# Quiz 2 Solutions

## Question 1 ($8$ points)

You play a video game where at the beginning of the game you are given either a regular sword or a magic sword. This is random, with a $25\%$ chance of getting the magic sword and a $75\%$ chance of getting the regular sword. At the end of the game you get only a single chance to fight the final boss. If you have the magic sword, you have a $50\%$ chance of beating the final boss, while with the regular sword, you have only a $25\%$ chance of beating the final boss. Suppose I know that you did actually beat the final boss; given this information what is the chance that you got the magic sword at the beginning of the game?

**Solution:** Let $M$ be the event of getting the Magic sword in the beginning and $\neg M$ be the event that you got the regular sword. Let $B$ be the event that you beat the boss at the end of the game and $\neg B$ be the event that you did not beat the boss.

Then the problem specifies that $\Pr(M) = \frac{1}{4}$, $\Pr(B|M) = \frac{1}{2}$ and $\Pr(B|\neg M) = \frac{1}{4}$.

Using the law of total probability:

$$\Pr(B) = \Pr(B|M)\Pr(M) + \Pr(B|\neg M)\Pr(\neg M) = \frac{1}{2} \times \frac{1}{4} + \frac{1}{4} \times \frac{3}{4} = \frac{5}{16}.$$

Using Bayes Theorem:

$$\Pr(M|B) = \frac{\Pr(B|M)\Pr(M)}{\Pr(B)} = \frac{\frac{1}{2} \times \frac{1}{4}}{\frac{5}{16}} = \frac{\frac{1}{8}}{\frac{5}{16}} = \frac{2}{5} = 40\%.$$

## Question 2 ($5 + 7 = 12$ points)

I roll a fair die with the numbers 1 through 6 on it 10 times. Each roll is independent. Let $R_i$ denote the event that the $(i+1)$-th roll is **exactly** twice as much as the $i$-th roll.

1. Are $R_1$ and $R_2$ independent, positively correlated or negatively correlated? Why? Show an explicit calculation of the relevant probabilities to justify your answer.

**Solution:** Let's first compute $\Pr(R_i)$ by considering all possible cases where $R_1$ occurs. If the first roll is 1, 2 or 3; then the second roll has to be 2, 4 or 6 respectively. If the first roll is $> 3$, then $R_1$ will not occur. The total number of possible first and second rolls is $6 \times 6 = 36$. Therefore, $\Pr(R_i) = 3/36 = 1/12$ (since the same reasoning works for any $R_i$).

Next, let's find $\mathbb{P}(R_1 \cap R_2)$. For both $R_1$ and $R_2$ to occur, roll 2 must be twice roll 1 and roll 3 must be twice roll 2. If roll 1 is 1, roll 2 must be 2 and roll 3 must be 4. If roll 1 is 2 or bigger, there is no way for both $R_1$ and $R_2$ to happen. The total number of possible first, second and third rolls is $6 \times 6 \times 6$. Consequently, $\Pr(R_1 \cap R_2) = \frac{1}{6\times6\times6}$, whereas $\Pr(R_1) \times \Pr(R_2) = \frac{1}{12} \times \frac{1}{12} = \frac{1}{4\times6\times6}$. So $\Pr(R_1 \cap R_2) < \Pr(R_1) \times \Pr(R_2)$ and hence the events are **negatively correlated**.

2. We are still in the setting of the previous part, where you roll a fair die 10 times. What is the expected number of times you see two **consecutive** rolls, where the later roll is **exactly** twice the earlier roll?

   So, for example, if we got $3, 1, 2, 4, 4, 2, 5, 6, 3, 6$ as the outcome, the number of times there were two consecutive rolls, where the later roll was exactly twice the earlier roll, would be 3 (roll numbers: (2 and 3), (3 and 4) and (9 and 10)).

   **Solution:** Let $X_i$ be the indicator variable that takes 1 if $R_i$ happens and $0$ otherwise. We wish to find $E[\sum_{i=1}^{9} X_i]$. By linearity of expectation (and using the fact that we already calculated $\Pr(R_i) = \frac{1}{12}$ in the previous part)

$$\begin{aligned}\mathbb{E}[\sum_{i=1}^{9} X_i] &= \sum_{i=1}^{9} \mathbb{E}[X_i] \\ &= \sum_{i=1}^{9} \mathbb{P}(R_i) \\ &= 9 \times \frac{1}{12} = \frac{9}{12} = \frac{3}{4} = 0.75.\end{aligned}$$