Problem Solving session 1

Question 1

Let Q(x, y, z) be the statement $x + y = z$. What are the truth values of the statements $\forall x \forall y \exists z Q(x, y, z)$ and $\exists z \forall x \forall y Q(x, y, z)$ where the domain of all variables consists of all real numbers?

Question 2

What is the negation of the statement:

$$(p \vee q) \wedge (q \vee r) \wedge (r \vee p).$$

Question 3

Let $F(x, y)$ be the predicate "x and y are friends". Then what does the following mean (in English). You can assume friendship is symmetric.

$$\exists x \forall y \forall z ((F(x, y) \wedge F(x, z) \wedge (y \neq z)) \implies \neg F(y, z))$$

Question 4

The quantification $\exists! x, P(x)$ denotes the proposition "There exists a unique natural number x such that $P(x)$ is true". Assume we are working in the domain of natural numbers. Such statements are quite common in mathematics. Express this quantification using universal and existential quantifications, logical operators and the notion of equality of numbers $(=)$.

Question 5

What is the converse, inverse and contrapositive of the statement "If a quadrilateral is a rectangle, then it has two pairs of parallel sides." Which are true?

Question 6

Consider the following implications. Is it possible for exactly 3 of them to be true.

1. $p \implies q$
2. $\neg p \implies \neg q$
3. $q \implies p$
4. $\neg q \implies \neg p$

Question 7

Prove that the sum of two rational numbers is rational.

Question 8

Prove that if r is irrational then so is $\sqrt{r}$.

Question 9

Prove that it is possible for a and b to be irrational but for a^b to be rational. (a is allowed to be equal to be b.)

Question 10 (Bonus): hat puzzle

There are three logicians, Alice, Bob, and Carla, each wearing a hat. They can see the colors of the others' hats but not their own. They know that their hats were drawn from a group of three red hats and two blue ones. Alice is asked what color her hat is and she responds, "I don't know." Bob is asked what color his hat is and he also says, "I don't know." When Carla is asked what color her hat is, she answers correctly. What color is her hat and how did she know?

Question 11 (Challenge Problem)

Suppose you start at 1, and are allowed one of two moves at every given step:

- Add 1 to your current number. So $x \to x + 1$.
- Take the reciprocal of your current number $x \to \frac{1}{x}$.

Prove that starting at 1 you can reach any positive rational number using just these two moves!