Variation Of Parameter :-

variation of parameters is a method in which we have to find a particular solution to a non-homogenous linear differential equation. It works for any non-homogenous term, unlike the methods of undetermined co-efficient which required specific forms (exponential, sine & cosine (trignometric), polynomial)

working Rule :

1. Find out the complementary form

   i.e $\quad c_1 y_1 + c_2 y_2$

2. Particular Integral $\Rightarrow U(x) y_1 + V(x) y_2$

   where U & V are the function of 'x'

3. Find U & V be using the formulae

$$U(x) = \int \frac{-y_2 R(x)}{y_1 y_2' - y_1' y_2} \quad , \quad V(x) = \int \frac{y_1 R(x)}{y_1 y_2' - y_1' y_2}$$

$$D^2 + P(x) D + Q(x) y = R(x)$$

Sol

neter :-

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ue to find a
homogenous linear
for any non-
e methods of
ch required spec
(trignometric),

form

$v(x) y_2$
unction of 'x'

formulae

$\dfrac{y_1 R(x)}{y_1 y_2' - y_1' y_2}$

$y = C.F + P.I$

↳ auxillary eqn.

$y_c = c_1 y_1 + c_2 y_2$

$R(x) =$ R.H.S of differential eqn.

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

$$U(x) = -\int \frac{y_2 R(x)}{W(x)} dx$$

$$V(x) = \int \frac{y_1 R(x)}{W(x)} dx$$

Q# Solve $y'' + y = \sec x$ by variation of parameters method.

Sol

For complementary func.

$(D^2+1) = 0$

$m^2 + 1 = 0$

$m = \pm i$ $\quad a + ib$

$\quad 0 + i(1)$

C.F

$e^{ax}(c_1 \cos bx + c_2 \sin bx)$

$e^{(0)x}(c_1 \cos(1) + c_2 \sin(1)x)$

$\underbrace{c_1 \cos x}_{y_1} + \underbrace{c_2 \sin x}_{y_2}$

...ameter:-

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...have to find a
...-homogeneous linear
...for any non-
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...ne (trignometric),

...form

$v(x)y_2$
function of 'x'

formulae

$\int \frac{y_1 R(x)}{y_1y_2' - y_1'y_2}$

$y = C.F + P.I$
↳ auxillary eqn.

$y_c = c_1y_1 + c_2y_2$

$R(x)$ = R.H.S of differential eqn.

$w(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$

$u(x) = -\int \frac{y_2 R(x)}{w(x)} dx$

$v(x) = \int \frac{y_1 R(x)}{w(x)} dx$

Q# Solve $y'' + y = \sec x$ by variation of parameters method.

Sol For complementary func.

$(D^2+1) = 0$

$m^2+1 = 0$

$m = \pm i$ $a+ib$ $0+i(1)$

C.F

$e^{ax}(c_1\cos bx + c_2\sin bx)$

$e^{(0)x}(c_1\cos(1) + c_2\sin(1)x)$

$\underbrace{c_1\cos x}_{y_1} + \underbrace{c_2\sin x}_{y_2}$

For P.I

$$P.I = u(x)\, y_1 + v(x)\, y_2 \quad \text{—— (i)}$$

$y_1 = \cos x$, $y_2 = \sin x$

$y_1' = -\sin x$, $y_2 = \cos x$

$$w(x) = y_1 y_2' - y_1' y_2$$

$$w(x) = (\cos x)(\cos x) - (-\sin x)(\sin x)$$

$$w(x) = \cos^2 x + \sin^2 (x)$$

$$w(x) = 1$$

$$u(x) = -\int \frac{y_2 R(x)}{w(x)}\, dx = -\int \sin x \cdot \frac{\sec x}{1}\, dx$$

$$= -\int \frac{\sin x}{\cos x}\, dx = -\int \tan x \, dx$$

$$\boxed{u(x) = -\ln \sec x}$$

$$v(x) = \int \frac{R(x)}{w(x)}\, dx = \int \cos \cdot \frac{\sec x}{1}\, dx$$

$$= \int \cos \cdot \frac{1}{\cos x}\, dx$$

$$\boxed{v(x) = x}$$

eq (i) $\Rightarrow$

$$P.I = (-\ln \sec x)\cos x + x \sin x$$

Now,

$$y = CF + PI$$

$$\boxed{y = c_1 \cos x + c_2 \sin x + (\ln \sec x)\cos x + x \sin x}$$

Ans

Ques: $\dfrac{d^2y}{dx^2} + 4y = 4\tan 2x$

Sol

For complementary function:

$$(D^2 + 4)y = 0$$

$$D^2 + 4 = 0$$

$$D^2 = -4$$

$$D = \pm 2i \qquad \begin{matrix} a + ib \\ 0 + i(2) \end{matrix}$$

$$C.F = e^{ax}(c_1 \cos bx + c_2 \sin bx)$$

$$e^{0(x)}(c_1 \cos 2x + c_2 \sin 2x)$$

$$C.F = \underbrace{c_1 \cos 2x}_{y_1} + \underbrace{c_2 \sin 2x}_{y_2}$$

<u>For Particular Integral</u>:

$$P.I = U(x)y_1 + V(x)y_2$$

$y_1 = \cos 2x$, $y_2 = \sin 2x$

$y_1' = -2\sin 2x$, $y_2' = 2\cos 2x$

$$W(x) = (\cos 2x)(2\cos 2x) - (-2\sin 2x)(\sin 2x)$$

$$W(x) = 2\cos^2 2x + 2\sin^2 2x$$

$$w(x) = 2(1)$$

$$u(x) = -\int \frac{y_1 R(x)}{W(x)} = -\int \frac{\sin 2x\,(4\tan 2x)}{2}$$

$$u(x) = -2\int \frac{\sin^2 2x}{\cos 2x}\,dx$$

$$u(x) = -2\left(\int \frac{(1-\cos^2 x)}{\cos 2x}\,dx\right)$$

$$= -2\left(\int \frac{1}{\cos 2x}\,dx - \int \cos 2x\,dx\right)$$

$$= -2\int \sec 2x + 2\int \cos 2x\,dx$$

$$\boxed{u(x) = -\ln(\sec 2x + \tan 2x) + \sin 2x}$$

$$v(x) = \int \frac{y_1 R(x)}{W(x)} = \int \frac{\cos 2x - (4\tan 2x)}{2}$$

$$v(x) = 2\int \sin 2x\,dx$$

$$\boxed{v(x) = -\cos 2x}$$

$$PI = [-\ln(\sec 2x + \tan 2x) + \sin 2x]\cos 2x + (-\cos 2x)\cdot\sin 2x$$

Now,

$$y = CF + PI$$

$$\boxed{y = C_1\cos 2x + C_2\sin 2x + [\sin 2x - \ln(\sec 2x + \tan 2x)]\cos 2x - \cos 2x\,\sin 2x}$$

$$U(x) = -\int \frac{y_1 R(x)}{W(x)} = -\int \frac{\sin 2x\,(4\tan 2x)}{2}$$

$$U(x) = -2\int \frac{\sin^2 2x}{\cos 2x}\,dx$$

$$U(x) = -2\left(\int \frac{(1-\cos^2 2x)}{\cos 2x}\,dx\right)$$

$$= -2\left(\int \frac{1}{\cos 2x}\,dx - \int \cos 2x\,dx\right)$$

$$= -2\int \sec 2x + 2\int \cos 2x\,dx$$

$$\boxed{U(x) = -\ln(\sec 2x + \tan 2x) + \sin 2x}$$

$$V(x) = \int \frac{y_1 R(x)}{W(x)} = \int \frac{\cos 2x - (4\tan 2x)}{2}$$

$$V(x) = 2\int \sin 2x\,dx$$

$$\boxed{V(x) = -\cos 2x}$$

$$P.I = \left[-\ln(\sec 2x + \tan 2x) + \sin 2x\right]\cos 2x + (-\cos 2x)\cdot\sin 2x$$

Now,

$$y = CF + PI$$

$$\boxed{y = C_1\cos 2x + C_2\sin 2x + \left[\sin 2x - \ln(\sec 2x + \tan 2x) + s\ldots\right]\cos 2x - \cos 2x\,\sin 2x}$$

$$U(x) = -\int \frac{y_1 R(x)}{W(x)} = \ldots 2x)$$

$$U(x) = -2\int \frac{\sin^2 2x}{\cos 2x}\,dx$$

$$U(x) = -2\left(\int \frac{(1-\cos^2 2x)}{\cos 2x}\,dx\right)$$

$$= -2\left(\int \frac{1}{\cos 2x}\,dx - \int \cos 2x\,dx\right)$$

$$= -2\int \sec 2x + 2\int \cos 2x\,dx$$

$$\boxed{U(x) = -\ln(\sec 2x + \tan 2x) + \sin 2x}$$

$$V(x) = \int \frac{y_1 R(x)}{W(x)} = \int \frac{\cos 2x - (4\tan 2x)}{2}$$

$$V(x) = 2\int \sin 2x\,dx$$

$$\boxed{V(x) = -\cos 2x}$$

$$PI = [-\ln(\sec 2x + \tan 2x) + \sin 2x]\cos 2x + (-\cos 2x)\cdot \sin 2x$$

So,

$$y = CF + PI$$

$$= C_1 \cos 2x + C_2 \sin 2x + [\sin 2x - \ln(\sec 2x + \tan 2x) + \sin 2x]\cos 2x - \cos 2x \sin 2x$$

$$a_n x^n \frac{d^n y}{dx^n} + a_n \ldots$$

where $a_0, a_1, \ldots$

$$x^3 \frac{d^3 y}{dx^3} + \frac{x^2 d^2 y}{dx^2} \ldots$$

Cauchy-Euler ... to solve a ... differential eq... are power... solve using transfering differential ...

let, $e^z = \ldots$

$\ln e^z = \ldots$

$z = \ldots$

Taking deri...

$$\frac{dz}{dx}$$

Multiplying

$$\frac{dy}{dx} = \ldots$$

Cauchy - Euler ODEs :-

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_0 y = \phi(x)$$

where $a_0, a_1, a_2 \dots$ are constant

$$x^3 \frac{d^3 y}{dx^3} + x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + y = xe^x$$

Cauchy-Euler method is a special technique used to solve a particular type of 2nd order linear differential eq. where the variable co-efficients are power of independent variable. It can be solve using a characteristic equations by transfering it into a constant co-efficient differential equation.

let,

$$e^z = x$$

$$\ln e^z = \ln x$$

$$z = \ln x$$

Taking derivative with 'x'

$$\frac{dz}{dx} = \frac{1}{x}$$

Multiplying both side by dy/dz

$$\frac{dy}{dx} = \left(\frac{dy}{dz}\right)\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{x} \quad \text{—— (1)}$$

$$x\frac{dy}{dx} = \frac{dy}{dz} \quad \left[\because D = d/dz\right]$$

$$\boxed{x\frac{dy}{dx} = Dy}$$

For $x^2\frac{d^2y}{dx^2}$:

Taking derivative with 'x'

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\underset{u}{\frac{dy}{dz}}\,\underset{v}{\frac{1}{x}}\right)$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) + \frac{dy}{dz}\left(\frac{-1}{x^2}\right)$$

Solving $1/x \frac{d}{dx}\left(\frac{dy}{dz}\right)$:

$$= \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)\frac{dz}{dz}$$

$$= \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= \frac{1}{x}\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)$$

$$= \frac{1}{x}\frac{d^2y}{dz^2}\left(\frac{1}{x}\right) \quad \left[\because \frac{dz}{dx} = 1/x\right]$$

$$= \frac{1}{x^2}\frac{d^2y}{dz^2} \quad \left[\because \frac{d}{dz} = D\right]$$

$$= 1/x^2\, D^2y$$

$$x^2\frac{d^2y}{dx^2} =$$

Similarly,

x^3

x^4

Ques: x

Given:

let,

$D =$

Subs

$$x\frac{dy}{dx} = \frac{dy}{dz} \qquad \left[\because D = d/dz\right]$$

$$\boxed{x\frac{dy}{dx} = Dy}$$

For $x^2\frac{d^2y}{dx^2}$:

Taking derivative with 'x'

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\underbrace{\frac{dy}{dz}\,\frac{1}{x}}_{u\cdot v}\right)$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) + \frac{dy}{dz}\left(\frac{-1}{x^2}\right)$$

Solving $1/x \, \frac{d}{dx}\left(\frac{dy}{dz}\right)$:

$$= \frac{1}{x}\,\frac{d}{dx}\left(\frac{dy}{dz}\right)\frac{dz}{dz}$$

$$= \frac{1}{x}\,\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= \frac{1}{x}\,\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)$$

$$= \frac{1}{x}\,\frac{d^2y}{dz^2}\left(\frac{1}{x}\right) \qquad \left[\because \frac{dz}{dx} = 1/x\right]$$

$$= \frac{1}{x^2}\,\frac{d^2y}{dz^2} \qquad \left[\because \frac{d}{dz} = D\right]$$

$$= 1/x^2 \, D^2y$$

$x^2\frac{d^2y}{dx^2} = D^2y$

Similarly,

$x^3 \; d^3y$

$x^4 \; d^4y$

Ques : x^2 d

Given :

let,

$D = \frac{d}{dz}$

Substitu

$$x\frac{dy}{dx} = \frac{dy}{dz} \qquad \left[\because D = d/dz\right]$$

$$\boxed{x\frac{dy}{dx} = Dy}$$

For $x^2\frac{d^2y}{dx^2}$:

Taking derivative with 'x'

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\underset{u\,.\,v}{\frac{dy}{dz}\frac{1}{x}}\right)$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) + \frac{dy}{dz}\left(\frac{-1}{x^2}\right)$$

Solving $1/x \; \frac{d}{dx}\left(\frac{dy}{dz}\right)$:

$$= \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)\frac{dz}{dz}$$

$$= \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= \frac{1}{x}\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)$$

$$= \frac{1}{x}\frac{d^2y}{dz^2}\left(\frac{1}{x}\right) \qquad \left[\because \frac{dz}{dx} = 1/x\right]$$

$$= \frac{1}{x^2}\frac{d^2y}{dz^2} \qquad \left[\because \frac{d}{dz} = D\right]$$

$$= 1/x^2 \; D^2y$$

$$x^2\frac{d^2y}{dx^2} = D^2y - Dy = D(D-1)y$$

Similarly,

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

$$x^4\frac{d^4y}{dx^4} = D(D-1)(D-2)(D-3)y$$

Ques : $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = 0$

Given : $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = 0$ — (i)

let,

$$x = e^z$$

$$\ln x = \ln e^z$$

$$\ln x = z$$

$D = \frac{d}{dz}$; $x\frac{dy}{dx} = Dy$; $x^2\frac{d^2y}{dx^2} = (D^2-D)y$

Substituting the value in eq(i)

$\cancel{(D^2-D)y - 3Dy + 4y = ze^{2z}}$

eq (i) ⇒ $(D^2-D)y - Dy - 3y = 0$

$$(D^2 - D - D - 3)y = 0$$

$$(D^2 - 2D - 3)y = 0 \qquad \left[\because D = \frac{d}{dz}\right]$$

$m^2 - 2m - 3 = 0 \rightarrow$ auxillary eq.

$m = 3, -1$

C.F

$C_1 e^{3z} + C_2 e^{-z}$

$= C_1 e^{3\ln x} + C_2 e^{-\ln x}$

$= C_1 e^{\ln x^3} + C_2 e^{\ln x^{-1}}$

$= C_1 x^3 + C_2 \, 1/x$

Ques: $(x^2 D^2 - 3xD + 4)y = 2x^2$

Sol

Let $x = e^z \Rightarrow x^2 = e^{2z}$

$\ln x = \ln e^z$

$\ln x = z$

$D = d/dz$; ....

Substituting the value in eq (2)

$(D^2 - D)y - 3Dy + 4y = 2e^{2z}$

C.F

$(D^2 - D - 3D + 4)y = 0$

$(D^2 - 4D + 4)y = 0$

$m^2 - 4m + 4 = 0$

eq.

$m = 2, 2$

C.F

$c_1 e^{2z} + z c_2 e^{2z}$

$y_c = c_1 e^{2\ln x} + \ln x \, c_2 e^{2\ln x}$

$= c_1 e^{\ln x^2} + \ln x \, c_2 e^{\ln x^2}$

$y_c = c_1 x^2 + \ln x \, c_2 x^2$

Particular Integral

$$y_p = \frac{1}{f(D)} = 2e^{2z}$$

$$= \frac{1}{D^2 - 4D + 4} 2e^{2z}$$

$$= \frac{1}{(2)^2 - 4(2) + 4} 2 \cdot e^{2z} = \frac{1}{0} \text{ (case fail)}$$

$$= 2z \cdot \frac{1}{2D - 4} e^{2z}$$

$$= 2z \frac{1}{2(2) - 4} e^{2z} = \frac{1}{0} \text{ (case fail)}$$

$$= 2z^2 \cdot \frac{1}{2} e^{2z}$$

$y_p = z^2 \cdot e^{2z} = (\ln x)^2 \cdot e^{2z}$

$y_p = (\ln x)^2 \cdot x^2$

Therefore,

$$y = C.F + CP$$

$$\boxed{y = c_1x^2 + \ln c_2x^2 + x^2(\ln x)^2}$$

Partial Differential Equation :-

They are those equation which contain partial Differential co-efficient, Independent variable & Dependent variable. The independent variable denoted by x & y, The dependent variable denoted by z.

$\frac{\partial z}{\partial x} = p$; $\frac{\partial z}{\partial y} = q$; $\frac{\partial z^2}{\partial x^2} = r$; $\frac{\partial z^2}{\partial x \partial y} = s$

$\frac{\partial^2 z}{\partial y^2} = t$

case 1 :
If arbitrary const. $\leq$ I.V
p and q order will be 1

case 2 :
If arbitrary const > I.V
p, q, r, s, t

Formation of PDE :-
Two primary method
1. By eleminating arbitrary constant
2. By eleminating arbitrary function

Ques: form the PDE by eleminating the arbitrary contant.

$$z = (x-a)^2 + (y-b)^2$$

Sol

Given → $z = (x-a)^2 + (y-b)^2$ — (i)

Differentiate eq (i) w.r.t x & y respectively

$\frac{\partial z}{\partial x} = 2(x-a) + 0 \Rightarrow p = 2(x-a)$ — (2)

$\frac{\partial z}{\partial y} = 2(y-b) \Rightarrow q = 2(y-b)$ — (3)

squaring eq (2) & (3) & adding them.

eq(2) $\Rightarrow$ $p^2 = 4(x-a)^2$

eq(3) $\Rightarrow$ $q^2 = 4(y-b)^2$

$\Rightarrow \frac{p^2+q^2}{4} = (x-a)^2 + (y-b)^2$

By using eq (1)

$\frac{p^2+q^2}{4} = z$

Ques: Find the PDE of the family of spheres having their centres on the line $x = y = z$

Sol

General eq. of sphere is

$(x-a)^2 + (y-b)^2 + (z-c)^2 = k^2$ — (1)

the centre is (a, b, c) & radius k

Differentiate eq (1) partially w.r.t x

— (2)

(3)

Term.

mily of
on the

—(1)

k

$$\underset{\text{indep.}}{\frac{\partial}{\partial x}(x-a)^2} + \underset{\text{indep.}}{\frac{\partial}{\partial x}(y-b)^2} + \underset{\text{dep}}{\frac{\partial}{\partial x}(z-c)^2} = \frac{\partial}{\partial x}(0)$$

$$\frac{\partial}{\partial x}(x-a)^2 + 0 + \frac{\partial}{\partial x}(z-c)^2 = 0$$

$$2(x-a) + 2(z-c)\frac{\partial z}{\partial x} = 0$$

[$\because a=b=c$]

$$2(x-a) + 2(z-a)p = 0$$

$$\boxed{a = \frac{x+zp}{1+p}} \text{ — (i)}$$

Differentiate eq (1) partially w.r.t 'y'

$$0 + 2(y-b) + 2(z-c)\cdot\frac{\partial z}{\partial y} = 0$$

$$2(y-b) + 2(z-c)\frac{\partial z}{\partial y} = 0$$

$a=b=c$

$$2(y-a) + 2(z-a)q = 0$$

$$\boxed{a = \frac{y+qz}{1+q}} \text{ — (ii)}$$

compare (i) & (ii)

$$\frac{x+zp}{1+p} = \frac{y+qz}{1+q}$$

$$(x+zp)(1+q) = (y+qz)(1+p)$$

$$x + qx + pz + \cancel{pzq} = y + py + qz + \cancel{pqz}$$

$$x + qx + pz - y \mp py \mp qz = 0$$

$$\boxed{x - y = (z-x)q + (y-z)p}$$

Ans: $\frac{r}{t} = \frac{1+p^2}{1+q^2}$

Ques: Find the PDE of all spheres where radii are the same

Given that : $(x-a)^2 + (y-b)^2 + (z-c)^2 = k^2$ — (i)

Here,
centre is (a, b, c) & radius is k

Differentiating eq (ii) w.r.t 'x'

$$2(x-a) + 0 + 2(z-c) \cdot \frac{\partial z}{\partial x} = 0$$

$$2x - 2a + 2z \frac{\partial z}{\partial x} - 2c \frac{\partial z}{\partial x} = 0$$

Again diff w.r.t 'x'

$$2 + 2\left(z \cdot \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}\right) - 2c \frac{\partial^2 z}{\partial x^2} = 0$$

$$1 + (z \cdot r + p \cdot p) - cr = 0$$

$$1 + zr + p^2 = cr$$

$$\frac{1 + zr + p^2}{r} = c$$

Differentiating eq (ii) w.r.t 'y'

$$0 + 2(y-b) + 2(z-c) \cdot \partial z / \partial y = 0$$

$$2y - 2b + 2z\, \partial z / \partial y - 2c\, \partial z / \partial y = 0$$

Again diff w.r.t 'y'

spheres where

$(z-c)^2 = K^2$ — (i)

is k

0

0

$c\frac{\partial^2 z}{\partial x^2} = 0$

)

$= 0$

$$2 + 2\left(z \cdot \partial^2 z/\partial y^2 + \partial z/\partial y \cdot \partial z/\partial y\right) - 2c\frac{\partial^2 z}{\partial y^2} = 0$$

$$1 + (z \cdot t + q \cdot q) - ct = 0$$

$$1 + zt + q^2 = ct \Rightarrow 1 + zt + q^2/t = c$$

Comparing both values of c,

$$\frac{1 + zr + p^2}{r} = \frac{1 + zt + q^2}{t}$$

$$t + \cancel{zrt} + p^2 t = r + \cancel{zrt} + q^2 r$$

$$t(1 + p^2) = r(1 + q^2)$$

$$\boxed{\frac{r}{t} = \frac{1 + p^2}{1 + q^2}}$$

Lagrange Method :-

Solution of 1st order PDE is known as Lagrange Method equation is defined as:

$$Pp + Qq = R \quad \text{—— (1)}$$

Such That :

$p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$ and P, Q & R are always the function of x, y & z (P, Q, R) can also be constants. If the given eq. is in it standard form then we can write lagrange auxillary eq. as:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

On the solution of auxillary eq. we will have.

$$U(x, y, z) = C_1 \text{ and } V(x, y, z) = C_2 \quad \text{—— (2)}$$

Finally the solution of the eq. (1) will be represented as:

$$\phi(U, V) = 0 \longrightarrow \text{Solution.}$$

$D^2 - 3D + 3 = 0$

$m^2 - 3m + 3 = 0$

$$\frac{dx}{P} = \frac{dy}{Q}$$

$$\int x dx = \int y dy$$

$$x^2 = y^2$$

nown as
red as:

& R are alwa

an also be
standard form
xillary eq: as:

will have.

c_2 —— 2

l be represented

Q: $\underbrace{\frac{y^2 z}{x}}_{P}\, p + \underbrace{xz}_{Q}\,q = \underbrace{y^2}_{R}$

Sol:

$P = \frac{y^2 z}{x}$, $Q = xz$, $R = y^2$

Auxillary eq:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y^2 z/x} = \frac{dy}{xz} = \frac{dz}{y^2}$$

(1) $\Rightarrow \frac{x\,dx}{y^2 z} = \frac{dy}{xz}$

$x^2\,dx = y^2\,dy$

Integrating B.S

$\int x^2\,dx = \int y^2\,dy$

$\frac{x^3}{3} = \frac{y^3}{3} + C_1$

$\frac{x^3}{3} - \frac{y^3}{3} = C_1$

(2) $\Rightarrow \frac{x\,dx}{y^2 z} = \frac{dz}{y^2}$

$x\,dx = z\,dz$

Integrating B.S

$\int x\,dx = \int z\,dz$

$\frac{x^2}{2} = \frac{z^2}{2} + C_2$

$\frac{x^2}{2} - \frac{z^2}{2} = C_2$

(3) $\Rightarrow \frac{dy}{xz} = \frac{dz}{z^2}$

$\times$

Hence,

$$\phi\left(\frac{x^3}{3} - \frac{y^3}{3},\ \frac{x^2}{2} - \frac{z^2}{2}\right) = 0$$

Q: $p \tan x \cdot q \tan y = \tan z$

Sol

$P = \tan x \quad Q \tan y = R = \tan z$

Auxillary eq:

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

(1) $\Rightarrow \dfrac{dx}{\tan x} = \dfrac{dy}{\tan y}$

I.B.S

$\int \cot x \, dx = \int \cot y \, dy$

$\ln \sin x = \ln \sin y + \ln C_1$

$\ln \sin x - \ln \sin y = \ln C_1$

$\ln\left(\dfrac{\sin x}{\sin y}\right) = \ln C_1$

$C_1 = \sin x / \sin y$

(2) $\Rightarrow \dfrac{dx}{\tan x} = \dfrac{dz}{\tan z}$

I.B.S

$\int \cot x \, dx = \int \cot z$

$\ln \sin x = \ln \sin z + \ln C_2$

$\ln\left(\dfrac{\sin x}{\sin z}\right) = \ln C_2$

$C_2 = \dfrac{\sin x}{\sin z}$

Hence,

$$\phi\left(\frac{\sin x}{\sin y}, \frac{\sin x}{\sin y}\right) = 0$$

Q $= yP + y$

Sol $P = y$

Auxillary eq

$\dfrac{dx}{y} =$

(1) $\Rightarrow \dfrac{dx}{y} =$

$dx =$

I.B.S

$\int dx =$

$x = y +$

$x - y =$

Hence,

Q: $yp + yq = z^2 + 1$

Sol

$P = y \quad , \quad Q = y \quad , \quad R = z^2 + 1$

Auxiliary eq:

$$\frac{dx}{y} = \frac{dy}{y} = \frac{dz}{z^2+1}$$

① $\Rightarrow \dfrac{dx}{y} = \dfrac{dy}{y}$

$dx = dy$

I.B.S

$\int dx = \int dy$

$x = y + c_1$

$x - y = c_1$

② $\dfrac{dy}{y} = \dfrac{dz}{z^2+1}$

I.B.S

$\int \frac{1}{y}\, dy = \int \frac{1}{z^2+1}\, dz$

$\ln y = \tan^{-1} z + c_2$

$\ln y - \tan^{-1} z = c_2$

③ $\dfrac{dx}{y} = \dfrac{dz}{z^2+1}$ ✗

Hence,

$$\phi\left(x - y,\ \ln y - \tan^{-1} z\right) = 0$$

Methods of Multipliers :-

If we have an auxillary eq. such as :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{—— (1)}$$

where P, Q & R are functions of x, y & z by ~~suit~~ choosing suitable multiplies for eq (1), we can rewrite eq (1) by the help of well known principle of algebra that say :

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R}$$

If $P_1 P + Q_1 Q + R_1 R = 0$, then denominator of eq (2) will be zero. that will give as a eqn :

$$P_1 dx + Q_1 dy + R_1 dz = 0$$

Ques: $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

Sol

$P = x(y^2 - z^2)$

$Q = y(z^2 - x^2)$

$R = z(x^2 - y^2)$

as:

x, y & z

eq (1), we

f well

ay:

ator of eq (2)

a eqn:

$y^2)$

Auxillary eq:

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$$

considering (x, y, z) as multiplier

$$\frac{x\,dx + y\,dy + z\,dx}{x^2y^2 - x^2z^2 + y^2z^2 - y^2x^2 + x^2z^2 - y^2z^2}$$

$$x\,dx + y\,dy + z\,dx = 0$$

Integrate B.S

$$\int x\,dx + \int y\,dy + \int z\,dx = \int 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_1$$

Again considering $(1/x, 1/y, 1/z)$ as a multiplier.

$$\frac{1/x \cdot dx + 1/y\,dy + 1/z\,dz}{1/z \cdot x(y^2-z^2) + 1/y\,|y(z^2-x^2) + 1/z \cdot z(x^2-y^2)}$$

$$\frac{1/x\,dx + 1/y\,dy + 1/z\,dz}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2}$$

$$1/x\,dx + 1/y\,dy + 1/z\,dz = 0$$

Integrate B.S

$$\int 1/x\,dx + \int 1/y\,dy + \int 1/z\,dz = \int$$

$$\ln x + \ln y + \ln z = \ln C_2$$

$\ln(x.y.z) = \ln c_2$

$xyz = c_2$

Hence,

$-\phi(x^2/2, y^2/2, z^2/2, xyz) = 0$

Ques: $z(x+y)p + z(x-y)q = x^2+y^2$

Sol

$P = z(x+y)$

$Q = z(x-y)$

$R = x^2+y^2$

Auxillary eq:

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}$$

considering $(-x, y, z)$ as multiplier.

$$\frac{-xdx + ydy + zdz}{-x(z(x+y)) + y.z(x-y) + z(x^2+y^2)}$$

$$\frac{-xdx + ydy + zdz}{-x^2z - xyz + xyz - y^2z + x^2z + y^2z}$$

$$-xdx + ydy + zdz = 0$$

Integrate B.S

$$\int -x\,dx + \int y\,dy + \int z\,dz = 0$$

$$-\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_1$$

Again considering $(y, x, -z)$ as multiplier.

$$\frac{y\,dx + x\,dy - z\,dz}{y(xz+yz) + x(xz-yz) - z(x^2+y^2)}$$

$$\frac{y\,dx + x\,dy - z\,dz}{xyz + y^2z + x^2z - xyz - x^2z - y^2z}$$

$$y\,dx + x\,dy - z\,dz = 0$$

Integrate B.S

$$\int y\,dx + \int x\,dy - \int z\,dz = \int 0$$

$$\frac{y^2}{2} + \frac{x^2}{2} - \frac{z^2}{2} = C_2$$

Hence :

$$\phi\left(-x^2/2 + y^2/2 + z^2/2,\ y^2/2 + x^2/2 - z^2/2\right) = 0$$

$$\int -x\,dx + \int y\,dy + \int z\,dz = 0$$

$$-\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_1$$

Again considering $(y, x, -z)$ as multiplier

$$\frac{y\,dx + x\,dy - z\,dz}{y(xz+yz) + x(xz-yz) - z(x^2+y^2)}$$

$$\frac{y\,dx + x\,dy - z\,dz}{xyz + y^2z + x^2z - xyz - x^2z - y^2z}$$

$$y\,dx + x\,dy - z\,dz = 0$$

Integrate B.S

$$\int y\,dx + \int x\,dy - \int z\,dz = \int 0$$

$$\frac{y^2}{2} + \frac{x^2}{2} - \frac{z^2}{2} = C_2$$

Hence:

$$\phi\left(-x^2/2,\ y^2/2 + z^2/2,\ y^2/2 + x^2/2 - z^2/2\right) = 0$$

Linear Homogenous PDE :-

An equation of the form :

$$\underbrace{a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \dots + a_n \frac{\partial^n z}{\partial y^n} = f(x,y)}$$

Linear Homogenous PDE of n order with const. coefficient

1. It is called Homogenous bcz all terms contains derivative of same order

$$\left.\begin{array}{l} \dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = \\[2ex] \dfrac{\partial^3 z}{\partial x^3} + \dfrac{\partial^3 z}{\partial x^2 \partial y} + \dfrac{\partial^3 z}{\partial y^3} = \end{array}\right\} \text{High order Homogenous PDE}$$

2. It is called linear b/c max. power (degree) of each derivative is 1

completer solution = complementary function + Particular Integral

PDE :-

$+ a_n \frac{\partial^n z}{\partial y^n} = f(x,y)$

-rder with const. coeff

all terms containing

High order Homogenous PDE

power (degree) of

-tary + Particular Integral

Rules for finding complementary function :

$$a_0 \frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = f(x,y)$$

Put $D = \partial/\partial x$; $D' = \partial/\partial y$

$$(a_0 D^2 + a_1 D D' + a_2 D'^2) z = 0$$

put $D = m$; $D' = 1$

$$a_0 m^2 + a_1 m + a_2 = 0$$

This is an auxillary equation

CASE #01 : Roots are real & distinct

i.e m_1, m_2, m_3 are distinct

$$c.f = f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x)$$

e.g $m = 1, 2, 3$

$$C.F = f_1(y + x) + f_2(y + 2x) + f_3(y + 3x)$$

CASE #02 : Roots are repeated i.e m_1, m_1, m_2

$$c.f = f_1(y + m_1 x) + x f_2(y + m_1 x) + f_3(y + m_2 x)$$

e.g $-2, -2, 0$

$$c.f = f_1(y - 2x) + x f_2(y - 2x) + f_3(y)$$

CASE #03 Roots are complex / imginary.

i.e $m = \alpha + i\beta$

$$C.F = f_1(y + (\alpha + i\beta)x) + f_2(y + (\alpha - i\beta)x)$$

e.g $1 \pm i$

$$C.F = f_1(y + (1+i)x) + f_2(y + (1-i)x)$$

Rules for Finding Particular Integral :

1. $f(x,y) = e^{ax+by}$

$$P.I = \frac{1}{f(D,D')} e^{ax+by}$$

$$P.I = \frac{1}{f(a,b)} e^{ax+by}$$

if $f(a,b) = 0$

$$P.I = x \cdot \frac{1}{f'(a,b)} e^{ax+by} \quad (\text{w.r.t } x)$$

OR

$$y \cdot \frac{1}{f'(a,b)} e^{ax+by} \quad (\text{w.r.t } y)$$

2. $f(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

$$P.I = \frac{1}{f(D,D')} \sin(ax+by)$$

OR $\cos(ax+by)$

Substitute

Ques :

put

Ques :

put

CASE #03 Roots are complex / imaginary.

i.e $m = \alpha + i\beta$

$$C.F = f_1(y + (\alpha + i\beta)x) + f_2(y + (\alpha - i\beta)x)$$

e.g $1 \pm i$

$$C.F = f_1(y + (1+i)x) + f_2(y + (1-i)x)$$

Rules for Finding Particular Integral:

1. $f(x,y) = e^{ax+by}$

$$P.I = \frac{1}{f(D,D')} e^{ax+by}$$

$$P.I = \frac{1}{f(a,b)} e^{ax+by}$$

if $f(a,b) = 0$

$$P.I = x \cdot \frac{1}{f'(a,b)} e^{ax+by} \quad (\text{w.r.t } x)$$

OR

$$y \cdot \frac{1}{f'(a,b)} e^{ax+by} \quad (\text{w.r.t } y)$$

2. $f(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

$$P.I = \frac{1}{f(D,D')} \sin(ax+by)$$

OR

$$P.I = \frac{1}{f(D,D')} \cos(ax+by)$$

substitute :

$$D^2 = -a^2$$
$$D'^2 = -b^2$$
$$D \cdot D' = -ab$$

Ques : $r - s - 2t' = 0$

$$= \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 x}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = 0$$

$$D^2 - DD' - 2D'^2 = 0$$

put $D = m$; $D' = 1$

$m^2 - m - 2 = 0$; $m = 2, -1$

$z = f_1(y + 2x) + f_2(y - x)$ Ans

Ques : $r - 3as + 2a^2 t = 0$

$$D^2 - 3aDD' + 2a^2 D'^2 = 0$$

put $D = m$; $D' = 1$

$$m^2 - 3am + 2a^2 = 0$$
$$m^2 - 2am + am + 2a^2 = 0$$
$$m(m - 2a) - a(m - 2a) = 0$$
$$(m - 2a)(m - a) = 0$$
$$m = a, 2a$$

$$z = f_1(y + ax) + f_2(y + ax)$$

...ry.

$(\alpha - i\beta)x)$

$1 - i)x)$

gral :

)

oy)

substitute :

$D^2 = -a^2$

$D'^2 = -b^2$

$D \cdot D' = -ab$

Ques : $r - s - 2t' = 0$

$$= \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 x}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = 0$$

$$D^2 - DD' - 2D'^2 = 0$$

put $D = m$; $D' = 1$

$m^2 - m - 2 = 0$; $m = 2, -1$

$z = f_1(y + 2x) + f_2(y - x)$ Ans

Ques : $r - 3as + 2a^2 t = 0$

$$D^2 - 3aDD' + 2a^2 D'^2 = 0$$

put $D = m$; $D' = 1$

$m^2 - 3am + 2a^2 = 0$

$m^2 - 2am + am + 2a^2 = 0$

$m(m - 2a) - a(m - 2a) = 0$

$(m - 2a)(m - a) = 0$

$m = a, 2a$

$z = f_1(y + ax) + f_2(y + ax)$

metric	mmc	MNMC	Better
(a) MIPS	453.125	1450	MNMC,

Ques : $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = e^{x+2y}$

Sol

<u>For C.F :</u>

$D^2 - D'^2 = 0$

Put $D = m$; $D' = 1$

$m^2 - 1 = 0$

$m = \pm 1$

$C.F = f_1(y+x) + f_2(y-x)$

<u>For P.I :</u>

$P.I = \frac{1}{f(D, D')} e^{ax+by}$

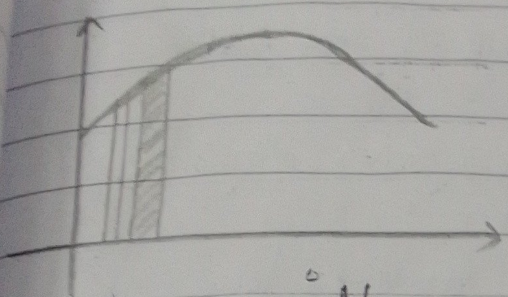
$= \frac{1}{D^2 - D'^2} e^{\cancel{ax+by}\ x+2y}$

$= \frac{1}{(1)^2 - (2)^2} e^{x+2y}$

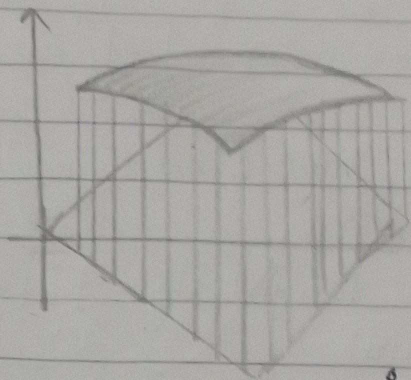
$= \frac{1}{-3} e^{x+2y}$

$z = \{ f_1(y+x) + f_2(y-x) + \left(\frac{-1}{3} e^{x+2y} \right)$

Double & Tripple Integral :-



Single variable



Multi variable

Double & Tripple integrals are extension of single variable integral to higher dimension. They are externals tools for dealing with func. of multi varable calculating areas, volumes, masses

* $\iint \rightarrow$ for areas in 2D and volume under surface

* $\iiint \rightarrow$ volume in 3D & other quantity is fixed

Gemoterically, it represents volume of the solid object above region and below surface.

* Double integral over Rectangular Region :-

$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dA$

* Double integral over non-Rectangular Region :

$$\int_{y=c}^{d} \int_{x=h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_{y=c}^{d} \left[\int_{x=h_1(y)}^{h_2(y)} f(x,y)\,dx \right] dy$$

$$\int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_{x=a}^{b} \left[\int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx$$

The Double Integral to Solve An Area :

consider a solid consisting of a point b/w the plane $z = 1$ & the region in x-y plane

Volume $= \iint 1\,dA = \iint dA$ —— (A)

volume = Base Area × height

volume = Base Area × 1

volume = Base Area —— (B)

comparing (A) & (B)

Region :-

$\iint_a^{b} f(x,y)\, dA$

lar Region :

dx] dy

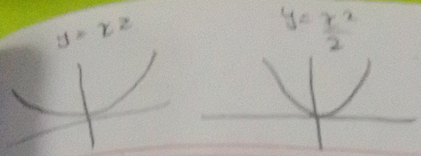
y) dy] dx

An Area :

point b/w

plane

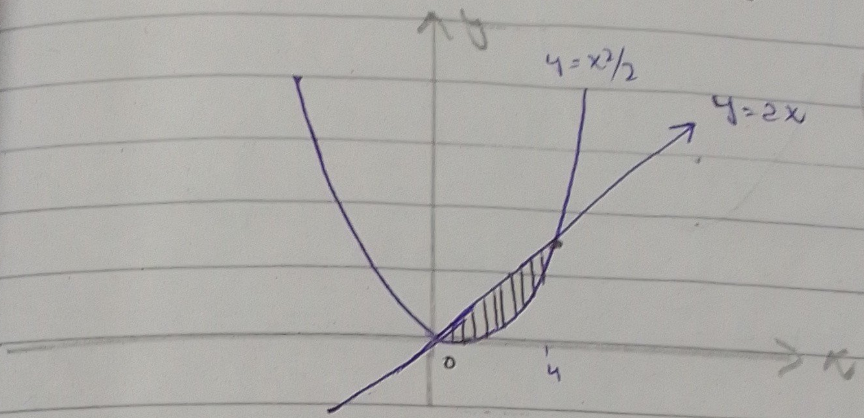
)



$y = mx + b$
↓ slope = $\frac{rise}{run}$
↳ y-intercept

Area = $\iint dA$

Ques: Use double integral to find the area of region R enclosed b/w the parabola $y = x^2/2$ and the line $y = 2x$



$y = 2x + 0$
↓
$m = \frac{2 \text{ (rise)}}{1 \text{ (run)}}$

$y = x^2/2$, $y = 2x$

comparing,

$x^2/2 = 2x$

$x = 4$

$x = 0 \to 4$

$$\int_{x=0}^{4}\int_{y=x^2/2}^{2x} 1 \, dy\, dx$$

$$\int_{x=0}^{4} y \Big|_{x^2/2}^{2x} dx$$

$$\frac{2x^2}{2} - \frac{x^3}{6}\Big|_0^4 =$$

$$= \frac{16}{3} \text{ units}$$

$\int^4$

Region :-

$\int_a^b f(x,y)\, dA$

ar Region :

$dx \,]\, dy$

$y)\, dy \,]\, dx$

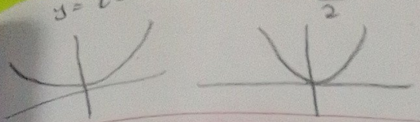
An Area:

point b/w

plane

)

$y = x^2$ $\quad$ $y = \frac{x^2}{2}$

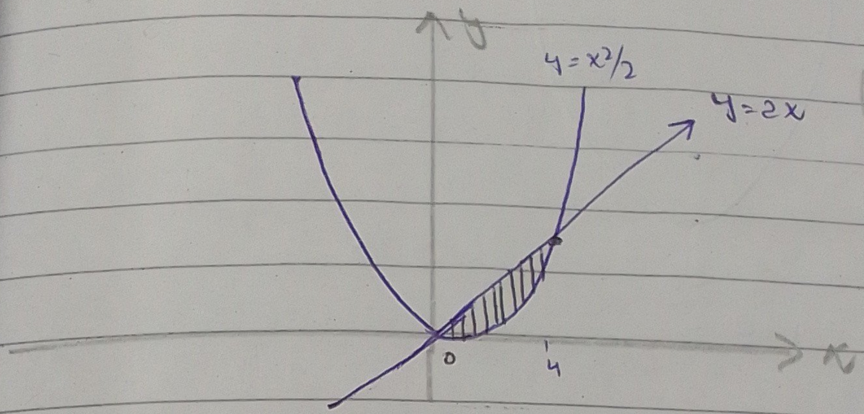


$y = mx + b$

slope $= \frac{\text{rise}}{\text{run}}$ $\quad$ ↳ y-intercept

Area $= \iint dA$

Ques: Use double integral to find the area of region R enclosed b/w the parabola $y = x^2/2$ and the line $y = 2x$



$y = 2x + 0$

$m = \frac{2\ (\text{rise})}{1\ (\text{run})}$

$y = x^2/2 \quad , \quad y = 2x$

comparing,

$x^2/2 = 2x$

$x = 4$

$x = 0 \to 4$

$$\int_{x=0}^{4}\int_{y=x^2/2}^{2x} 1 \, dy\, dx$$

$$\int_{x=0}^{4} y \Big|_{x^2/2}^{2x} dx$$

$$\int_{x=0}^{4} (2x - x^2/2)\, dx$$

$$\frac{2x^2}{2} - \frac{x^3}{6}\Big|_0^4 =$$

$$= \frac{16}{3} \text{ units}$$

Region :-

$\int_a^b f(x,y)\, dA$

ar Region :

dx] dy

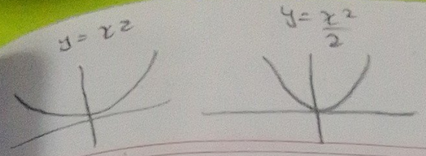
) dy] dx

An Area:

point b/w

plane

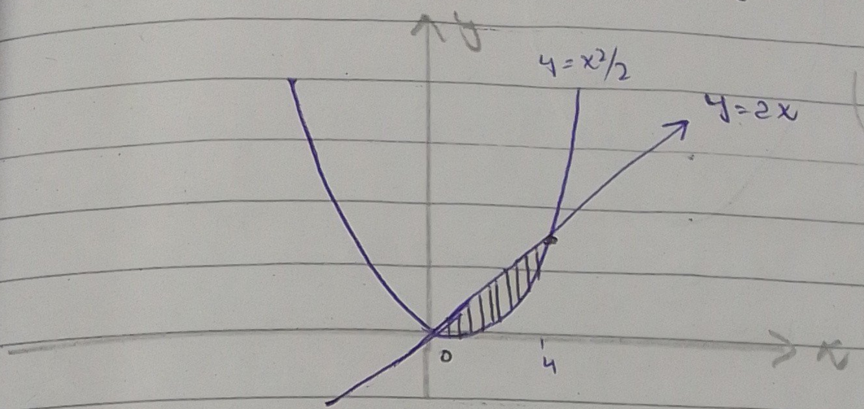
)



$y = mx + b$
↓ slope $= \frac{\text{rise}}{\text{run}}$
↳ y-intercept

Area $= \iint dA$

Ques: Use double integral to find the area of region R enclosed b/w the parabola $y = x^2/2$ and the line $y = 2x$



($y = 2x + 0$
↓
$m = \frac{2 \text{ (rise)}}{1 \text{ (run)}}$)

$y = x^2/2$, $y = 2x$

comparing,

$x^2/2 = 2x$

$x = 4$

$x = 0 \to 4$

$$\int_{x=0}^{4} \int_{y=x^2/2}^{2x} 1 \, dy\, dx$$

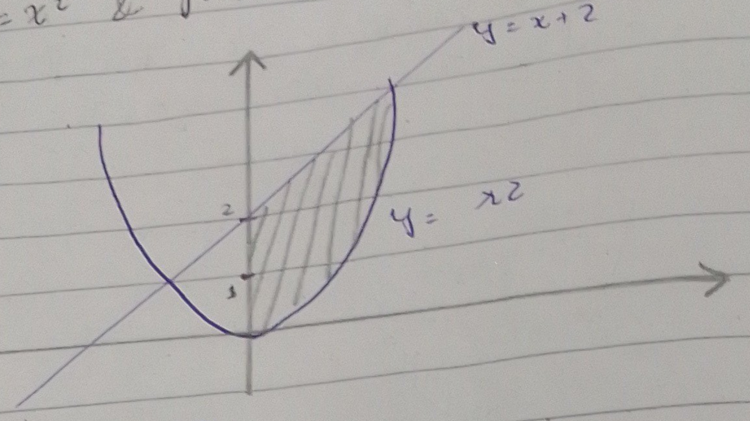
$$\int_{x=0}^{4} y \Big|_{x^2/2}^{2x} dx$$

$$\int_{x=0}^{4} (2x - x^2/2)\, dx$$

$$\frac{2x^2}{2} - \frac{x^3}{6} \Big|_0^4 =$$

$= \frac{16}{3}$ units

Ques: Find the area of the region R enclosed by $y = x^2$ & $y = x+2$ in 1st Quadrant.



$y = x^2$, $y = x+2$

comparing

$x^2 = x+2$

$x^2 - x - 2 = 0$

$x = -1, 2$

$x = 0 \rightarrow 2$

$$\Rightarrow \int_{x=0}^{2} \int_{y=x^2}^{x+2} 1 \, dy\,dx$$

If z (func. of x & y) not given then use 1 otherwise func will be use

$$\Rightarrow \int_{x=0}^{2} y \Big|_{x^2}^{x+2} dx$$

$$\Rightarrow \int_{x=0}^{2} (x+2-x^2)\, dx$$

$$\Rightarrow \frac{x^2}{2} + 2x - \frac{x^3}{3} \Big|_0^2 = \frac{10}{3} \text{ units}^2$$

Ques: Evalu
region

R enclosed
drant.

) not
otherwis
se

Ques: Evaluate $\iint 6xy\, dA$ where R is bounded by region $y=0$; $x=2$ & $y=x^2$

$$\int_{x=0}^{2}\int_{y=0}^{x^2} 6xy\, dy\, dx$$

$$\int_{x=0}^{2} \frac{6xy^2}{2}\Big|_0^{x^2} dx = \int_{x=0}^{2} 3xy^2\Big|_0^{x^2} dx$$

$$\int_{x=0}^{2} 3x x^4\, dx = \int_{x=0}^{2} 3x^5\, dx$$

$$\frac{3x^6}{\cancel{6}\,2}\Big|_0^2 = \frac{1}{2}x^6\Big|_0^2$$

$$= \frac{1}{2}(2)^6 - \frac{1}{2}(2)^6$$

$= 32$ units^2

Moment Of Inertia :-

$I_x = \iint_R y^2 \, \delta(x,y) \, dA$ about x-axis

$I_y = \iint_R x^2 \, \delta(x,y) \, dA$ about y-axis

$I_0 = I_x + I_y$

↳ resultant inertia

where δ (mass density) = mass / Area .

It represents the resistance of a body to rotational motion arround on axis. It is also called as rotational inertia, polar moment of inertia and angular mass. The above eq Inertia of rotating body is given as The above equation.

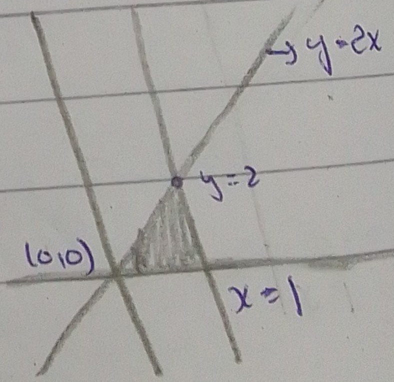
Ques: A thin plate ~~of~~ covers the triangular region bounded by x-axis, the line $x = 1$ & the line $y = 2x$ in first quadrant. The plate density at every point $6x + 6y + 6$. Find the moment of inertia of co-ordinate axes.

Given

$x = 1$

$y = 2x$

$6x + 6y + 6$ (plate density)



$x = 0 \rightarrow 1$, $y = 0 \rightarrow 2x$

$$I_x = \int_{x=0}^{1}\int_{y=0}^{2x} y^2(6x+6y+6)\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{2x} (6xy^2 + 6y^3 + 6y^2)\,dy\,dx$$

$$= \int_{x=0}^{1} \frac{6xy^3}{3} + \frac{6y^4}{4} + \frac{6y^3}{3}\Bigg|_{y=0}^{2x} dx$$

$$= \int_{x=0}^{1} 2x(2x)^3 + \frac{3}{2}(2x)^4 + 2(2x)^3\,dx$$

$$= \int_{x=0}^{1} 16x^4 + 24x^4 + 16x^3\,dx$$

$$= \frac{16x^5}{5} + \frac{24x^5}{5} + \frac{16x^4}{4}\Bigg|_{x=0}^{1}$$

$$= \frac{16(1)^5}{5} + \frac{24(1)^5}{5} + 4(1)$$

$$\boxed{I_x = 12}$$

$$I_y = \int_{x=0}^{1}\int_{y=0}^{2x} x^2(6x+6y+6)\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{2x} 6x^3 + 6x^2y + 6x^2\,dy\,dx$$

$$= \int_{x=0}^{1} 6x^3y + \frac{6x^2y^2}{2} + 6x^2y\Bigg|_{y=0}^{2x} dx$$

$$= \int_{x=0}^{1} 6x^3(2x) + 3x^2(2x)^2 + 6x^2(2x)\,dx$$

$$= \int_{x=0}^{1} 12x^4 + 12x^4 + 12x^3 \, dx$$

$$= \frac{12x^5}{5} + \frac{12x^5}{5} + \frac{12x^4}{4} \Big|_{x=0}^{1}$$

$$= \frac{12}{5} + \frac{12}{5} + 3$$

$$\boxed{I_y = 39/5}$$

Ques :- Find moment of inertial of a thin triangular plate bounded by y-axis with with the lines $y = x$ & $y = 2 - x$. If mass density function is $6x + 3y + 3$

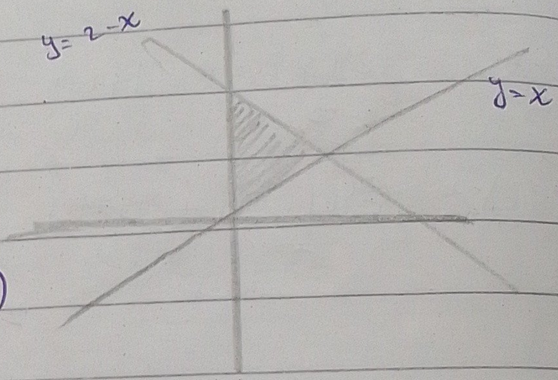
Sol

<u>Given</u>

$y = x$

$y = 2 - x$

$6x + 3y + 3$ (mass density func)



By comparing

$x = 2 - x \Rightarrow 2x = 2 \Rightarrow \boxed{x = 1}$

$x = 0 \rightarrow 1 \quad , \quad y = x \rightarrow 2 - x$

For Ix :

$$I_x = \int_{x=0}^{1} \int_{y=x}^{2-x} y^2(6x + 3y + 3) \, dy \, dx$$

$$= \int_0^1 \int_{y=x}^{2-x} 6xy^2 + 3y^3 + 3y^2 \Big|_{y=x}^{2-x} dy \, dx$$

$$= \int_{x=0}^{1} \left(\frac{6xy^3}{3} + \frac{3y^4}{4} + \frac{3y^3}{3} \right) \Big|_{y=x}^{2-x} dx \Rightarrow \int_{x=0}^{1} 2x(2-x-x)^3 + \frac{3}{4}(2-x-x)^4 + (2-x-x)^3 dx$$

$$= \int_{x=0}^{1} 16x + 12 + 8 \, dx \Rightarrow \int_{x=0}^{1} 16x + 20 \, dx \Rightarrow \frac{16x^2}{2} + 20x \Big|_0^1 \Rightarrow \boxed{I_x = 28}$$

$$= \int_0^1 6x(2-x-x) + \frac{3(2-x-x)^2}{2} + 3(2-x-x) \, dx$$

$$= \int_0^1 12x + 6 + 6 \, dx \Rightarrow \int_0^1 12x + 12 \, dx$$

$$= \frac{12x^2}{2} + 12x \Big|_0^1$$

$$= 6 + 12 = 18 \Rightarrow \boxed{I_x = 18}$$

For I_y:

$$I_y = \int_{x=0}^{1} \int_{y=x}^{2-x} x^2 (6x + 3y + 3) \, dy \, dx$$

$$= \int_{x=0}^{1} \int_{y=x}^{2-x} 6x^2 + 3x^2 y + 3x^2 \, dy \, dx$$

$$= \int_{x=0}^{1} \left(6x^2 y + \frac{3x^2 y^2}{2} + 3x^2 y \right) \Big|_x^{2-x} dx$$

$$= \int_{x=0}^{1} 6x^2(2-x-x) + \frac{3}{2}(x^2)(2-x-x)^2 + 3x^2(2-x-x) \, dx$$

$$= \int_{x=0}^{1} (12x^2 + 6x^2 + 6x^2) \, dx$$

$$= \int_{x=0}^{1} 24x^2 \, dx \Rightarrow \frac{24x^3}{3} \Big|_0^1$$

$$= 8(1-0)^3 \Rightarrow \boxed{I_y = 8}$$

For I_0

$$I_0 = I_x + I_y$$

$$= 28 + 8$$

$$\boxed{I_0 = 36}$$

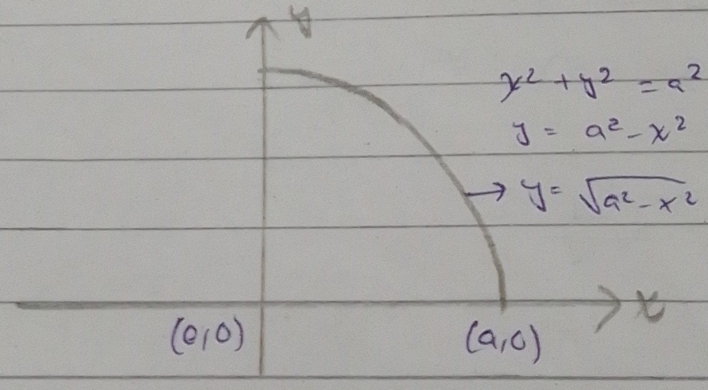
Centroid :-

Centroid means the geometric centre of an object.
Centroid is an special case of centre of gravity when mass density function is constant.
Double integrals can be use to calculate the centroid of simple 2D cube.

$$\bar{x} = \frac{\iint_R x\,dA}{\text{Area of region}}$$

$$\bar{y} = \frac{\iint_R y\,dA}{\text{Area of Region}}$$

Ques: Calculate centroid of region that lies in 1^{st} quadrant of xy plane bounded by circle.



Now,

$$\bar{x} = \frac{\iint_R x\,dA}{\iint_R dA}$$

of an object
of gravity
ont.
ulate the

1st
ircle.

$\bar{y} = \bar{x}$ (by symmetry)

Area of region $= \frac{1}{4}\pi a^2$ (Area of circle $= \pi r^2$)

for $\bar{x}$:

$\iint_R x\,dA = \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} x\,dy\,dx \Rightarrow$

$\Rightarrow \int_{x=0}^{a} xy \Big|_0^{\sqrt{a^2-x^2}} dx$

$\Rightarrow \int_0^a \frac{-2x\sqrt{a^2-x^2}}{2}\,d$

$\Rightarrow -\frac{1}{2}\frac{(a^2-x^2)^{1/2+1}}{1/2+1}\Big|_0^a$

$\Rightarrow -\frac{1}{\cancel{2}}\frac{(a^2-\cancel{a}x^2)^{3/2}}{3/\cancel{2}}\Big|_0^a$

$\Rightarrow -\frac{1}{3}(a^2-a^2)^{3/2} - \left(-\frac{1}{3}(a^2-0)\right)^{3/2}$

$\Rightarrow 0 + 1/3\ a^{2(3/2)}$

$\Rightarrow 1/3\,a^3$

$\bar{x} = \frac{\iint_R x\,dA}{\text{Area of region}}$

$= \frac{1/3\,a^{\cancel{3}}}{1/4\,\pi a^{\cancel{2}}}$

$$\bar{x} = \frac{4a}{3\pi}$$

also,

$$\bar{y} = \frac{4a}{3\pi}$$ centre

centroid $(\bar{x}, \bar{y}) = \left(4a/3\pi , 4a/3\pi\right)$

Vector Integration :-

let, $R(U) = R_1(u)\,\hat{i} + R_2(u)\,\hat{j} + R_3(u)\,\hat{k}$

be a vector suppose continuous in a specific inertia, then

$$\int R(U)\,du = \hat{i}\int R_1(U)\,du + \hat{j}\int R_2(U)\,du + \hat{k}\int R_3(U)\,du$$

If

$$R(U)\,du = \frac{dS(U)}{dU}\,dU, \text{ then}$$

$$\int R(U)\,du = S(U) + C$$

$$\int_a^b R(U)\,du = S(b) - S(a)$$

Ques. $R(U) =$ ...
Find ...
(a) $\int R(U)$ ...

(a) $\int R(U)\,du$ ...

$\int R(U)\,dU =$ ...

$\Rightarrow \hat{i}\left(\frac{U^2}{2} - \frac{U^3}{\ldots}\right.$ ...

(b) $\int_1^2 R(U)\,du$ ...

$= \hat{i}\left(\frac{2^2}{2}\right.$ ...

$= \left[\left(2 - \frac{8}{3}\right.\right.$ ...

$= \left[-\frac{2}{3}\hat{i}\right.$ ...

$= -\frac{5}{6}$ ...

Ques: $R(U) = (U - U^2)\hat{i} + 2U^3\hat{j} - 3\hat{k}$

find ~~$\int R\,dU$~~

(a) $\int R(U)\,dU$ (b) $\int_1^2 R(U)\,dU$

(a) $\int R(U)\,dU \Rightarrow$ Given that $R(U) = (U - U^2)\hat{i} + 2U^3\hat{j} - 3\hat{k}$

$$\int R(U)\,dU = \hat{i}\int (U - U^2)\,dU + \hat{j}\int 2U^3\,dU - \hat{k}\int 3\,dU$$

$$\Rightarrow i\left(\frac{U^2}{2} - \frac{U^3}{3}\right) + \frac{2U^4}{4}\hat{j} - 3U\hat{k}$$

(b) $$\int_1^2 R(U)\,dU = i\left(\frac{U^2}{2} - \frac{U^3}{3}\right) + \frac{2U^4}{4}\hat{j} - 3U\hat{k}\Big|_1^2$$

$$= \hat{i}\left(\frac{2^2}{2} - \frac{2^3}{3}\right) + \frac{2(2)^4}{4}\hat{j} - 3(2)\hat{k} - \frac{1^2}{2}\hat{i} - \frac{1^3}{3} + \frac{2(1)}{4}\hat{j} - 3(1)\hat{k}$$

$$= \left[\left(2 - \frac{8}{3}\right)\hat{i} + 8\hat{j} - 3\hat{k}\right] - \left[\left(\frac{1}{2} - \frac{1}{3}\right)\hat{i} + \frac{1}{2}\hat{j} - 3\hat{k}\right]$$

$$= \left[-\frac{2}{3}\hat{i} + 8\hat{j} - 6\hat{k}\right] - \left[\frac{1}{6}\hat{i} + \frac{1}{2}\hat{j} - 3\hat{k}\right]$$

$$= -\frac{5}{6}\hat{i} + \frac{15}{2}\hat{j} - 3\hat{k} + C$$

$\hat{k}$

specific

$(U)\,dU$

Ques: If acceleration $\vec{a}$ of a particle at any time 't' $t \geq 0$ is given by $\vec{a} = 12\cos 2t\,\hat{i} - 8\sin 2t\,\hat{j} + 16t\,\hat{k}$. If the velocity 'v' & displacement 'r' are zero at $t = 0$ then find the velocity and displacement at any time 't'

A/c to question,

$$\vec{a} = 12\cos 2t\,\hat{i} - 8\sin 2t\,\hat{j} + 16t\,\hat{k}$$

As we know that,

$$\vec{v} = \int \vec{a}\,dt$$

$$\vec{v} = \int 12\cos 2t\,\hat{i}\,dt + \int -8\sin 2t\,\hat{j} + \int 16t\,\hat{k}$$

$$= 12\hat{i}\int \cos 2t\,dt - 8\hat{j}\int \sin 2t\,dt + 16\hat{k}\int t\,dt$$

$$= \cancel{12}^{6}\,\hat{i}\,\frac{\sin 2t}{\cancel{2}} - 8\hat{j}\frac{(-\cos 2t)}{2} + 16\hat{k}\frac{t^2}{2}$$

$$\vec{v} = 6\sin 2t\,\hat{i} + 4\cos 2t\,\hat{j} + 8t^2\,\hat{k}$$

we also know that,

$$\vec{r} = \int \vec{v}\,dt$$

$$\vec{r} = \int 6\sin 2t\,\hat{i}\,dt + \int 4\cos 2t\,\hat{j}\,dt + \int 8t^2\,\hat{k}\,dt$$

$$= 6\hat{i}\int \sin 2t\,dt + 4\hat{j}\int \cos 2t\,dt + 8\hat{k}\int t^2\,dt$$

$$= 6\hat{i}\frac{(-\cos 2t)}{2} + 4\hat{j}\frac{\sin 2t}{2} + 8\hat{k}\frac{t^3}{3}$$

$$\vec{r} = -3\cos 2t\,\hat{i} + 2\sin 2t\,\hat{j} + \frac{8}{3}t^3\,\hat{k}$$