Solutions to Problem Set 1

Problem 1. A real number r is called *sensible* if there exist positive integers a and b such that $\sqrt{a/b} = r$. For example, setting $a = 2$ and $b = 1$ shows that $\sqrt{2}$ is sensible. Prove that $\sqrt[3]{2}$ is not sensible. (Consider only positive real roots in this problem)

Solution. The proof is by contradiction. Assume for the purpose of contradiction that $\sqrt[3]{2}$ is sensible. Then there exist positive integers a and b such that $\sqrt{a/b} = \sqrt[3]{2}$. Squaring both sides of this equation gives $a/b = \sqrt[3]{4}$, which implies that $\sqrt[3]{4}$ is rational.

Since $\sqrt[3]{4}$ is rational, we can write it as a fraction x/y in lowest-terms, where x is an integer and y is a positive integer. Therefore, we have:

$$\begin{aligned} \sqrt[3]{4} &= x/y \\ 4 &= x^3/y^3 \\ 4y^3 &= x^3 \end{aligned}$$

In the last equation, the left side is even, and so the right side must be even. Since x^3 is even, x itself must be even. This implies that the right side is actually divisible by 8, and so the left side must also be divisible by 8. Therefore, y^3 is even, and so y itself must be even.

The fact that both x and y are even contradicts the fact that x/y is a fraction in lowest terms. Therefore, $\sqrt[3]{2}$ is not sensible. ■

Problem 2. Translate the following sentence into a predicate formula:

> There is a student who has e-mailed exactly two other people in the class, besides possibly herself.

The domain of discourse should be the set of students in the class; in addition, the only predicates that you may use are equality and $E(x, y)$, meaning that "x has sent e-mail to y."

Solution. A good way to begin tackling this problem is by trying to translate parts of the sentance. Begin by trying to assert that student x has emailed students y and z:

$$E(x, y) \land E(x, z).$$

Now we want to say that y and z not the same student, and neither of them is x either:

$$x \neq y \land x \neq z \land y \neq z,$$

where $x \neq y$ abbreviates $\neq (x = y)$.

Now, we must think of a way to say that the only people x might have e-mailed are x, y and z:

$$\forall s,\ E(x, s) \longrightarrow s = x \lor s = y \lor s = z.$$

Finally, we can say there is some student who emailed exactly two other two students by existentially quantifying x, y and z. So the complete translation is

$$\exists x \exists y \exists z.\ E(x, y) \land E(x, z) \land \tag{1}$$
$$x \neq y \land x \neq z \land y \neq z \land \tag{2}$$
$$\forall s,\ E(x, s) \longrightarrow s = x \lor s = y \lor s = z. \tag{3}$$

■

Problem 3. Express each of the following predicates and propositions in formal logic notation. The domain of discourse is the nonnegative integers, $\mathbb{N}$.

In addition to the propositional operators, variables and quantifiers, you may define predicates using addition, multiplication, and equality symbols, but no *constants* (like 0, 1, ...). For example, the proposition "n is an even number" could be written

$$\exists m.\ (m + m = n).$$

(a) n is the sum of three perfect squares.

Solution.

$$\exists x \exists y \exists z.\ (x \cdot x + y \cdot y + z \cdot z = n)$$

■

Since the constant 0 is not allowed to appear explicitly, the predicate "$x = 0$" can't be written directly, but note that it could be expressed in a simple way as:

$$x + x = x.$$

Then the predicate $x > y$ could be expressed

$$\exists w.\ (y + w = x) \wedge (w \neq 0).$$

Note that we've used "$w \neq 0$" in this formula, even though it's technically not allowed. But since "$w \neq 0$" is equivalent to the allowed formula "$\neg(w + w = w)$," we can use "$w \neq 0$" with the understanding that it abbreviates the real thing. And now that we've shown how to express "$x > y$", it's ok to use it too.

(b) $x > 1$.

Solution. The straightforward approach is to define $x = 1$ as $\forall y.\ xy = y$ and then express $x > 1$ as $\exists y.\ (y = 1) \wedge (x > y)$. ■

(c) n is a prime number.

Solution.

$$\text{IS-PRIME}(n) ::= (n > 1) \wedge \neg(\exists x \exists y.\ (x > 1) \wedge (y > 1) \wedge (x \cdot y = n))$$

■

(d) n is a product of two distinct primes.

Solution.

$$\exists x \exists y.\ \neg(x = y) \wedge (n = x \cdot y) \wedge \text{IS-PRIME}(x) \wedge \text{IS-PRIME}(y).$$

■

(e) There is no largest prime number.

Solution. Of course this is a true statement, so it could be expressed by the logically equivalent formula "$1 = 1$," but even if we didn't know this, we could transcribe the statement directly as:

$$\neg\left(\exists p.\ \text{IS-PRIME}(p) \wedge (\forall q.\ \text{IS-PRIME}(q) \longrightarrow p \geq q)\right)$$

■

(f) (Goldbach Conjecture) Every even natural number $n > 2$ can be expressed as the sum of two primes.

Solution. We can define $n > 2$ with the formula $\exists y.\ (y = 1) \wedge (x > y + y)$. Likewise, $n = 2k$ can be expressed as $n = k + k$. Then we can express the Conjecture with:

$$\forall n.\ ((n > 2 \wedge \exists k.\ n = 2k) \longrightarrow \exists p \exists q.\ \text{IS-PRIME}(p) \wedge \text{IS-PRIME}(q) \wedge (n = p + q)))$$

■

(g) (Bertrand's Postulate) If $n > 1$, then there is always at least one prime p such that $n < p < 2n$.

Solution.

$$\forall n.\ ((n > 1) \longrightarrow (\exists p.\ \text{IS-PRIME}(p) \wedge (n < p) \wedge (p < 2n)))$$

■

Problem 4. If a set, A, is finite, then $|A| < 2^{|A|} = |\mathcal{P}(A)|$, and so there is no surjection from set A to its powerset. Show that this is still true if A is infinite. *Hint:* Remember Russell's paradox and consider $\{x \in A \mid x \notin f(x)\}$ where f is such a surjection.

Solution. We prove there is no surjection by contradiction: suppose there was a surjection $f : A \to \mathcal{P}(A)$ for some set A. Let $W ::= \{x \in A \mid x \notin f(x)\}$. So by definition,

$$(x \in W) \longleftrightarrow (x \notin f(x)) \tag{4}$$

for all $x \in A$. But $W \subseteq A$ by definition and hence is a member of $\mathcal{P}(A)$. This means $W = f(a)$ for some $a \in A$, since f is a surjection to $\mathcal{P}(A)$. So we have from (4), that

$$(x \in f(a)) \longleftrightarrow (x \notin f(x)) \tag{5}$$

for all $x \in A$. Substituting a for x in (5) yields a contradiction, proving that there cannot be such an f. ■

Problem 5. (a) Prove that

$$\exists z.\ [P(z) \wedge Q(z)] \longrightarrow [\exists x.\ P(x) \wedge \exists y.\ Q(y)] \tag{6}$$

is valid. (Use the proof in the subsection on Validity in Week 2 Notes as a guide to writing your own proof here.)

Solution. *Proof.* Assume

$$\exists z.\ [P(z) \wedge Q(z)]. \tag{7}$$

That is, $P(z) \wedge Q(z)$ holds for some element, z, of the domain. Let c be this element; that is, we have $P(c) \wedge Q(c)$.

In particular, $P(c)$ holds by itself. So we conclude (by Existential Generalization)$\exists x\ P(x)$. We conclude $\exists y\ Q(y)$ similarly. Hence,

$$\exists x.\ P(x) \wedge \exists y.\ Q(y) \tag{8}$$

holds.

This shows that (8) holds in any interpretation in which (7) holds. Therefore, (7) implies (8) in all interpretations, that is, (6) is valid. □

■

(b) Prove that the converse of (6) is not valid by describing a counter model as in Week 2 Notes.

Solution. *Proof.* We describe a counter model in which, (8) is true and (7) is false. Namely, let the domain, D, be $\{\pi, e\}$, $P(x)$ mean "$x = \pi$," and $Q(y)$ mean "$y = e$." Then, $\exists x.\ P(x)$ is true (let x be π) and likewise $\exists y.\ Q(y)$ is true (let y be e), so (8) holds.

On the other hand, $Q(\pi)$ is not true, so $P(\pi) \wedge Q(\pi)$ is not true. Likewise $P(e) \wedge Q(e)$ is not true. Since these are the only elements of D, it is not true that there is an element, z, of D, such that $P(z) \wedge Q(z)$, That is, (7) is not true. □

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Problem 6. **(a)** Give an example where the following result fails:

False Theorem. *For sets A, B, C, and D, let*

$$L ::= (A \cup C) \times (B \cup D),$$
$$R ::= (A \times B) \cup (C \times D).$$

Then $L = R$.

Solution. If $A = D = \emptyset$ and B and C are both nonempty, then $L = C \times B \neq \emptyset$, but $R = \emptyset$. ■

(b) Identify the mistake in the following proof of the False Theorem.

Bogus proof. Since L and R are both sets of pairs, it's sufficient to prove that $(x, y) \in L \longleftrightarrow (x, y) \in R$ for all x, y.

The proof will be a chain of iff implications:

$(x, y) \in L$	iff
$x \in A \cup C$ and $y \in B \cup D$,	iff
either $x \in A$ or $x \in C$, and either $y \in B$ or $y \in D$,	iff
($x \in A$ and $y \in B$) or else ($x \in C$ and $y \in D$),	iff
$(x, y) \in A \times B$, or $(x, y) \in C \times D$,	iff
$(x, y) \in (A \times B) \cup (C \times D) = R$.	

□

Solution. The mistake is in the third "iff." If [$x \in A$ or $x \in C$, and either $y \in B$ or $y \in D$], it does not necessarily follow that $(x, y) \in (A \times B) \cup (C \times D)$. It might be that (x, y) is in $A \times D$ instead. This happens, for example, if $A = \{1\}, B = \{2\}, C = \{3\}, D = \{4\}$, and $(x, y) = (1, 4)$. ■

(c) Fix the proof to show that $R \subseteq L$.

Solution. Replacing the third "iff" by "which will be true when," yields a correct proof that $(x, y) \in L$ will be true when $(x, y) \in R$, which implies that $R \subseteq L$. ■