Statistical Sciences
UNIVERSITY OF TORONTO

# STA457H1/STA2202HF L5101 – Time Series Analysis

**PROBLEM SET # 1**
**Due by October 2nd, 2019**

## Problem 1 (10)

Let $r_t$ be a log return. Suppose that $r_1, r_2, \ldots$ are i.i.d. $N(\mu, \sigma^2)$.

a) What is the distribution of $r_t(3) = r_t + r_{t-1} + r_{t-2}$.

b) What is the covariance between $r_t(k)$ and $r_t(k+l)$ for some integer $t, k$ and $l$.

## Problem 2 (20)

Suppose you bought an asset at initial price $P_0 = 20$ and the asset price follows a lognormal geometric random walk where

$$P_t = P_0 \exp(r_t + r_{t-1} + \cdots + r_1)$$

and $r_i$ are i.i.d. $N(0.03, 0.005^2)$.

a) Simulate the annual price of the asset for the next 10 years i.e. $(P_1, P_2, \ldots, P_{10})$ and plot it against time.

b) Simulate $P_{10}$ 2000 times and estimated the expected value i.e $\mathbb{E}[P_{10}]$.

c) Compare your simulated result with the actual expected value of $P_{10}$ .

## Problem 3 (10)

Suppose in a normal plot that the sample quantiles are plotted on the vertical axis, rather than on the horizontal axis as in our lectures.

(a) What is the interpretation of a convex pattern?
(b) What is the interpretation of a concave pattern?
(c) What is the interpretation of a convex-concave pattern?
(d) What is the interpretation of a concave-convex pattern?

## Problem 4 (20)

Suppose that $X_1,\ X_2,\ \ldots$ is a lognormal geometric random walk with parameters $(\mu, \sigma^2)$. More specifically, suppose that $X_k = X_0\, exp(r_1 + r_2 + \cdots + r_k)$, where $X_0$ is a fixed constant and $r_1, r_2, \ldots$ are i.i.d. $N\,(\mu, \sigma^2)$.

(a) Find $P(X_2 > 1.5X_0)$.
(b) Find a formula for the 0.8 quantile of $X_k$ for all $k$.
(c) What is the expected value of $X_k^2$ for any $k$? (Find a formula giving the expected value as a function of $k$).
(d) Find the variance of $X_k$ for any $k$.

## Problem 5 (20)

Suppose that $Y_1, \ldots, Y_n$ are i.i.d. $N(\mu, \sigma^2)$, where $\mu$ is *known*. Show that the MLE of $\sigma^2$ is

$$n^{-1} \sum_{i=1}^{n} (Y_i - \mu)^2.$$

## Problem 6 (20)

(Mixture models). Let $X_1 \sim N(0, \sigma_1^2)$ and $X_2 \sim N(0, \sigma_2^2)$, two independent normal distribution. Let $Y$ also be another independent random variable with a Bernoulli distribution, that is $P(Y = 1) = p$ and $P(Y = 0) = 1\text{-}p$, for some $0 < p < 1$.

(a) What is the mean and the variance of $Z = YX_1 + (1 - Y)X_2$?
(b) Are the tails of its distribution heavier or lighter when compared to a normal distribution with the same mean and variance? If so, for what values of p? Give also an intuitive explanation of your mathematical derivation.