Quiz 1 Solutions

Question 1 (5 points)

Consider the following statement (assume all variables are **natural numbers**):

$$\exists x \forall y \exists z, x + y > z.$$

- What is the negation of the statement?
- Which one is true, the statement or it's negation? Very briefly explain why.

Solution:

$$\neg(\exists x \forall y \exists z, x + y > z) = \forall x \exists y \forall z, x + y \leq z$$

The original statement is true. Since we pick x before z, we can let $z = x$. Thus, $x+y \geq x+1 > x = z$.

Question 2 (10 points)

Using induction, prove, for any natural number n that the sum of the first n odd numbers is n^2, i.e.,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Solution: Base case: The claim holds for $n = 1$, since $(2 \times 1 - 1) = 1 = 1^2$.

Inductive hypothesis: Assume the claim holds for $n = k$, i.e. $1 + 3 + 5 + \cdots + (2k - 1) = k^2$

Inductive step: Consider $n = k + 1$.

$$\begin{aligned} 1 + 3 + \cdots + (2k - 1) + (2(k + 1) - 1) &= 1 + 3 + \cdots + (2k - 1) + (2(k + 1) - 1) \\ &= k^2 + (2(k + 1) - 1) && \text{(I.H.)} \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2 \end{aligned}$$

So the claim holds for $k+1$. By induction, this completes the proof that the claim holds for all natural numbers.

Question 3 (5 points)

A particle starts out at the origin $(0, 0)$ of a large (1000×1000) grid. At every step, it either takes two steps right and one step up, or it takes two steps up and one step right. You can pick either of these two moves at each step. So, for example, from $(0, 0)$ it can get to $(1, 2)$ or $(2, 1)$ in the next step.

Can the particle end up at $(101, 101)$ after any number of steps? Why/ Why not? Prove your claim (if yes, explain how; if not, prove that it cannot).

Solution: No, the particle can never reach $(101, 101)$.

The reason is that in each step the sum of the co-ordinates increases by 3. Since initially the sum of the co-ordinates is a multiple of 3, so the sum of the coordinates of the particle will always be multiple of three (since adding 3 to a multiple of 3, results in a multiple of 3).

But $101 + 101 = 202 = 67 \times 3 + 1$ is not a multiple of 3, so we cannot reach this position.