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The Historical Connection of Fourier analysis to Music

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Abstract: This paper will discuss the relevance between mathematics and music throughout a few periods of history. The paper will first discuss how the Ancient Chinese hired mathematicians in order to "perfect the music" used in the court rooms. Mathematics was typically used in music to develop ratios and intervals that are found in music. This paper will then discuss the history of Fourier analysis, as well as give a brief history of Jean Baptiste Fourier. The Fourier analysis was used to find naturally occurring harmonics, to model sound, and to define sound by breaking it up into pieces. Many examples of the Fourier series and Fourier transform can be seen in relation to music. Some more simple examples will also be demonstrated, in order to understand how the Fourier series can model sound waves. While there are many other examples of how math has been used in music, these two aspects will be the main focus of this paper, with favorability placed on discussing the importance and relevance of the Fourier series. However, due to the inability to find sources, Fourier's derivation of the series can only be mentioned in a simplistic manner. To find more examples of math and music more time and research would need to be done.

Keywords: math and music, Chinese math and music, Fourier series, Fourier analysis, Fourier transform, Fourier series coefficients, modeling music

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Introduction

There is no doubt that studying mathematics helped contribute to musical properties. Musical sheets are full of notes that include scales, intervals, and ratios between notes. Considerably, mathematical contributions helped utilize these different concepts when practicing or writing music. In ancient China, mathematicians were hired to find equal temperament in order to "perfect" the music. Clearly, this was not a simple quality that could be represented without the help of mathematics. The history and math behind this issue is quite interesting. Throughout history, Fourier analysis was also used in music. This technique of analysis can be used to identify the naturally occurring harmonics of music. In essence, this is the basis for all musical composition. Clearly, math and music share attributes that can help in musical representation.

History of Math and Music in China

The Ming Dynasty (1368-1644) was the era in China in which the first successful work on the mathematical problem of equal temperament was accomplished. To understand why the work of equal temperament was desired, the history of math and music must be considered. As early as 2700 B.C., the Chinese were occupied with establishing the gong pitch. People had struggled with the mathematical complexities of calculating the eleven tones that the gong pitch should rise above (Joseph, 2011). The importance of this matter was not for mere enjoyment. The Chinese used music as an important component of rituals in the courts (Joseph, 2011). With this in mind, it is not unreasonable to imagine that many Chinese believed that the downfall of previous dynasties was due to a flaw in the music used in court. Therefore, it was imperative that each new dynasty established the "correct" ritual music in order to prolong the survival of the dynasty (Joseph, 2011).

The musician Zhu Zaiyu (1536-1611) was the first person to solve the problem of equal temperament. Descending from a founder of the Ming dynasty, Zhu grew up in wealth. However, after his father was accused of treason against the emperor and placed on house arrest, Zhu decided to dedicate himself to scholarly pursuits (Joseph, 2011). Zhu believed that by introducing equal temperament into the court, the Ming dynasty could be restored. Equal temperament is a system of tuning in which each pair of adjacent notes has an identical frequency ratio (Joseph, 2011). Zhu was the first person to obtain an accurate solution of equal temperament in 1584.

The Math Involved in Equal Temperament

In equal temperament, an interval—such as an octave—is divided into a series of equal steps. For octaves, the ratio of tones between octaves is 1:2 (Joseph, 2011). In equal temperament, the distance between each step of the scale is the same length. Zhu concludes that in a twelve-tone equal temperament system, the ratio of frequencies between two adjacent semitones is $\sqrt[12]{2}$ (Joseph, 2011). Zhu began with a familiar Pythagorean result for a right-angled isosceles triangle of length 10 *cun*. Therefore, the length of the hypotenuse is

$$\sqrt{10^2+10^2} = 10\sqrt{2} = 10 \times 1.41421356\ldots$$

Ignoring the length of the fundamental, which in this case would be 10 *cun*, $\sqrt{2}$ represents the note that is the midpoint between the octave of the fundamental 1and the terminal 2 (Joseph, 2011). The length between the calculated midpoint and the terminal point can be calculated using the following procedure:

$$\sqrt{10 \times 10\sqrt{2}} = 10\sqrt[4]{2} \text{ (Joseph, 2011).}$$

Doing the calculation again for a terminal of 3 provides:

$$\sqrt[3]{10 \times 10 \times 10\sqrt[4]{2}} = 10\sqrt[12]{2}.$$

In Zhu's calculations, he calculates $\sqrt{2}$, the $\sqrt{\sqrt{2}}$, and the $\sqrt[3]{\sqrt{\sqrt{2}}}$ (Joseph, 2011). Therefore, the ratio to divide an octave into twelve equal parts is $\sqrt[12]{2}$. More generally, the smallest interval in an equal tempered scale is the ratio $r^n = p$, so $r = \sqrt[n]{p}$. In this equation, the ratio r divides the ratio p—which equals 2/1 in an octave—into n equal parts (Joseph, 2011). There are thoughts that Zhu Zaiyu's work was transmitted to Europe.

History of the Fourier Analysis

Pythagoras not only worked on the most well–known Pythagorean Theorem, but he also studied music and the arithmetical relationships between pitches. It has been said that he discovered the relationship between number and sound (Hammond, 2011). With this connection to Pythagoras, it shows that a relationship between math and music was established as early as the sixth century B.C. Pythagoras found ratios relating to harmonizing tones (Hammond, 2011). A musical tuning system is based on his discoveries.

As seen earlier, ratios are apparent in music. When a note is played on an instrument, listeners hear the played tone as the fundamental, as well as a combination of its harmonics sounding at the same time (Hammond, 2011). Pythagoras discovered the idea of harmonics, causing many others to explore this idea. A French mathematician Marin Mersenne (1588-1648), defined harmonics that Pythagoras had already found. Going back to the ratios discussed earlier, Mersenne defined six harmonics as ratios of the fundamental frequency, 1/1, 2/1 (the ratio we used to find a twelve-tone equal temperament of an octave), 3/1, 4/1,5/1, and 6/1 (Hammond, 2011).

In the 18th century, calculus was used in discussions of vibrating strings. Brook Taylor found an equation representing the vibrations of a string based on the intitial equation, and he found that the sine curve was a solution to this equation (Hammond, 2011). D'Alembert was also led the the differenital equation of Taylor,

$$\frac{\partial^2 y}{\partial x^2} = a^2 \frac{\partial^2 y}{\partial t^2}$$

where the x-axis is in the direction of the string and y is the displacement at time *t* (Hammond, 2011). Euler's response to this equation said that the string could lie along any curve and would require mulitple expressions to model the curve (Hammond, 2011). Bernoulli disagreed with Euler and came up with the following equation:

$a_1 \sin(x)\cos(t) + a_2 \sin(2x)\cos(2t) + a_3 \sin(3x)\cos(3t) + \cdots$ (Hammond, 2011).

In this equation, setting *t*=0 would give the initial position of the string. Bernoulli claimed that his solution was general and should include Euler's and D'Alembert's solutions, leading to the problem of expanding arbitraty functions with trigonometric series (Hammond, 2011). Every mathematician stayed clear of this possibility until the work of Fourier.

Jean Baptiste Fourier (1768-1830) was a French mathematician. When he was only nine years old, his mother died and his father followed a year later. Fourier studied mathematics at a military school in Auxerre, where he first demonstrated talents in literature, but mathematics soon became his real interest (O'Connor & Robertson, 1997). He was educated by Benvenistes (Hammond, 2011). Instead of taking his religious vows to joing the priesthood, Fourier left and became a teacher at Benedictine college. During the French Revolution, although he did not like the affairs that were taking place, he became entangled in a revolutionary committee. In 1794, he was nominated to study in Paris and was educated by Lagrange, Monge, and Laplace (O'Connor & Robertson, 1997). Due to his involvement in the revolutionary committee, Fourier was also arrested a number of times. In 1798, Fourier joined Napoleon's army in the invasion of Egypt as a scientific advisor (O'Connor & Robertson, 1997). He acted as an adminstrator for the

French poltical institutions and administrations were set up. He also helped establish educational facilities in Egypt, along with carrying out archaeological explorations (O'Connor & Robertson, 1997). After traveling to Egypt with Napoleon, Fourier returned to France in 1801. Under Napoleon's request, he traveled to Grenoble. While here, Fourier did some of his most important work on the theory of heat. He then published a paper about heat waves in 1807 (Hammond, 2011). Fourier examined the problem of describing the evolution of the "temperature T (x, t) of a thin wire of length π, stretched between x = 0 and x = π, with a constant zero temperature at the ends: T (0, t) = 0 and T (π, t) = 0; he proposed that the initial temperature T (x, 0) = f(x) could be expanded in a series of sine functions" (Walker). While his studies are now held in high regards, during his time this work was very controversial. The controversy was caused by his expansion of functions as trigonometric series, as well as his derivation of the equations (O'Connor & Robertson, 1997). He stated that the wave equation could be solved with a sum of trigonometric functions, also known as the Fourier Series.

Even though Fourier published a paper on these findings, the exact derivation of the series appears to still remain quite abstract. The following information provides a slight insight into the work of Fourier. Fourier found an equation for the annulus of radius *R* as follows:

$$\frac{\partial z}{\partial t} = \frac{K}{CD}\frac{\partial^2 z}{\partial x^2} - \frac{hl}{CDS}z,$$

where *x* is the angular variable on the annulus (Grattan-Guinness & Ravetz, 1972). He then began his solution with the transformation $z = e^{-ht}v$ which converts into the diffusion equation

$$\frac{\partial v}{\partial t} = K\frac{\partial^2 v}{\partial x^2},$$

where *K* represents what was previously *K/CD* (Grattan-Guinness & Ravetz, 1972). From here, Fourier found the solution form that applied to the diffusion equation as

$$e^{-kn^2 t}\sin nx \text{ or } e^{-kn^2 t}\cos nx,$$

implying that the general solution would be a linear combination of these terms (Grattan-Guinness & Ravetz, 1972). Including the intitial temperature, the equation would then lead to the full Fourier series. Using integration, Fourier was also able to find the coeffiecients to the equation. This is the basic understanding of how Fourier derived his series that modeled heat and sound waves.

The Fourier Series

The Fourier Series is key to the decomposition of a signal into sinusiodal components. The series is as follows

$$f(x) \approx \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos\frac{n\pi x}{a} + B_n \sin\frac{n\pi x}{a})$$

for a 2a-periodic funtion f(x) with the following coefficients:

$$A_0 = \frac{1}{a}\int_{-\pi}^{\pi} f(x)dx,$$

$$A_n = \frac{1}{a}\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi x}{a}dx,$$

and $B_n = \frac{1}{a}\int_{-\pi}^{\pi} f(x)\sin\frac{n\pi x}{a}dx$ (Hammond, 2011).

Solving for these coefficients will be demonstrated later in this paper. The idea of the Fourier series is that as $\lim_{n\to\infty} f(x)$ will have enough terms so that it will converge to the

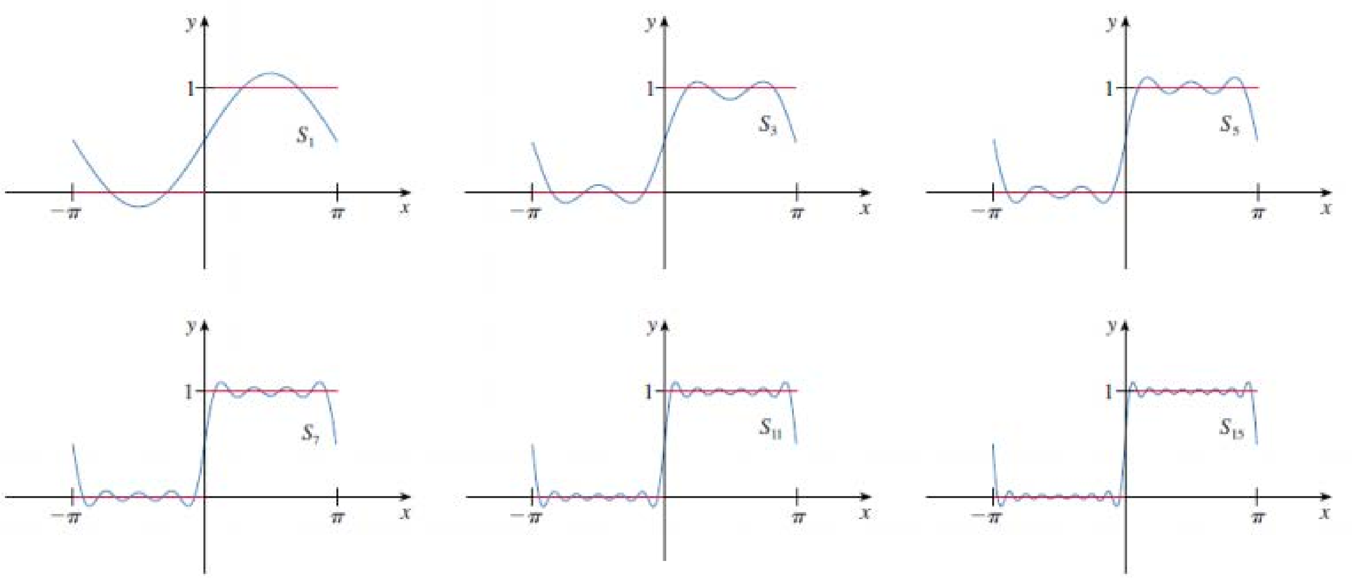

Figure A

function. Figure A "shows a simple piecewise equation in red (the horizontal lines), and the partial sums in blue (summed to a given n, i.e. the waves) of the Fourier series of the function for n=1, 3, 5, 7, 11, 15. As n grows, the Fourier Series gives a closer approximation of the actual function" (Hammond, 2011).

Let's examine the concept of the Fourier series in the terms of plucking a string, causing vibration. If the string is plucked in a precise manner to only vibrate at the fundamental harmonic, then this pattern can be represented by single sinusoid with frequency v_0 and the amplitude of the oscillation (Lenssen & Needell, 2014). So the frequency-domain representation $F(v)$ only has one spike at $v = v_0$ with a height equal to the amplitude of the wave. However, typically they have more than one frequency. Taking this into account, the frequency domain can be represented by an infinite series, the Fourier Series, with the harmonics weighted in such a way that they represent the motion of the string (Lenssen & Needell, 2014). This provides the basis for the Fourier transform.

The Fourier Transform

Through the Fourier transform, one can obtain the frequency-domain representation of a time-domain function. The transform is also invertible (Lenssen & Needell, 2014). First, let's observe the continuous Fourier transform. Let w_k be the angular frequency and

$$w_k \equiv 2\pi k v.$$

Then the relationship between the time-domain function *f* and its frequency-domain function *F* is

$$F(w_k) \equiv \int_{-\infty}^{\infty} f(t)e^{-2\pi i k t}dt, \text{ where } k \in (-\infty, \infty)$$ (Lenssen & Needell, 2014).

The sinusoidal components can be found within

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$ (Lenssen & Needell, 2014).

The Fourier transform can be used to turn musical signals into frequencies and amplitudes. A mathematician from Dalhousie University in Canada, Dr. Jason Brown, put the Fourier transform to work on the Beatles song, *A Hard Day's Night*. The opening chord that sounds much like a distinct chang or bell has been a mystery to numerous musicians. As such, many have tried to reproduce this chord without much luck. Brown used computer technology to run the Fourier transform on a one second clip of the chord. Doing so gave him a list of over 29,000 frequencies (Hammond, 2011). Looking for the fundamental frequencies, he observed the frequencies with the highest amplitude since these would be most likely to be fundamental frequencies. He then compared these frequencies to an A of 220Hz. Using the half step frequency change for equal tempered instruments, he found he found how many half steps each one was from A (Hammond, 2011). This was then easily converted into a list of notes being played. He then assigned notes to instruments, and values of half steps that were not close to integers could be accounted for by out of tune instruments. Ultimatley, Brown was able to replicate the sound of the chord (Hammond, 2011). The Fourier transform is also used to compose music. Computer generated music began to originate in Paris in the 1970's, and the Fourier transform helps composers create entirely new sounds (Hammond, 2011). Certainly, the Fourier transform has many practical applications for music.

The Fourier transform is also important when trying to transfer music onto a CD. To observe the benefits of the Fourier transform for this purpose, first how the Fourier transform translates a sample of music needs to be observed. This can be seen in Figure B. The sample of music in Figure B was taken from that of a piano sound, with it's Fourier transform is shown to the right of the sample. As one can see, the Fourier transform is clearly periodized (D., 2012). One then wonders how to obtain the original signal from the sampled version. To do this, we need to multiply by a lowpass filter to get rid of the unwanted copy (D., 2012). The same type of event occurs in data aquistion between the sampler and the signal being sampled.

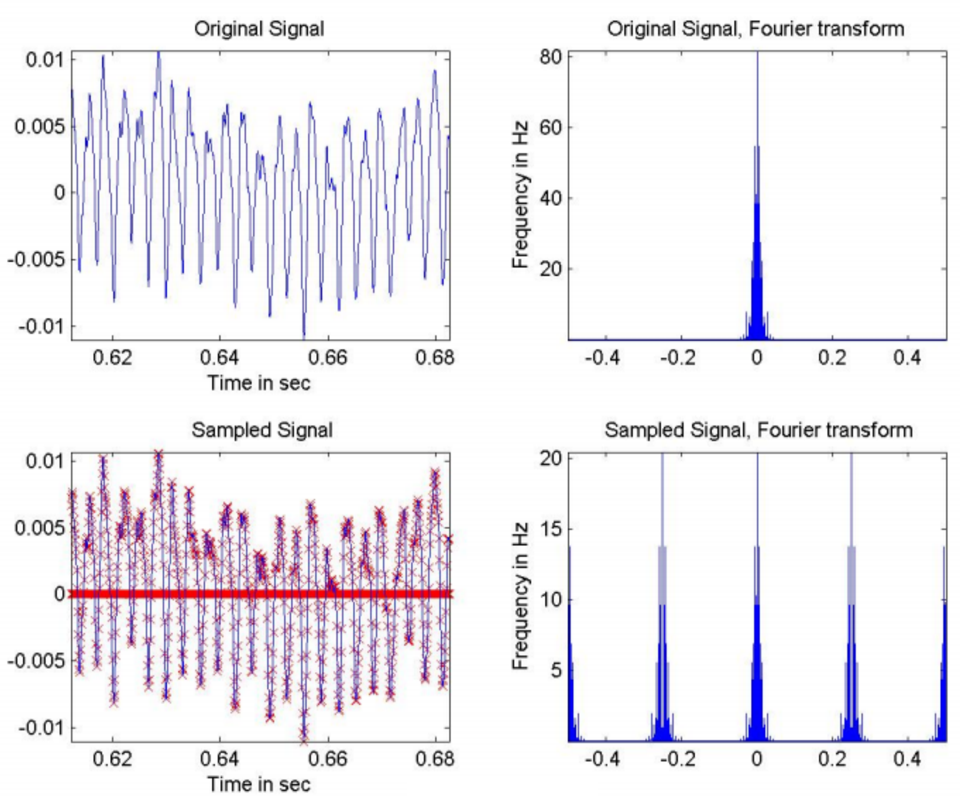

Figure B

One limitation of discrete-time sampling is an effect called aliasing. An example of this can be seen in older movies, like when watching a wagon moving forward but the wheels of the wagon appear to be going backwards (D., 2012). In this case, the sampler is the camera. Therefore, this phenomena occurs when the wagon wheel's spokes spinning approaches the rate of the sampler, which for a camera is approximately 30 frames per second (D., 2012). In Figure C, the sinusoid of 330Hz appears to have a much lower frequency of 30Hz, and the sampling rate of 320Hz maps the frequency of 330Hz to 10Hz (D., 2012). An example of this can be seen in the square wave observed in Figure A. The Fouier series equation for this square wave is $f(x) = \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{L}$, the derivation of this equation will shown later in the paper. This equation can also be expressed as $f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2\pi(2n-1)x)$ (D., 2012). The square wave does not have a finite number of frequencies but rather a spectrum. Therefore, the square wave cannot be sampled properly without aliasing. As an example, assume sampler frequency equals 44100Hz and the square wave has a fundamental

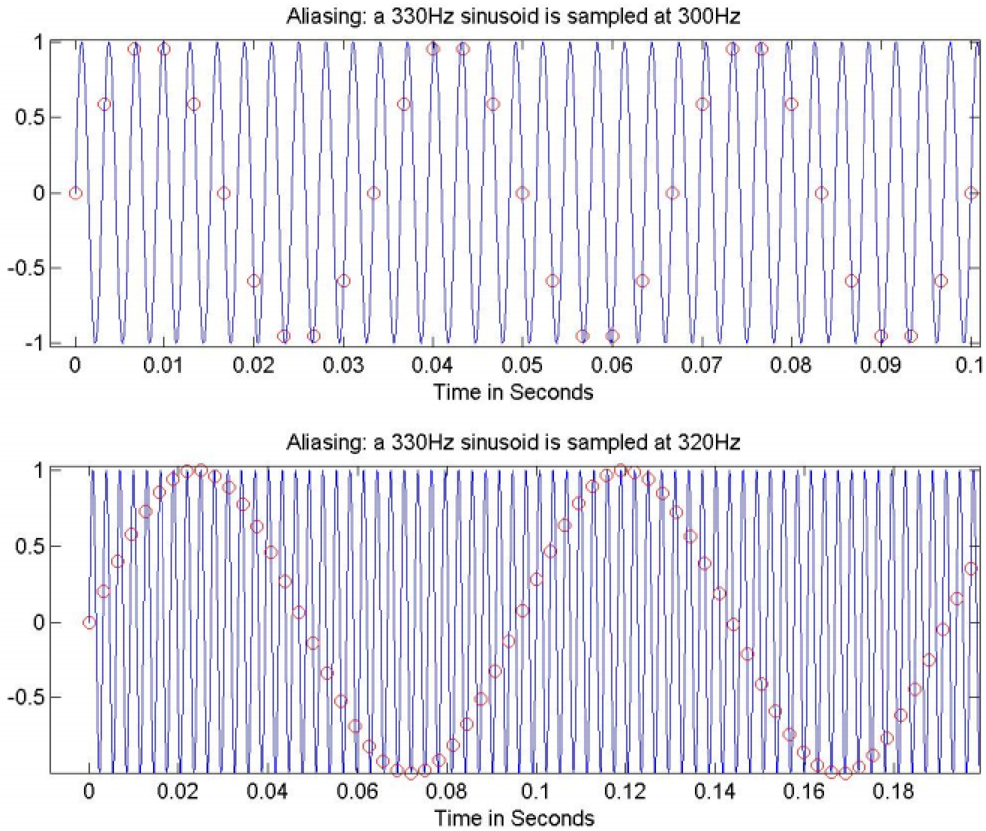

Figure C

frequncy of 700Hz. Then the Fourier series is $f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(1400\pi(2n-1)x)$ which will result in 700 oscillations per second (D., 2012). For this example, the highest frequency that is still below the Nyquist frequency of 22050Hz is the 31st harmonic, or 21700Hz (D., 2012). Therfore, it will show up as an alias at (23100 – 44100) = –21000Hz. Usually, aliasing is avoided in sampling, but aliasing is sometimes used for various sound effects (D., 2012). An anti-alias filter is pre-filtering of analog signals before sampling that can remove undesired aliasing components (D., 2012). It is through this sampling and filtering process that music is actually encrypted onto CDs. Truly, the relevance of the Fourier transform with relation to music, delves into the curiosity of why more musicians to not also major in mathematics.

The Fourier Series and Sound

As we have seen, the Fourier series can model sound by modeling frequency and amplitude. These waves can be represented by sinusoidal equations. Sounds are made of pure tones and other linear combinations such as chords (Hammond, 2011). Vibrating strings and air columns of instruments obey this wave representation using the Fourier series. Note that pure tones are simple tones that can be represented by a single trigonometric function. When these tones are produced by a computer they are very simple and empty sounding. An instrument cannot produce such pure tones, therefore other linear combinations must be added to the pure tones.

The Fourier Series and Harmonics

When an instrument plays a note, the fundamental frequency is heard as well as harmonics. The amplitude of each harmonic is the difference we’re hearing from an instrument (Hammond, 2011). Harmonics are integer multiples of the fundamental frequency. So the fundamental frequncy is the pitch of the note heard and the harmonics are the tonal color of the sound (Hammond, 2011). White noise is created when many equal-amplitude frequencies are sounded at the same time. The harmonic series is the series of tones that are created by multiplying the fundamental frequency by integers (Hammond, 2011). The integers in the harmonic series are related to the harmonic ratios discussed earlier. The numerator of the ratios is a multiple of the fundamental frequency, and the denominator is a number of octaves between the two to put the tone in the same octave of the fundamental frequency (Hammond, 2011).

For an example, look at the harmonic series on C, where C is the fundamental note. So the first harmonic plays C an octave higher. This means that the frequency ratio between octaves is 2:1 (Hammond, 2011). The second harmonic is a fifth higher than that with a frequency that is three times the fundamental note. Dividing by 2 puts the fifth in the same octave as the fundamental, because it is an octave too high and the fifth becomes the ratio 3:2 (Hammond, 2011). The perfect fourth then has the ratio 4:3. Multiplying by the fundamental n goes up the harmonic series, and multiplying by $\frac{1}{n}$ goes down the series (Hammond, 2011). These are the ratios Pythagoreas found when he was investigating strings. When a string is divided in two, the frequency doubles producing a tone and octave higher (Hammond, 2011). The harmonic series exists naturally in sound as Pythagoreas realized.

The Fourier Series and Modeling Music

As discussed, the Fourier series is a solution to the wave equation that can be used to model sound. The Fourier series is broken into trigonometric functions with various frequencies and amplitudes i.e. the fundamental and its harmonics. The fundamental can be represented by the first term after the constant A_0 in the Fourier Series (Hammond, 2011). The form of the

Fourier series has two functions for each term that form a single wave. The first non-constant term that represent sthe fundamental is

$$A_1 \cos\frac{\pi x}{a} + B_1 \sin\frac{\pi x}{a} \text{ (Hammond, 2011).}$$

The first harmonic in the Fourier series is then the term

$$A_2 \cos\frac{2\pi x}{a} + B_2 \sin\frac{2\pi x}{a},$$

and so on until the n-th term as

$$A_n \cos\frac{n\pi x}{a} + B_n \sin\frac{n\pi x}{a} \text{ (Hammond, 2011).}$$

The coefficients of the harmonics in the Fourier series give the amplitude of each harmonic determining the tonal quality.

Tuning Systems

If a composer wants to change keys, to have pure intervals new frequencies are needed based on ratios for the new key. In addition, some notes from the upper harmonics will sound jarring and there is no way of notating decreasing step sizes. To solve this problem, tuning systems have been developed throughout history for instruments (Hammond, 2011).

The Pythagorean tuning system is based on the perfect fifth interval using small integer ratios. This system fills in the chromatic scale with a series of fifths. In order to get a perfect octave a significantly smaller fifth is needed (Hammond, 2011). This system is a theoretical system and hasn't been put into practice because of the problems of variation and inconsistent fifths. Many historical systems have modified the Pythagorean system to keep some intervals pure and some intervals approximated (Hammond, 2011). However, many of these systems still had limits.

As discussed earlier in the paper with the Chinese, the current dominant system is equal temperament. In an octave, there are twelve chromatic steps which are half steps. Equal temperament divides the octave into twelve equal steps (Hammond, 2011). Since the frequency ratio for an octave is 2:1, this means that each half step has a frequency of

$$u_n = u_0 2^{\frac{n}{12}},$$

where u_0 is the fundamental frequency and *n* is the number of half steps from the fundamental note (Hammond, 2011). Using the equation above, we find that the interval between each half step is $2^{\frac{1}{12}}$, as we also saw with the Chinese. Also, the number of half steps *n* between two frequencies u_1 and u_2 is as follows:

$$u_1 = u_2 \left(2^{\frac{1}{12}}\right)^n,$$

$$\frac{u_1}{u_2} = (2^{\frac{1}{12}})^n,$$

$$\log_{2^{1/12}} \frac{u_1}{u_2} = \log_{2^{1/12}} (2^{\frac{1}{12}})^n,$$

$$n = \log_{2^{1/12}}(u_1/u_2)\ (\text{Hammond, 2011}).$$

Interval	Pythagorean Ratio	Cents	Eq. Temp. Approximation	Cents
Minor 2nd	256/243= 1.035	90	1.0595	100
Major 2nd	9/8 = 1.125	204	1.1225	200
Minor 3rd	32/27 = 1.185	294	1.1892	300
Major 3rd	81/64 = 1.266	408	1.26	400
Perfect 4th	4/3 = 1.333	498	1.335	500
Tritone	729/512 = 1.424	612	1.414	600
Perfect 5th	3/2 = 1.5	702	1.498	700
Minor 6th	128/81 = 1.58	792	1.587	800
Major 6th	27/16 = 1.688	906	1.682	900
Minor 7th	16/9 = 1.778	996	1.782	1000
Major 7th	243/128 = 1.898	1110	1.888	1100
Octave	2/1	1200	2	1200

Figure D

Since each half step is equal in size, a G in the key of D is the same frequency as a G in the key of A, which allows for modulation (Hammond, 2011). However, since the step size is an irrational number, the intervals are not pure to the harmonic series intervals. Technically, the intervals are not quite in tune with each other. Another calculation divides each half step into 100 cents, another frequency measure. So the frequency based on the fundamental u_0 and the number of cents difference is

$$u_n = u_0 2^{\frac{n}{1200}},$$

where *n* is the number of cents, so the number of half steps is 100*n* (Hammond, 2011). Figure D shows the frequency ratios for the two tuning systems in numbers and cents based on the intervals (Hammond, 2011). The Fourier series and transform can analyze these various tones.

Coefficients of the Fourier Series

As we saw in Figure A, as the number of sinusoidal functions summed together grew, the function tended towards the “square wave” function. That is

$$f(x) = \begin{cases} 1 \text{ for } 0 \leq x < \pi \\ -1 \text{ for } \pi \leq x \leq 2\pi \end{cases}.$$

As we have already seen, the Fourier series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$.

The following mathematical steps to find the Fourier series coefficients have been followed from Professor Lionheart's notes on the Fourier series. First, we will try to solve for the coefficient a_n. Let's multiply by $\cos mx$ and integrate both sides from 0 to 2π. So

$$\int_0^{2\pi} f(x)\cos mx\, dx =$$

$$\int_0^{2\pi} \frac{a_0}{2}\cos mx\, dx + \sum_{n=1}^{\infty}\left(\int_0^{2\pi} a_n \cos\frac{n\pi x}{L}\cos mx\ \ dx + \int_0^{2\pi} b_n \sin\frac{n\pi x}{L}\cos mx\ \ dx\right).$$

Finding the antiderivative of the piece, $\int_0^{2\pi}\frac{a_0}{2}\cos mx\, dx = \frac{a_0}{2}\left[\frac{\sin mx}{m}\right]$, and evaluating the antiderivative from 0 to 2π, it is clear that the first piece equals zero. This means

$$\int_0^{2\pi} f(x)\cos mx\, dx = \sum_{n=1}^{\infty}\left(a_n\int_0^{2\pi}\cos\frac{n\pi x}{L}\cos mx\ \ dx + b_n\int_0^{2\pi}\sin\frac{n\pi x}{L}\cos mx\ \ dx\right).$$

Remembering trigonometric rules, we see that $\cos(a+b) = \cos a\cos b - \sin a\sin b$ and $\cos(a - b) = \cos a\cos b + \sin a\sin b$. From these rules we can conclude that

$$2\cos\frac{n\pi x}{L}\cos m = \cos\left(\left(\frac{n\pi}{L}+m\right)x\right) + \cos\left(\left(\frac{n\pi}{L}-m\right)x\right) \text{ and}$$

$$\cos\frac{n\pi x}{L}\cos m = \frac{\cos\left(\left(\frac{n\pi}{L}+m\right)x\right)+\cos\left(\left(\frac{n\pi}{L}-m\right)x\right)}{2}.$$

Therefore,

$$a_n\int_0^{2\pi}\cos\frac{n\pi x}{L}\cos mx\ \ dx = \frac{a_n}{2}\int_0^{2\pi}\cos\left(\left(\frac{n\pi}{L}+m\right)x\right)dx + \frac{a_n}{2}\int_0^{2\pi}\cos\left(\left(\frac{n\pi}{L}-m\right)x\right)dx,$$

$$= \frac{a_n}{2}\left[\frac{\sin\left(\left(\frac{n\pi}{L}+m\right)x\right)}{\frac{n\pi}{L}+m}\right] + \frac{a_n}{2}\left[\frac{\sin\left(\left(\frac{n\pi}{L}-m\right)x\right)}{\frac{n\pi}{L}-m}\right] \text{ both pieces evaluated at 0 and } 2\pi.$$

Solving these antiderivatives, one can find that if $\frac{n\pi}{L} \neq m$, then the integral equals 0. However, if $\frac{n\pi}{L} = m$, then the integral equals $\int_0^{2\pi}\cos^2 mx\, dx = \frac{a_n}{2}\int_0^{2\pi}\cos 2mx + 1\, dx = \frac{a_n}{2}\left[\frac{\sin 2mx}{2m} + x\right]$ evaluated from 0 to $2\pi = \pi a_n$. So let's look at the second piece when $\frac{n\pi}{L} = m$ to see if we can solve for the coefficient a_n. With this condition,

$$b_n \int_0^{2\pi} \sin\frac{n\pi x}{L}\cos mx\ \ dx = b_n \int_0^{2\pi} \sin mx \cos mx\ \ dx = \ b_n\left[\frac{\sin^2 mx}{2m}\right] \text{evaluated from 0 to } 2\pi =$$

$$0.\ \text{Therefore,}$$

$$\int_0^{2\pi} f(x)\cos mx\, dx = \sum_{n=1}^{\infty}\left(a_n \int_0^{2\pi} \cos\frac{n\pi x}{L}\cos mx\ \ dx + b_n \int_0^{2\pi} \sin\frac{n\pi x}{L}\cos mx\ \ dx\right),$$

$$= \pi a_n.$$

Notice that π is half the length of the period and $L = \pi$, so $a_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\cos\frac{n\pi x}{L}dx$.

We can obtain the next coefficient b_n using similar methods. However, this time let's multiply by $\sin mx$ and integrate both sides.

$$\int_0^{2\pi} f(x)\sin mx\, dx =$$

$$\int_0^{2\pi} \frac{a_0}{2}\sin mx\ dx + \sum_{n=1}^{\infty}\left(\int_0^{2\pi} a_n \cos\frac{n\pi x}{L}\sin mx\ \ dx + \int_0^{2\pi} b_n \sin\frac{n\pi x}{L}\sin mx\ \ dx\right).$$

Focusing on the first piece,

$$\int_0^{2\pi} \frac{a_0}{2}\sin mx\ dx = \ \frac{a_0}{2}\left[\frac{-\cos mx}{m}\right] \text{ evaluating from 0 to } 2\pi = 0.$$

So ultimately, we are now focusing on

$$\int_0^{2\pi} f(x)\cos mx\, dx = \sum_{n=1}^{\infty}\left(a_n \int_0^{2\pi} \cos\frac{n\pi x}{L}\sin mx\ \ dx + b_n \int_0^{2\pi} \sin\frac{n\pi x}{L}\sin mx\ \ dx\right).$$

Since we are looking to solve for the coefficient b_n, let's first try to solve that second piece $b_n \int_0^{2\pi} \sin\frac{n\pi x}{L}\sin mx\ \ dx$ when $\frac{n\pi}{L} \neq m$ and when $\frac{n\pi}{L} = m$. Looking back to the trigonometric functions mentioned before, $\cos(a+b) = \cos a \cos b - \sin a \sin b$ and $\cos(a-b) = \cos a \cos b + \sin a \sin b$, one can see that $\sin a \sin b = \frac{\cos(a-b)-\cos(a+b)}{2}$. Therefore,

$$b_n\int_0^{2\pi}\sin\frac{n\pi x}{L}\sin mx\ dx = \frac{b_n}{2}\int_0^{2\pi}\cos\left(\left(\frac{n\pi}{L}-m\right)x\right)-\cos\left(\left(\frac{n\pi}{L}+m\right)x\right)\ dx,$$

$$=\frac{b_n}{2}\left[\frac{\sin\left(\left(\frac{n\pi}{L}-m\right)x\right)}{\frac{n\pi}{L}-m}\right]-\frac{b_n}{2}\left[\frac{\sin\left(\left(\frac{n\pi}{L}+m\right)x\right)}{\frac{n\pi}{L}+m}\right] \text{ both pieces evaluated at 0 and } 2\pi.$$

Evaluating these antiderivatives, one can see that when $\frac{n\pi}{L}\neq m$, the integral equals 0. However, since we want to solve for the coefficient b_n, it is not helpful for this piece to equal 0. Therefore, we observe the case where $\frac{n\pi}{L}=m$. With this stipulation, the integral becomes much easier to solve. Consequently,

$$b_n\int_0^{2\pi}\sin\frac{n\pi x}{L}\sin mx\ dx = b_n\int_0^{2\pi}\sin^2 mx\ dx,$$

$$=\frac{b_n}{2}\int_0^{2\pi}1-\cos 2mx\ dx=\frac{b_n}{2}\left[x-\frac{\sin 2mx}{2m}\right] \text{ evaluated from 0 to } 2\pi.$$

It can be seen easily now that the integral equals πb_n. Now that we have solved this, we can solve the last piece $a_n\int_0^{2\pi}\cos\frac{n\pi x}{L}\sin mx\ dx$ when $\frac{n\pi}{L}=m$. Therefore,

$$a_n\int_0^{2\pi}\cos\frac{n\pi x}{L}\sin mx\ dx = a_n\int_0^{2\pi}\cos mx\sin mx\ dx,$$

$$=\frac{a_n}{m}\left[\frac{\sin^2 mx}{2}\right] \text{evaluated from 0 to } 2\pi = 0.$$

Now it is clear that

$$\int_0^{2\pi}f(x)\sin mx\,dx =$$

$$\int_0^{2\pi}\frac{a_0}{2}\sin mx\,dx+\sum_{n=1}^{\infty}\left(\int_0^{2\pi}a_n\cos\frac{n\pi x}{L}\sin mx\ dx+\int_0^{2\pi}b_n\sin\frac{n\pi x}{L}\sin mx\ dx\right),$$

$$\int_0^{2\pi}f(x)\sin mx\,dx=\pi b_n.$$

Similar to the last coefficient that was solved for, it can be seen that π is half the length of the period and $L = \pi$, so $b_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\sin\frac{n\pi x}{L}\,dx$.

Last but not least, the coefficient a_0 needs to be solved for. To do this, we will simply integrate both sides of the original Fourier series equation from 0 to 2π, keeping in mind that L is equal to half the period which in this case is π. So

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi}\frac{a_0}{2}\,dx + \sum_{n=1}^{\infty}\left(a_n\int_0^{2\pi}\cos nx\;dx + b_n\int_0^{2}\sin mx\,dx\right) \text{where } n = \frac{n\pi}{L} \text{ and } m = \frac{m\pi}{L}.$$

Solving $a_n\int_0^{2\pi}\cos nx\;dx = a_n\left[\frac{\sin nx}{n}\right]$ evaluated from 0 to $2\pi = 0$. Also, evaluating $b_n\int_0^{2}\sin mx\,dx = b_n\left[\frac{-\cos mx}{m}\right]$ evaluated from 0 to $2\pi = 0$. Therefore,

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi}\frac{a_0}{2}\,dx,$$

$$\int_0^{2\pi} f(x)\,dx = \frac{a_0}{2}[x] \text{evaluated from 0 to } 2\pi = \pi a_0.$$

This means that $a_0 = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\;dx$. Note that one could calculate these integrals in terms of L. However, for the sake of clarity, the specific example of $L = \pi$ was used. Although the math behind finding the coefficients is time consuming, it is beneficial to see how the coefficients were derived from the Fourier series.

Example of Using the Fourier Series

Remembering the square wave seen earlier in the paper, let's us derive the Fourier representation of this specific wave. To do this, we will consider the square wave *f(x)* over the interval [0, 2L]. Notice that $f(x) = 2\left[H\left(\frac{x}{L}\right) - H\left(\frac{x}{L} - 1\right)\right] - 1$, where H(x) is the Heaviside step function (Weisstein, 2016). Since $f(x) = f(2L - x)$, the function is odd, so when examining the Fourier series, $a_0 = a_n = 0$ (Weisstein, 2016). Therfore, we know that

$$f(x) = \sum_{n=1}^{\infty} \left(b_n \sin \frac{m\pi x}{L}\right) \text{ and}$$

$$b_n = \frac{1}{L} \int_0^{2L} f(x) \sin \frac{n\pi x}{L} dx.$$

Evaluating this we find,

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{4}{n\pi} \sin^2\left(\tfrac{1}{2} n\pi\right) \\ &= \frac{2}{n\pi} [1 - (-1)^n] \\ &= \frac{4}{n\pi} \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd.} \end{cases} \end{aligned}$$

Therefore, the Fourier series for the square wave function is as follows:

$$f(x) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin \frac{m\pi x}{L} \text{ (Weisstein, 2016).}$$

Conclusions and Future Study

There is no doubt in my mind that mathematics has played an important role in the development of mathematics. As seen in early China, as well as later in time in Western cultures, equal temperament was required in music to help improve the sound of the music. However, it is clear that this could not have been accomplished without the help of mathematicians. Then, as was heavily discussed, the Fourier series added numerous contributions to music. By transforming samples of music, the Fourier transform has helped with recreating particular sounds in muscical compositions, as well as encrypting music onto CDs. The Fourier series has also been beneficial to model sound waves and helping create tuning systems. Clearly, from the mathematics demonstrated throughout this paper, the Fourier series also has a strong mathematical background. Without the progression of mathematics throughout time, it would not be as easy as it has now become to use the Fourier series to represent numerous waves. It would be very interesting to also study the contributions of the Fourier series in the areas of physics and other sciences. This is what is so beautiful about the Fourier series. It does not purely have one application. The Fourier series has helped with various problems throughout various fields. Therefore, maybe we should all brush up on our mathematical skills in order to apply these useful equations in numerous situations. To conclude, consider the following quote:

> Mathematical analysis is as extensive as nature itself; it defines all perceptible relations, measures times, spaces, forces, temperatures; this difficult science is formed slowly, but it preserves every principle which it has once acquired; it grows and strengthens itself incessantly in the midst of the many variations and errors of the human mind. . .Mathematical analysis can yet lay hold of the laws of these phenomena. It makes them present and measurable, and seems to be a faculty of the human mind destined to supplement the shortness of life and the imperfection of the senses. . .it follows the same course in the study of all phenomena; it interprets them by the same language, as if to attest the unity and simplicity of the plan of the universe, and to make still more evident that unchangeable order which presides over all natural causes (Grattan-Guinness & Ravetz, 1972).

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