Homework 6 Solutions

Question 2

Suppose you roll a pair of fair dice (they are standard dice with the numbers 1 through 6 on them).

1. What is the probability that the sum of the numbers equals 7?
2. Given that one of the numbers was a 3, what is the probability that the sum of the numbers equals 7?
3. Given that the sum is strictly more than 5, but strictly less than 9, what is the probability that the sum is 7?
4. Given that the sum is 7, what is the probability that both numbers rolled are odd?

Solution: For a single dice roll, let D_i be the event the result is i. For a roll of two dice, let S_j be the event that the sum of the dice is j.

1. $\Pr(S_7) = 2{\cdot}\Pr(D_1){\cdot}\Pr(D_6)+2{\cdot}\Pr(D_2){\cdot}\Pr(D_5)+2{\cdot}\Pr(D_3){\cdot}\Pr(D_4) = 2{\cdot}\frac{1}{6^2}+2{\cdot}\frac{1}{6^2}+2{\cdot}\frac{1}{6^2} = \frac{1}{6}$
2. $\Pr(S_7|\text{ One is a 3}) = \frac{\Pr(S_7 \cap \text{ One is a 3})}{\Pr(\text{One is a 3})} = \frac{2\cdot\Pr(D_3)\cdot\Pr(D_4)}{1-\Pr(D_3^c)^2} = \frac{2\cdot\frac{1}{6^2}}{1-(\frac{5}{6})^2} = \frac{\frac{2}{36}}{\frac{11}{36}} = \frac{2}{11}$
3. We begin by using the law of total probability.

$$\begin{aligned}\Pr(S_7) &= \Pr(S_7|S_{>5}\cap S_{<9})\cdot\Pr(S_{>5}\cap S_{<9}) + \Pr(S_7|S_{\le 5})\cdot\Pr(S_{\le 5}) + \Pr(S_7|S_{\ge 9})\cdot\Pr(S_{\ge 9})\\ &= \Pr(S_7|S_{>5}\cap S_{<9})\cdot\Pr(S_{>5}\cap S_{<9})\end{aligned}$$

Which implies

$$\begin{aligned}\Pr(S_7|S_{>5}\cap S_{<9}) = \frac{\Pr(S_7)}{\Pr(S_{>5}\cap S_{<9})} &= \frac{\Pr(S_7)}{\Pr(S_6\cup S_7\cup S_8)}\\ &= \frac{\Pr(S_7)}{\Pr(S_6)+\Pr(S_7)+\Pr(S_8)}\\ &= \frac{\frac{1}{6}}{\frac{5}{36}+\frac{1}{6}+\frac{5}{36}}\\ &= \frac{3}{8}\end{aligned}$$

4. $\Pr(S_7|\text{ Both numbers odd}) = 0$. The sum of two odd numbers is even, which implies the sum can never be 7.

Question 2

Consider a graph with 3 vertices A, B and C and three edges AB, BC and CA. We color every vertex in this graph using either red, blue or green and we decide the color of each vertex independently and uniformly at random; each vertex is red with 1/3 probability, blue with 1/3 probability and green with 1/3 probability.

1. What is the chance that at least two of the three vertices have the same color.

> **Solution:** This is $1-\Pr$(no two have the same color). There are $3^3 = 27$ equally likely outcomes and the number of ways to assign 3 different colors to the three vertices is $3! = 6$. Thus $\Pr$(no two have the same color)$= 6/27 = 2/9$. Thus $\Pr$(atleast two have the same color)$= 7/9$.

2. An edge is called **properly colored** if it's vertices do **not** have the same color. What is the chance that AC is properly colored.

> **Solution:** This is just $1 - \Pr$(A and C both get the same color). Whatever color A has, C has a $1/3$ chance of having the same color. Thus $\Pr$(A and C both get the same color)$= 1/3$. Thus the chance that AC is properly colored $= 1 - 1/3 = 2/3$.

3. What is the chance that all the edges are properly colored.

> **Solution:** This is exactly $1 - \Pr$(at least one edge is badly colored). But $\Pr$(at least one edge is badly colored) $= \Pr$(at least two of the three vertices have the same color) which we already calculated in part (1) to be $2/9$. Thus, the chance that all edges are properly colored is $1-7/9 = 2/9$.

4. Given that AB and BC are both properly colored, what is the chance that AC is properly colored?

> **Solution:** (**Note: This can be calculated in several ways, either directly or by using Bayes theorem**)
>
> $$\Pr(AC \text{ proper}|AB \text{ and } BC \text{ proper}) = \frac{\Pr(AB \text{ and } BC \text{ proper}|AC \text{ proper}) \times \Pr(AC \text{ proper})}{\Pr(AB \text{ and } BC \text{ proper})}$$
>
> $\Pr(AC \text{ proper})) = 2/3$ as calculated in part (2).
>
> To calculate $\Pr(AB \text{ and } BC \text{ proper}|AC \text{ proper})$, notice that this will occur if and only if B is colored differently to both A and C. A and C are known to be different, so there is a $1/3$ chance that B gets the remaining color. Thus $\Pr(AB \text{ and } BC \text{ proper}|AC \text{ proper}) = 1/3$.
>
> To calculate $\Pr(AB \text{ and } BC \text{ proper})$, consider B having any fixed color. Then there are $3^2 = 9$ possible equally likely outcomes for A and C. Of these, for both AB and BC to be properly

colored, both A and C must have some color different than B, leading to $2^2 = 4$ different possibilities (it is ok if A and C are the same color). Thus $\Pr(AB \text{ and } BC \text{ proper}) = 4/9$.

Plugging these into the Bayes formula we get:

$$\Pr(AC \text{ proper}|AB \text{ and } BC \text{ proper}) = \frac{\frac{1}{3} \times \frac{2}{3}}{\frac{4}{9}} = \frac{1}{2}.$$

Question 3

Show that if $\Pr(A|B) > \Pr(A)$ then $\Pr(B|A) > \Pr(B)$.

Solution:

$$\begin{aligned}
&\Pr(A|B) > \Pr(A) \\
&\implies \frac{\Pr(B|A)\Pr(A)}{\Pr(B)} > \Pr(A) && \text{(Bayes' Theorem)} \\
&\implies \Pr(B|A) > \frac{\Pr(A)\Pr(B)}{\Pr(A)} \\
&\implies \Pr(B|A) > \Pr(B)
\end{aligned}$$

$\square$

Question 4

(1) One bag, Bag A has 90 red balls and 10 white ball. Another, bag B, has 70 white balls and 30 red ball. One of the bags are chosen uniformly at random and a ball is drawn uniformly at random from within the bag. Given the ball was red, what is the chance that the ball was drawn from bag A?

Solution: By Bayes' theorem, we have $\Pr[A|red] = \frac{\Pr[red|A]\Pr[A]}{\Pr[red]}$. Because bag is chosen uniformly at random, $\Pr[A] = 1/2$. Since bag A has 90 balls, $\Pr[red|A] = 90/100$. $\Pr[red] = \Pr[red|A]\Pr[A] + \Pr[red|B]\Pr[B] = \frac{90}{100}\frac{1}{2} + \frac{70}{100}\frac{1}{2} = \frac{80}{100}$. Thus,

$$\Pr[A|red] = \frac{(90/100)(1/2)}{(80/100)} = 90/160 = \frac{9}{16}.$$

(2) A disease occurs in only 1% of the population and there is a test which detects the disease in 99% of patients who have the disease. When the test is administered to a uniformly randomly drawn person from the general public, it has a 3% chance of being positive. Suppose the test was given to an individual

drawn uniformly at random from the general population and it turns out to be positive. What is the chance that that person actually has the disease?

Solution: We use D to denote the event that a person has the disease, and let T be the event that the test is positive. We are given that $\Pr[D] = 0.01$, $\Pr[T] = 0.03$ and $\Pr[T|D] = 0.99$. Now we want to find out $\Pr[D|T]$. By Bayes' theorem,

$$\Pr[D|T] = \frac{\Pr[T|D]\Pr[D]}{\Pr[T]} = \frac{0.01 * 0.99}{0.03} = 0.33.$$

Question 5

Recall that an event is just a subset of the outcome set. Thus we can talk about events being disjoint.

You might intuitively think that if events are disjoint, then they should have "nothing to do with each other", i.e be independent. However this is false!

Give a formal example by defining a probability space and events A and B such that A and B are disjoint, but negatively correlated. Show your calculation demonstrating that A and B are negatively correlated.

Solution: For two events A and B, we know they are negatively correlated if $\Pr[A \cap B] < \Pr[A] \times \Pr[B]$. Since A, B are disjoint, we know $\Pr[A \cap B] = 0$. So consider any pair of disjoint events, each having non-zero probability, and they will be negatively correlated. (Thus any example of this form works!)

Question 6

It possible to have events A, B and C such that A is positively correlated with B, and B is positively correlated with C, but C is not positively correlated with A! Give an example of a probability space and events A, B and C such that this is the case.

Provide a formal example by defining a probability space and events A, B and C such that A is positively correlated with B, and B is positively correlated with C, but C is not positively correlated with A. Show your calculation demonstrating these correlations.

Solution: Consider an example where we are picking uniformly from $\{1, 2, 3\}$ and let $A = \{1\}$, $B = \{1, 3\}$, and $C = \{3\}$. Then, $\Pr[A] = \Pr[C] = \frac{1}{3}$, $\Pr[B] = \frac{2}{3}$, $\Pr[A|B] = \Pr[C|B] = \frac{1/3}{2/3} = \frac{1}{2} > \Pr[A] = \Pr[C]$, which means they are positively correlated, A and B and B and C are positively correlated. However, $\Pr[A|C] = \frac{|A \cap C|}{|C|} = \frac{0}{2} = 0 < \Pr[A]$, which means A and C are negatively correlated. $\square$

(**Note to students/ graders:** There are many possible examples. It's also OK to test for positive / negative correlation by comparing $\Pr[A \cup B]$ with $\Pr[A] \times \Pr[B]$.

Extra Practice Questions

Question 8

Suppose you have two fair dice. You roll them both together.

1. Given that exactly one of the numbers rolled is even, what is the chance that the other number is odd?
2. Given that at least one of the numbers rolled is even, what is the chance that the other number is odd?
3. Given that exactly one of the numbers rolled is even, what is the chance that sum of the two numbers is odd?
4. Given that at least one of the numbers rolled is even, what is the chance that sum of the two numbers is odd?

Solution: Let A be the event the first is even and B the event the second is even, and C at least one is even. Notice that since the number of even numbers on each die is the same as the number of odd, $\Pr[A] = \Pr[B] = 3/6 = 1/2$. Furthermore,

$$\Pr[C] = 1 - \Pr[\neg C]$$

Since A and B are identically independently distributed (IID):

$$\begin{aligned} &= 1 - \Pr[A]^2 \\ &= 1 - \left(\frac{1}{2}\right)^2 \\ &= \frac{3}{4} \end{aligned} \tag{1}$$

1. If exactly one die has an even number, then the other die must have an odd number. Thus, this happens with probability 1.

2.

$$\begin{aligned}
\Pr\left[(] A \wedge B | C\right) &= \frac{\Pr\left[(\neg A \vee \neg B) \wedge C\right]}{\Pr\left[C\right]} \\
&= \frac{\Pr\left[A \vee \neg B\right] + \Pr\left[\neg A \vee B\right]}{\Pr\left[C\right]}
\end{aligned}$$

Since A and B are IID:

$$\begin{aligned}
&= \frac{2\Pr\left[A \vee \neg B\right]}{\Pr\left[C\right]} \\
&= \frac{1/2}{3/4} \\
&= \frac{2}{3}
\end{aligned}$$

In other words, there are four possible ways for the even/odd parity to form for both dies. The one where they are both even is eliminated since we are assuming at least one of the dies is even. Thus, we have a sample space with three elements. There are two ways for the event we are looking for to happen. Therefore, we get a probability of $2/3$, since everything is uniformly distributed.

3. Since exactly one number rolled is even, the other must be odd, and since the sum of an even and an odd number is odd, the sum is odd with probability 1.

4. By the logic in the last question, the sum is only odd when at least one of the numbers is odd. Notice that since a given die rolling an even number has the same probability as it rolling an odd number, by symmetry the answer to this question is the same as the first question, $2/3$.

□

Question 9

When a test for steroids is given to soccer players, 90% of the players taking steroids test positive and 10% of the players not taking steroids test positive. Suppose that 5% of soccer players take steroids. What is the probability that a soccer player who tests positive takes steroids?

Solution: Let A be the event that a player tests positive and B be the event that a player takes steroids. Using Bayes formula,

$$\begin{aligned}
\mathbb{P}(B|A) &= \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|B^c)\mathbb{P}(B^c)} \\
&= \frac{0.9 \times 0.05}{0.9 \times 0.05 + 0.1 \times 0.95}
\end{aligned}$$