Homework 8 Solutions

Question 1

If a graph has maximum degree d prove that it can always be colored using $d + 1$ colors.

Solution: We proceed by induction on the vertices of the graph.
Base Case: If there are no vertices, then there are no edges and so we have a legal coloring.
Inductive Hypothesis: Suppose the claim holds for $n = k$ vertices and consider $n = k + 1$.
Inductive Step: For the newly added vertex, since it has at most d neighbors and we have $d+1$ colors, there must be a color we can give it that is not used by any of its neighbors. By the inductive hypothesis, all other vertices have a legal coloring, so we conclude the graph has a legal $d + 1$ coloring. $\square$

(**Note:** You can do induction on d instead of n, but this is a bit trickier!)

Question 2

(1) Are the following graphs isomorphic or not? Either prove that they are not, or label the vertices to show an isomorphism between the graphs.

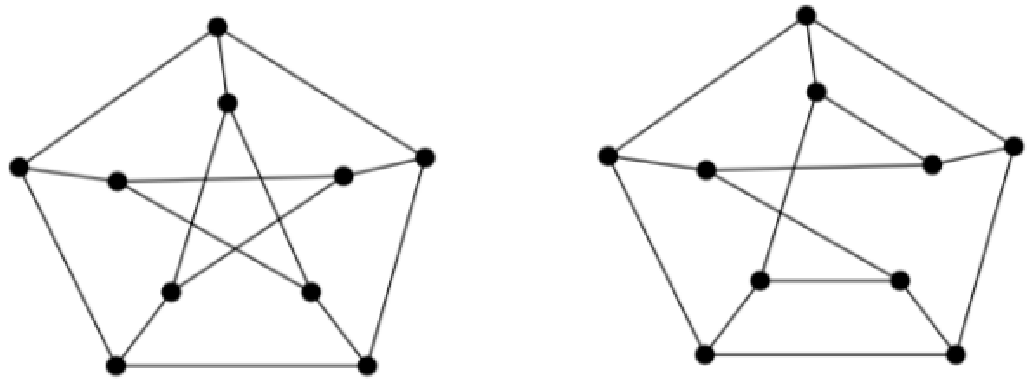


Figure 1

Solution: They are not isomorphic. To see this, notice that the graph on the left has no cycles of length 4. By symmetry, we can check this exhaustively by starting at any vertex (we can either check inner and outer vertices separately, but it actually makes no difference in this case). Then, looking at the outgoing paths, it takes at least 5 vertices to form a cycle. However, the graph on the right has two cycles of length 4. Therefore, they cannot be isomorphic. $\square$

(2) What is the chromatic number of the Petersen graph shown below? You must provide a proof of your answer.

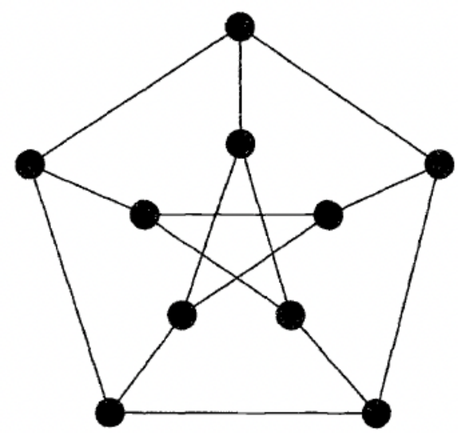


Figure 2: The Petersen Graph

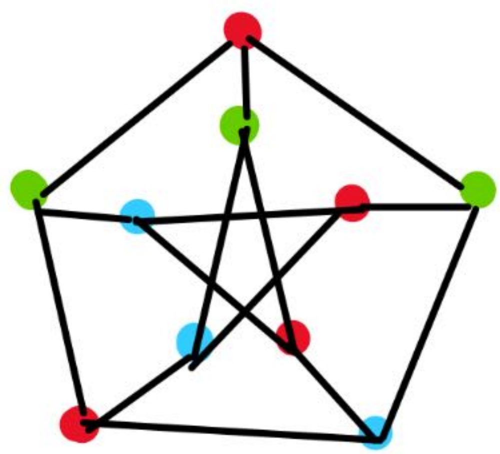


Figure 3

Solution: The chromatic number is 3. See figure 3 for an example of 3-coloring. To prove the chromatic number is 3, we need to show it is not 2-colourable. By theorem 2 from lecture note 8, it is equivalent to show the Petersen Graph is not bipartite. Since it has an odd cycle (e.g. the cycle that connects the outer 5 vertices), the Petersen Graph is not bipartite, and therefore not 2-colourable.

Question 3

State true or false. If true prove the claim. If false provide a counter example and a proof that your counterexample is indeed a counter example.

1. There exists a graph with 17 vertices and 138 edges.
2. A graph with 4 vertices and 6 edges exists that has the additional property that it can be drawn on a 2D sheet of paper without any crossing edges (**Hint:** the edges do not have to be straight lines!).

3. There is a graph with 10 vertices and 15 edges such that each vertex has the same degree.

4. If two graphs have exactly the same number of edges and exactly the same number of vertices, then they must be isomorphic.

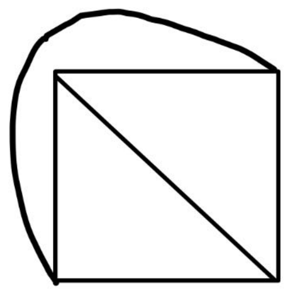


Figure 4

Solution:

1. False. The maximum number of edges one can have is $\frac{17\times16}{2} = 136$.

2. True. See figure 4 for an example. This can also be drawn diffferently, so that each edge is a straight line.

3. True. By theorem 1 from lecture note 8, we have $10k = 2 \times 15$ where k is the degree number of each vertex. So $k = 3$ and let's try to construct a graph with 10 vertices such that each vertex has degree 3. Denote the vertices by v_i for $1 \leq i \leq 10$. First, we can form a cycle by connecting the 10 vertices. This gives each vertex degree 2. Then, we add an edge between v_i and v_{i+5} for $1 \leq i \leq 5$. This increases the degree of each vertex by 1. The resulting graph is an example that satisfies the desired property. **Alternatively**, the Petersen graph is also an example of such a graph!

4. False. See figure 11 from lecture note 8 for a counter example. In fact even the degree sequences can be exactly the same, but the graphs may not be isomorphic.

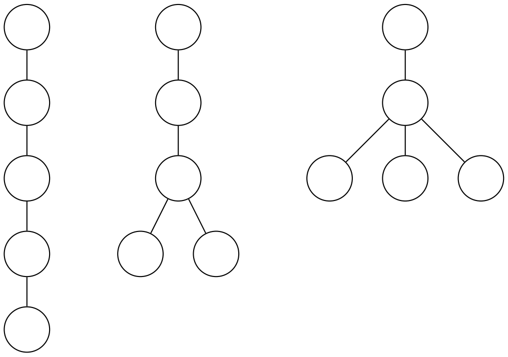
Question 4

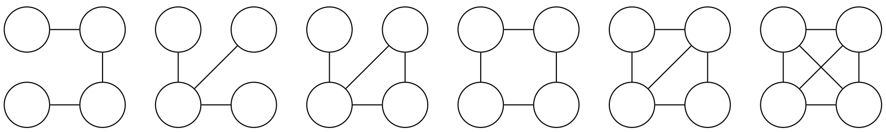
1. Draw all non-isomorphic trees on 5 vertices.

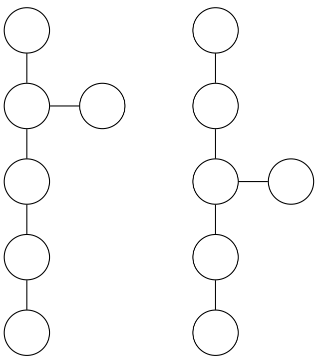
2. Draw all non-isomorphic connected graphs on 4 vertices.

3. Draw two graphs that are not isomorphic but have the same degree sequences.

Solution:

1. 

2. 

3. 

Question 5

State true or false. If true prove the claim. If false provide a counter example and a proof that your counterexample is indeed a counter example.

1. Given a graph G, if you keep the vertices the same, but add edges wherever G did not have edges and remove all of G's original edges, then you get the graph G^c that is the complement of G. See the figure for Honors Question 1 for an example. Claim: A graph and its complement cannot both be disconnected.

2. A connected graph that has the property that removing any edge makes it disconnected must be a tree.

3. If the chromatic number of two graphs is different then they cannot be isomorphic.

4. Every tree has a chromatic number of at most 2.

(**Notes:** Part 1 can also be proved by induction. For Part 4, we provide an inductive proof, but you can also do a non-inductive proof, where you define the coloring function by picking a root, and coloring red if the vertex is at even distance form the root, and blue otherwise.

In general, there are several proofs possible for each of these statements.)

Solution:

1. True. If $G = (V, E)$ is connected we are done, so we assume G is disconnected. Then G is comprised of a set of connected components C, where $|C| > 1$. Let $c : V \mapsto C$ be the function which maps a vertex to the connected component in G to which it belongs. Let $u, v \in G$ be arbitrary. Consider two cases:

   - If $c(u) \neq c(v)$, then there is an edge between them in G^c, implying there is a path between them in G^c.
   - If $c(u) = c(v)$, then $\exists w \in V$ s.t. $c(u) \neq c(w) \neq c(v)$, since $|C| > 1$. Thus, there is an edge between u and w in G^c, and an edge between w and v in G^c, which implies there is a path between u and v in G^c.

   Since u and v are arbitrary and there is a path between in both cases, G^c is connected. $\square$

2. True. Let $G = (V, E)$ be the connected graph described. Assume $\exists u, v \in V$ s.t. there is more than one path between u and v. Then $\exists e \in E$ s.t. e belongs to one of the paths between u and v, but not all others. If we remove e, there will still be at least one path between u and v left intact. Contradiction, since G should become disconnected by the removal of any edge. Therefore the path between u and v is unique. By Theorem 5 in lecture notes 9, since there is a unique path between any two of its vertices, G must be a tree. $\square$

3. True. Let $G = (V, E), G' = (V', E')$ be two isomorphic graphs. That is, there exists a bijection $f : G \mapsto G'$ which preserves adjacency. Let n be the chromatic number of G, then consider a coloring of G which uses n colors. We can then construct a coloring of G', where $f(v) = v'$ implies we use the color of v for v'. Since f preserves adjacency, this will be a valid n-coloring of G'. Let m be the chromatic number of G'. Then $m \leq n$. If $m < n$, then we could use the same proof by construction to show that there exists an m-coloring of G, which would be a contradiction. Therefore $m = n$ and G and G' have the same chromatic number, and we have proven the contrapositive of the provided claim. $\square$

4. True. Let n be the size of an arbitrary tree. Proof by induction on n.

   - Base cases: For $n = 1$, the tree has chromatic number 1. For $n = 2$, the tree has chromatic number 2. ✓
   - I.H.: Let $n = k$ and assume a tree of size k has chromatic number at most 2.
   - I.S.: Let $n = k + 1$ and consider a tree T of size $k + 1$. Since T is a tree, it must have a vertex v with $deg(v) = 1$. If we consider the subgraph T' of T without v or its incident edge, we have another tree of size k. By I.H. T' has chromatic number at most 2. Consider a coloring of T' with two colors c_1, c_2. WLOG, the color of the vertex adjacent to v in T is c_1. By coloring v in T with c_2 we have obtained a 2-coloring of T, and thus T has chromatic number at most 2. $\square$

Question 6

Let G be a graph on $2k$ vertices containing no triangles (i.e., no three vertices are all connected to each other). Show that G has at most k^2 edges, and give an example of a graph for which this upper bound is achieved.

Solution: We prove the statement by induction. For the base case, when $k = 1$, it's trivial that the max number of edges is 1. Now assume that this statement holds true for any i, $i \leq k$.

Let x, y be two vertices that are connected in G, where G is a $2k + 2$ vertex triangle free graph. If $d(v)$ is the degree of a vertex v, we see that $d(x) + d(y) \leq 2(k + 1) = 2k + 2$. This is because every vertex in the graph G is connected to at most one of x and y since it's triangle-free. Denote the graph H as we remove x, y and the edges connected with them from G. We know that H is a triangle-free graph with $2k$ vertices, so by the inductive hypothesis, H has k^2 edges. Then the number of edges in G is upper bounded as:

$$k^2 + d(x) + d(y) - 1 = k^2 + (2k + 2) - 1 = k^2 + 2k + 1 = (k + 1)^2.$$

We subtract one because the edge between x and y is counted twice.

(Notice that we picked x, y, such that there is an edge between them. But if no such edge exists in the graph G, then G simply has 0 edges, and this case is trivial.)

For an example of such a graph, consider the bipartite graph with each side has k vertices. Denote the two parts of this graph as A and B, $|A| = k$ and $|B| = k$. We add an edge between all $a \in A$ and all $b \in B$, so there are k^2 edges.