General notations (not always applied): tildes for variables, no tildes for constants or given values.

1 radiative transfer

The differential fom of the equation of radiative transfer is:

$$\frac{1}{c}\frac{\partial}{\partial t}I_\nu + \hat{\Omega}\cdot\nabla I_\nu + (k_{\nu,s} + k_{\nu,a})I_\nu = j_\nu + \frac{1}{4\pi}\int_\Omega I_\nu d\Omega, \tag{1}$$

where j_ν is the emission coefficient, $k_{\nu,s}$ is the scattering opacity, $k_{\nu,a}$ is the absorption opacity, and $\int_\Omega I_\nu d\Omega$ is radiation scattered from other direction onto a surface.

Consider a medium is uniform and isotropic and the radiation is not changing with time:

$$\hat{\Omega}\cdot\nabla I_\nu + (k_{\nu,s} + k_{\nu,a})I_\nu = j_\nu + \frac{1}{4\pi}\int_\Omega I_\nu d\Omega.$$

By defining source function $S_\nu = \frac{j_\nu}{k_{\nu,s}+k_{\nu,a}}$ and ignore the radiation from other places, we have:

$$S_\nu = I_\nu + \frac{1}{(k_{\nu,s} + k_{\nu,a})}\hat{\Omega}\cdot\nabla I_\nu.$$

If we follow the same procedures as in the textbook:

$$d(e^{\tau_\nu}I_\nu) = e^{\tau_\nu}S_\nu d\tau_\nu$$

where we set $d\tau_\nu = (k_{\nu,s} + k_{\nu,a})ds$. If we do the spherical integration on the LHS, we get $e^{\tau_\nu}\mathbb{E}(I_\nu) - I(0)$. It's no clearly to see the spherical mean of intensity is on LHS, but it is not so clear how to get source function solely on the RHS. I guess that would require some assumptions that allows the RHS to leave only S_ν.

2 HI gas and CMB

2.1 brightness temperature

Consider a H I gas in between CMB radiation and us, the brightness temperature is:

$$T_B(\tilde{\nu} = \nu, I_\nu) = \frac{h\nu/k}{\ln\left(1 + \frac{2h\nu^3}{c^2 I_\nu}\right)}. \tag{2}$$

Brightness temperature as a function of ν and I_ν with the frequency is given:

$$\tilde{\nu} = \nu_{\mathrm{H\,I}} = \frac{c}{\lambda_{\mathrm{H\,I}}} = \frac{3*10^{10}\,\mathrm{cm/s}}{21\,\mathrm{cm}} \simeq 1.5*10^9\,\mathrm{s}^{-1}.$$

For $\tilde{\tau}$ is small enough ($\tilde{\tau} = \tau < 0.2$), it is legal to neglect absorption. We thus have a radiative transfer equation:

$$I_\nu(\tilde{u}, \nu_{u\ell} = \nu_{\mathrm{H\,I}}) = I_\nu(0) + \frac{3}{16\pi} A_{u\ell} h \nu_{u\ell} \phi(\tilde{u}) N(\mathrm{H\,I}). \tag{3}$$

For convenience, we consider line center velocity for the velocity distribution $\phi(\tilde{u} = 0) = \frac{1}{\sqrt{2\pi}} \frac{c}{\nu_{u\ell}} \frac{1}{\sigma_\nu}$.

Some typical values for the 21 cm line are listed in the textbook:

$$A_{u\ell} = 2.8843 * 10^{-15}\,\mathrm{s}^{-1}$$
$$\nu_{\mathrm{H\,I}} = 1420.4 * 10^6\,\mathrm{s}^{-1}$$
$$\sigma_\nu \sim \ \mathrm{km/s}$$
$$N(\mathrm{H\,I}) \simeq 10^{20}\,\mathrm{cm}^{-2}.$$

Eq 3 turns out to be:

$$I_\nu(\tilde{u}, \nu_{u\ell} = \nu_{\mathrm{H\,I}}) = I_\nu(0) + \frac{3}{16\pi} 2.8843 * 10^{-15}\,\mathrm{s}^{-1} h 1420.4 * 10^6\,\mathrm{s}^{-1} \frac{1}{\sqrt{2\pi}} \frac{c}{1420.4 * 10^6} \frac{1}{\mathrm{km/s}} 10^{20}.$$

The rest part is to work out the CMB intensity. Since CMB should be a black body, the LTE condition could be applied here:

$$I_\nu(0) = B_{CMB}(\tilde{\nu} = \nu_{CMB}, \tilde{T} = T_{CMB}) = \frac{2h\nu^3}{c^2} \frac{1}{\exp{(h\nu/kT)} - 1}, \tag{4}$$

where these values are given:

$$T_{CMB} = 2.7255\,K$$
$$\nu_{CMB} = 160.23 * 10^9\,\mathrm{s}^{-1}$$

Combine Eq 3 and Eq 4, we arrive:

$$I_\nu(\tilde{u} = 0) = 3.83 * 10^{-15} + 1.36 * 10^{-17} \simeq 3.84 * 10^{-15}, \tag{5}$$

with the brightness temperature:
$$T_B \simeq 6200\,K.$$

It is much hotter than I believed before.

2.2 intensity at line center

From SI unit to Jy is $\times 10^{26}$ and we need to divide 4π to get solid angle in the nominator of the unit:
$$I_\nu = 3 * 10^7\,\mathrm{Jy\,sr}^{-1}.$$

3 Doublet metal lines

3.1 C IV energy levels

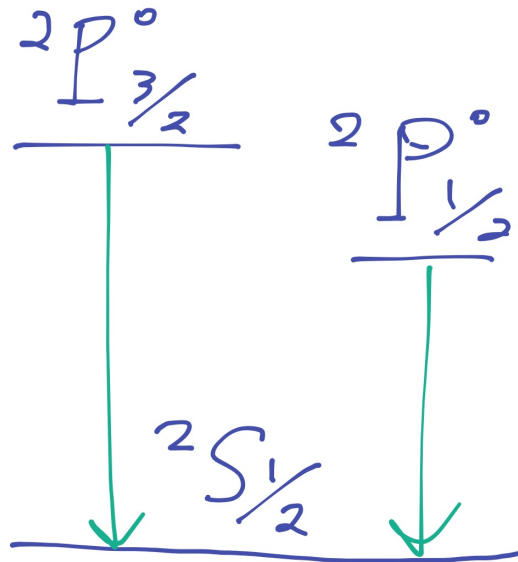Consider the ground state of electron configuration of C IV,

$$1s^2 2s^1$$

with the spectroscopic term

$$^2S_{1/2}.$$

By search the NIST database, we know there are two excited states contribute to C IV doublet

$$^2P^o_{3/2}, {}^2P^o_{1/2}.$$

See the diagram below:

3.2 optical depth

The statistical weight is defined as the degeneracy of a given state:

$$g = 2J + 1,$$

so we have $g = 4$ for 1548 Å line and $g = 2$ for 1550 Å line.

According to textbook Eq (9.9), the line-center optical depth is approximated as:

$$\tau_0 \propto N_\ell f_{\ell u} \lambda_{\ell u} \frac{1}{b}.$$

The oscillation strength can be written as proportional to the statistical weight, as listed in (6.18) and (6.19):

$$f_{\ell u} \propto \frac{g_u}{g_\ell} \frac{1}{\nu_{\ell u}^2} A_{u\ell}.$$

By assuming these two lines have similar Einstein coefficients, here is the final proportionality:

$$\tau_0 \propto \frac{g_u}{g_\ell} \lambda_{\ell u}^3$$

which gives:

$$\frac{\tau_{1548}}{\tau_{1550}} = 1.99.$$

3.3   weak lines, EW

Following the same line of thought, and given that we have (9.28) from the textbook,

$$\frac{W_2}{W_1} = \frac{f_{\ell u_2} \lambda_{\ell u_2}}{f_{\ell u_1} \lambda_{\ell u_1}}$$

the ratio of EW should be the same as the ratio of the optical depths

$$\frac{W_2}{W_1} = 1.99.$$

3.4   strong lines, EW

Following the same line of thought and given that we have (9.29) from the textbook

$$\frac{W_2}{W_1} \simeq \left[1 + \frac{\ln \left(f_{\ell u_2} \lambda_{\ell u_2} / f_{\ell u_1} \lambda_{\ell u_1} \right)}{\ln \tau_{\ell u_1} / \ln 2} . \right]^{1/2}$$

We already know how to calculate some part of this equation, which gives

$$\frac{W_2}{W_1} \simeq \left[1 + \frac{\ln 1.99}{\ln \tau_{\ell u_1} / \ln 2} . \right]^{1/2},$$

which is sensitive to what you choose the optical depth for $\tau_{\ell u_1}$. For the case $\tau_{\ell u_1} \sim 1.5$, we have $\frac{W_2}{W_1} \simeq 1.5$. In a sense we know the ratio is going to be lower.

3.5   Mg II and Ca II

For Mg II it has these two energy level transitions which corresponding to 2798 Å and 2803 Å:

$$^2P^o_{3/2} \to {}^2S_{1/2}$$
$$^2P^o_{1/2} \to {}^2S_{1/2};$$

similarly, for Ca II we have these two transitions for 3934 Å and 3969 Å:

$$^2P^o_{3/2} \to {}^2S_{1/2}$$
$$^2P^o_{1/2} \to {}^2S_{1/2}.$$

We clearly notice the changes of the spectroscopic terms from all three (C IV, Mg II, Ca II) ions are the same, it just the n for the energy levels are different. In this case, though we still have $\frac{\lambda^3_{\ell u_2}}{\lambda^3_{\ell u_1}}$ different, but the ratio of statistical weights are the same. This fact gives that, for weak lines, the ratio of EW are the same ~ 2 for all three ions given that the ratio of wavelengths are similar. While for stronger lines, it depends on the optical depth of one of the lines, but the ratio of EW is not going to be so much different if you look at Fig 9.3, since the term of optical depth is in the log.
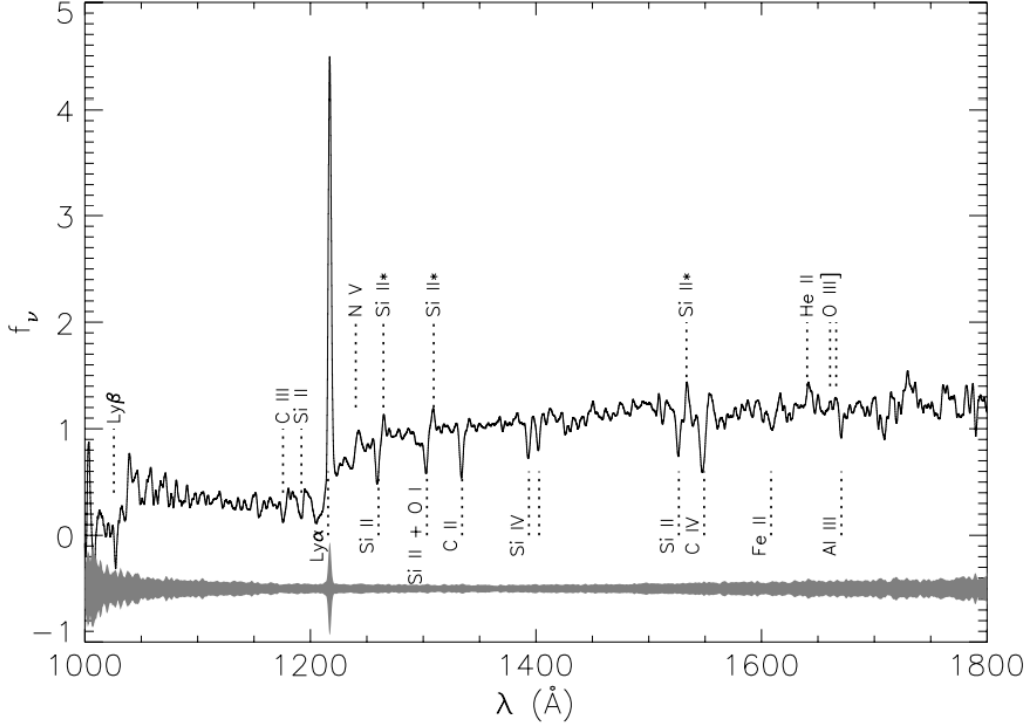
4 composite spectrum

JONES, STARK, & ELLIS

Figure 3. Composite spectrum of 81 galaxies in our sample with $3.5 < z < 4.5$ and apparent magnitude $z'_{AB} < 26$. The effective mean redshift for the sample averaged over wavelength is $\bar{z} = 3.9$. The strongest spectral features are labeled. The gray filled region shows the $\pm 1\sigma$ error at each point, determined from the scatter of individual spectra used to create the composite. The error spectrum peak at 1216 Å is due to large scatter in the intrinsic distribution of Lyα equivalent widths. The error is lowest at $\sim$1300–1500 Å where the continuum signal-to-noise ratio is $\simeq$30. The error increases at shorter wavelengths where the instrument throughput is lower and at longer wavelengths where sky emission is much stronger.

For Si II 1260.4 Å,

$$^2D_{3/2} \rightarrow {}^2P^o_{1/2}$$
$$A = 2.57 * 10^9 \, \text{s}^{-1}$$

while for Si II* 1264.7 Å,

$$^2D_{5/2} \rightarrow {}^2P^o_{1/2}$$
$$A = 3.04 * 10^9 \, \text{s}^{-1}.$$

The apparent difference is the difference in J, which means the difference in the degeneracies or the ratio of statistical weights ($g = 2J + 1$).

4.1   why Si II* emission and Si II absorption

The apparent difference between Si II and Si II* is:

$$\frac{g_2}{g_1} = \frac{4}{2}, \text{for Si II}$$
$$\frac{g_2}{g_1} = \frac{6}{4}, \text{for Si II} *$$

This gives no sense at first, but we can think that if the degeneracies is higher than the systems have the tendency to go that direction. This sense gives Si II* tends to drop to lower level while Si II tends to go up to upper level, by comparing these two lines.

But a more reasonable argument is, if we take a look at the transitions of the energy levels:

$$\text{Si II} : E_i = 0.00 \rightarrow E_f = 79338.50$$
$$\text{Si II}^* : E_i = 287.24 \rightarrow E_f = 79355.02$$

Notice it is much easier to go down to ground state for Si II than Si II*, which makes for sense for Si II happens to have more absorptions.