HDR/STDA masterclass

19/7/16

1. HW 6 discussion
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   HW 7
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HW 7 discussion summary:

Sensitivity of results to parameters can be tested by:

- Rule N: significance of eigenvalues (HW5)
- robustness of eigenvectors to random noise (HW6)
- ~~…~~ changing: time interval, spatial domain, observational error, ...
  → effect on interpreted EOFs, PCs, λ.

HDR/STDA Masterclass SSA algorithm

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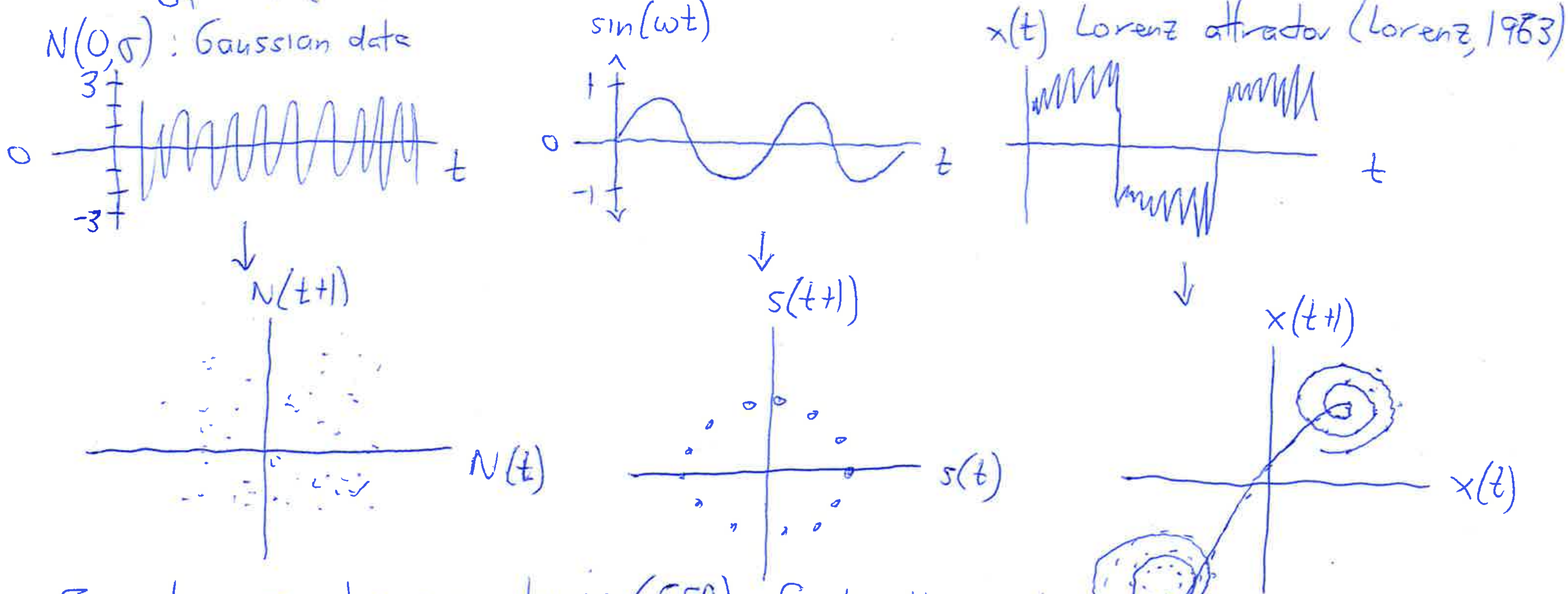
- Based on Broomhead + King (1986); Strogatz (1988); Ghil et al (2002)

Background: Given a time series of data $T(t)$ and an embedding dimension M, phase spaces can be illustrated as a scatter plot of $T(t+1)$ vs $T(t)$:



Singular spectrum analysis (SSA) finds the patterns in the autocovariance of the time series over a given bandwidth (embedding dimension):

- non parametric (except for M)
- data adaptive (trends, oscillations, random variation

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1. for a time series $x(t) = [x_1 \ x_2 \ x_3 \cdots x_n]$ with $\bar{x} = 0$:

2. We form a trajectory matrix for a chosen embedding dimension m:

$$X = \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_{n+1-m} \\ x_2 & x_3 & x_4 & \cdots & \vdots \\ \vdots & & \ddots & & \vdots \\ x_m & & & \cdots & x_n \end{bmatrix}}_{m \text{ rows}}$$

3. Form the <u>autocovariance matrix</u> R:

$$R = \frac{XX^T}{n-m+1}$$

(subtract mean of each row of X to give each autocovariance)

HW7 problem 1:

1. $x = [0 \ -1 \ 0 \ 1 \ 0]$, $n=5$

2. $m=2$:

$$X = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

$$R = \frac{\begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}}{5-2+1}$$

$$= \frac{1}{4}\begin{bmatrix} 0\cdot 0 + -1\cdot -1 + 0\cdot 0 + 1\cdot 1 & 0\cdot -1 + -1\cdot 0 + 0\cdot 1 + 1\cdot 0 \\ -1\cdot 0 + 0\cdot -1 + 1\cdot 0 + 0\cdot 1 & -1\cdot -1 + 0\cdot 0 + 1\cdot 1 + 0\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

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4. The eigenvalue problem is: $E\Lambda = RE$

→ characteristic equation $f(\lambda) = 0$

→ eigenvalue determination.

→ eigenvector determination:

→ principal components:

$$A = E^T X$$

Problem | HW7

$$\begin{vmatrix} \frac{1}{2}-\lambda & 0 \\ 0 & \frac{1}{2}-\lambda \end{vmatrix} = 0$$

$$\left(\frac{1}{2}-\lambda\right)\left(\frac{1}{2}-\lambda\right) = 0$$

$$\lambda = \frac{1}{2}, \frac{1}{2}$$

for $RE = \Lambda E$

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$$

$$\rightarrow E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$A = E^T X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 0 + 0\cdot -1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}.$$

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5. Reconstructed components are formed by convolution of principal components with eigenvectors: $q = a * e$

The convolution can be written as the sum of the inverse diagonals of a matrix formed as the products of a and e:

for example: $a = [a_{11} \;\; a_{12} \;\; a_{13} \cdot a_{14}]$

$e = [e_{11} \;\; e_{21}]$

HW7 problem 1:

q_1:

$k=1, \; k=2, \; k=3, \; k=4, \; k=5$

$$\begin{array}{c|cc} & e_{11} & e_{21} \\ \hline a_{11} & a_{11}e_{11} & a_{11}e_{21} \\ a_{12} & a_{12}e_{11} & a_{12}e_{21} \\ a_{13} & a_{13}e_{11} & a_{13}e_{21} \\ a_{14} & a_{14}e_{11} & a_{14}e_{21} \end{array}$$

$k_1, \; k_2, \; k_3, \; k_4, \; k_5$

$$q_1 = \begin{array}{|cc} 0 \cdot 1 & 0 \cdot 0 \\ -1 \cdot 1 & -1 \cdot 0 \\ 0 \cdot 1 & 0 \cdot 0 \\ 1 \cdot 1 & 1 \cdot 0 \end{array} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}^T$$

$q_2 =$

The RC's are scaled (Ghil et al 2002):

For column in Q: $1, 2, 3, \ldots t \ldots n$:

$1 \le t \le m-1$: q_t / t

$m \le t \le n-m+1$: q_t / m

$n-m+2 \le t \le n$: $q_t / (n-t+1)$.

$$Q = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix}$$

Scalings for $m=2$:

$k=1$ $q = q/1$

$k=2,3,4$ $q = q/2$

$k=5$ $q = q/1$.

Scaled Q:

$$\begin{bmatrix} 0/1 & -1/2 & 0/2 & 1/2 & 0/1 \\ 0/1 & -1/2 & 0/2 & 1/2 & 0/1 \end{bmatrix}$$

$$x = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 \end{bmatrix}.$$

6. Reconstitution of x:

$$x \stackrel{!}{=} \sum_{i=1}^{m} q_i$$