

Problem Set 2

Due: September 26

Reading: Course notes on induction.

Problem 1. Use induction to prove that the following inequality holds for all integers $n \geq 1$.

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)} \geq \frac{1}{2n+2}$$

Problem 2. This term in 6.042, we're constantly trying to divide a class of n students into groups of either 4 or 5 students.

(a) Let's try to use strong induction prove that a class with $n \geq 8$ students can be divided into groups of 4 or 5.

Proof. The proof is by strong induction. Let $P(n)$ be the proposition that a recitation with n students can be divided into teams of 4 or 5.

First, we prove that $P(n)$ is true for $n = 8, 9$, or 10 by showing how to break classes of these sizes into groups of 4 or 5 students:

$$\begin{aligned} 8 &= 4 + 4 \\ 9 &= 4 + 5 \\ 10 &= 5 + 5 \end{aligned}$$

Next, we must show that $P(8), \dots, P(n)$ imply $P(n+1)$ for all $n \geq 10$. Thus, we assume that $P(8), \dots, P(n)$ are all true and show how to divide up a class of $n+1$ students into groups of 4 or 5. We first form one group of 4 students. Then we can divide the remaining $n-3$ students into groups of 4 or 5 by the assumption $P(n-3)$. This proves $P(n+1)$, and so the claim holds by induction. \square

This proof contains a critical logical error. (In fact, the claim is false!) Identify the first sentence in the proof that does not follow and explain what went wrong.

(b) Provide a correct strong induction proof that a class with $n \geq 12$ students can be divided into groups of 4 or 5.

Problem 3. The game of *Mini-nim* is defined as follows: Some positive number of sticks are placed on the ground. Two players take turns removing one, two, or three sticks. The player to remove the last one loses.

Use strong induction to show that:

The second player has a winning strategy if the number of sticks, equals $4k + 1$ for some $k \in \mathbb{N}$; otherwise, the first player has a winning strategy.

Problem 4. Consider the following equivalent way of viewing the subset take-away game from the in-class problem on Friday, Week 2: for a fixed, finite set, A , let \mathcal{S} initially be all the proper subsets of A . Players alternately choose a set $B \in \mathcal{S}$ and remove B and all sets that contain B from \mathcal{S} ; they then continue playing on the updated \mathcal{S} . The player that chooses the last set in \mathcal{S} wins.

(a) Use the well-ordering property to show that, in any game, one of the players must have a winning strategy. *Hint:* Consider games whose initial set, \mathcal{S} , is an arbitrary collection of subsets of, A , not necessarily all the proper subsets of A . Reach a contradiction by considering a minimum size game with no winning strategy for either player. What is a useful measure of size of a game?

(b) If the whole set A is a possible move in a game, explain why the 1st player must have a winning strategy.

Student's Solutions to Problem Set 2

Your name:

Due date: September 26

Submission date:

Circle your TA: David Jelani Sayan Hanson

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

got help from:¹

and referred to:²

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
Total	