So a connected graph like $K_{3,3}$ with no cut edges is planar iff it is a map. ■