Midterm Solutions

Note: For problems 1, 2 and 3 you do not need to provide a formal proof, but do show your calculations/ give a brief justification

Question 1 (2 points)

What is the last digit of 123^{407}. Show your calculation.

Solution: The last digit of $123^{407} = 123^{407} \mod 10$.

$$123^{407} \mod 10 = (123 \mod 10)^{407} \mod 10 = 3^{407} \mod 10.$$

We calculate $3^i \mod 10$ and obtain that at $i = 1, 2, 3, 4$ it is $3, 9, 7, 1$ respectively after which it cycles. Thus

$$3^{407} \mod 10 = 3^{407 \mod 4} \mod 10 = 3^3 \mod 10 = 7$$

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Question 2 (4 points)

5 couples (that is 10 people) are to be seated in a a row of 10 seats. How many ways are there to seat these 10 people so that each person is sitting adjacent to their partner. The people are (obviously) distinct individuals! Provide a short justification for your answer.

Note: You can leave your answer in reasonably simple terms (you do not have to give a single number as the answer.)

Solution: First we can imagine each of the couples as a single entity. The idea is that since each couple must sit together, we can first determine the order of the couples, treating each couple as a single entity. There are $5!$ such orderings.

But for each of these $5!$ (i.e. once we have fixed the order in which the ‘couples as a single units’ will be seated) we can choose to seat each couple in one of two ways (for instance, if I have decided to seat a particular couple in seats 3 and 4, there is still a choice of which of those two people I should seat at seat 3 and which at seat 4). Thus for each of the $5!$ orderings of the ‘couples as single unit’, there 2^5 distinct ways of seating the people.

Hence the total number of ways is $5! \times 2^5$.

Question 3 (4 points)

A natural number n has **exactly** 3 distinct divisors. If $n < 100$, what are the possible values of n? List the values of $n < 100$ which have exactly 3 distinct divisors. Provide a short justification for your answer.

(**Hint:** How many distinct prime factors can such an n have?)

Solution: If n had two distinct prime factors, say p and q, then the numbers $1, p, q, pq$ are all divisors of n, and they are all distinct. So n can have at most 1 prime factor. Since $n = 1$ is not a valid possibility (as 1 has only 1 divisor), hence n must have exactly one prime factor. **That means n is the power of a single prime.**

For a prime p, the number p^i has $i + 1$ divisors: $p^0, p^1, p^2, \ldots, p^i$. To get $i + 1 = 3$ we thus need $i = 2$. **This means that n must be the square of a prime number.** So the possible n are just the squares of primes. Since $n < 100$, there are exactly four such n: $2^2, 3^2, 5^2, 7^2$.

Question 4 (10 points)

Prove the following via a combinatorial argument:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

You will not get significant credit if you do the proof via induction, or by other methods. Your proof should involve counting the same thing in two different ways.

(**Hint:** Imagine you have to select any 2 students from a set of n undergraduates and n masters students.)

Solution: We can pick any 2 students from the collective of n undergraduates and n masters in two ways. First, we can just consider them as a single type of student, giving $\binom{2n}{2}$ possible choices, which is the left-hand side of the equation above.

But we can also consider separately the scenarios where we pick two undergraduates, totaling $\binom{n}{2}$ choices; or picking two masters, totaling again $\binom{n}{2}$ choices; or one undergraduate and one masters student, totaling n^2 choices. Adding up these three terms gives the right-hand side. These three cases are clearly disjoint and cover all the possibilities, which proves the claimed identity.

Side-Note: if we had n students in total, k of whom were master's students, then the reasoning above gives the equality

$$\binom{n}{2} = \binom{k}{2} + \binom{n-k}{2} + k(n-k),$$

After writing the question, I realized that you can actually use this on Question 6, to make the calculation even simpler :)

Question 5 (10 points)

Consider the equation:

$$3x^4 - 9x^3 + 5x^2 - 6x + 8 = 0$$

Prove there is no natural number n that satisfies this equation.

(**Hint:** If n is a natural number, what can $n^2 \mod 3$ be? Make sure to prove your claim.)

Solution: Claim: If n is a natural number, $n^2 \mod 3$ is either 0 or 1.

We will first prove this claim. You can also prove this easily by using Fermat's little theorem, but here is an even simpler proof:
There are only 3 possible values for $n \bmod 3$:

- $n \bmod 3 = 0 \implies n^2 \mod 3 = 0 \cdot 0 = 0 \mod 3$
- $n \bmod 3 = 1 \implies n^2 \mod 3 = 1 \cdot 1 = 1 \mod 3$
- $n \bmod 3 = 2 \implies n^2 \mod 3 = 2 \cdot 2 = 4 = 1 \mod 3$

Thus $n^2 \bmod 3 \in \{0, 1\}$.

Now we consider the provided equation mod 3:

$$\begin{aligned}(3x^4 - 9x^3 + 5x^2 - 6x + 8) \mod 3 &= 0 \mod 3\\ \implies (5x^2 + 8) \mod 3 &= 0 \mod 3\\ \implies (2x^2 + 2) \mod 3 &= 0 \mod 3\end{aligned}$$

(This simplification follows immediately from the sum and product rules for modular arithmetic).

If $x^2 \mod 3 = 0$, then $2x^2 + 2 = 2 \neq 0 \mod 3$. If $x^2 \mod = 1$, then $2x^2 + 2 = 2 + 2 = 4 = 1 \neq 0 \mod 3$. Therefore there is no natural number solution to the equation. □

Question 5 (10 points)

Consider the following solitaire 'game'. You start with a stack of n pennies. At each move you pick a stack that has at least two pennies in it and split it into two non-empty stacks; your score for that move is the product of the numbers of pennies in the two stacks. Thus, if you split a stack of 10 pennies into a stack of 3 and a stack of 7, you get $3 \cdot 7 = 21$ points at that step. The game is over when you have n stacks of one penny each. Your total score is the sum of scores for all your moves.

Prove that no matter how you play, your total score at the end will be $\frac{1}{2}n(n-1)$.

(**Hint**: Use strong induction. Start by thinking about just the first step: the n size pile is broken into a k size pile and an $n-k$ size pile.)

Solution: The solution is not that long, but there is a detailed discussion below (including a way to avoid the algebra and a discussion of the common errors):

The first observation is that $\frac{1}{2}n(n-1) = \binom{n}{2}$. This observation is **not necessary** to solve the problem, but it can slightly simplify your calculation.

We proceed by strong induction.

In the base case, if $n = 1$ then the game is already over and we got $0 = \frac{1}{2}1 \cdot 0$ points.

For the inductive step, let n be an arbitrary number, and assume that the statement holds for all $i < n$.

Our first step will split the stack in two of size k and $n-k$ for some k, and net us $k(n-k)$ points. Now every move onwards will either happens with the coins in the first stack, or in the coins in the second stack, and the points we get are independent of each other. In other words, we can imagine that we first do all moves with the coins in the first stack, then do all moves with the coins in the second stack.

By the inductive hypothesis, we get $\binom{k}{2}$ points from the moves with the coins in the first stack, and $\binom{n-k}{2}$ points from the second stack. Thus in total we make

$$k(n-k) + \binom{k}{2} + \binom{n-k}{2}$$

points. By the side-note in Question 4 (or by direct algebraic verification), this quantity equals $\binom{n}{2}$, as desired.

Note: This equality could also be verified algebraically (without observing the connection to $\binom{n}{2}$), without appealing to Question 4. You get a $k(n-k)$ score in the first move, and by IH you get a $\frac{k(k-1)}{2}$ score from the left pile and a $\frac{(n-k)(n-k-1)}{2}$ score from the right pile. You then just need to verify that:

$$k(n-k) + \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} = \frac{n(n-1)}{2}$$

Thus the observation that $\frac{1}{2}n(n-1) = \binom{n}{2}$, simplifies the calculation because it lets us give a combinatorial proof of the identity, **but is not critical to the solution**.

Some common errors:

1. Assumed that the first split was of n coins into 1 and $n-1$ coins. Then used the claim for $n-1$. But this is only a very special case, and we are asked to prove the claim, no matter how the game plays out. This why we use strong induction.

2. Some sort of argument that a particular way of splitting the coins is optimal (gives max score); for example splitting evenly. Then arguing that in this case the score is $\frac{n(n-1)}{2}$. However again, the question was to prove that actually, the score is the same ($\frac{n(n-1)}{2}$) in all cases, no matter how we play out the game, not just in one special case.