Problem 1

So let me list some of the values here:

- $(R_* \mid X = 1, m = 20, MK) = 1890\,\mathrm{e^-/s}$
- $(m_{sky} \mid MK, m = m_{sky}, r_{pix}) = 20.3\,\mathrm{mag/arsec^2}$
- $(r_{pix} \mid MK) = 0.135\,\mathrm{arcsec/pix}$
- $(RN^2 \mid MK) = 5\,\mathrm{e^-}$
- $(G \mid MK) = 1.2\,\mathrm{e^-/DN}$

Here are the questions,

1. **R_{sky} of detected in R-band?**

   We knew,
   $$(R_* \mid X = 1, m = 20, MK) = 1890\,\mathrm{e^-/s}. \tag{1}$$
   Now we took off the condition on R magnitude,
   $$(R_* \mid X = 1, m, MK) = 1890 \times 10^{-\frac{m-20}{2.5}} \tag{2}$$
   since $R_* \propto 10^{-\mathrm{mag}/2.5}$.

   Condition on pixel size to get R_{sky}
   $$\begin{aligned}(m_{sky} \mid MK, m = m_{sky},) &= (m_{sky} \mid MK, m = m_{sky}, r_{pix}) - 2.5\log_{10}(r_{pix} \mid MK)^2 \\ &= 20.3 - 5 \times \log_{10} 0.135 \\ &= 24.6\,\mathrm{mag/pixel}\end{aligned} \tag{3}$$
   Use the information of sky magnitude on a pixel to get the rate of detection on a pixel, which is
   $$\begin{aligned}(R_* \mid X = 1, m = m_{sky}, MK) &= 1890 \times 10^{-\frac{m_{sky}-20}{2.5}} \\ &= 1890 \times 10^{-(24.6-20)/2.5} = 26.13\,\mathrm{e^-/pixel/s}\end{aligned} \quad \square \tag{4}$$

2. **Find $R_* \mid m = 26, X = 1.2$.**

   As suggested by the instructor of fundamental astrophysics- never use magnitude,
   $$\begin{aligned}(R_* \mid X = 1.2, m = 26) &\propto 10^{-\frac{26-20}{2.5}} \times \exp\left(-cX\right) \\ &\propto 10^{-\frac{26-20+kX}{2.5}} \\ &\propto 10^{-\frac{26-20+0.1*1.2}{2.5}}.\end{aligned} \tag{5}$$

Remember to bring back your constant

$$(R_* \mid X = 1.2, m = 26) = 1890 \times 10^{-\frac{26-20+0.1*1.2}{2.5}} = 6.74\,\mathrm{e}^-/\mathrm{s}. \tag{6}$$

3. What's the time range for sky-dominated? Condition on radius of 7 pixels aperture.

   Since we have more electrons from sky, we need to wait for enough time to reach source-dominated. The effect of sky decrease as $t^{-1/2}$ because there's a t in the nominator in S/N equation while there's only $t^{1/2}$ in the denominator.

   The uncertainty term came from Poisson distribution, which is

$$\sigma = \sqrt{R_* t + R_{sky} n_{pix} t + (RN)^2 n_{pix} + DN \times n_{pix} t} \tag{7}$$

   Though the description in the question is unclear about '7' pixels, (Does it mean it contains 7 pixels or its radius has a range of 7 pixelscales?) I tried both cases and I thought it should be $n_{pix} = 49$.

   Therefore,

$$\begin{aligned} \sigma &= \sqrt{R_* t + R_{sky} n_{pix} t + (RN)^2 n_{pix} + DN \times n_{pix} t} \\ &= \sqrt{6.74t + 26.13 * 49 * t + 5 * 49 + 1.2 * 49 * t}. \end{aligned} \tag{8}$$

   What we want is sky-dominated, so $\sigma > R_* t =$ signal. Allow me to make some sloppy approximations,

$$\begin{aligned} \sigma &> 6.7t \\ \sqrt{6.74t + 26.13 * 49 * t + 5 * 49 + 1.2 * 49 * t} &> \\ \sqrt{7t + 25 * 50 * t + 250 + 70 * t} &\gtrsim \\ \sqrt{1327 * t + 250} &\gtrsim \end{aligned} \tag{9}$$

   Square both sides,

$$\begin{aligned} 1327t + 250 &\gtrsim 40t^2 \\ \Rightarrow 6 &\gtrsim t^2 - 30t = (t-15)^2 - 225 \\ \Rightarrow 230 &\gtrsim (t-15)^2. \end{aligned}$$

   So roughly before $15 + 15$ seconds is sky-dominated,

$$t \lesssim 30\,\mathrm{s}. \tag{10}$$

4. **Explain how S/N scale with seeing?**

   Seeing is cased by the turbulence in the atmosphere. Therefore, the effect of seeing will contribute to the width of PSF. To a first order approximation, let's say seeing $\propto \sigma_{PSF}$.

   So that we arrive this scaling relation,

$$n_{pix} \propto r_{measure}^2 \propto \sigma_{PSF}^2 \propto \text{seeing}^2. \tag{11}$$

   Recall the equation of S/N,

$$S/N = \frac{R_* t}{\sqrt{R_* t + R_{sky} n_{pix} t + (RN)^2 n_{pix} + DN \times n_{pix} t}} \tag{12}$$

   Let's make some assumptions and drop those small terms, e.g., assume the time is long enough to drop RN and we don't care DN.

   So,

$$\begin{aligned} S/N &\propto \frac{R_*}{\sqrt{R_* + R_{sky} n_{pix}}} \\ &\propto \frac{\sqrt{R_*}}{\sqrt{1 + \frac{R_{sky}}{R_*}\text{seeing}^2}} \\ &\propto \frac{1}{\sqrt{1 + \frac{R_{sky}}{R_*}\text{seeing}^2}}. \end{aligned} \tag{13}$$

   For our case, S/N roughly scales as

$$\begin{aligned} &(S/N \mid X = 1.2, m_* = 26, m_{sky} = 20.3, MK) \\ &\qquad \propto \frac{1}{\sqrt{1 + \frac{1}{4}\text{seeing}^2}}. \end{aligned} \tag{14}$$

5. **What is the exposure time to reach SN = 20?**

   Before solving $S/N \mid X = 1.2, m_* - 26, m_{sky} = 20.3$, we need to know $R_{sky} \mid X = 1.2$.

$$(R_{sky} \mid X = 1.2) = 1890 * 10^{-\frac{20.3-20+0.12}{2.5}} \times 0.135^2 = 23.395\,\mathrm{e}^-/\mathrm{s}. \tag{15}$$

   So the problem is just solving,

$$\begin{aligned} S/N = 20 &= \frac{6.74t}{\sqrt{6.74t + 23.395 * 49t + 5 * 49 + 1.2 * 49t}} \\ &\sim \frac{7t}{\sqrt{7t + 1000t + 250 + 70t}}. \end{aligned} \tag{16}$$

$$
\begin{aligned}
\Rightarrow 1077t + 250 \sim 49/400t^2 \sim 1/8t^2 \\
\Rightarrow t^2 - 8600t - 2000 \sim 0 \\
\Rightarrow (t-4300)^2 - 4300^2 \sim 2000 \\
\Rightarrow t \sim 4300 + \sqrt{4300^2 + 2000} \sim 9000.
\end{aligned}
$$

So it's roughly $t \sim 9\,000$ seconds. I carefully solved it numerically,

Snippet 1: Solve t

```
import sympy as sp

# construct the signal to noise equation using lambda
sn = lambda R_s, R_sky, RN, DN, npix, t :
        R_s * t /
      ( R_s * t + R_sky * t * npix + RN**2 * npix + DN * t * npix )**(1/2)

# solve polynomial with sympy.solve
expr = sn(6.74, 23.395, 5**(1/2), 1.2, 7**2, t)
sp.solve(sp.Eq(expr, 20), t)
```

and I found $t = 10\,671$ seconds.

6. **Find the exposure time on the HST.**

   I checked this filter and thought it's close enough to R band we used: http://svo2.cab.inta-csic.es/svo/theory/fps3/index.php?id=HST/WFC3_UVIS2.F625W

   After plugging those numbers, I got $t = 2059.6179 \sim 2\,000$ seconds.

7. **Try out some parameters you are interested.**

   - 0.4" : $\sim 9\,500$ s
   - 4" : $\sim 700\,000$ s
   - F600LP : $\sim 1\,000$ s
   - F850LP : $\sim 5\,000$ s
   - Narrow F631N : $\sim 120\,000$ s
   - Medium F621M : $\sim 5\,000$ s

   If the aperture is larger, n_{pix} in the denominator would be larger; therefore, longer time to reach $S/N = 20$. But I have no idea why the results of changing filters would be so heterogeneous. Maybe it relates to some intrinsic properties of the filters.

   So I went back to check the filter table-

These are filters :)			
Filter	Transmission	Width	Exposure Time
F600LP	0.99	4 000Å	1 000 s
F850LP	0.96	1 500Å	5 000 s
F631N	0.86	43.1Å	120 000 s
F621M	0.99	631Å	5 000 s
F625W	0.95	1575Å	2 000 s

The numbers in the table are kind of making sense for me since I would also expect narrower filters would have longer exposure times. If I am really optimistic, I probably would guess in this way,

$$\begin{aligned} 1/t &\propto E \\ &\propto \int_{\lambda_{min}}^{\lambda_{max}} T_{\text{trans}} f_\lambda d\lambda \\ &\propto T_{\text{trans}} \Delta\lambda / \lambda_{\text{eff}}. \end{aligned} \tag{17}$$

Let me try a few $t * T_{\text{trans}}\Delta\lambda/\lambda_{\text{eff}}$. Taking the numbers in the names of filters as λ_{eff},

- F600LP: $\sim 1000 * 0.99 * 4000/600 \sim 7000$
- F850LP: ~ 8000.
- F631N: ~ 7000
- F621M: ~ 5000
- F625W: ~ 5000

It's interesting to see our guess brings those heterogeneous times to the numbers with the same order of magnitude. I guess by doing the proper integration, I probably could get the same $1/t$ numbers.