1: 21 cm H I gas

Show:

$$M_{\mathrm{H\,I}} = \frac{16\pi m_H}{3A_{u\ell}h\nu_{u\ell}} D_L^2 F_{obs} \tag{1}$$

If we decompose the equation into:

$$\begin{aligned} M_{\mathrm{H\,I}} &= \left(\frac{16\pi}{3A_{u\ell}h\nu_{u\ell}}\right) m_H (D_L^2 F_{obs}) \\ &= \left(\frac{16\pi}{3A_{u\ell}h\nu_{u\ell}} \int [I_\nu - I_\nu(0)]d\nu\right) m_H \times (\text{area}) \\ &= (\text{column density})\,(\text{mass per particle})(\text{area}). \end{aligned}$$

It follows the line of thought from (8.16):

$$\begin{aligned} \int [I_\nu - I_\nu(0)]d\nu &= \frac{3}{16\pi} A_{u\ell}h\nu_{u\ell} N_{\mathrm{H\,I}} \\ \Rightarrow N_{\mathrm{H\,I}} &= \frac{16\pi}{3A_{u\ell}h\nu_{u\ell}} \int [I_\nu - I_\nu(0)]d\nu \end{aligned} \tag{2}$$

where we integrate the intensity over line to get the expression of column density.

Some unit conversions could help to think this problem:

$$\begin{aligned} &I_\nu = L_\nu / (\text{area (of the object)}) \\ &L_\nu \propto D_L^2 F_\nu, \end{aligned}$$

thus

$$D_L^2 F_{obs} = D_L^2 \int F_\nu d\nu = D_L^2 \int \frac{L_\nu}{D_L^2} d\nu = \int I_\nu d\nu \times (\text{area}). \tag{3}$$

Reverse the logic:

$$\begin{aligned} M_{\mathrm{H\,I}} &= N_{\mathrm{H\,I}} \times m_H \times (\text{area}) \\ &= \left(\frac{16\pi}{3A_{u\ell}h\nu_{u\ell}} \int [I_\nu - I_\nu(0)]d\nu\right) m_H \times (\text{area}) \\ &= \left(\frac{16\pi}{3A_{u\ell}h\nu_{u\ell}}\right) m_H (D_L^2 F_{obs}) \end{aligned} \tag{4}$$

8.4: phases of H I

(a): radiative transfer

The radiative transfer equation is:

$$dI_\nu = -I_\nu \kappa_\nu ds + j_\nu ds. \tag{5}$$

Consider the case for 21 centimeter signals, we have:

$$j_\nu \simeq \frac{3}{16\pi} A_{u\ell} h\nu_{u\ell} n(\mathrm{H\,I})\phi_\nu$$
$$\kappa_\nu \simeq \frac{3}{32\pi} A_{u\ell} \frac{hc\lambda_u \ell}{kT_{\mathrm{spin}}} n(\mathrm{H\,I})\phi_\nu.$$

The integral form of the radiative transfer equation is:

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau')} S_\nu d\tau'. \tag{6}$$

The interpretation of this equation is the intensity at τ_ν is equal to initial intensity attenuated by a exponential factor $e^{-\tau_\nu}$ and the emission of the gas in between integral over the optical depth in between and attenuated by a exponential factor $e^{-(\tau_\nu - \tau')}$. The source function $S_\nu = j_\nu / \kappa_\nu$ depends on the material in between.

We could imagine $I_\nu(0) = I_\nu^{\mathrm{sky}} + I_\nu^{\mathrm{radio}}$. We could also imagine the integral should be decomposed into cold gas area and warm gas area.

For case one, the factorization should be:

$$\int_0^{\tau_\nu} = \int_0^{\tau_w} + \int_{\tau_w}^{\tau_c + \tau_w}.$$

For case two, the factorization is:

$$\int_0^{\tau_\nu} = \int_0^{\tau_c} + \int_{\tau_c}^{\tau_c + \tau_w}.$$

The source function for 21 centimeter signal is:

$$S_\nu(\tilde{T}_{spin}, \nu_{u\ell}, \lambda_{u\ell}) = \frac{j_\nu}{\kappa_\nu} \simeq \frac{2kT_{spin}\nu_{u\ell}}{c\lambda_{u\ell}}. \tag{7}$$

Combine all information, we have:

$$\begin{aligned} &I_\nu(\tilde{\tau_\nu} = \tau_w + \tau_c) \\ &\quad = (I_\nu^{\mathrm{sky}} + I_\nu^{\mathrm{radio}})e^{-(\tau_c + \tau_w)} \\ &\quad + \int_0^{\tau_1} e^{-(\tau_c+\tau_w)-\tau'} \frac{2kT_1\nu_{u\ell}}{c\lambda_{u\ell}} d\tau' + \int_{\tau_1}^{\tau_w+\tau_c} e^{-(\tau_c+\tau_w)-\tau'} \frac{2kT_2\nu_{u\ell}}{c\lambda_{u\ell}} d\tau', \end{aligned} \tag{8}$$

where

$$
\begin{aligned}
\tau_1 &= \tau_w \mathbb{I}(\text{case1}) + \tau_c \mathbb{I}(\text{case2}) \\
T_1 &= T_w \mathbb{I}(\text{case1}) + T_c \mathbb{I}(\text{case2}) \\
T_2 &= T_w \mathbb{I}(\text{case2}) + T_c \mathbb{I}(\text{case1}) \\
&,
\end{aligned} \tag{9}
$$

where $\mathbb{I}$ is the indicator function.

Write the equation into flux density form, simply multiply by the beamsize Ω:

$$
\begin{aligned}
F_\nu^* &= \Omega I_\nu(\tilde{\tau_\nu} = \tau_w + \tau_c) \\
&= \Omega(I_\nu^{\text{sky}} + I_\nu^{\text{radio}})e^{-(\tau_c+\tau_w)} \\
&+ \Omega \int_0^{\tau_1} e^{-(\tau_c+\tau_w)-\tau'} \frac{2kT_1\nu_{u\ell}}{c\lambda_{u\ell}} d\tau' + \Omega \int_{\tau_1}^{\tau_w+\tau_c} e^{-(\tau_c+\tau_w)-\tau'} \frac{2kT_2\nu_{u\ell}}{c\lambda_{u\ell}} d\tau'.
\end{aligned} \tag{10}
$$

Note: the source function S_ν has the same notation as the S_ν (the flux density from the source in the absence of any intervening absorption) in the question. So I denote S_ν^{radio} to be the S_ν given in the question.

(b): off

If the telescope is pointing at the sky, there would be no photons from the source. Re-write Eq 10 to fit this scenario:

$$
\begin{aligned}
F_\nu^{\text{off}} &= \Omega I_\nu(\tilde{\tau_\nu} = \tau_w + \tau_c) \\
&= \Omega I_\nu^{\text{sky}} e^{-(\tau_c+\tau_w)} \\
&+ \Omega \int_0^{\tau_1} e^{-(\tau_c+\tau_w)-\tau'} \frac{2kT_1\nu_{u\ell}}{c\lambda_{u\ell}} d\tau' + \Omega \int_{\tau_1}^{\tau_w+\tau_c} e^{-(\tau_c+\tau_w)-\tau'} \frac{2kT_2\nu_{u\ell}}{c\lambda_{u\ell}} d\tau'.
\end{aligned} \tag{11}
$$

(c): S_ν^{radio} known

Re-write Eq 10 to include S_ν^{radio}:

$$
\begin{aligned}
F_\nu^* &= \Omega I_\nu(\tilde{\tau_\nu} = \tau_w + \tau_c) \\
&= (S_\nu^{\text{radio}} + \Omega I_\nu^{\text{sky}})e^{-(\tau_c+\tau_w)} \\
&+ \Omega \int_0^{\tau_1} e^{-(\tau_c+\tau_w)-\tau'} \frac{2kT_1\nu_{u\ell}}{c\lambda_{u\ell}} d\tau' + \Omega \int_{\tau_1}^{\tau_w+\tau_c} e^{-(\tau_c+\tau_w)-\tau'} \frac{2kT_2\nu_{u\ell}}{c\lambda_{u\ell}} d\tau'.
\end{aligned} \tag{12}
$$

Now the flux of source and flux of sky are given:

$$\begin{aligned}\tilde{F}^{\text{off}} &= F^{\text{off}} \\ \tilde{F}^{*} &= F^{*}\end{aligned} \tag{13}$$

The obvious choice is to subtract between Eq 10 and Eq 11, so we get rid of the emission from the material in between.

$$F^{*}_{\nu} - F^{\text{off}}_{\nu} = S^{\text{radio}}_{\nu} e^{-(\tilde{\tau}_c + \tilde{\tau}_w)}. \tag{14}$$

A little bit algebra:

$$\tilde{\tau}_c + \tilde{\tau}_w = -\log\left(\frac{F^{*}_{\nu} - F^{\text{off}}_{\nu}}{S^{\text{radio}}_{\nu}}\right). \tag{15}$$

Strange enough, we don't need the geometry of case 1 or case 2.

(d): column density

The first way is to just integrate the equation given in the textbook:

$$\frac{dN(\text{H I})}{dv} \simeq \frac{32\pi}{3\lambda^2} \frac{k}{hcA_{u\ell}} [T^{\text{on}}(v) - T^{\text{sky}}(v)]$$

with the given linear relationship between antenna temperature and intensity. So the column density can be found by measurements is proved.

The second way is to start with (8.14):

$$I_{\nu} = I^{\text{sky}}_{\nu} + I^{\text{radio}}_{\nu} + \frac{3}{16\pi} A_{u\ell} h\nu_{u\ell} \phi_{\nu} N_{\text{H I}}.$$

Slight changes:

$$I^{\text{off}}_{\nu} - I^{\text{sky}}_{\nu} = \frac{3}{16\pi} A_{u\ell} h\nu_{u\ell} \phi_{\nu} N_{\text{H I}}.$$

The line profile is given:

$$\phi_{\nu} = \frac{1}{\sqrt{2\pi}\sigma_V} \frac{c}{\nu_{u\ell}} e^{-u^2/2\sigma_V^2}.$$

For optical thin, the absorption is almost negligible, so the geometric factor is not parameterized in the equation. Finally, the total column density could be obtained by integrate out the line profile:

$$N_{\text{H I}} = \frac{16\pi}{3A_{u\ell} h\nu_{u\ell}} \frac{\int I^{\text{off}}_{\nu} - I^{\text{sky}}_{\nu} d\nu}{\int \phi_{\nu} d\nu}, \tag{16}$$

where I^{sky}_{ν}, u and ℓ, and ϕ_{ν} are known. So the total column density could be found by measurements is proved.

(e) effective spin temperature

Write the flux density equation again:

$$\begin{aligned}
F_\nu^* &= \Omega I_\nu(\tilde{\tau_\nu} = \tau_w + \tau_c) \\
&= (S_\nu^{\text{radio}} + \Omega I_\nu^{\text{sky}}) e^{-(\tau_c + \tau_w)} \\
&+ \Omega \int_0^{\tau_1} e^{-(\tau_c+\tau_w)-\tau'} \frac{2kT_1\nu_{u\ell}}{c\lambda_{u\ell}} d\tau' + \Omega \int_{\tau_1}^{\tau_w+\tau_c} e^{-(\tau_c+\tau_w)-\tau'} \frac{2kT_2\nu_{u\ell}}{c\lambda_{u\ell}} d\tau'.
\end{aligned} \tag{17}$$

Slightly re-write right hand side:

$$\begin{aligned}
&F_\nu^* - (S_\nu^{\text{radio}} + \Omega I_\nu^{\text{sky}}) e^{-(\tau_c + \tau_w)} \\
&= T_1 \Omega \int_0^{\tau_1} e^{-(\tau_c+\tau_w)-\tau'} \frac{2k\nu_{u\ell}}{c\lambda_{u\ell}} d\tau' + T_2 \Omega \int_{\tau_1}^{\tau_w+\tau_c} e^{-(\tau_c+\tau_w)-\tau'} \frac{2k\nu_{u\ell}}{c\lambda_{u\ell}} d\tau' \\
&= T_{\text{eff}} \Omega \int_0^{\tau_w+\tau_c} e^{-(\tau_c+\tau_w)-\tau'} \frac{2k\nu_{u\ell}}{c\lambda_{u\ell}} d\tau'
\end{aligned}$$

The last line we assume an effective temperature which describes the combination of the behaviours of T_1 and T_2. This implies:

$$T_{\text{eff}} = \frac{T_1 \int_0^{\tau_1} e^{-(\tau_c+\tau_w)-\tau'} \frac{2k\nu_{u\ell}}{c\lambda_{u\ell}} d\tau' + T_2 \int_{\tau_1}^{\tau_w+\tau_c} e^{-(\tau_c+\tau_w)-\tau'} \frac{2k\nu_{u\ell}}{c\lambda_{u\ell}} d\tau'}{\int_0^{\tau_w+\tau_c} e^{-(\tau_c+\tau_w)-\tau'} \frac{2k\nu_{u\ell}}{c\lambda_{u\ell}} d\tau'}. \tag{18}$$

Note $\tau_w + \tau_c$ is known, ℓ and u are known. Also from (d):

$$N_{\text{H I}} = \frac{16\pi}{3A_{u\ell}h\nu_{u\ell}} \frac{\int I_\nu^{\text{off}} - I_\nu^{\text{sky}} d\nu}{\int \phi_\nu d\nu}, \tag{19}$$

where

$$I_\nu^{\text{off}} = I_\nu^{\text{sky}} e^{\tau_w+\tau_c} + T_{\text{eff}} \int_0^{\tau_w+\tau_c} e^{-(\tau_c+\tau_w)-\tau'} \frac{2k\nu_{u\ell}}{c\lambda_{u\ell}} d\tau'. \tag{20}$$

Without making too complicate mathematics, from Eq 19 and Eq20 we can see the effective temperature also relates to the total column density.