Homework 3 Solutions

Question 1

An odd number of people stand in a room and each person has one ball. When I blow a whistle, each person tosses the ball to their nearest neighbour. Show that somebody ends up without a ball. You can assume that there is more than one person and that all pairwise distances between the people are distinct.

(**Hint:** Recall from class that we argued that there will be a pair of people who will throw the ball to each other. Prove this fact on your own. Now try to do an inductive proof where you consider two cases: no body throws a ball to either member of this pair, or else someone does throw a ball to one of them.)

Solution: First we show that for any group (number of members can be even or odd) where the pairwise distances are unique, there must be a pair that throws their balls at each other. Consider all pairs of people and the distances between them. There must be a smallest distance. Consider a pair of people that has that distance between them. Since we are given that all pairwise distances are unique, they must throw their balls at each other, because for both of them, the other person is the closest person.

Now we will prove the claim from the question. Let n be the number of people in the group, where n is given to be odd and greater than 1. We will use induction on n.

Let $n = 3$. We are given that all pairwise distances between the people are distinct, so there are two people who throw their balls to each other. The third person will then throw their ball to a member of this pair without receiving a ball in return.

Assume our claim holds for $n = k > 3$, where k is odd. Let $n = k + 2$. Consider the pair in the group of $k + 2$ people that throws to each other. There are two cases:

- No one else from the group throws to them. Using our inductive hypothesis, we know the group of size k which excludes this pair has somebody who ends up without a ball. Then the group of size $k + 2$ including the isolated pair also has somebody who ends up without a ball.
- At least one person from the group throws to them. Then we know the group excluding this pair has fewer balls than when the game started, meaning at least one person must end up without a ball.

Since k is odd $\implies$ $k + 2$ is odd, we have shown our claim is true for all odd n, where $n > 3$.

Alternate Solution: Let $P = \{p_1, ..., p_n\}$ be the members of the group, where n is given to be odd and greater than 1. We will prove our claim using induction on n.

Let $n = 3$. We are given that all pairwise distances between the people are distinct, so out of the set of three pairwise distances between the people, there exists a least pairwise distance which is unique. The people who comprise this pair will then throw to each other, and the third person throws their ball to a member of this pair without receiving a ball in return.

Assume our claim holds for $n = k > 3$, where k is odd. Let $n = k + 2$. For each $p_i, 1 \leq i \leq n$, there is another member of the group which is closest at distance d_i from p_i. Consider D, the set of minimum distances for each member of the group. D must have a maximum, d_{max}. We consider two cases:

- $\exists! i, 1 \leq i \leq n$, s.t. $d_i = d_{max}$. In this case p_i will throw the ball away, but will not be thrown any ball since $\forall j \neq i, d_j < d_{max}$
- $\exists! i, j, 1 \leq i, j \leq n, i \neq j$, s.t. $d_i = d_j = d_{max}$. Then, p_i, p_j will exchange balls, and nobody will throw them a ball. Consider $P' = P - \{p_i, p_j\}$. $|P'| = k + 2 - 2 = k$, and so by our inductive hypothesis, somebody in P' ends up without a ball. Since somebody in P' ends up without a ball, and p_i, p_j throw to each other in isolation, somebody P will also end up without a ball (it will be the same person).

No other cases are possible since we are given that all pairwise distances are unique.

Since k is odd $\implies k + 2$ is odd, we have shown our claim is true for all odd n, where $n > 3$.

(**Note to graders:** There are other proofs possible as well, based on the idea that there cannot be any 'cycle of throws'. However one does need to be a little careful to formalize such proofs correctly.)

Question 2

There are 7 greeting cards, each of a different colour and 7 envelopes, which are also of these exact 7 distinct colours. Each card must go in a different envelope. How many ways are there to put the cards into the envelopes such that exactly 4 of them are in envelopes of the same colour as the card?

Solution: There are $\binom{7}{4}$ ways to pick the 4 cards which will match their envelope color. Consider the number of ways we can arrange the remaining 3 cards. The remaining envelopes must have the same colors as the remaining cards, since we have already matched 4 by color and we are given all 7 cards have matching envelopes. Thus we only have 2 choices of cards to assign to the first of our remaining envelopes, and the remainder of our assignments are forced (we are asked the number of ways *exactly* 4 cards match their envelope). Thus there are

$$\binom{7}{4} \cdot 2 = 70$$

ways to arrange the cards in the requested way.

(**Note to graders:** The 2 term can also be calculated by looking at the 6 possible orderings of the remaining 3 cards into the corresponding 3 envelopes, and excluding all the cases that have any pairs.)

Question 3

Part 1: Prove that

$$2\binom{n}{1} + 2^2\binom{n}{2} + 2^3\binom{n}{3} + \cdots + 2^{n-1}\binom{n}{n-1} + 2^n\binom{n}{n} = 3^n - 1$$

Solution: By Binomial theorem, $3^n = (1+2)^n = \sum_{i=0}^{n} \binom{n}{i} 2^i = 1\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + 2^3\binom{n}{3} + \cdots + 2^{n-1}\binom{n}{n-1} + 2^n\binom{n}{n}$. By subtracting 1 from both sides, we get the equality.

(**Note to graders/ students:** There are at least 2 other proofs. One is a **combinatorial proof**, say by looking at ternary strings. The other is **a proof by induction**, using Pascal's identity in the inductive step. It's a good excercise to work out all 3 proofs on your own.)

Part 2: Prove that
Prove, via a combinatorial argument, that:

$$\binom{n}{k} \cdot \binom{k}{t} = \binom{n}{t} \cdot \binom{n-t}{k-t}$$

Note: you will not get credit if you simply use the formula for $\binom{n}{k}$; rather you must use a combinatorial argument, i.e., show that both the LHS and RHS represent ways of counting the number of different things of a certain kind.

Solution: Suppose you want to form a committee of k people from n people, and inside the committee, we have a leading committee of t people.

LHS:

1. First we choose k people out of the total of n people to form the committee, which gives us $\binom{n}{k}$ ways.
2. Then we choose t people out of the people that are already selected to be in the committee, which has $\binom{k}{t}$ different choices.
3. Thus, in total we have $\binom{n}{k} \cdot \binom{k}{t}$

RHS:

1. First we choose t people out of all the people so these t people form the leading committee.
2. Then we choose the remaining $k-t$ people to form the committee of k people from the remaining people, which has $\binom{n-t}{k-t}$ ways.
3. Thus, in total we have $\binom{n}{t} \cdot \binom{n-t}{k-t}$

(**Note to graders:** There are many ways to express this idea, for instance in the language of sets.)

Question 4

How many subsets of a 100 element set have an odd number of elements?

Solution:

(**Note to graders:** There are several proofs. I will sketch some of them:)

One key intuition is that there are 2^{100} subsets in total and we expect half of them to be odd in size. So the answer should be 2^{99}. But we know need to prove this.

Proof 1: Bijective proof:

We show that there is a bijection between **all** subsets of a 99 element set, say $\{1, 2, ..., 99\}$ and **odd size subsets** of a 100 element set, say $\{1, 2, ..., 100\}$. Since the total number of subsets of a 99 element set is 2^{99}, this completes the proof.

The key idea is to look at the subsets of the 99 element set. For each of them, if it has an odd number of elements, we know it's already an odd subset of the 100 element set. If it's even, we just add the missing element from the 100 element set to make it odd.

Formally speaking, the bijection is given by $f(X) = X$ if X has an odd number of elements and $f(X) = X \cup \{100\}$ if X has an even number of elements.

(**Note to students/ graders:** It is relatively easy to check that this is indeed a bijection. While you don't have to prove this formally here to get points on the question, it is a good exercise.)

(**Note to students/ graders:** Of course, you can also write the bijection in the other direction: from **odd size subsets** of a 100 element set to **all** subsets of a 99 element set: that would just be the inverse of this function f.)

Proof 2: Induction proof:

We prove that the number of subsets of odd number elements of a n element set is 2^{n-1} by induction. First, when $n = 1$, there is only one subset with an odd number of elements, which proves the base case.

Now assume for any $n - 1$ element set, the number of odd subsets is 2^{n-1} by induction. Given a set S of n elements, let's take one element out of the set, and by the inductive hypothesis, we know this set of $n - 1$ elements, call it S' has 2^{n-2} subsets of odd number elements.

Note these sets are all subsets of the original S. So there itself we get 2^{n-2} odd size sets of S. And in fact we have counted all the odd size subsets of S that are missing the one element that we removed to get S'.

By adding the 1 element that we removed, the odd size subsets of the $n - 1$ elements are now even size and the even size subsets are now odd size, which gives us another 2^{n-2} subsets of odd number.

By combining them together, we have shown that the number of subsets of odd number elements of a n element set is 2^{n-1}. Let $n = 100$, we have 2^{99} such subsets.

(**Note to students/ graders:** This is ultimately a very similar proof idea as Proof 1. I skipped some small details in the argument, but you should be able to work these out.)

Proof 3: Proof using Binomial Theorem:

Notice that the number of odd subsets of a 100 element set is:

$$\binom{100}{1} + \binom{100}{3} + \binom{100}{5} + ... + \binom{100}{99}.$$

Meanwhile it is easy to argue via a bijection, or by setting $a = b = 1$ in the binomial theorem that:

$$\binom{100}{1} + \binom{100}{2} + \binom{100}{3} + \binom{100}{4} ... + \binom{100}{100} = 2^{100}.$$

But if we plug $a = -1, b = 1$ into the binomial theorem, this yields:

$$\binom{100}{0} - \binom{100}{1} + \binom{100}{2} - \binom{100}{3} + \binom{100}{4} - \binom{100}{5} ... = 0.$$

or equivalently:

$$\binom{100}{1} + \binom{100}{3} + \binom{100}{5} + ... + \binom{100}{99} = \binom{100}{0} + \binom{100}{2} + \binom{100}{4} + ... + \binom{100}{100}.$$

The LHS and RHS are the number of odd and even size subsets respectively, and this concludes the proof that each of these are 2^{99}, since together they are 2^{100}.

Question 5

How many distinct sets of solutions are there to the equation

$$x + y + z = 13$$

where x, y, z are non-negative integers, i.e., they are each in the set $\{0, 1, 2, \ldots\}$. Note that we consider $x = 6, y = 3, z = 4$ a different solution from $x = 3, y = 4, z = 6$.

(**Hint:** Think about it like this. There are 13 ones and you have to allocate them as belonging to x, y or z. Think of the stars and bars type reasoning we saw in class)

Solution: Using the stars and bars strategy, we consider the 1's which add up to 13 to be our stars, and the separation of x, y, and z to be our bars. There are then

$$\binom{13+2}{2} = 105$$

distinct sets of solutions to the given equation.

(**Note to students:** Make sure you understand why there is a bijection between solutions to this equation and strings of 13 stars and 2 bars! We don't necessarily expect you to write a formal proof of this bijection for this question, but you should understand why this way of counting works.)

Question 6

I claim to have found a set of 100 ten digit natural numbers, with the remarkable property that there are no two different subsets of this set such that the sum of the numbers in the two subsets is the same. Prove that this is impossible.

Solution:

Let A be the proposed set and n be the number of subsets it has. Let m be the number of possible distinct sums for those numbers. Since $|A| = 10$, $n = 2^{100} = 1024^{10} > 1000^{10} = 10^{30}$. Now, notice that the largest value for any element in A is $10^{10} - 1 < 10^{10}$, and so we can get the largest possible sum by multiplying this by the number of elements in A, giving $m < 10^{10} \cdot 100 = 10^{12}$.

However, notice that $m < 10^{12} < 10^{30} < n$, which means there are far more sums for subsets of A than there are distinct sums, which leads us to conclude at least two subsets have the same sum. $\square$

Extra Practice Questions

Question 7

Consider n factories and n ports. No three locations (any combination factories and ports) lie on the same straight line. Each factory is to be connected to exactly one port by means of a single straight line road. Show that there is a way to do this such that no two roads cross one another.

(**Hint:** Consider the total length of all n roads. Consider a road network that minimizes this total length (why must such a road network exist?). Argue that if this road network has a pair of crossing roads, we can construct another road network with lower total road length, that does not have this intersection.)

Solution: In each road network, we can compute the total length by simply adding all the line roads together. Note there are $n!$ possible road networks, as for the first factory, we can select one from n ports, for the second one, we can select one from $n-1$,..., by continuing this process, we know the total number of possible road networks is $n!$. Since the number of road networks is finite, we know there exists a matching that minimizes the total road length. Denote this road network as R. Assume that R has a pair of crossing roads, and denote them as (f_i, p_i) and (f_j, p_j), where f means factories and p means ports. Consider the quadrilateral formed by points f_i, p_i, f_j, p_j. Since (f_i, p_i) and (f_j, p_j) has an intersection inside this quadrilateral, we know (f_i, p_i) and (f_j, p_j) are the two diagonal lines.

Since they are diagonal lines and (f_j, p_i) and (f_j, p_i) are side lines, we know the total length of (f_i, p_i) and (f_j, p_j) is larger than the total length of (f_j, p_i) and (f_j, p_i), which contradicts that R minimizes the total road length. Therefore, R contains no intersections. We have thus shown that there does exist a road configuration that has no intersections, namely R.

Question 8

Prove that:

$$\binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{2}^2 + \binom{n}{n}^2 = \binom{2n}{n} - 1$$

This follows from Vandermonde's identity, but you should try to do a proof from scratch. Try to prove this in two ways, once by following the first hint below (which leads to an algebraic proof), and the second time by a combinatorial argument.

(**Hint:** $(1+x)^{2n} = (1+x)^n(x+1)^n$. Use the binomial theorem twice on the RHS. Now look at the coefficient of x^n on both sides.)

For the combinatorial argument it might be easier to prove the following equivalent identity:

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n-1}\binom{n}{1} + \binom{n}{n}\binom{n}{0} = \binom{2n}{n}$$

Solution:

- **algebraic proof:**

$$(1+x)^{2n} = (1+x)^n(1+x)^n = (\sum_{k=0}^{n}\binom{n}{k}x^k)(\sum_{j=0}^{n}\binom{n}{j}x^j) = \sum_{0\le k,j\le n}\binom{n}{k}\binom{n}{j}x^{k+j}.$$

  For the coefficient of x^n on the LHS, by binomial theorem we know it's $\binom{2n}{n}$. For the right hand side, we need $k + j = n$ so that $j = n - k$. Summing the coefficients of all x^n on the RHS we obtain $\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n}\binom{n}{k}^2$. Rearranging yields the identity we wish to prove.

- **combinatorial argument:** consider a set $S = \{x_1, x_2, ..., x_n, y_1, ..., y_n\}$ with $2n$ elements. The number of subsets with size n is $\binom{2n}{n}$. Besides, to form a set with size n is to select k elements from $\{x_1, x_2, ..., x_n\}$ and $n-k$ elements from $\{y_1, ..., y_n\}$. The total number of ways of doing so equals $\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}$. Since we are counting the same thing, we obtain the desired identity.

Question 9

Prove the multinomial theorem. This states that for any natural numbers r and n:

$$(x_1 + x_2 + \cdots + x_r)^n = \sum \frac{n!}{n_1!n_2!n_3!\ldots n_r!}x_1^{n_1}x_2^{n_2}\cdots x_r^{n_r}$$

where the sum is over all the ways of writing $n = n_1 + n_2 + \cdots + n_r$, where the n_i are all non negative integers.

(**Hint:** Follow the way we proved the binomial theorem in class. Think about the product on the LHS. How many times would the term $x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$ appear?)

Solution: Follow the proof of the binomial theorem. How many times would the term $x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$ appear?

First, out of n buckets you select which n_1 buckets to pick x_1 from. Then from $n - n_1$ buckets you to select which n_2 buckets you pick x_2 from and so on.

So the number of times that the term $x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$ appears would be:

$$\binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \binom{n-n_1-n_2}{n_3} ... \times \binom{n-n_1-n_2 ... - n_{r-1}}{n_r}.$$

But if you use the formula for $\binom{n}{k}$ and expand this out, you get

$$\frac{n!}{n_1! n_2! n_3! \ldots n_r!}$$

and this completes the proof.

Question 10

Consider an equilateral triangle of side length n, which is divided into unit triangles, as shown. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example of one such path is illustrated in figure 2 for $n = 5$. Determine the value of $f(2023)$.

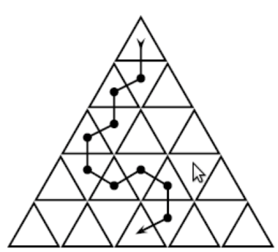

Figure 1

(**Hint:** The path must enter each row via some triangle. Number the upward pointing small triangles in each row. Note that there is 1 triangle you can use to go from row 1 to 2, 2 triangles you can use to go from row 2 to 3 and so on. Suppose I tell you a path uses triangle number a_1 in row 1 to go from row 1 to 2, triangle number a_2 in row 2 to go from row 2 to 3 and so on. Then I claim that I have completely specified the path. For example we can map the path in the figure to the string 1113)

Solution: Following the hint we know we only need to count the number of valid strings. Given length n, our string is of length $n-1$. The first bit only takes 1 possible value, the second bit takes 2 possible values,..., the final bit takes $n-1$ possible values. So there are $(n-1)!$ ways to form the string. $f(2023) = 2022!$.