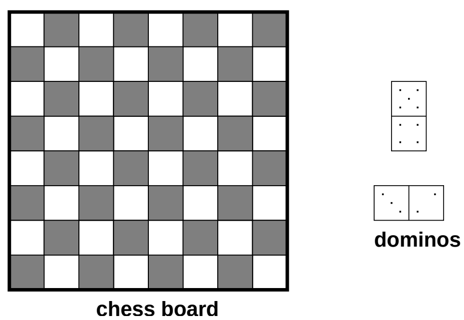
Problem Set 5

Due: October 21

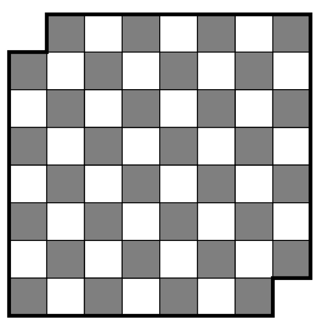
Reading: Course notes on number theory.

Problem 1. Suppose that one domino can cover exactly two squares on a chessboard, either vertically or horizontally.

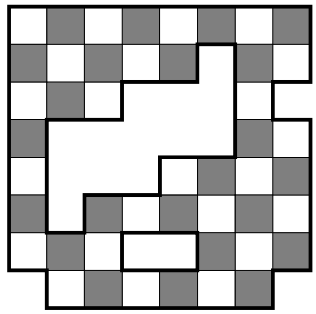
(a) Can you tile an 8×8 chessboard with 32 dominos?



(b) Can you tile an 8×8 chessboard with 31 dominos if opposite corners are removed?



(c) Now suppose that an assortment of squares are removed from a chessboard. An example is shown below.



Given a truncated chessboard, show how to construct a bipartite graph G that has a perfect matching if and only if the chessboard can be tiled with dominos.

(d) Based on this construction and Hall's theorem, can you state a necessary and sufficient condition for a truncated chessboard to be tilable with dominos? Try not to mention graphs or matchings!

Problem 2. Prove that $\gcd(ka, kb) = k \cdot \gcd(a, b)$ for all $k > 0$.

Problem 3. Suppose that $a \equiv b \pmod{n}$ and $n > 0$. Prove or disprove the following assertions:

(a) $a^c \equiv b^c \pmod{n}$ where $c \geq 0$

(b) $c^a \equiv c^b \pmod{n}$ where $a, b, \geq 0$

Problem 4. An inverse of k modulo $n > 1$ is an integer, k^{-1}, such that

$$k \cdot k^{-1} \equiv 1 \pmod{n}.$$

Show that k has an inverse iff $\gcd(k, n) = 1$. *Hint: We saw how to prove the above when n is prime.*

Problem 5. Here is a long run of composite numbers:

$$114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126$$

Prove that there exist arbitrarily long runs of composite numbers. Consider numbers a little bigger than $n!$ where $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$.

Problem 6. Take a big number, such as 37273761261. Sum the digits, where every other one is negated:

$$3 + (-7) + 2 + (-7) + 3 + (-7) + 6 + (-1) + 2 + (-6) + 1 \quad = \quad -11$$

As it turns out, the original number is a multiple of 11 if and only if this sum is a multiple of 11.

(a) Use a result from elsewhere on this problem set to show that $10^k \equiv -1^k \pmod{11}$.

(b) Using this fact, explain why the procedure above works.

Problem 7. Let $S_k = 1^k + 2^k + \ldots + (p-1)^k$, where p is an odd prime and k is a positive multiple of $p - 1$. Use Fermat's theorem to prove that $S_k \equiv -1 \pmod{p}$.

Student’s Solutions to Problem Set 5

Your name:

Due date: October 21

Submission date:

Circle your TA: David Jelani Sayan Hanson

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Total	

[1]People other than course staff.

[2]Give citations to texts and material other than the Fall ’02 course materials.