ST433 COMPUTATIONAL METHODS IN FINANCE AND INSURANCE

1. Project guidelines

Your project must be submitted to the Departmental Office COL 6.11 by

9 June 2014, 11:30am.

All consequences regarding late submission can be found on the School's website

`http://www.lse.ac.uk/resources/calendar/academicRegulations/regulationsForTaughtMastersDegrees.htm`

In particular "five marks out of 100 will be deducted for coursework submitted within 24 hours of the deadline and a further five marks will be deducted for each subsequent 24-hour period (working days only) until the coursework is submitted. After five working days, coursework will only be accepted with the permission of the Chair of the Sub-Board of Examiners".

- Please submit two spiral bound copies of your project report. In addition, please upload your report as a PDF-file and all other files containing your computer codes, figures, etc. via Moodle. If your candidate number is 123456 then name your PDF file CN123456report.pdf. You can check your candidate number on 'LSE for You'. Name the other electronic files similarly and make sure that *none* of the electronic files contains your name.
- Make sure that the title page of your printed report contains your candidate number but not your name. Make a copy of this title page, write your name on it and submit this separately along with the two bound copies. (This is for the administrators to keep track of your reports and will not be forwarded to the examiners.)
- You must also hand in a completed and signed copy of the Plagiarism Statement (available from the Departmental Office). You are required to read the information on plagiarism on the following website:

  `http://www.lse.ac.uk/resources/calendar/academicRegulations/RegulationsOnAssessmentOffences-Plagiarism.htm`

  Note in particular the first paragraph on this website:

  "All work for classes and seminars as well as scripts (which include, for example, essays, dissertations and any other work, including computer programs) must be the student's own work.

> Quotations must be placed properly within quotation marks or indented and must be cited fully. All paraphrased material must be acknowledged. Infringing this requirement, whether deliberately or not, or passing of the work of others as the work of the student, whether deliberately or not, is plagiarism. "
>
> Note that all reports will be submitted to Turnitin for textual similarity review and the detection of plagiarism.

Keep in mind that you are asked to write a report and not just a question-answer style exercise set solution. The report should be maximum 25 single sided A4 pages including plots and the bibliography but excluding the title page and the codes. The font size must be at least 11pt and the line spacing should be 1.5. The left hand margin should be 3.5cm and the right hand margin should be 2.0cm. Footnotes are not allowed. In addressing the problems, explain carefully in the main body of the text what you are doing to solve these tasks and any observations you make during the simulations. You are free and in fact welcome to make further studies in the context of each problem. This will have a positive influence on your final mark.

Your project will be marked by two members of staff, and then read by the external examiner. Marks are awarded for the overall structure and presentation of the project as well as the quality of research carried out. Great emphasis will be put on your ability to communicate your results and methods.

You are free to use any platform for your simulations or numerical studies. Collaboration for writing the code is *not* allowed. You will need to write the codes as well as the report yourself. It is in your own interest to contain comments in your codes.

The project consists of 4 problems. You must solve ALL problems.

Layout:

The elements described below are required parts of the thesis. Once again, please do not include your name on any part of the thesis.

Title page: This should include "Department of Statistics 2014", the title "Project in Computational Finance and Insurance", your candidate number, and the statement "Submitted for the Master of Science, London School of Economics".

Main sections: the body of your work, laid out in sections relating to the individual questions.

Bibliography: plagiarism is viewed as a serious offence and you should take great care to cite all sources used in your project.

Codes: Please include your codes in an appendix at the end of your thesis.

2. Project description

2.1. **Black-Scholes economy.** Recall that the stock price, S, under the risk neutral measure in a Black-Scholes model is given by

$$S_t = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right),$$

where $\sigma > 0$ is the stock volatility, r is the interest rate, and B is a standard Brownian motion.

Problem 1. (1) *Describe how you would simulate the price process for a given set of parameters σ, S_0 and r.*

(2) *Suppose $S_0 = 40$ and $r = 0.03$ and create a histogram for $\frac{1}{\sigma}\ln S_1$ for three different values of $\sigma \in (0,1)$. Interpret these histograms. (Make sure you have enough simulations to produce meaningful histograms.)*

(3) *Using the parameters from the previous part compute the price of a European call option with strike $K = 40$ and maturity $T = 1$ using a Monte Carlo simulation.*

(4) *Fix a σ and construct the following two plots of European call option prices: one for a sufficiently large number of call options with fixed maturity $T = 1$ but varying strike K, and other with fixed strike $K = 40$ and varying maturity $T \in (0,1)$. When plotting option prices against the strike make sure your graph includes in-the-money as well as out-of-the-money options. Interpret these plots.*

Problem 2. *Now, suppose that the volatility is stochastic. More precisely, the stock price and the volatility process are the solutions to the following system:*

$$\begin{aligned} dS_t &= S_t\left(rdt + \sqrt{V_t}dW_t\right) \\ dV_t &= \alpha(\beta - V(t)) + \gamma\sqrt{V_t}dB_t \end{aligned}$$

where B and W are two standard Brownian motions with correlation coefficient $\rho \in (-1,1)$, i.e. $d[W,B]_t = \rho dt$. The above stochastic volatility model is called the Heston model*. Suppose $S_0 = 40$ and $r = 0.03$. Let $V_0 = \sigma^2$, where σ is the value you fixed in Part 4 of Problem 1. Choose a reasonable set of values for alpha, $\beta, \rho(\neq 0)$ and γ yourself.*

(1) *Describe the standard first order Euler scheme for $\ln S$ and V. Note that since the correlation coefficient between W and B is ρ, you can write $W = \rho B + \sqrt{1-\rho^2}\beta$ where β is another standard Brownian motion independent of B, i.e. you can simulate W as a convex combination of independent Brownian motions.*

(2) *Modify your Euler scheme from above so that*

$$V_{t_{i+1}} = \max\left\{0, V_{t_i} + \alpha(\beta - V_{t_i})(t_{i+1} - t_i) + \gamma\sqrt{V_{t_i}}(B_{t_{i+1}} - B_{t_i})\right\}.$$

Describe the advantage(s) of this modification over the standard scheme.

(3) *Produce a graph for the prices of European call options with strike $K = 40$ and maturity $T \in (0, 1]$ in this model. Compare this plot with the analogous one from Part 4 of Problem 1.*

(4) *Let $K_n = 30 + n$ for $n = 0, 1, \dots 20$. Compute the European call option price with strike K_n and maturity $T = 1$ for each n.*

(5) *Let $\tilde{C}_n$ denote the option prices you found in the previous part for each n and suppose that $\tilde{C}_n$ is the option price data that you have for European call options maturing at time 1 with strikes K_n. Your goal now is to find the corresponding implied volatilities for the model considered in Problem 1. That is, find the value of σ_n that achieves $C(\sigma_n, 0.03, 40, K_n) = \tilde{C}_n$ for each n, where $C(a, b, c, d)$ is the Black-Scholes price for the call option on a stock with volatility a, interest rate b, current stock price c, and strike d. Plot the implied volatilities against the strike. What do you see?*

(6) *Consider the down-and-out European call option with maturity $T = 1$, strike $K = 40$, and down-and-out barrier 30. Using the implied volatility σ_{10} from the previous part perform a Monte-Carlo simulation to price this option in the Black-Scholes model of Problem 1. Repeat your simulation by choosing 3 other values for the barrier from the interval $[25, 40)$.*

(7) *You will now compute prices of this barrier option by solving the corresponding PDE. Let $u(t, x)$ be the time-t price of the down-and-out barrier option with strike K, maturity $T = 1$, and barrier L in the Black-Scholes model of Problem 1. Then, u solves the following PDE:*

$$
\begin{aligned}
u_t(t,x) + rxu_x(t,x) + \frac{\sigma^2}{2}x^2u_{xx}(t,x) - ru(t,x) &= 0; \\
u(1,x) &= (x-K)^+ \\
u(t,x) &= 0,\ t \in [0,1],\ x \le L.
\end{aligned}
$$

Use the Crank-Nicolson scheme to solve the above PDE using the parameters from the previous part (Note that you first need to make a change of variables to turn the above PDE into a heat equation). Using your solution obtain 3-D plots of the option prices as a function of time and stock price for every value of the barrier (You will have 4 plots in total). Comment on your results. What is the hedging strategy when $L = 30$?

2.2. **Ruin probabilities in finite time.** Suppose that the cash balances of an insurance company is given by

$$U_t = u + ct - \sum_{i=1}^{N_t} Y_i$$

where N is a Poisson process with intensity λ and Y_is are iid exponential random variables with mean α^{-1} and independent of N. c is the constant rate of premiums satisfying $c \geq \lambda\alpha^{-1}$. We define the ruin time before T as

$$\tau_T := \min\{t \in [0,T] : U_t < 0\},$$

where $\tau_T = T$ by definition if the above set is empty.

Problem 3. (1) *Explain how you would generate an exponential random variable if you are given a random variable with uniform distribution over* $[0,1]$. *Then, simulate 1000 realisations of an exponential random variable with mean* 1 *and plot its histogram.*

(2) *In this exercise you aim to analyse the dependence of* U *on the model parameters* u, c, α *and* λ. *You will produce 4 different sets of graphs and in each graph you must plot 4 simulated paths of* U *on* $[0,10]$ *by varying one of the parameters while keeping the rest unchanged. Interpret what you see. (One way to simulate a Poisson process is via simulation of its inter-arrival times as exponential random variables.)*

(3) *Choose a set of parameters* (u, α, c, λ). *Using Monte Carlo simulation estimate the probabilities* τ_1, τ_5 *and* τ_{10}. *Discuss the dependence of ruin probabilities on* u, c *and* λ *by repeating your simulations.*

(4) *Let* $\alpha = 4, \lambda = 2$, *and* $c = 0.5$. *For* $u \in \{0.5, 2, 4\}$ *create histograms for* $-U_{\tau_5}$. *Interpret what you see.*

(5) *Now suppose that the cash balances have a Brownian term due to investments, i.e.*

$$U_t = u + ct + \sigma W_t - \sum_{i=1}^{N_t} Y_i,$$

where $\sigma > 0$ *is a constant and* W *is a standard Brownian motion independent of* Y_i*s and* N. *Observe that there is no need to change the premium rate since* W *has zero mean. Repeat Part 2 with an additional graph depicting the dependence on* σ.

(6) *Estimate the ruin probabilities* τ_1, τ_5 *and* τ_{10} *using your initial set of parameters* (u, α, c, λ) *from Part 3 and choosing* $\sigma \in (0,1]$. *Also discuss the dependence of ruin probabilities on* σ, u *and* λ *by running additional simulations.*

(7) *Let* $\alpha = \lambda = 2$, $c = 1.1$ *and* $\sigma = 0.3$. *For* $u \in \{0.5, 2, 4\}$ *create histograms for* $-U_{\tau_5}$. *Interpret what you see and compare with what you found in Part 4. Repeat for* $\sigma = 1$.

2.3. **Systemic risk.** In this section we consider a simple model for interbank borrowing and lending. Let N be the number of banks and X^i denote the log reserves of Bank i. We suppose $X_0^i = 0$ for all i and

$$dX_t^i = a(\bar{X}_t - X_t^i)dt + \sigma dB_t^i,$$

where $a \geq 0$, $\sigma > 0$, $\bar{X}_t = \frac{1}{N}\sum_{j=1}^{N} X_t^j$, and B^is are independent Brownian motions. Note that although the Brownian motions are independent, this is an interacting system of banks due to the presence of $\bar{X}$ in the drift term for every bank. In particular, we can view $\frac{a}{N}(X_t^j - X_t^i)$ as the rate of borrowing for Bank i from Bank j.

Problem 4. *Suppose $\sigma = 1$.*

(1) *Suppose $N = 10$ and $a = 0$. Simulate X^i over the time interval $[0, 10]$ and produce a single graph for the paths of log reserves of every bank.*
(2) *Repeat the previous part by taking $a = 10$. Comment on your plot. What happens if you increase/decrease a?*
(3) *Let $D = -0.5$ and define $\tau_i = \inf\{t : X_t^i \leq D\} \wedge 10$. τ_i is the default time for bank i. Compute $\mathbb{E}[\tau_i]$ by Monte-Carlo simulation and discuss the impact of a, σ and N on this expectation.*
(4) *According to your simulations how close are the default times for different banks? How is this affected by the values of a, σ, and N?*
(5) *Let τ_i be as above and define $L = \sum_{i=1}^{N} \mathbf{1}_{[\tau_i < 10]}$, i.e. L is the number of banks defaulting before time-10. Compute the mean and variance of L via a Monte-Carlo simulation and discuss the impact of a, σ and N on these values.*
(6) *Repeat the last three parts for $D = -0.25$ and $D = -0.75$.*