Final Exam

Question 1 (5 points)

A liar speaks the truth with probability $4/5$ but lies with probability $1/5$. He tosses a biased coin that has only $1/6$ chance of coming up heads and then reports back to you that the toss was heads. Given that he reported that the toss was heads, what is the probability that the toss really was heads?

(**Hint:** Bayes' Theorem)

Solution: Consider the events $H =$ toss was actually heads. $T =$ toss was actually tails. $S_H =$ the liar says it was heads. $S_T =$ the liar says it was tails. We are interested in $Pr[H|S_H]$.

We are interested in $\Pr[H|S_H]$. Notice that $\Pr[S_H|H] = \frac{4}{5}$ since this is the chance he is truthful, whereas $\Pr[S_H|T] = \frac{1}{5}$, which is the chance that he lies! $\Pr[H] = \frac{1}{6}$, as told to us.

We also need to calculate $\Pr[S_H]$, which we can do by the law of total probability.

$$\Pr[S_H] = \Pr[S_H|H] \times \Pr[H] + \Pr[S_H|T] \times \Pr[T] = \frac{4}{5} \times \frac{1}{6} + \frac{1}{5} \times \frac{5}{6} = \frac{4}{30} + \frac{5}{30} = \frac{9}{30}.$$

Then, by Bayes Theorem,

$$\Pr[H|S_H] = \frac{\Pr[S_H|H] \times \Pr[H]}{\Pr[S_H]} = \frac{\frac{4}{5} \times \frac{1}{6}}{\frac{9}{30}} = \frac{\frac{4}{30}}{\frac{9}{30}} = \frac{4}{9}.$$

Question 2 (5 points)

$$\frac{20! \cdot 22!}{9! \cdot 10! \cdot 11! \cdot 12!}$$

Prove that the above expression is a natural number.

Do **not** just expand out each of the factorials (this will **not** earn you significant points), or indeed attempt to calculate what the natural number is in any way. You shouldn't need to do any significant calculations at all: just follow the hint!

(**Hint:** What is the formula for $\binom{n}{k}$?)

Solution:

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}.$$

Notice that

$$\frac{20! \cdot 22!}{9! \cdot 10! \cdot 11! \cdot 12!} = \frac{20!}{9! \cdot 11!} \times \frac{22!}{10! \cdot 12!} = \binom{20}{9} \times \binom{22}{10}.$$

But $\binom{n}{k}$ is the number of distinct ways to select k out of n objects, which must be a natural number! Thus $\binom{20}{9}$ and $\binom{22}{10}$ are both natural numbers. So their product is a natural number, which proves the claim.

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Question 3 (10 points)

Consider the following solitaire 'game'. You start with a stack of n pennies. At each move you pick a stack that has at least two pennies in it and split it into two non-empty stacks; your score for that move is the product of the numbers of pennies in the two stacks. Thus, if you split a stack of 10 pennies into a stack of 3 and a stack of 7, you get $3 \cdot 7 = 21$ points at that step. The game is over when you have n stacks of one penny each. Your total score is the sum of scores for all your moves.

Prove that no matter how you play, your total score at the end will be $\frac{1}{2}n(n-1)$.

(**Hint**: This is literally exactly the same question as the midterm! Use strong induction.)

Solution: Note: This is (obviously) exactly the same solution as **Q.5 from the midterm!**.

The first observation is that $\frac{1}{2}n(n-1) = \binom{n}{2}$. This observation is **not necessary** to solve the problem, but it can slightly simplify your calculation.

We proceed by strong induction.

In the base case, if $n = 1$ then the game is already over and we got $0 = \frac{1}{2}1 \cdot 0$ points.

For the inductive step, let n be an arbitrary number, and assume that the statement holds for all $i < n$. That is, assume that **no matter how we play the game**, if we start with $i < n$ coins in a single pile, we will end up with a total score of $\frac{i(i-1)}{2} = \binom{i}{2}$.

Our first step will split the stack in two of size k and $n-k$ for some k, and net us $k(n-k)$ points. Now every move onwards will either happen with the coins in the first stack, or in the coins in the second stack, and the points we get are independent of each other. In other words, we can imagine that we first do all moves with the coins in the first stack, then do all moves with the coins in the second stack.

By the inductive hypothesis, we get $\binom{k}{2}$ points from the moves with the coins in the first stack, and $\binom{n-k}{2}$ points from the second stack. Thus in total we make

$$k(n-k) + \binom{k}{2} + \binom{n-k}{2} \text{points.}$$

Notice that the LHS as counting the number of ways to pick 2 students from a group of n distinct students. This is because we can imagine k of the n students are undergrads, and the rest are graduate students. Now we can either select one undergrad and one grad student ($k(n-k)$ ways), or two undergrads ($\binom{k}{2}$ ways) or two grad students ($\binom{n-k}{2}$ ways). This combinatorial argument shows that

$$k(n-k) + \binom{k}{2} + \binom{n-k}{2} = \binom{n}{2}.$$

as desired.

(**Note:** You can also do a direct algebraic verification that $k(n-k)+\frac{k(k-1)}{2}+\frac{(n-k)(n-k-1)}{2} = \frac{n(n-1)}{2}$, as desired.)

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Question 4 ($5+5=10$ points)

A standard deck of cards has 13 cards of each suit: the ace, 2, 3,...,10, Jack, Queen and King. There are also 4 different suits: hearts (H), clubs (C), diamonds (D) and spades (S). (There are 52 different cards in total.) A **hand** is a selection of exactly 5 different cards and the **order of the cards in the hand does not matter**. In both calculations below, you **do not** have to express your answer as a single number.

(1) A **full house** is a hand which has cards of exactly 2 different ranks where 3 cards are of one rank and the remaining two cards are both of the other rank. For example, $\{$7H, 7D, 7S, 2D, 2C$\}$ is a full house. But $\{$7H, 7D, 7S, 3D, 10C$\}$ is not a full house. How many different hands are there that are full houses?

(2) How many different hands are there which contain at least one card from each of the 4 suits. **Note:** this is a separate calculation from the previous part: these hands need not be full houses!

(**Hint:** Depending on how you do the calculations, you may need to be careful of over-counting!)

Solution: There are **many possible ways of counting that may lead to equal but different looking expressions**. I will provide (what I think is) the simplest method here:

1. First select which 3 suits will form the three cards that are the same rank and then pick which 2 suits will will form the other two that are the same rank. There are $\binom{4}{3} \times \binom{4}{2}$ ways in total. Next, we need to decide which rank will be used in the triple and which rank will be used in the pair. They can't be the same rank, since there are only 4 cards in total of a given rank in a deck. The total number of ways of doing this is $\binom{13}{1} \times \binom{12}{1}$ (or you can think of this as $2 \times \binom{13}{2}$ if you wish). This leads to a final answer of

$$\binom{4}{3} \times \binom{4}{2} \times \binom{13}{1} \times \binom{12}{1} = 4 \times \frac{4 \times 3}{2} \times 13 \times 12 = 3744.$$

(**Note:** you don't need to evaluate this expression as a single number. Also, any answer that evaluates to 3744 will get full points.)

2. Any such hand will have 4 cards from 4 different suits and exactly one more card which will be from one of the four suits. Thus, exactly one of the four suits will have two cards from it. We can first decide which of the four suits has two cards. This can be done in $\binom{4}{1}$ ways. Now we select a rank for each of the three non-repeated suits. There are 13 ways to make the selection for each of these suits, so 13^3 ways. Finally for the repeated suit we have to pick two different ranks, this can be done in $\binom{13}{2}$ ways (notice the order does not matter). That means the total number of such hands is

$$\binom{4}{1} \times 13^3 \times \binom{13}{2} = 4 \times 13^3 \times \frac{13 \times 12}{2} = 2 \times 13^4 \times 12 = 685464.$$

(**Note:** you don't need to evaluate this expression as a single number. Also, any answer that evaluates to 685464 will get full points.)

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For part (2), here is another alternative calculation (going card by card) that you may have tried. However we need to be careful while taking care of the over counting here.

Since the order doesn't matter, I will fix that the last two cards chosen will be of the same suit. Let's choose the first card. There are 52 ways. But now we can't use this suit, so for the next card there are $52 - 13 = 39$ ways and for the next one there are $39 - 13 = 26$ ways. Finally we have two cards left and we have ruled out 3 suits of cards. We have 13 cards remaining and must pick one (13 ways) and then another (12 ways). This leads to a total count of

$$52 \times 39 \times 26 \times 13 \times 12.$$

But we have over counted. Every possible rearrangement of these cards is a valid way to pick. That might suggest to you that we should divide by $5!$. But this is incorrect! (And here is something that should tip you off from dividing by 5!; the product above is clearly not divisible by 5.)
The reason that dividing by 5! is incorrect is that in our counting we have not actually considered every possible rearrangement of these 5 cards. We have only been listing rearrangements where the first 3 cards are the ones that are the unique ones from their suit and the last two cards are both from the same suit. That is, for example, for the selected set $\{5H, 7D, 6C, 4S, 2S\}$ we have counted ordered tuples like $(5H, 7D, 6C, 4S, 2S)$ and $(7D, 5H, 6C, 2S, 4S)$, but we never counted an ordered tuple like $(2S, 7D, 5H, 6C, 4S)$ in our count. Thus, we need to divide by the number of different re-orderings for a given such set of 5 cards that we have counted as distinct. We put the first 3 cards as the ones that are the unique ones from their suit and the last two cards are both from the same suit. Thus we have counted the same selection exactly are $3! \times 2! = 12$ times. This yields a final count of

$$\frac{52 \times 39 \times 26 \times 13 \times 12}{12}.$$

Notice that the numerator can be expressed as $(4 \times 13) \times (3 \times 13) \times (2 \times 13) \times 13 \times 12 = (4 \times 3 \times 2) \times 13^4 \times 12 = 24 \times 13^4 \times 12$. So dividing by 12 gives the final answer of $2 \times 13^4 \times 12$, as we obtained from the previous method (and so all is well with the world; we can sleep in peace, or rather move on to the next question :))

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Question 5 (10 points)

Place the numbers $1, 2, 3, ..., n^2$ (without duplication) in any order into an $n \times n$ grid, with one number per square. Assume $n > 1$. Prove that there must exist two (horizontally or vertically) adjacent squares whose values differ by **at least** $(n+1)/2$. **Note:** for two squares to be adjacent, they must share an edge; **diagonally adjacent does not count!**

(**Hint:** 1 and n^2 will be in two different squares. Notice that there is always a "path" where you go from adjacent square to adjacent square, by which you can go from any square to any other square within $(2n-2)$ steps. You do **not** need to prove this observation.)

Solution: **Note:** This is very similar to **HW 5, Q.1**.

Denote the integer placed on the square (a, b) of the grid as $G_{(a,b)}$. Note (as stated in the hint) that we can get from any square (a_i, b_i) to any square (a_j, b_j) on the grid, within at most $2n-2$ moves, which consist of going to a adjacent square horizontally or vertically, or diagonally (**Note:** you don't have to prove this, but the proof is in fact really easy; given any such (a_i, b_i) and (a_j, b_j) you can explicitly write down such a path: first equalize the row number and then the column number).

Now denote the square that has 1 as G_{a_1,b_1} and the square that has n^2 as G_{a_2,b_2}, and a path P from G_{a_1,b_1} to G_{a_2,b_2}, where P consists of at most $(2n-2)$ moves between neighbours. **Suppose, by way of contradiction**, that every pair of neighbours has a gap in value that is strictly $< \frac{n+1}{2}$. Then, in each step of this path, the value is increased by strictly $< \frac{n+1}{2}$. Since the path has at most $2n-2$ steps. Then at the end of the path **the value will be strictly less** than

$$1 + (2n-2) \times \frac{(n+1)}{2} = 1 + \frac{2(n-1)(n+1)}{2} = 1 + (n^2 - 1) = n^2.$$

So it's impossible to reach G_{a_2,b_2} from G_{a_1,b_1} using at most $(2n-2)$ steps between neighbours. Contradiction! Thus, the assumption that the gap between every pair of neighbours is strictly$<\frac{n+1}{2}$ is false, and so we know there exists a pair of adjacent squares whose value differ by at least $\frac{n+1}{2}$.

Question 6 (10 points)

What is the last digit of $777^{777^{777}}$? Make sure you justify your answer.

Solution: **Note:** This is very similar to **Practice Final, Q.6**.

The last digit of a natural number k, is just $k \mod 10$. First we notice that $777 \mod 10 = 7$, so computing the last digit of $777^{777^{777}}$ is same as computing the last digit of $7^{777^{777}}$. The remainders when we divide 7^i by 10 are 7, 9, 3 and 1 for $i = 1, 2, 3, 4$ respectively. The cycle length is 4 and in particular $7^4 \mod 10 = 1$, so we need to determine the remainder when 777^{777} is divided by 4.

Since $777 = 194 * 4 + 1$, we know $777 \mod 4 = 1$, so $777^{777} \mod 4 = 1$. Thus, $777^{777^{777}} \mod 10 = 7^1 \mod 10 = 7$.

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Question 7 ($5+5=10$ points)

I roll a fair die with the numbers 1 through 6 on it 5 times. Each roll is independent. Let E_i denote the event that the $(i+1)$-th roll is **exactly 2 more than** than the i-th roll. For example in the sequence $(1, 3, 5, 4, 6)$ only E_1, E_2 and E_4 are true.

(1) Are E_1 and E_2 independent, positively correlated or negatively correlated? Show an explicit calculation of the relevant probabilities to justify your answer.

(2) Are E_1 and E_3 independent, positively correlated or negatively correlated? Show an explicit calculation of the relevant probabilities to justify your answer.

(**Hint:** The calculation will be easier if you keep all probabilities as fractions instead of decimals.)

Solution: Note: This is very similar to **Quiz 2, Q.2.1**.

1. They are negatively correlated. To see why, let's first compute $\mathbb{P}[E_1]$ by considering all possible cases where E_1 occurs. Let the outcome space be the ordered tuples of rolls. There are 6^2 possibilities for the first two rolls (all equally likely). The only cases where E_1 would hold are $(1, 3)$; $(2, 4)$; $(3, 5)$ and $(4, 6)$ (notice fixing the first roll automatically fixes the second). Therefore, $\mathbb{P}[E_1] = \frac{4}{6^2}$. The same reasoning works for any E_i, so $\Pr[E_i] = \frac{1}{9}$ for any i.

   Next, let's find $\mathbb{P}[E_1 \cap E_2]$. There are 6^3 equally likely possibilities for the sequence of the first 3 rolls. One can enumerate all possible pairs of the first three rolls where both E_1 and E_2 occur: $(1, 3, 5)$ and $(2, 4, 6)$ are the only options. Notice that the first roll fixes the subsequent two rolls and if the first roll is ≥ 3, then E_1 and E_2 cannot both occur since that would mean that the second roll has to be ≥ 5 and the third one ≥ 7. Consequently, $\mathbb{P}[E_1 \cap E_2] = \frac{2}{6^3}$, whereas

   $$\Pr[E_1] \times \Pr[E_2] = \frac{4}{6^2} \times \frac{4}{6^2} = \frac{16}{6 \times 6^3} > \frac{2}{6^3} = \mathbb{P}[E_1 \cap E_2].$$

   So $\mathbb{P}[E_1 \cap E_2] < \Pr[E_1] \times \Pr[E_2]$ and hence the events E_1 and E_2 are negatively correlated.

2. Intuitively these events should be independent since E_1 depends on the first two rolls and E_3 on the third and fourth rolls and each individual roll is independent. However we can explicitly calculate the probabilities as asked, exactly like in part (1).

   We already reasoned above that $\Pr[E_1] = \Pr[E_3] = \frac{4}{6^2}$. So $\Pr[E_1] \times \Pr[E_3] = \frac{4 \times 4}{6^4}$.

   What about $\Pr[E_1 \cap E_3]$. There are 6^4 equally likely sequences of the first 4 rolls. There are 4 ways to make E_1 true: the first two rolls could be $(1, 3)$; $(2, 4)$; $(3, 5)$ and $(4, 6)$. But for each of these 4 possibilities, there are 4 ways of making E_3 true: the third and fourth roll could be $(1, 3)$; $(2, 4)$; $(3, 5)$ and $(4, 6)$. Thus there are $4 \times 4 = 16$ favorable outcomes in total. Hence,

   $$= \Pr[E_1] \times \Pr[E_3] = \frac{16}{6^4} = \Pr[E_1 \cap E_3].$$

   So, the events E_1 and E_3 are independent.

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Question 8 (10 points)

Prove that no matter what graph you consider, there is always a way to color the vertices using only 10 colors such that at least 90% of the graph's edges have two differently colored vertices.

(**Hint:** Probabilistic proof!)

Solution: Note: This is very similar to **Practice Final, Q.6**.

Let $G = (V, E)$ be a graph. We will randomly assign one of the 10 colors to the vertices independently with uniform probability. For every $e = \{i, j\} \in E$, let X_e be an indicator s.t. $X_e = 1$ if i and j have different colors and it is 0 otherwise. Since we sample uniformly, given any color for i, there is a $9/10$ chance j will have a different color. Therefore, $\Pr[X_e = 1] = 9/10$.

Let $X = \sum_{e \in E} X_e$ be the total number of edges with differently colored endpoints. Thus, by linearity of expectation,

$$\begin{aligned}\mathbb{E}[X] &= \sum_{e \in E} \mathbb{E}[X_e] \\ &= \sum_{e \in E} \frac{9}{10} \\ &= \frac{9}{10}|E|\end{aligned}$$

In order for the expectation to be $\frac{9}{10}|E|$, there must be some coloring which gets at least $\frac{9}{10}|E|$ to have differently colored endpoints. This completes the proof.

Question 9 ($5 + 5 = 10$ points)

(1) Consider any tree T. Prove that the average of the degrees of the vertices of T must always be < 2. (You need to prove this for any arbitrary tree, that is for any connected graph that does not have cycles).

(2) Suppose **every** vertex in a graph G has degree $\geq k$. Prove that G must have some path of length $\geq k$. (This is a different problem from the question in part (1): here G can be any graph, it need not be a tree.)

(**Hints:** There are **very short proofs** for both parts. **For (1):** use the relationship between number of edges and number of vertices in a tree. Also, for any graph, how is the number of edges related to the sum of the vertex degrees?
For (2): Consider a longest path!)

Solution: There are slightly different proofs possible for both parts. It's very plausible that you can prove both by induction; if you did provide a **correct** inductive proof, that would also get full credit.

1. Let the tree have V vertices and E edges. Let the sum of degrees of the vertices be S. Then because thsi si a tree we know that
$$E = V - 1.$$
and we also know that for any graph (by the Handshaking lemma)
$$2E = S.$$
So $S = 2(V-1) = 2V - 2$. The average of the degrees of the vertices of the tree is $\frac{S}{V} = 2 - \frac{2}{V} < 2$. This completes the proof (as long as $V > 0$; if $V = 0$, the average of degrees is not well-defined.)

2. The graph has a finite number of paths (since you cannot repeat edges or vertices in a path, hence if the graph has n vertices, then the number of paths is surely $\leq n!$ which is finite). Each path has finite length. Thus a longest path (may not be unique) exists. Say $(v_1, v_2, ..., v_l)$ is some such longest path.

   Suppose this path has some length strictly $< k$. Consider the last vertex v_l on this path. Since the path has length $< k$, it has $< k$ edges, and hence in total it contains $\leq k$ vertices. Thus, apart from v_l, there are at most $k-1$ other vertices on this path. But every vertex in the graph has degree $\geq k$. So v_l has at least k neighbours, and at most $k-1$ of them can already be on this path. Hence there must exist some vertex v that is a neighbour of v_l, but that is not on the path. But then $(v_1, v_2, ..., v_l, v)$ would be a strictly longer path! Contradiction.

   Thus any longest path in this graph must have length at least k. This completes the proof.

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Question 10 (Extra Credit)

Attempt only if you have finished the exam and want to work on some more challenging problems! You can earn a maximum of 10 points which may allow you to make up for some errors/ typos on the other questions. However, you **do not need** to solve this problem even to score full points on this test.

Please do not work on this problem until you have seriously attempted/solved everything else on the test.

Question:

There is a row of n coins on a table, each having some positive value. Assume n is an even number. You and I take turns to pick a coin from either end of the row and pocket it. **You get to go first.** You and I both see the entire row of remaining coins at all times, but in our turns we can each take a coin only from either end of the row of remaining coins.

In this question your goal is to **prove that if you are playing first, there exists a way for you to guarantee that you get at least half of the total money on the table (no matter how I play).**

(**Hint:** *There is an* ***elegant and quite short proof*** of this fact that involves coming up with a quite simple strategy for the first player. This simple strategy may not be optimal, but it does guarantee you get at least half the money. *Find this strategy* and prove that it guarantees that you (the person who plays first) get at least half the money!)

Example:

Say the row of coins is

$$2, 3, 10, 4$$

then you can pick either 2 or 4 on turn one.

Notice that, in this example, **if you take the** 4 **on the first turn**, then row of coins in my turn will be

$$2, 3, 10$$

I can take the 10, leaving your best option to be just taking the 3 next. So you get just $4 + 3 = 7$ dollars.

However, **if you take the** 2 **on the first turn**, I will have to play with the row of coins being

$$3, 10, 4$$

and now whatever I do, you take the 10 next, netting yourself $2 + 10 = 12$ dollars! *So taking the highest available coin is not always a good strategy.*

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Solution: Let the coin values be $(c_1, c_2, ..., c_{2n})$. We will show that Player 1 can forcibly take all coins with odd index (i.e. $c_1, c_3, c_5, ..., c_{2n-1}$) or all coins with even index (as per their choice). It should be intuitively quite clear that player 1 can do this, we provide a formal proof by induction on n.

If $n = 2$ we can take the only odd-indexed (or even-indexed, respectively). Otherwise, if we want to take all the odd-indexed coins, take c_1. Then Player 2 must either take c_2 or c_{2n}. In the first case, the row of coins is now $c_3, c_4, ..., c_{2n}$, so we can continue (by induction) taking the odd-indexed coins here. In the second case, the row of coins is now $c_2, c_3, ..., c_{2n-2}, c_{2n-1}$. For this new row of coins, the even-indexed coins are $c_3, c_5, ..., c_{2n-3}, c_{2n-1}$), so if we take the even- indexed coins (which we can do, by induction) we get all odd-indexed coins from the original row.

Similarly, it is possible for player 1 to guarantee getting all the even indexed coins by starting with c_{2n} and then always picking some even indexed coin in every turn.

Now we can describe a very simple strategy: if $c_1+c_3+...+c_{2n-3}+c_{2n-1} \geq c_2+c_4++c_{2n-2}+c_{2n}$, take all odd-indexed coins, otherwise take all even-indexed coins. By the previous paragraphs we can perform this action. But at least one out of the sum of all the even position coins, or the sum of all the odd position coins, must be at least least half of the total value of all the coins. Thus we can manage to secure for ourselves (as player 1) at least half of the total amount of money!

Note that **this strategy is not necessarily the optimal way to play**, but it is very simple and it does get you at least half the total amount of money. We can use **dynamic programming** to construct an algorithm which, given the sequence of coins, calculates the optimal strategy. **This is the sort of thing which you will see in an algorithms class!**