Homework 9 Solutions

Question 1

State true or false. If true, prove the claim. If false, provide a counter example and a proof that your counterexample is indeed a counter example.

1. All trees are bipartite graphs.
2. All bipartite graphs are trees.

Solution:

1. True. According to q6.4 in hw 8, we know that for each tree the chromatic number is at most 2. To see a graph with chromatic number at most 2 is bipartite, we can split the vertices according to coloring. In a properly colored graph, for any two vertices that are colored the same, there is no edge between them.
2. False. Consider the $4-$cycle graph C_4. C_4 is bipartite but it's not a tree.

Question 2

State true or false. If true, prove the claim. If false, provide a counter example and a proof that your counterexample is indeed a counter example.

1. All two-colourable graphs are bipartite.
2. All bipartite graphs are two-colourable.
3. All bipartite graphs have no odd length cycles.

Solution:

1. True. Same as question 1, we can split the vertices into two disjoint set according to their color. For vertices with same color, there are no edges between them.
2. True. If a graph is bipartite, we know that vertices can be split into two disjoint sets A and B, so we can color all vertices in A using one color and all vertices in B using another. By the definition of bipartite graph, there are no edges between two vertices with the same color.
3. True. If G is bipartite with vertex sets A and B, every step along a walk takes you either from A to B or from A to B. To end up where you started, therefore, we must take an even number of steps.

Question 3

Prove that the following are equivalent. A proof is outlined in the lecture notes but you should prove the equivalence yourself.

1. A graph G is a tree: that is it is a connected acyclic graph.
2. G has n vertices (for some value of n), is connected and has exactly $n-1$ edges.

Solution:

Let $G = (V, E)$ be a graph. Since both definitions require G to connected, we will assume G is connected. We proceed by strong induction on $n = |V|$, the number of vertices in the graph.

Base Case: $n = 1$.
Denote $V = \{v\}$. There are no possible edges (recall we only care about simple graphs). Thus, it satisfies both definitions. Conversely, if G contains $n-1 = 0$ nodes, then it must be connected.
Inductive Hypothesis: Given any connected graph with $n \leq k$ nodes, it is a tree if and only if it has $n-1$ nodes.
Inductive Step: Consider $n = k+1$. We will consider the forward and backward cases separately.

- **Forward:** Suppose G is a connected tree with at least two nodes. Then, $\exists e \in E$. Let G' be the graph G with e removed. Since G is connected, G' contains two components, G_1 and G_2. Let n_1 and n_2 be the number of nodes in G_1 and G_2, respectively, and note that $0 < n_1, n_2 < n$ and $n_1 + n_2 = n$.

  By theorem 5, for every pair of vertices in G, there is a unique path connecting them. Thus, for G', every pair of vertices has at most two paths connecting them. Thus, by definition of a component, G_1 and G_2 each have exactly one path connecting each pair of nodes. Therefore, G_1 and G_2 are trees, and so by the inductive hypothesis, the total nodes in both is $(n_1 - 1) + (n_2 - 1) = n - 2$. Hence, adding back e gives us the total number of edges in G, which is $n-1$ as desired.

- **Backward:** Suppose G is a connected and has $n-1$ edges. Then, by the pigeonhole principle, since G contains n nodes, G must have a node that is connected to exactly one edge. Denote that node and edge as v and e, respectively.
  Let G' be the resulting graph from removing v and the edge it is connected to. Since v had degree 1, G' must be connected and contains $(n-1) - 1 = n - 2$ edges and $n-1$ nodes. Therefore, by the inductive hypothesis, G' is a tree. Then, it follows that G must also be a tree, as adding back v and e cannot create any cycles. $\square$

Question 4

Prove that the following are all equivalent. Some of these equivalences are proved in the lecture notes, but you should prove them yourselves.

1. A graph G is a tree: that is it is a connected acyclic graph.

2. There is a unique path between any two vertices in G.
3. G has no cycles, but if you add an edge between any two non adjacent vertices of G to G you now have a cycle in the new graph.

Solution:

First, notice that if G only contains one node, then it has no edges and so satisfies all three definitions. We proceed assuming G has at least two nodes.
To see how (1) and (2) are equivalent, consider the following:
Since G contains at least two nodes, $\exists u, v \in V$ s.t. $u \neq v$.

- **Forward:** Assume $G = (V, E)$ is a tree.
  Let P'_{uv} be a path between u and v. If $P'_{uv} \neq P_{uv}$, then we could create a cycle, which would contradiction the claim that G is a tree. Thus, $P'_{uv} = P_{uv}$, and so we conclude P_{uv} is unique. Since our choice of u and v was arbitrary, only requiring they be distinct, this concludes the forward direction.
- **Backward:** Assume there is a unique path connecting any two vertices in G.
  Since u and v are connected by a path, and were chosen arbitrarily, we conclude G is connected. Furthermore, since the path is unique, we conclude G is acyclic, as no cycle could contain u or v, for any choice of (distinct) u and v.

This concludes the proof for (1) and (2) being equivalent.

In a similar way, we will next show that (2) and (3) are equivalent, which by transitivity will show (1) and (3) are equivalent.

- **Forward:** Assume there is a unique path connecting any two vertices in G.
  Suppose we want to add some edge $e = (u, v) \notin E$. Then, since there is already a path connecting u and v, lets call it P_{uv}, adding in e would allow us to create a cycle that starts at u, traverses P_{uv}, then crosses e to get back to u. This concludes the forward direction.
- **Backward:** Assume (3).
  Since G has no cycles, there can be at most one path connecting any pair of vertices. Conversely, there must be at least one path connecting any pair of vertices, as otherwise there must be some $u, v \in V$ s.t. adding (u, v) to E only results in there being one path connecting u to v, and so no cycle could have been added. Therefore, there must be a unique path connecting any pair of vertices in G.

This concludes the proof. $\square$

Question 5

Calculate the following for rooted complete binary trees. You must prove the correctness of your calculations.

1. the maximum possible number of leaves in a complete binary tree of height h.
2. the maximum possible total number of vertices in a complete binary tree of height h.

Solution:

1. 2^h. We prove this by induction. For $h = 0$, clearly one can have only $2^0 = 1$ leaf. Assume it is true for $h = k$ where $k \geq 0$, we will show it is also correct for $h = k + 1$. By the induction hypothesis, we know the maximum possible number of leaves in a complete binary tree of height k is 2^k. Let's increase the height of the tree by 1 from it. Since each of the original leaf can have at most 2 children, the maximum possible number of leaves after we grow the tree is $2 \times 2^k = 2^{k+1}$.
2. $\sum_{i=0}^{h} 2^i = 2^{h+1} - 1$. Again we prove this by induction. For $h = 0$, one can have 1 vertex. Assume it is true for $h = k$ where $k \geq 0$, we will show it is also correct for $h = k + 1$. By the induction hypothesis, we know the maximum possible number of vertices in a complete binary tree of height k is $\sum_{i=0}^{k} 2^i$. Let's increase the height of the tree by 1 from it. Since each of the original leaf can have at most 2 children, the maximum possible number of vertices after we grow the tree is $\sum_{i=0}^{k} 2^i + 2^{k+1} = \sum_{i=0}^{k+1} 2^i$.

Question 6

We will call degree 0 vertices of a graph, isolated vertices.

Claim: Consider **any** graph G. If G has no isolated vertices then it must be connected.

Consider the following inductive proof (induction on the number of vertices, n) that supposedly proves this claim for all graphs.

1. Clearly the claim is true in the case of a single vertex $(n = 1)$. Since in fact the vertex must be isolated, there is nothing to prove.
2. The claim is true for $n = 2$. If neither vertex is isolated then they must both be degree 1 and thus they have an edge between them, so the graph is connected.
3. Assume that the claim is true for every n vertex graph.
4. Consider an arbitrary n vertex graph G with no isolated vertex. By the inductive hypothesis, it must be connected.
5. Now let's add a vertex v to G to create the new graph G'. G was an arbitrary n vertex graph that had no isolated vertex, so we can, in this way, construct any arbitrary $n + 1$ vertex graph that has no isolated vertex.
6. But G' has no isolated vertices, so v must be connected by an edge to at least one other vertex, u of G.
7. G was a connected graph, so it consists of a single connected component.

8. In G', v is in the same connected component as u since v has an edge to u.
9. So u and v are in the same connected component in G'. The other vertices apart from u were all in the same connected component as u in G, and since we didn't remove any vertices or edges, they will also be in the same connected component as u in G'.
10. Thus G' has a single connected component: the one that contains u. Thus G' is connected, which completes the proof.

However this claim is incorrect!

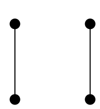
1. Provide a counterexample demonstrating that the claim is false.
2. Point out at exactly which step the argument goes wrong. Explain why the step in the argument that you point out does not hold.

Important Note: This example points out the issue with starting with a smaller graph and trying to inductively prove the claim for the bigger graph! These sorts of issues can often occur when you are trying to prove some statement **for any arbitrary graph** in this sort of way. Often it is safer to start with the $n+1$ vertex (larger) graph that you are trying to prove something about. Then remove a vertex (or possibly multiple vertices/ edges in general) to get an n vertex graph. Argue that the (smaller) n vertex graph, by the inductive hypothesis, has the desired property. Now use this to argue that the (bigger) $n+1$ vertex graph has the desired property. $Q.7$ on the previous homework is a typical example of this type of argument.

Notice if you want to prove merely that for all values of n **there exists** an n size graph with a certain property, you can just take an n size example graph with the property and construct from it an $n+1$ size example graph with the property. The issue in this example has to do with trying to prove a property **for all** graphs.

Solution:

1. Let G be a graph with two disjoint copies of K_2. (K_2 is the complete graph on 2 vertices, i.e. two vertices joined by an edge.)

2. The issue happens on step 5. In that step we add a vertex to an existing graph (which is technically fine) and claim that every graph of $n+1$ vertices can be constructed this way (which is incorrect). For example, when constructing the graph of the counterexample this way, we add a vertex to a 3-vertex graph G to form the 4-vertex graph G'; although G' has no isolated vertices, G has.

   Note: although the part of adding a vertex to an existing graph is fine, it is a good indication that the argument may not encompass all graphs, so beware of such steps in proofs.