Homework 5 Solutions

Question 1

Place the integers $1, 2, 3, ..., n^2$ (without duplication) in any order onto an $n \times n$ grid, with one integer per square. Assume $n > 1$. Show that there exist two (horizontally, vertically, or diagonally) adjacent squares whose values differ by at least $n + 1$.

Solution: We prove this problem by contradiction. Assume that there doesn't exist a pair of adjacent squares whose value differ by at least $n + 1$.

Denote the integer placed on the square (a, b) of the grid as $G_{(a,b)}$. Note that from any square (a_i, b_i) to any square (a_j, b_j) on the grid, we can use at most $n - 1$ moves, which consist of going to a adjacent square horizontally, vertically, or diagonally. (**Note to graders:** A formal proof that there is always a path using at most $n - 1$ such moves is not required).

Now denote the square that has 1 as G_{a_1,b_1} and the square that has n^2 as G_{a_2,b_2}, and the path P from G_{a_1,b_1} to G_{a_2,b_2}. In each step of this path, the value is increased by at most n, by our assumption, and path has at most $n - 1$ steps. So it's impossible to reach G_{a_2,b_2} from G_{a_1,b_1} using n steps as $1 + n(n - 1) < n^2$. Thus, the assumption is false, so we know there exists a pair of adjacent squares whose value differ by at least $n + 1$.

Question 2

Are there any natural numbers n such that $5^n + 6^n$ is divisible by 7? Either prove there are no such numbers or find them.

Solution:
First, we evaluate the possible reminders of $6^i \mod 7$. Notice that $6^1 \mod 7 = 6$ and $6^2 \mod 7 = 1$, so we know that for $5^n + 6^n$ to be divisible by 7, we must have $5^n \mod 7 = 6$ when n is even or $5^n \mod 7 = 1$ when n is odd. But for $5^i \mod 7$, we have

$$\begin{aligned}
5^1 &= 5 \mod 7 \\
5^2 &= 4 \mod 7 \\
5^3 &= 6 \mod 7 \\
5^4 &= 2 \mod 7 \\
5^5 &= 3 \mod 7 \\
5^6 &= 1 \mod 7.
\end{aligned}$$

We know that for larger n, the remainder repeats this pattern. Thus the only cases where $5^i + 6^i \mod 7 = 0$ are:

- $i \mod 6 = 0$ and i is odd. But if $i \mod 6 = 0$, then i is a multiple of 6 and hence it is a multiple of 2. So i is even, not odd. So there is no number possible in this case.
- $i \mod 6 = 3$ and i is even. But if $i \mod 6 = 3$, then we know that there is some integer k such that $i = 6k + 3 = 3(2k + 1)$. But then i is 3 times an odd number, so $i \mod 2 = 1$. i is odd, not even. So there is no number possible in this case.

Thus, $5^n + 6^n$ is never divisible by 7, for any values of n.

Question 3

Give a combinatorial proof of the identity:

$$\binom{n+k}{2} = \binom{n}{2} + \binom{k}{2} + kn.$$

Note, you will not get significant credit if you just use the formula for binomial coefficients or give a different, non-combinatorial proof.

Solution: Suppose we have n undergraduate students and k master's students, and we want to select a pair of students.

If we simply ignore the undergraduate/master's labels, we have $n + k$ students to select from, giving $\binom{n+k}{2}$ possible pairs; this is the LHS.

We can also consider the labels. We have $\binom{n}{2}$ ways of choosing a pair of undergraduate students, plus $\binom{k}{2}$ ways of choosing a pair of master's students, plus nk pairs with one undergraduate and one master's student. Adding up these three terms gives the RHS.

Question 4

There are 100 doors in a row, all doors are initially closed. A person walks through all doors multiple times and toggle (if open then close, if closed then open) them in the following way: In the first walk, the person toggles every door. In the second walk, the person toggles every second door, i.e., 2nd, 4th, 6th, 8th,... In the third walk, the person toggles every third door, i.e. 3rd, 6th, 9th,... This goes on till the 100th walk, where just the 100th door is toggled. Which doors are open in the end?

Solution: Let $k \in \{1, 2, 3, \ldots, 100\}$ and suppose that the distinct factors of k are $k_1, k_2, \ldots, k_h$, for some $h > 0$. Then, for every factor, the door gets toggled because every $k_i \leq 100$ and $k/k_i \in \mathbb{N}$. Any door is left open if it is toggled an odd number of times, so every door whose number contains an odd number of factors is left open. And if a door is toggled an odd number of times, then it will be closed at the end.

We will now prove that the numbers with an odd number fo factors are exactly the perfect squares.

Suppose the total number fo distinct factors is odd. Notice we can can write k as $k = 2^{c_1} \cdot 3^{c_2} \cdot \ldots$, for all primes below 100. For a given divisor of k, each prime factor p_i could appear 0 to c_i times (inclusive). Thus, the total number of distinct factors is $\prod_{p_i \leq 100}(c_i + 1)$. However, since this term must be odd, each prime number must appear an even number of times, which means the doors that are left open corresponds exactly to the perfect squares.

On the other hand, suppose k was a perfect square. Then it is easy to argue that $k = 2^{c_1} \cdot 3^{c_2} \cdot \ldots$ and each c_i must be even. But then the number of divisors of k is $\prod_{p_i \leq 100}(c_i + 1)$, which is odd, since each c_i is even.

Hence the numbers less than 100 with an odd number of factors (which are exactly the door numbers that are open at the end) are exactly the perfect squares up to 100, namely 1, 4, 9, 16, 25, 36, 49, 64, 81, and 100. $\square$

(**Note to graders**: There is another way of proving that perfect square $\iff$ odd number fo divisors, using a bijection between divisors of $k < \sqrt{k}$ and divisors of $k > \sqrt{k}$.)

Question 5

Suppose that S is a set with n elements. How many ordered pairs (A, B) are there such that A and B are subsets of S with $A \subseteq B$ (that is A is a subset of B)? (**Hint:** Show that each element of S belongs to A, $B - A$, or $S - B$.)

Solution: Following the hint, let $x \in S$.

Claim: $x \in A$, $x \in B - A$, or $x \in S - B$.

If $x \notin S - B$, then either $x \notin S$ or $x \in S \cap B = B$. By definition $x \in S$, so $x \in B$. If $x \notin B - A$, then $x \notin B$ or $x \in A \cap B = A$, but we already know $x \in B$, so $x \in A$, proves our claim.

Thus, we can find the number of possible choices of (A, B), by looking at the ways we can place each choice of x into A, $B - A$, and $S - B$.

Since there are n values in S, and each value has three options, there are a total of 3^n possible distinct choices for (A, B). $\square$

(**Note to graders and students**: Another way of counting these is to say that we must first pick the set B. There are 2^n options for B, but these have different sizes. There are $\binom{n}{i}$ ways of picking a subset B of size i. But once we have picked a subset B of size i, there are further 2^i ways of picking a subset A of B. This reasoning leads to a total count of

$$\sum_{i=0}^{n} 2^i \binom{n}{i}$$

and as per a previous homework we know this is 3^n.)

Question 6

How many binary bit strings of length n are there, where $n \geq 4$, which contain exactly two occurrences of 01? Prove your claim.

(**Hint:** Argue that such strings must be of the form $1^a0^b1^c0^d1^e0^f$.)

Solution: Let's consider building all possible strings from two substrings 01 and 01. Between 01 and 01, we cannot have any 01 because that will give us more than two occurrences. However, we can have arbitrary number of 1's followed by any number of 0's. So we get the initial form 01^c0^d1 where $c, d \geq 1$ (here 1^c means c repetitions of 1). Next let's consider growing strings from the left side. Similarly we cannot have any 01 but arbitrary number of 1's followed by any number of 0's. Applying the same argument to the right side gives us the final form $1^a0^b1^c0^d1^e0^f$ with $a, f \geq 0$ and $b, c, d, e \geq 1$. (**Note to graders:** This isn't completely formal, but a reasonable explanation of why the strings must look like this is enough.)

Since our string has length n, we have the additional constraint $a+b+c+d+e+f = n$. Using a change of variables (setting $b = b'+1$, etc...) we obtain the equivalent equation $a+(b'+1)+(c'+1)+(d'+1)+(e'+1)+f = n$ for $a, b', c', d', e', f \geq 0$ and simplifying we get $a+b'+c'+d'+e'+f = n-4$. Finally, we can apply the stars and bars formula to get $\binom{n-4+5}{5} = \binom{n+1}{5}$.

(**Note to graders:** You can get the stars and bars result without doing the change of variables as well; for instance by imagining 'sticking' a star at the right side of the first four bars.)

Extra Practice Questions

Question 7

Prove that $n^2 - 1$ is divisible by 8 whenever n is an odd positive integer.

Solution: Since n is odd, $\exists\, k \in \mathbb{Z}$ s.t. $n = 2k + 1$. Thus,

$$\begin{aligned} n^2 - 1 &= (2k+1)^2 - 1 \\ &= 4k^2 + 4k \\ &= 4k(k+1) \end{aligned}$$

Either, $2|k$ or $2|k+1$, so $2|k(k+1)$, which implies $8|4k(k+1)$. $\square$

Question 8

Prove that, for every natural number $n \geq 2$, $n^2 - 2$ is not divisible by 4.

Solution: Suppose $4 \mid n^2 - 2$, then $n^2 - 2 = 4k$ for some natural number k. So $n^2 = 4k + 2 = 2(2k + 1)$. Notice $\gcd(2, 2k+1) = 1$, which means $2k + 1$ cannot contribute any factor of 2. Additionally, since n^2 is a square and 2 is prime, we must have $2 \mid n$ and $4 \mid n^2$. We get a contradiction because we cannot find the extra factor of 2.

Question 9

What is the last digit of 777^{446}?

Solution: We know the last digit of 777^{446} is the same as the last digit of 7^{446}, so we find the cycle of $7^i \mod 10$:

$$\begin{aligned} 7^1 &= 7 \mod 10 \\ 7^2 &= 9 \mod 10 \\ 7^3 &= 3 \mod 10 \\ 7^4 &= 1 \mod 10 \\ 7^5 &= 7 \mod 10 \end{aligned}$$

Since the cycle is of length 4, and $446 \mod 4 = 2$, we have the last digit of 777^{446} is 9.

Question 10

A length n string is a palindrome if and only if it's i-th letter is the same as it's $n - i + 1$-th letter. Define palindromes recursively. Make sure to test that your definition is correct by using a few examples. For example DAD and $DEED$ are both palindromes. Make sure to include the base case(s) of your recursive definition.

Solution: We define $P_n \subset \Sigma^*$ to be the set of all palindromes of length n over the alphabet Σ. Then

$$P_n = \{\sigma + \omega + \sigma | \sigma \in \Sigma \wedge \omega \in P_{n-2}\}$$

where $P_0 = \{\epsilon\}$ (set of empty string), $P_1 = \{\sigma | \sigma \in \Sigma\}$.

We will now check our definition against the provided examples.

- $D \in \Sigma$ is the first and last character of DAD, and $A \in P_1$ since $A \in \Sigma$. Thus $DAD \in P_3$. ✓
- $D \in \Sigma$ is the first and last character of $DEED$, and we need to show $EE \in P_2$. $E \in \Sigma$ is the first and last character EE, and $\epsilon \in P_0$. Thus $DEED \in P_4$. ✓