Homework 2 Solutions

Question 1

Prove the following equality using induction on n, where r is a real number:

$$(1-r)(1+r+r^2+\cdots+r^{n-1}) = 1-r^n \text{ for all } n \in \mathbb{N}\,. \tag{1}$$

The series in the second parenthesis is a geometric series, i.e. it is the sum of a bunch of terms where each term is multiplied by a constant factor, in this case r. Use the above formula to simplify the sum:

$$3^n + 2\cdot 3^{n-1} + 2^2\cdot 3^{n-2} + \cdots + 2^{n-2}\cdot 3^2 + 2^{n-1}\cdot 3 + 2^n$$

Geometric series show up a lot when bounding the running times of algorithms, especially 'divide and conquer' style algorithms that break a problem into smaller pieces, solve those smaller problems and then put the solutions together to get a solution of the original problem. You will see more of this in your algorithms class.

Solution: For $n = 1$, the equality holds. Assume it is true for $n = k$, we will show it holds for $n = k+1$.

$$\begin{aligned}(1-r)(1+r+\cdots+r^{k-1}+r^k) &= (1-r)(1+r+\cdots+r^{k-1}) + (1-r)r^k \\ &= 1-r^k+r^k-r^{k+1} \\ &= 1-r^{k+1}.\end{aligned}$$

In the second equality we uses the induction hypothesis. To compute the sum, we have

$$\begin{aligned}3^n + 2\cdot 3^{n-1} + 2^2\cdot 3^{n-2} + \cdots + 2^{n-2}\cdot 3^2 + 2^{n-1}\cdot 3 + 2^n &= 3^n(1+\frac{2}{3}+(\frac{2}{3})^2+\cdot+(\frac{2}{3})^n) \\ &= 3^n(1-\frac{2}{3})(1+\frac{2}{3}+(\frac{2}{3})^2+\cdot+(\frac{2}{3})^n)/(1-\frac{2}{3}) \\ &= 3^n\{(1-\frac{2}{3})(1+\frac{2}{3}+(\frac{2}{3})^2+\cdot+(\frac{2}{3})^n)\}/(\frac{1}{3}) \\ &= 3^{n+1}(1-(\frac{2}{3})^{n+1}) \\ &= 3^{n+1}-2^{n+1}.\end{aligned}$$

Question 2

Use induction to prove that, for any natural number n:

$$1+\frac{1}{2^2}+\frac{1}{3^2}\cdots+\frac{1}{n^2} < 2$$

(**Hint:** This is a case where strengthening the hypothesis might help. Try to prove instead the following:)

$$1+\frac{1}{2^2}+\frac{1}{3^2}\cdots+\frac{1}{n^2} \leq 2-\frac{1}{n}$$

Solution: For $n = 1$, the inequality holds. Assume it is true for $n = k$, we will show it holds for $n = k + 1$. By the induction hypothesis we have

$$1 + \frac{1}{2^2} + \frac{1}{3^2} \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}. \tag{2}$$

Since

$$\frac{1}{k} - \frac{1}{(k+1)^2} = \frac{k^2 + 2k + 1 - k}{k(k+1)^2} = \frac{k^2 + k + 1}{k(k+1)^2}$$

and

$$\frac{k^2 + k + 1}{k(k+1)} > 1$$

so we know that

$$\frac{1}{k} - \frac{1}{(k+1)^2} = \frac{k^2 + k + 1}{k(k+1)^2} > \frac{1}{k+1}.$$

Plugging this back to (2) we obtain

$$\begin{aligned} 1 + \frac{1}{2^2} + \frac{1}{3^2} \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} &\leq 2 - (\frac{1}{k} - \frac{1}{(k+1)^2}) \\ &\leq 2 - \frac{1}{k+1}. \end{aligned}$$

Question 3

Prove that any natural number greater than or equal to 8 can be written as the sum of some number of 3's and $5's$, i.e for any natural number $n \geq 8$, there are whole numbers x and y such that $n = 3x + 5y$. Whole numbers are just the natural numbers, with 0 also included as a whole number. (**Hint:** You can do a proof by induction. What happens if you know three consecutive numbers can be written in this way?)

Solution: For the base case, notice $8 = 3 + 5$. Assume the statement is true for $n = k$ where $k \geq 8$, we will show it holds for $n = k+1$. By the induction hypothesis, we know $k = 3x+5y$. Let's consider 2 cases:

- If $y \geq 1$, then $k + 1 = 3x + 5y + 1 = 3(x + 2) + 5(y - 1)$. Since $y \geq 1$, $y - 1$ is still a whole number.
- If $y = 0$, then $k + 1 = 3x + 1 = 3(x - 3) + 5 \times 2$. Since we need $x \geq 3$ to ensure $k \geq 8$, $x - 3$ is a whole number.

Perhaps a simpler solution would be to follow the hint:

$P(n)$=" n can be written as $3x + 5y$, where x and y are whole numbers".

Base Case: Check that $P(8)$, $P(9)$ and $P(10)$ are true by hand (this is easy).

Now I.H: $P(k)$ where k is any fixed natural number.

We will prove that $P(k) \implies P(k+3)$. Suppose $P(k)$ is true, then there exist whole numbers x and y, such that $k = 3x + 5y$.

But then $k+3 = 3(x+1)+5y$ and $(x+1)$ and y are both whole numbers. Thus $P(k+3)$ is true.

Thus we know $P(8)$, $P(9)$ and $P(10)$ are true and we know that $P(k) \implies P(k+3)$. Make sure you understand why this means that $P(n)$ is true for all $n \geq 8$.

Question 4

Use the well ordering principle to prove the following formula holds for all natural numbers, n:

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Note: for full credit, you must not just do any proof by induction, but rather explicitly use the well ordering principle.

(**Hint:** Consider a set of natural numbers where the above formula does not hold, and show that it cannot have a least element. You can more or less exactly follow the proof method in Lecture 2 notes, section 5.2, proposition 11.)

Solution: Let S be the set of natural numbers s.t. $\forall n \in S$,

$$1^2 + 2^2 + \cdots + n^2 \neq \frac{n(n+1)(2n+1)}{6}$$

Assume S is non-empty (or else we are already done!). Then by the well ordering principle, S contains a least element. Let $l \in S$ be this least element, that is $\forall n \in S, l \leq n$. For $n = 1, 1 = 1(1+1)(2+1)/6 = 1$, so $l > 1$. Thus we have $l - 1 \in \mathbb{N}$ but $l - 1 \notin S$, that is, the formula fails for l, but holds for $l - 1$:

$$1^2 + 2^2 + \cdots + (l-1)^2 = \frac{(l-1)l(2l-1)}{6}$$

By adding l^2 to both sides and simplifying, we get

$$\begin{aligned} 1^2 + 2^2 + \cdots + (l-1)^2 + l^2 &= \frac{(l-1)l(2l-1)}{6} + l^2 \\ &= \frac{l(l+1)(2l+1)}{6} \end{aligned}$$

This implies $l \in S$, i.e. the formula holds for l as well! which is a contradiction. In brief, by assuming that there are values where the formula doesn't hold, we looked at the least such value, l. So then the formula should hold for $l-1$ (notice, we checked that $l \neq 1$). But we showed that if the formula holds for $l-1$, it must hold for l as well. This is a contradiction, since the formula was not supposed to hold for l!

(**Note to graders:** Here we essentially proved $P(l-1)$ implies $P(l)$. Then we reasoned that since the statement is true for $l-1$, thus it is true for l, which is a contradiction.

It is completely equivalent to reason that we know (by definition) that the statement is false for l, but then it could not have been true for $l-1$ (since that would imply it's true for l). Thus the statement is false for $l-1$, which is a contradiction since l was the minimal value for which it was supposed to be false!)

Question 5

Say you have finitely many red and blue points on a plane with the interesting property: every line segment that joins two points of the same color contains a point of the other color. Prove that all the points lie on a single straight line.

(**Hint:** Consider the set of real numbers that are the areas of all the triangles formed by all the triples of points. Since there are finitely many points, there must be only finitely many triangles. So this is a finite set of real numbers and it must have a minimum element.

Now suppose some three points are not on the same line, then they form a triangle. But then argue that there must be a strictly smaller triangle! Why does this cause a contradiction?)

Solution: Proof by contradiction. Assume that not all the points lie on a single straight line. Then, there exists at least one triangle formed by the line segments of three points that are not on the same line. Consider any such triangle. By the pigeonhole principle, at least two of the points that form this triangle are the same color (three points and only two colors).

Based on our givens, there must exist a point of the other color between these two points of the same color. Now consider a triangle formed by this point and two points from the smallest triangle. This new triangle is strictly smaller than the original triangle which we were considering!

However this process can now be repeated with this smaller triangle, ad infinitum, leading to an infinitely long sequence of triangles with strictly decreasing areas. However we had a finite number of trinagles in total (since the point set is finite). Thus tehre must be a minimum area triangle! This yeilds a contradiction.

Therefore our assumption was incorrect, and we have shown that all the points lie on a single straight line.

(This argument can also be stated explicitly in terms of the well ordering principle, say by considering the set of areas of all possible triangles. This would be a finite set of real numbers so if it is non-empty, it must have a minimum...)

Question 6

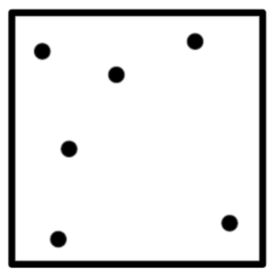


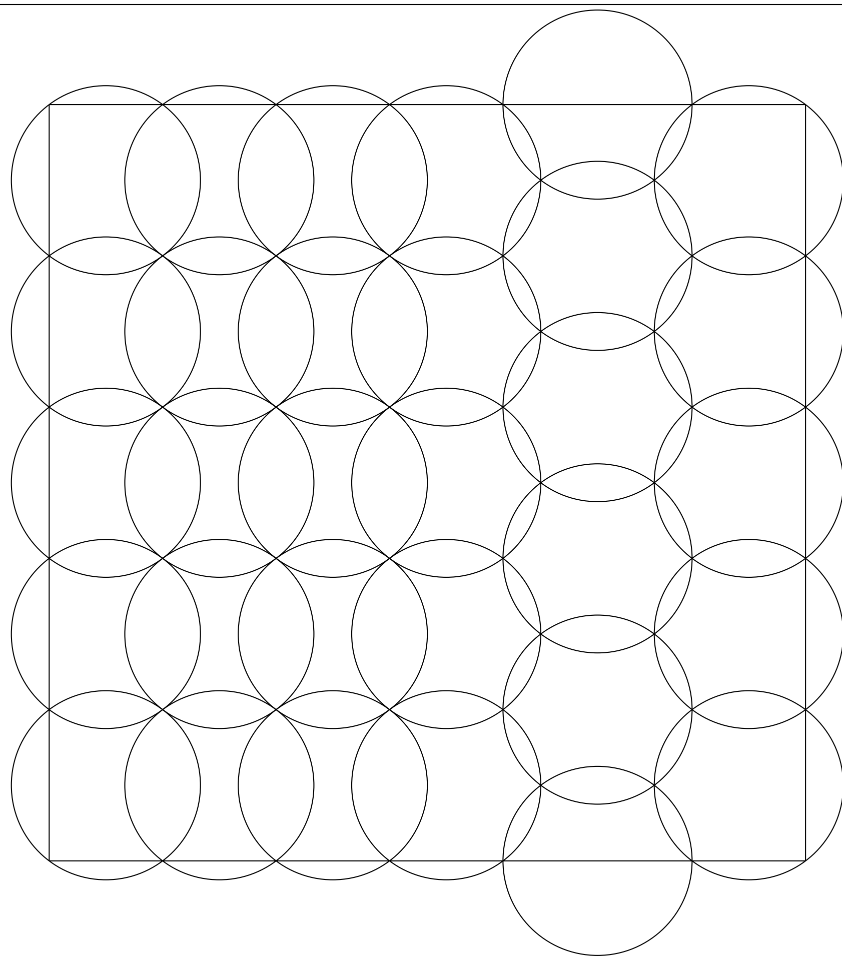
Figure 1: Sparse set of points in a unit square

Consider a unit square (side length is 1 unit). A set of points S is called *sparse* if every pair of two points in S are at least $\frac{1}{4}$ distance away from each other (See figure 1). Prove that there exists a constant C such that no sparse set of points within the square can contain more than C many points.

That is, prove, for some explicit number C, that no sparse set of points within the square can contain more than C many points (proving this for any constant C is ok, but try to prove it for as small a value of C as possible).

(**Hint:** Use the Pigeon Hole Principle. Treat the points as pigeons, figure out how you should define the pigeonholes).

Solution: Consider the following covering of the unit square:



The covering consists of 31 circles of diameter $\frac{1}{4}$. We will prove C exists and that $C \leq 31$. First Assume $C > 31$. Then, there exists an arrangement of 32 points within the unit square such that no two points are less than $\frac{1}{4}$ distance apart. Since the circles covering the unit square are of diameter $\frac{1}{4}$, no two points can be in the same circle. But since we have 32 points and only 31 circles, by the pigeonhole principal we know that at least two points must exist in the same circle. Contradiction, therefore $C \leq 31$. By proving C has an upper bound, we have also shown that C exists.

Note: The key point is to divide up the square into small regions that:

- together cover the whole square and
- are small enough that any two points in a single small region would be less than 1/4 apart.

Now if you can do this using C small regions, and the point set is sparse, then you certainly cannot have more than C points in total, since you can't put two points into a single small region.

Note: You could for example divide the unit square into a grid of small enough squares (calculate how small is small enough) and use that to get a (simpler, though weaker) upper bound on the number of points you can fit in the unit square given that the point set has to be sparse. This is perfectly fine, as for this question, any constant upper bound C suffices.

Note: You can assume points on the boundary are in the square or not, this doesn't change the solution.

Extra Practice Questions

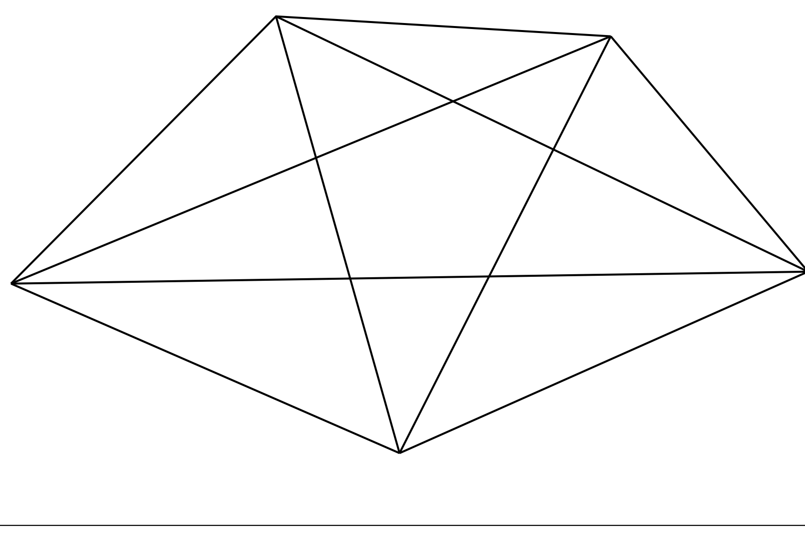
Question 7

Two players are playing two-moves chess, i.e. chess in which each player makes two moves in a single turn. Prove that, with perfect play, the player that plays with the black figures (and plays second) cannot win the game. (**Hint:** You don't need to know much about chess, only the rules about how the pieces move)

Solution: Suppose there exists a winning strategy for black (the second player). Then white (the first player) moves the knight piece twice, achieving the same configuration as before white had moved. But in this case, white has effectively become the second player, and so white has a winning strategy as well as black. Both players cannot have a winning strategy, so we have a contradiction.

Question 8

Prove by induction: A convex n-gon has $n(n-3)/2$ diagonals. A convex n-gon is a shape with n angles such that each interior angle is less than or equal to 180 degrees. A diagonal is a line segment connecting any two non-adjacent vertices. For example, a triangle is a "3-gon" and a pentagon is a "5-gon". Thus a square "4-gon" has $\frac{4\times1}{2} = 2$ diagonals and a pentagon has $\frac{5\times2}{2} = 5$ diagonals (see figure below).



Solution:
Base case: If $n = 3$, then there are no diagonals because for every pair of vertices, there is an edge connecting them.

Inductive hypothesis: For some integer $h \geq 3$, the convex h-gon has $h(h-3)/2$ diagonals.

Inductive step: Let $n = h+1$. Consider adding an $(h+1)$-th vertex to an h-gon such that the newly formed $(h+1)$-gon is convex. That is, the new vertex is outside the perimeter of the existing h-gon, and forms $h-2$ new diagonals. All the new diagonals must be interior, since otherwise the $(h+1)$-gon would not be convex. We also have a new diagonal formed by the side which connected the vertices adjacent to the newly added vertex. Thus a total of $(h-2)+1 = h-1$ new diagonals are formed. By the inductive hypothesis, the h-gon contained $h(h-3)/2$ diagonals, so the new $(h+1)$-gon contains

$$\frac{h(h-3)}{2} + h - 1 = \frac{h^2 - 3h + 2h - 2}{2} = \frac{h^2 - h - 2}{2} = \frac{(h-2)(h+1)}{2}$$
$$= \frac{(h+1)((h+1)-3)}{2}$$
$$= \frac{n(n-3)}{2} \qquad \square$$

Question 9

Show that if 25 people play in a ping pong tournament (each game is one versus one and we do not know anything else about the tournament structure: as in who plays how many matches) then, prove that at the end of the tournament, the number of people who played an even number of games is odd.

(**Hint**: Let n_i be the number of games player i played. What can we say about the sum of all the n_i?)

Solution:
For every player i, let n_i be the number of games they play. Since each game consists of two players, $n_1 + \ldots + n_{25}$ is even, as each game is counted twice. Notice for each player i, we can write $n_i = 2m_i + h_i$, where m_i is a natural number (or 0) and $h_i \in \{0, 1\}$. Then, we get that the total number of odd players is $h_1 + \ldots + h_{25}$, as each h_i is odd whenever a given player i plays an odd number of games. Furthermore,

$$\sum_{i=1}^{25} n_i = \sum_{i=1}^{25} (2m_i + h_i)$$
$$= 2\sum_{i=1}^{25} m_i + \sum_{i=1}^{25} h_i$$

Since the entire expression must be even and the left term is even, then the right term must be even.

Therefore, since the right term gives the total number of players who play an odd number of games, we conclude the total number of players who play an odd number of games is even. But since the total number of players is 25, which is odd, this means that the number of players who play an even number of games must be odd. $\square$