Reinforcement Learning: An Introduction

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Chapter 4

Dynamic Programming

The term dynamic programming (DP) refers to a collection of algorithms that can be used to compute optimal policies given a perfect model of the environment as a Markov decision process (MDP). Classical DP algorithms are of limited utility in reinforcement learning both because of their assumption of a perfect model and because of their great computational expense, but they are still important theoretically. DP provides an essential foundation for the understanding of the methods presented in the rest of this book. In fact, all of these methods can be viewed as attempts to achieve much the same effect as DP, only with less computation and without assuming a perfect model of the environment.

Starting with this chapter, we usually assume that the environment is a finite MDP. That is, we assume that its state, action, and reward sets, $\mathcal{S}$, $\mathcal{A}(s)$, and $\mathcal{R}$, for $s \in \mathcal{S}$, are finite, and that its dynamics are given by a set of probabilities $p(s', r|s, a)$, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$, $r \in \mathcal{R}$, and $s' \in \mathcal{S}^+$ ($\mathcal{S}^+$ is $\mathcal{S}$ plus a terminal state if the problem is episodic). Although DP ideas can be applied to problems with continuous state and action spaces, exact solutions are possible only in special cases. A common way of obtaining approximate solutions for tasks with continuous states and actions is to quantize the state and action spaces and then apply finite-state DP methods. The methods we explore in Chapter 9 are applicable to continuous problems and are a significant extension of that approach.

The key idea of DP, and of reinforcement learning generally, is the use of value functions to organize and structure the search for good policies. In this chapter we show how DP can be used to compute the value functions defined in Chapter 3. As discussed there, we can easily obtain optimal policies once we have found the optimal value functions, v_* or q_*, which satisfy the Bellman optimality equations:

$$\begin{aligned} v_*(s) &= \max_a \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \max_a \sum_{s',r} p(s', r|s, a) \Big[r + \gamma v_*(s') \Big] \qquad (4.1) \end{aligned}$$

or

$$\begin{aligned} q_*(s,a) &= \mathbb{E}\Big[R_{t+1} + \gamma \max_{a'} q_*(S_{t+1}, a') \;\Big|\; S_t = s, A_t = a\Big] \\ &= \sum_{s',r} p(s', r|s,a)\Big[r + \gamma \max_{a'} q_*(s', a')\Big], \end{aligned} \tag{4.2}$$

for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$, and $s' \in \mathcal{S}^+$. As we shall see, DP algorithms are obtained by turning Bellman equations such as these into assignments, that is, into update rules for improving approximations of the desired value functions.

4.1 Policy Evaluation

First we consider how to compute the state-value function v_π for an arbitrary policy π. This is called *policy evaluation* in the DP literature. We also refer to it as the *prediction problem.* Recall from Chapter 3 that, for all $s \in \mathcal{S}$,

$$\begin{aligned} v_\pi(s) &\doteq \mathbb{E}_\pi\big[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots \;\big|\; S_t = s\big] \\ &= \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s] && (4.3) \\ &= \sum_a \pi(a|s) \sum_{s',r} p(s', r|s,a)\Big[r + \gamma v_\pi(s')\Big], && (4.4) \end{aligned}$$

where $\pi(a|s)$ is the probability of taking action a in state s under policy π, and the expectations are subscripted by π to indicate that they are conditional on π being followed. The existence and uniqueness of v_π are guaranteed as long as either $\gamma < 1$ or eventual termination is guaranteed from all states under the policy π.

If the environment's dynamics are completely known, then (4.4) is a system of $|\mathcal{S}|$ simultaneous linear equations in $|\mathcal{S}|$ unknowns (the $v_\pi(s)$, $s \in \mathcal{S}$). In principle, its solution is a straightforward, if tedious, computation. For our purposes, iterative solution methods are most suitable. Consider a sequence of approximate value functions $v_0, v_1, v_2, \ldots$, each mapping $\mathcal{S}^+$ to $\mathbb{R}$. The initial approximation, v_0, is chosen arbitrarily (except that the terminal state, if any, must be given value 0), and each successive approximation is obtained by using the Bellman equation for v_π (3.12) as an update rule:

$$\begin{aligned} v_{k+1}(s) &\doteq \mathbb{E}_\pi[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s] \\ &= \sum_a \pi(a|s) \sum_{s',r} p(s', r|s,a)\Big[r + \gamma v_k(s')\Big], \end{aligned} \tag{4.5}$$

for all $s \in \mathcal{S}$. Clearly, $v_k = v_\pi$ is a fixed point for this update rule because the Bellman equation for v_π assures us of equality in this case. Indeed, the sequence $\{v_k\}$ can be shown in general to converge to v_π as $k \to \infty$ under the same conditions that guarantee the existence of v_π. This algorithm is called *iterative policy evaluation.*

To produce each successive approximation, v_{k+1} from v_k, iterative policy evaluation applies the same operation to each state s: it replaces the old value of s with a new value obtained from the old values of the successor states of s, and the expected immediate rewards, along all the one-step transitions possible under the policy being evaluated. We call this kind of operation a *full backup*. Each iteration of iterative policy evaluation *backs up* the value of every state once to produce the new approximate value function v_{k+1}. There are several different kinds of full backups, depending on whether a state (as here) or a state–action pair is being backed up, and depending on the precise way the estimated values of the successor states are combined. All the backups done in DP algorithms are called *full* backups because they are based on all possible next states rather than on a sample next state. The nature of a backup can be expressed in an equation, as above, or in a backup diagram like those introduced in Chapter 3. For example, Figure 3.4a is the backup diagram corresponding to the full backup used in iterative policy evaluation.

To write a sequential computer program to implement iterative policy evaluation, as given by (4.5), you would have to use two arrays, one for the old values, $v_k(s)$, and one for the new values, $v_{k+1}(s)$. This way, the new values can be computed one by one from the old values without the old values being changed. Of course it is easier to use one array and update the values "in place," that is, with each new backed-up value immediately overwriting the old one. Then, depending on the order in which the states are backed up, sometimes new values are used instead of old ones on the right-hand side of (4.5). This slightly different algorithm also converges to v_π; in fact, it usually converges faster than the two-array version, as you might expect, since it uses new data as soon as they are available. We think of the backups as being done in a *sweep* through the state space. For the in-place algorithm, the order in which states are backed up during the sweep has a significant influence on the rate of convergence. We usually have the in-place version in mind when we think of DP algorithms.

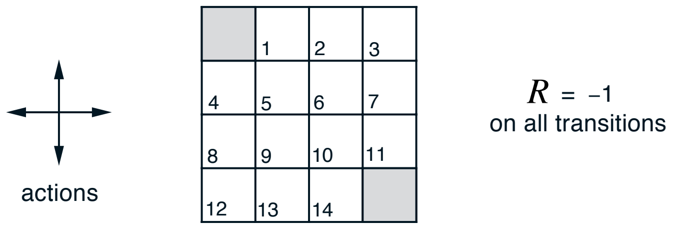
Another implementation point concerns the termination of the algorithm. Formally, iterative policy evaluation converges only in the limit, but in practice it must be halted short of this. A typical stopping condition for iterative policy evaluation is to test the quantity $\max_{s\in\mathcal{S}} |v_{k+1}(s) - v_k(s)|$ after each sweep and stop when it is suf-

```
Input $\pi$, the policy to be evaluated
Initialize an array $V(s) = 0$, for all $s \in \mathcal{S}^+$
Repeat
    $\Delta \leftarrow 0$
    For each $s \in \mathcal{S}$:
        $v \leftarrow V(s)$
        $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)\big[r + \gamma V(s')\big]$
        $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)
Output $V \approx v_\pi$
```

Figure 4.1: Iterative policy evaluation.

ficiently small. Figure 4.1 gives a complete algorithm for iterative policy evaluation with this stopping criterion.

Example 4.1 Consider the 4×4 gridworld shown below.



The nonterminal states are $\mathcal{S} = \{1, 2, \ldots, 14\}$. There are four actions possible in each state, $\mathcal{A} = \{$`up`, `down`, `right`, `left`$\}$, which deterministically cause the corresponding state transitions, except that actions that would take the agent off the grid in fact leave the state unchanged. Thus, for instance, $p(6, -1 \mid 5, \texttt{right}) = 1$, $p(7, -1 \mid 7, \texttt{right}) = 1$, and $p(10, r \mid 5, \texttt{right}) = 0$ for all $r \in \mathcal{R}$. This is an undiscounted, episodic task. The reward is -1 on all transitions until the terminal state is reached. The terminal state is shaded in the figure (although it is shown in two places, it is formally one state). The expected reward function is thus $r(s, a, s') = -1$ for all states s, s' and actions a. Suppose the agent follows the equiprobable random policy (all actions equally likely). The left side of Figure 4.2 shows the sequence of value functions $\{v_k\}$ computed by iterative policy evaluation. The final estimate is in fact v_π, which in this case gives for each state the negation of the expected number of steps from that state until termination. ■

Exercise 4.1 In Example 4.1, if π is the equiprobable random policy, what is $q_\pi(11, \texttt{down})$? What is $q_\pi(7, \texttt{down})$?

Exercise 4.2 In Example 4.1, suppose a new state 15 is added to the gridworld just below state 13, and its actions, `left`, `up`, `right`, and `down`, take the agent to states 12, 13, 14, and 15, respectively. Assume that the transitions *from* the original states are unchanged. What, then, is $v_\pi(15)$ for the equiprobable random policy? Now suppose the dynamics of state 13 are also changed, such that action `down` from state 13 takes the agent to the new state 15. What is $v_\pi(15)$ for the equiprobable random policy in this case?

Exercise 4.3 What are the equations analogous to (4.3), (4.4), and (4.5) for the action-value function q_π and its successive approximation by a sequence of functions $q_0, q_1, q_2, \ldots$?

Exercise 4.4 In some undiscounted episodic tasks there may be policies for which eventual termination is not guaranteed. For example, in the grid problem above it is possible to go back and forth between two states forever. In a task that is otherwise perfectly sensible, $v_\pi(s)$ may be negative infinity for some policies and states, in which case the algorithm for iterative policy evaluation given in Figure 4.1 will not terminate. As a purely practical matter, how might we amend this algorithm to assure termination even in this case? Assume that eventual termination *is* guaranteed

v_k for the Random Policy

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

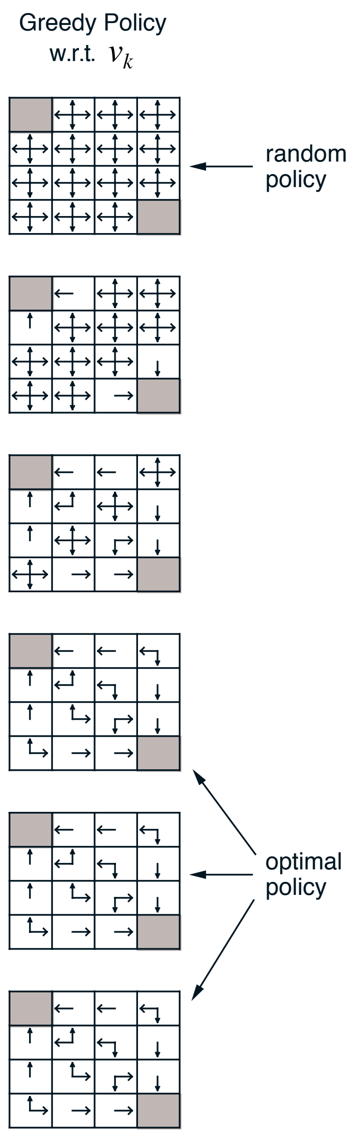


Figure 4.2: Convergence of iterative policy evaluation on a small gridworld. The left column is the sequence of approximations of the state-value function for the random policy (all actions equal). The right column is the sequence of greedy policies corresponding to the value function estimates (arrows are shown for all actions achieving the maximum). The last policy is guaranteed only to be an improvement over the random policy, but in this case it, and all policies after the third iteration, are optimal.

under the optimal policy.

4.2 Policy Improvement

Our reason for computing the value function for a policy is to help find better policies. Suppose we have determined the value function v_π for an arbitrary deterministic policy π. For some state s we would like to know whether or not we should change the policy to deterministically choose an action $a \neq \pi(s)$. We know how good it is to follow the current policy from s—that is $v_\pi(s)$—but would it be better or worse to change to the new policy? One way to answer this question is to consider selecting a in s and thereafter following the existing policy, π. The value of this way of behaving is

$$\begin{aligned} q_\pi(s,a) &= \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_\pi(s')\Big]. \end{aligned} \tag{4.6}$$

The key criterion is whether this is greater than or less than $v_\pi(s)$. If it is greater—that is, if it is better to select a once in s and thereafter follow π than it would be to follow π all the time—then one would expect it to be better still to select a every time s is encountered, and that the new policy would in fact be a better one overall.

That this is true is a special case of a general result called the *policy improvement theorem*. Let π and π' be any pair of deterministic policies such that, for all $s \in \mathcal{S}$,

$$q_\pi(s, \pi'(s)) \geq v_\pi(s). \tag{4.7}$$

Then the policy π' must be as good as, or better than, π. That is, it must obtain greater or equal expected return from all states $s \in \mathcal{S}$:

$$v_{\pi'}(s) \geq v_\pi(s). \tag{4.8}$$

Moreover, if there is strict inequality of (4.7) at any state, then there must be strict inequality of (4.8) at at least one state. This result applies in particular to the two policies that we considered in the previous paragraph, an original deterministic policy, π, and a changed policy, π', that is identical to π except that $\pi'(s) = a \neq \pi(s)$. Obviously, (4.7) holds at all states other than s. Thus, if $q_\pi(s,a) > v_\pi(s)$, then the changed policy is indeed better than π.

The idea behind the proof of the policy improvement theorem is easy to understand. Starting from (4.7), we keep expanding the q_π side and reapplying (4.7) until we get $v_{\pi'}(s)$:

$$\begin{aligned}
v_\pi(s) &\leq q_\pi(s, \pi'(s)) \\
&= \mathbb{E}_{\pi'}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s] \\
&\leq \mathbb{E}_{\pi'}\big[R_{t+1} + \gamma q_\pi(S_{t+1}, \pi'(S_{t+1})) \bigm| S_t = s\big] \\
&= \mathbb{E}_{\pi'}[R_{t+1} + \gamma \mathbb{E}_{\pi'}[R_{t+2} + \gamma v_\pi(S_{t+2})] \mid S_t = s] \\
&= \mathbb{E}_{\pi'}\big[R_{t+1} + \gamma R_{t+2} + \gamma^2 v_\pi(S_{t+2}) \bigm| S_t = s\big] \\
&\leq \mathbb{E}_{\pi'}\big[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 v_\pi(S_{t+3}) \bigm| S_t = s\big] \\
&\vdots \\
&\leq \mathbb{E}_{\pi'}\big[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \bigm| S_t = s\big] \\
&= v_{\pi'}(s).
\end{aligned}$$

So far we have seen how, given a policy and its value function, we can easily evaluate a change in the policy at a single state to a particular action. It is a natural extension to consider changes at *all* states and to *all* possible actions, selecting at each state the action that appears best according to $q_\pi(s, a)$. In other words, to consider the new *greedy* policy, π', given by

$$\begin{aligned}
\pi'(s) &\doteq \arg\max_a q_\pi(s, a) \\
&= \arg\max_a \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \arg\max_a \sum_{s', r} p(s', r|s, a)\Big[r + \gamma v_\pi(s')\Big],
\end{aligned} \tag{4.9}$$

where $\arg\max_a$ denotes the value of a at which the expression that follows is maximized (with ties broken arbitrarily). The greedy policy takes the action that looks best in the short term—after one step of lookahead—according to v_π. By construction, the greedy policy meets the conditions of the policy improvement theorem (4.7), so we know that it is as good as, or better than, the original policy. The process of making a new policy that improves on an original policy, by making it greedy with respect to the value function of the original policy, is called *policy improvement*.

Suppose the new greedy policy, π', is as good as, but not better than, the old policy π. Then $v_\pi = v_{\pi'}$, and from (4.9) it follows that for all $s \in \mathcal{S}$:

$$\begin{aligned}
v_{\pi'}(s) &= \max_a \mathbb{E}[R_{t+1} + \gamma v_{\pi'}(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \max_a \sum_{s', r} p(s', r|s, a)\Big[r + \gamma v_{\pi'}(s')\Big].
\end{aligned}$$

But this is the same as the Bellman optimality equation (4.1), and therefore, $v_{\pi'}$ must be v_*, and both π and π' must be optimal policies. Policy improvement thus must give us a strictly better policy except when the original policy is already optimal.

So far in this section we have considered the special case of deterministic policies. In the general case, a stochastic policy π specifies probabilities, $\pi(a|s)$, for taking each action, a, in each state, s. We will not go through the details, but in fact all the ideas of this section extend easily to stochastic policies. In particular, the policy improvement theorem carries through as stated for the stochastic case. In addition, if there are ties in policy improvement steps such as (4.9)—that is, if there are several actions at which the maximum is achieved—then in the stochastic case we need not select a single action from among them. Instead, each maximizing action can be given a portion of the probability of being selected in the new greedy policy. Any apportioning scheme is allowed as long as all submaximal actions are given zero probability.

The last row of Figure 4.2 shows an example of policy improvement for stochastic policies. Here the original policy, π, is the equiprobable random policy, and the new policy, π', is greedy with respect to v_π. The value function v_π is shown in the bottom-left diagram and the set of possible π' is shown in the bottom-right diagram. The states with multiple arrows in the π' diagram are those in which several actions achieve the maximum in (4.9); any apportionment of probability among these actions is permitted. The value function of any such policy, $v_{\pi'}(s)$, can be seen by inspection to be either -1, -2, or -3 at all states, $s \in \mathcal{S}$, whereas $v_\pi(s)$ is at most -14. Thus, $v_{\pi'}(s) \geq v_\pi(s)$, for all $s \in \mathcal{S}$, illustrating policy improvement. Although in this case the new policy π' happens to be optimal, in general only an improvement is guaranteed.

4.3 Policy Iteration

Once a policy, π, has been improved using v_π to yield a better policy, π', we can then compute $v_{\pi'}$ and improve it again to yield an even better π''. We can thus obtain a sequence of monotonically improving policies and value functions:

$$\pi_0 \xrightarrow{\text{E}} v_{\pi_0} \xrightarrow{\text{I}} \pi_1 \xrightarrow{\text{E}} v_{\pi_1} \xrightarrow{\text{I}} \pi_2 \xrightarrow{\text{E}} \cdots \xrightarrow{\text{I}} \pi_* \xrightarrow{\text{E}} v_*,$$

where $\xrightarrow{\text{E}}$ denotes a policy *evaluation* and $\xrightarrow{\text{I}}$ denotes a policy *improvement*. Each policy is guaranteed to be a strict improvement over the previous one (unless it is already optimal). Because a finite MDP has only a finite number of policies, this process must converge to an optimal policy and optimal value function in a finite number of iterations.

This way of finding an optimal policy is called *policy iteration.* A complete algorithm is given in Figure 4.3. Note that each policy evaluation, itself an iterative computation, is started with the value function for the previous policy. This typically results in a great increase in the speed of convergence of policy evaluation (presumably because the value function changes little from one policy to the next).

Policy iteration often converges in surprisingly few iterations. This is illustrated by the example in Figure 4.2. The bottom-left diagram shows the value function for the equiprobable random policy, and the bottom-right diagram shows a greedy policy for this value function. The policy improvement theorem assures us that these policies are better than the original random policy. In this case, however, these policies

```
1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Repeat
       $\Delta \leftarrow 0$
       For each $s \in \mathcal{S}$:
           $v \leftarrow V(s)$
           $V(s) \leftarrow \sum_{s',r} p(s', r|s, \pi(s))\big[r + \gamma V(s')\big]$
           $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$  (a small positive number)

3. Policy Improvement
   *policy-stable* $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
       $a \leftarrow \pi(s)$
       $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r|s, a)\big[r + \gamma V(s')\big]$
       If $a \neq \pi(s)$, then *policy-stable* $\leftarrow$ $false$
   If *policy-stable*, then stop and return $V$ and $\pi$; else go to 2
```

Figure 4.3: Policy iteration (using iterative policy evaluation) for v_*. This algorithm has a subtle bug, in that it may never terminate if the policy continually switches between two or more policies that are equally good. The bug can be fixed by adding additional flags, but it makes the pseudocode so ugly that it is not worth it.

are not just better, but optimal, proceeding to the terminal states in the minimum number of steps. In this example, policy iteration would find the optimal policy after just one iteration.

Example 4.2: Jack's Car Rental Jack manages two locations for a nationwide car rental company. Each day, some number of customers arrive at each location to rent cars. If Jack has a car available, he rents it out and is credited \$10 by the national company. If he is out of cars at that location, then the business is lost. Cars become available for renting the day after they are returned. To help ensure that cars are available where they are needed, Jack can move them between the two locations overnight, at a cost of \$2 per car moved. We assume that the number of cars requested and returned at each location are Poisson random variables, meaning that the probability that the number is n is $\frac{\lambda^n}{n!}e^{-\lambda}$, where λ is the expected number. Suppose λ is 3 and 4 for rental requests at the first and second locations and 3 and 2 for returns. To simplify the problem slightly, we assume that there can be no more than 20 cars at each location (any additional cars are returned to the nationwide company, and thus disappear from the problem) and a maximum of five cars can be moved from one location to the other in one night. We take the discount rate to be $\gamma = 0.9$ and formulate this as a continuing finite MDP, where the time steps are days, the state is the number of cars at each location at the end of the day, and

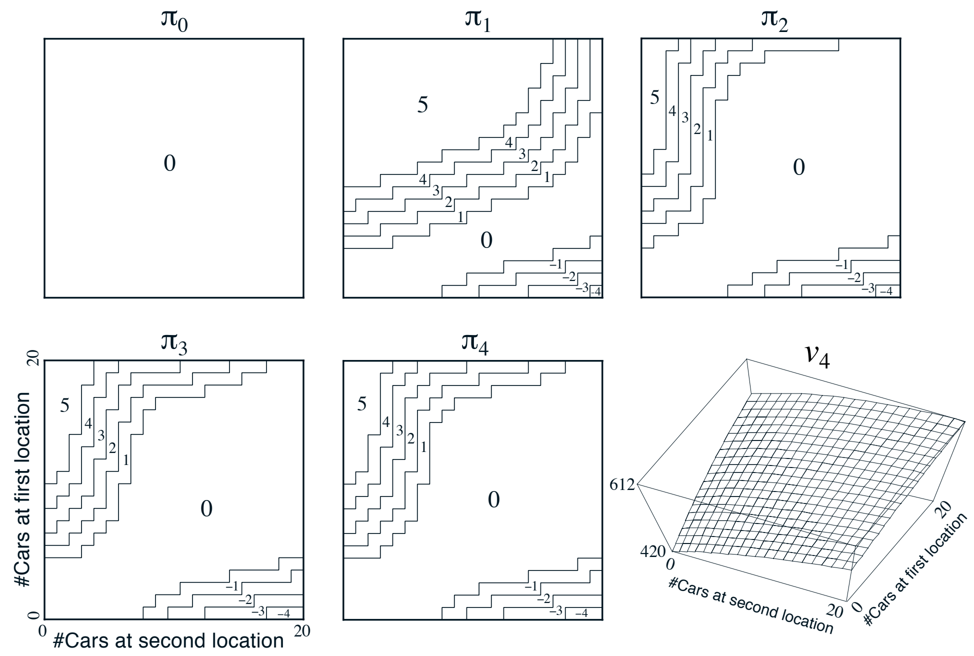


Figure 4.4: The sequence of policies found by policy iteration on Jack's car rental problem, and the final state-value function. The first five diagrams show, for each number of cars at each location at the end of the day, the number of cars to be moved from the first location to the second (negative numbers indicate transfers from the second location to the first). Each successive policy is a strict improvement over the previous policy, and the last policy is optimal.

the actions are the net numbers of cars moved between the two locations overnight. Figure 4.4 shows the sequence of policies found by policy iteration starting from the policy that never moves any cars. ■

Exercise 4.5 (programming) Write a program for policy iteration and re-solve Jack's car rental problem with the following changes. One of Jack's employees at the first location rides a bus home each night and lives near the second location. She is happy to shuttle one car to the second location for free. Each additional car still costs \$2, as do all cars moved in the other direction. In addition, Jack has limited parking space at each location. If more than 10 cars are kept overnight at a location (after any moving of cars), then an additional cost of \$4 must be incurred to use a second parking lot (independent of how many cars are kept there). These sorts of nonlinearities and arbitrary dynamics often occur in real problems and cannot easily be handled by optimization methods other than dynamic programming. To check your program, first replicate the results given for the original problem. If your computer is too slow for the full problem, cut all the numbers of cars in half.

Exercise 4.6 How would policy iteration be defined for action values? Give a complete algorithm for computing q_*, analogous to Figure 4.3 for computing v_*. Please pay special attention to this exercise, because the ideas involved will be used throughout the rest of the book.

Exercise 4.7 Suppose you are restricted to considering only policies that are *ϵ-soft*, meaning that the probability of selecting each action in each state, s, is at least $\epsilon/|\mathcal{A}(s)|$. Describe qualitatively the changes that would be required in each of the steps 3, 2, and 1, in that order, of the policy iteration algorithm for v_* (Figure 4.3).

4.4 Value Iteration

One drawback to policy iteration is that each of its iterations involves policy evaluation, which may itself be a protracted iterative computation requiring multiple sweeps through the state set. If policy evaluation is done iteratively, then convergence exactly to v_π occurs only in the limit. Must we wait for exact convergence, or can we stop short of that? The example in Figure 4.2 certainly suggests that it may be possible to truncate policy evaluation. In that example, policy evaluation iterations beyond the first three have no effect on the corresponding greedy policy.

In fact, the policy evaluation step of policy iteration can be truncated in several ways without losing the convergence guarantees of policy iteration. One important special case is when policy evaluation is stopped after just one sweep (one backup of each state). This algorithm is called *value iteration*. It can be written as a particularly simple backup operation that combines the policy improvement and truncated policy evaluation steps:

$$\begin{aligned} v_{k+1}(s) &\doteq \max_a \mathbb{E}[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s, A_t = a] \qquad (4.10) \\ &= \max_a \sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_k(s')\Big], \end{aligned}$$

for all $s \in \mathcal{S}$. For arbitrary v_0, the sequence $\{v_k\}$ can be shown to converge to v_* under the same conditions that guarantee the existence of v_*.

Another way of understanding value iteration is by reference to the Bellman optimality equation (4.1). Note that value iteration is obtained simply by turning the Bellman optimality equation into an update rule. Also note how the value iteration backup is identical to the policy evaluation backup (4.5) except that it requires the maximum to be taken over all actions. Another way of seeing this close relationship is to compare the backup diagrams for these algorithms: Figure 3.4a shows the backup diagram for policy evaluation and Figure 3.7a shows the backup diagram for value iteration. These two are the natural backup operations for computing v_π and v_*.

Finally, let us consider how value iteration terminates. Like policy evaluation, value iteration formally requires an infinite number of iterations to converge exactly to v_*. In practice, we stop once the value function changes by only a small amount in a sweep. Figure 4.5 gives a complete value iteration algorithm with this kind of termination condition.

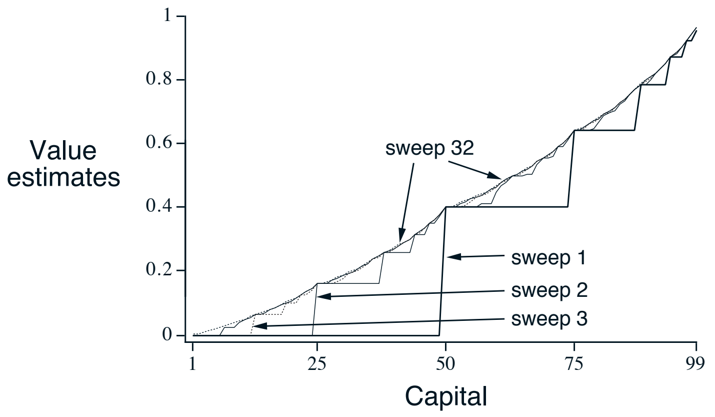
```
Initialize array $V$ arbitrarily (e.g., $V(s) = 0$ for all $s \in \mathcal{S}^+$)

Repeat
   $\Delta \leftarrow 0$
   For each $s \in \mathcal{S}$:
        $v \leftarrow V(s)$
        $V(s) \leftarrow \max_a \sum_{s',r} p(s',r|s,a)\big[r + \gamma V(s')\big]$
        $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)

Output a deterministic policy, $\pi$, such that
   $\pi(s) = \arg\max_a \sum_{s',r} p(s',r|s,a)\big[r + \gamma V(s')\big]$
```

Figure 4.5: Value iteration.

Value iteration effectively combines, in each of its sweeps, one sweep of policy evaluation and one sweep of policy improvement. Faster convergence is often achieved by interposing multiple policy evaluation sweeps between each policy improvement sweep. In general, the entire class of truncated policy iteration algorithms can be thought of as sequences of sweeps, some of which use policy evaluation backups and some of which use value iteration backups. Since the max operation in (4.10) is the only difference between these backups, this just means that the max operation is added to some sweeps of policy evaluation. All of these algorithms converge to an optimal policy for discounted finite MDPs.

Example 4.3: Gambler's Problem A gambler has the opportunity to make bets on the outcomes of a sequence of coin flips. If the coin comes up heads, he wins as many dollars as he has staked on that flip; if it is tails, he loses his stake. The game ends when the gambler wins by reaching his goal of \$100, or loses by running out of money. On each flip, the gambler must decide what portion of his capital to stake, in integer numbers of dollars. This problem can be formulated as an undiscounted, episodic, finite MDP. The state is the gambler's capital, $s \in \{1, 2, \ldots, 99\}$ and the actions are stakes, $a \in \{0, 1, \ldots, \min(s, 100-s)\}$. The reward is zero on all transitions except those on which the gambler reaches his goal, when it is $+1$. The state-value function then gives the probability of winning from each state. A policy is a mapping from levels of capital to stakes. The optimal policy maximizes the probability of reaching the goal. Let p_h denote the probability of the coin coming up heads. If p_h is known, then the entire problem is known and it can be solved, for instance, by value iteration. Figure 4.6 shows the change in the value function over successive sweeps of value iteration, and the final policy found, for the case of $p_h = 0.4$. This policy is optimal, but not unique. In fact, there is a whole family of optimal policies, all corresponding to ties for the argmax action selection with respect to the optimal value function. Can you guess what the entire family looks like? ■

Exercise 4.8 Why does the optimal policy for the gambler's problem have such a



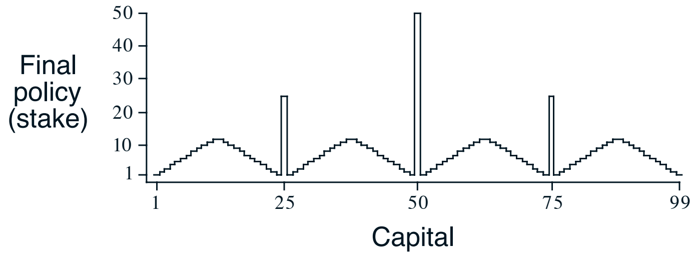


Figure 4.6: The solution to the gambler's problem for $p_h = 0.4$. The upper graph shows the value function found by successive sweeps of value iteration. The lower graph shows the final policy.

curious form? In particular, for capital of 50 it bets it all on one flip, but for capital of 51 it does not. Why is this a good policy?

Exercise 4.9 (programming) Implement value iteration for the gambler's problem and solve it for $p_h = 0.25$ and $p_h = 0.55$. In programming, you may find it convenient to introduce two dummy states corresponding to termination with capital of 0 and 100, giving them values of 0 and 1 respectively. Show your results graphically, as in Figure 4.6. Are your results stable as $\theta \to 0$?

Exercise 4.10 What is the analog of the value iteration backup (4.10) for action values, $q_{k+1}(s, a)$?

4.5 Asynchronous Dynamic Programming

A major drawback to the DP methods that we have discussed so far is that they involve operations over the entire state set of the MDP, that is, they require sweeps of the state set. If the state set is very large, then even a single sweep can be prohibitively expensive. For example, the game of backgammon has over 10^{20} states. Even if we could perform the value iteration backup on a million states per second, it would take over a thousand years to complete a single sweep.

Asynchronous DP algorithms are in-place iterative DP algorithms that are not organized in terms of systematic sweeps of the state set. These algorithms back up the values of states in any order whatsoever, using whatever values of other states happen to be available. The values of some states may be backed up several times before the values of others are backed up once. To converge correctly, however, an asynchronous algorithm must continue to backup the values of all the states: it can't ignore any state after some point in the computation. Asynchronous DP algorithms allow great flexibility in selecting states to which backup operations are applied.

For example, one version of asynchronous value iteration backs up the value, in place, of only one state, s_k, on each step, k, using the value iteration backup (4.10). If $0 \leq \gamma < 1$, asymptotic convergence to v_* is guaranteed given only that all states occur in the sequence $\{s_k\}$ an infinite number of times (the sequence could even be stochastic). (In the undiscounted episodic case, it is possible that there are some orderings of backups that do not result in convergence, but it is relatively easy to avoid these.) Similarly, it is possible to intermix policy evaluation and value iteration backups to produce a kind of asynchronous truncated policy iteration. Although the details of this and other more unusual DP algorithms are beyond the scope of this book, it is clear that a few different backups form building blocks that can be used flexibly in a wide variety of sweepless DP algorithms.

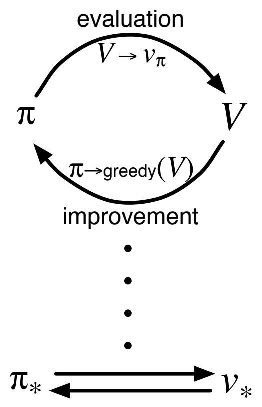
Of course, avoiding sweeps does not necessarily mean that we can get away with less computation. It just means that an algorithm does not need to get locked into any hopelessly long sweep before it can make progress improving a policy. We can try to take advantage of this flexibility by selecting the states to which we apply backups so as to improve the algorithm's rate of progress. We can try to order the backups to let value information propagate from state to state in an efficient way. Some states may not need their values backed up as often as others. We might even try to skip backing up some states entirely if they are not relevant to optimal behavior. Some ideas for doing this are discussed in Chapter 8.

Asynchronous algorithms also make it easier to intermix computation with real-time interaction. To solve a given MDP, we can run an iterative DP algorithm *at the same time that an agent is actually experiencing the MDP*. The agent's experience can be used to determine the states to which the DP algorithm applies its backups. At the same time, the latest value and policy information from the DP algorithm can guide the agent's decision-making. For example, we can apply backups to states as the agent visits them. This makes it possible to *focus* the DP algorithm's backups onto parts of the state set that are most relevant to the agent. This kind of focusing is a repeated theme in reinforcement learning.

4.6 Generalized Policy Iteration

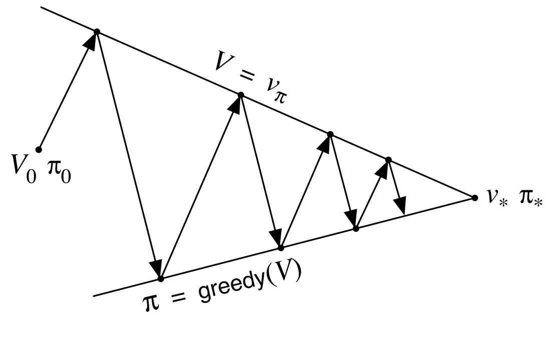
Policy iteration consists of two simultaneous, interacting processes, one making the value function consistent with the current policy (policy evaluation), and the other making the policy greedy with respect to the current value function (policy improvement). In policy iteration, these two processes alternate, each completing before the other begins, but this is not really necessary. In value iteration, for example, only a single iteration of policy evaluation is performed in between each policy improvement. In asynchronous DP methods, the evaluation and improvement processes are interleaved at an even finer grain. In some cases a single state is updated in one process before returning to the other. As long as both processes continue to update all states, the ultimate result is typically the same—convergence to the optimal value function and an optimal policy.

We use the term *generalized policy iteration* (GPI) to refer to the general idea of letting policy evaluation and policy improvement processes interact, independent of the granularity and other details of the two processes. Almost all reinforcement learning methods are well described as GPI. That is, all have identifiable policies and value functions, with the policy always being improved with respect to the value function and the value function always being driven toward the value function for the policy, as suggested by the diagram to the right. It is easy to see that if both the evaluation process and the improvement process stabilize, that is, no longer produce changes, then the value function and policy must be optimal. The value function stabilizes only when it is consistent with the current policy, and the policy stabilizes only when it is greedy with respect to the current value function. Thus, both processes stabilize only when a policy has been found that is greedy with respect to its own evaluation function. This implies that the Bellman optimality equation (4.1) holds, and thus that the policy and the value function are optimal.



The evaluation and improvement processes in GPI can be viewed as both competing and cooperating. They compete in the sense that they pull in opposing directions. Making the policy greedy with respect to the value function typically makes the value function incorrect for the changed policy, and making the value function consistent with the policy typically causes that policy no longer to be greedy. In the long run, however, these two processes interact to find a single joint solution: the optimal value function and an optimal policy.

One might also think of the interaction between the evaluation and improvement processes in GPI in terms of two constraints or goals—for example, as two lines in two-dimensional space as suggested by the diagram to the right. Although the real geometry is much more complicated than this, the diagram suggests what happens in the real case. Each process drives the value function or policy toward one of the lines repre-



senting a solution to one of the two goals. The goals interact because the two lines are not orthogonal. Driving directly toward one goal causes some movement away from the other goal. Inevitably, however, the joint process is brought closer to the overall goal of optimality. The arrows in this diagram correspond to the behavior of policy iteration in that each takes the system all the way to achieving one of the two goals completely. In GPI one could also take smaller, incomplete steps toward each goal. In either case, the two processes together achieve the overall goal of optimality even though neither is attempting to achieve it directly.

4.7 Efficiency of Dynamic Programming

DP may not be practical for very large problems, but compared with other methods for solving MDPs, DP methods are actually quite efficient. If we ignore a few technical details, then the (worst case) time DP methods take to find an optimal policy is polynomial in the number of states and actions. If n and k denote the number of states and actions, this means that a DP method takes a number of computational operations that is less than some polynomial function of n and k. A DP method is guaranteed to find an optimal policy in polynomial time even though the total number of (deterministic) policies is k^n. In this sense, DP is exponentially faster than any direct search in policy space could be, because direct search would have to exhaustively examine each policy to provide the same guarantee. Linear programming methods can also be used to solve MDPs, and in some cases their worst-case convergence guarantees are better than those of DP methods. But linear programming methods become impractical at a much smaller number of states than do DP methods (by a factor of about 100). For the largest problems, only DP methods are feasible.

DP is sometimes thought to be of limited applicability because of the *curse of dimensionality*, the fact that the number of states often grows exponentially with the number of state variables. Large state sets do create difficulties, but these are inherent difficulties of the problem, not of DP as a solution method. In fact, DP is comparatively better suited to handling large state spaces than competing methods such as direct search and linear programming.

In practice, DP methods can be used with today's computers to solve MDPs with millions of states. Both policy iteration and value iteration are widely used, and it is not clear which, if either, is better in general. In practice, these methods usually converge much faster than their theoretical worst-case run times, particularly if they are started with good initial value functions or policies.

On problems with large state spaces, *asynchronous* DP methods are often preferred. To complete even one sweep of a synchronous method requires computation and memory for every state. For some problems, even this much memory and computation is impractical, yet the problem is still potentially solvable because only a relatively few states occur along optimal solution trajectories. Asynchronous methods and other variations of GPI can be applied in such cases and may find good or optimal policies much faster than synchronous methods can.

4.8 Summary

In this chapter we have become familiar with the basic ideas and algorithms of dynamic programming as they relate to solving finite MDPs. *Policy evaluation* refers to the (typically) iterative computation of the value functions for a given policy. *Policy improvement* refers to the computation of an improved policy given the value function for that policy. Putting these two computations together, we obtain *policy iteration* and *value iteration*, the two most popular DP methods. Either of these can be used to reliably compute optimal policies and value functions for finite MDPs given complete knowledge of the MDP.

Classical DP methods operate in sweeps through the state set, performing a *full backup* operation on each state. Each backup updates the value of one state based on the values of all possible successor states and their probabilities of occurring. Full backups are closely related to Bellman equations: they are little more than these equations turned into assignment statements. When the backups no longer result in any changes in value, convergence has occurred to values that satisfy the corresponding Bellman equation. Just as there are four primary value functions (v_π, v_*, q_π, and q_*), there are four corresponding Bellman equations and four corresponding full backups. An intuitive view of the operation of backups is given by *backup diagrams*.

Insight into DP methods and, in fact, into almost all reinforcement learning methods, can be gained by viewing them as *generalized policy iteration* (GPI). GPI is the general idea of two interacting processes revolving around an approximate policy and an approximate value function. One process takes the policy as given and performs some form of policy evaluation, changing the value function to be more like the true value function for the policy. The other process takes the value function as given and performs some form of policy improvement, changing the policy to make it better, assuming that the value function is its value function. Although each process changes the basis for the other, overall they work together to find a joint solution: a policy and value function that are unchanged by either process and, consequently, are optimal. In some cases, GPI can be proved to converge, most notably for the classical DP methods that we have presented in this chapter. In other cases convergence has not been proved, but still the idea of GPI improves our understanding of the methods.

It is not necessary to perform DP methods in complete sweeps through the state set. *Asynchronous DP* methods are in-place iterative methods that back up states in an arbitrary order, perhaps stochastically determined and using out-of-date information. Many of these methods can be viewed as fine-grained forms of GPI.

Finally, we note one last special property of DP methods. All of them update estimates of the values of states based on estimates of the values of successor states. That is, they update estimates on the basis of other estimates. We call this general idea *bootstrapping*. Many reinforcement learning methods perform bootstrapping, even those that do not require, as DP requires, a complete and accurate model of the environment. In the next chapter we explore reinforcement learning methods that do not require a model and do not bootstrap. In the chapter after that we explore methods that do not require a model but do bootstrap. These key features and properties are separable, yet can be mixed in interesting combinations.

Bibliographical and Historical Remarks

The term "dynamic programming" is due to Bellman (1957a), who showed how these methods could be applied to a wide range of problems. Extensive treatments of DP can be found in many texts, including Bertsekas (2005, 2012), Bertsekas and Tsitsiklis (1996), Dreyfus and Law (1977), Ross (1983), White (1969), and Whittle (1982, 1983). Our interest in DP is restricted to its use in solving MDPs, but DP also applies to other types of problems. Kumar and Kanal (1988) provide a more general look at DP.

To the best of our knowledge, the first connection between DP and reinforcement learning was made by Minsky (1961) in commenting on Samuel's checkers player. In a footnote, Minsky mentioned that it is possible to apply DP to problems in which Samuel's backing-up process can be handled in closed analytic form. This remark may have misled artificial intelligence researchers into believing that DP was restricted to analytically tractable problems and therefore largely irrelevant to artificial intelligence. Andreae (1969b) mentioned DP in the context of reinforcement learning, specifically policy iteration, although he did not make specific connections between DP and learning algorithms. Werbos (1977) suggested an approach to approximating DP called "heuristic dynamic programming" that emphasizes gradient-descent methods for continuous-state problems (Werbos, 1982, 1987, 1988, 1989, 1992). These methods are closely related to the reinforcement learning algorithms that we discuss in this book. Watkins (1989) was explicit in connecting reinforcement learning to DP, characterizing a class of reinforcement learning methods as "incremental dynamic programming."

4.1–4 These sections describe well-established DP algorithms that are covered in any of the general DP references cited above. The policy improvement theorem and the policy iteration algorithm are due to Bellman (1957a) and Howard (1960). Our presentation was influenced by the local view of policy improvement taken by Watkins (1989). Our discussion of value iteration as a form of truncated policy iteration is based on the approach of Puterman and Shin (1978), who presented a class of algorithms called *modified policy iteration*, which includes policy iteration and value iteration as special cases. An analysis showing how value iteration can be made to find an optimal policy in finite time is given by Bertsekas (1987).

Iterative policy evaluation is an example of a classical successive approximation algorithm for solving a system of linear equations. The version of the algorithm that uses two arrays, one holding the old values while the other is updated, is often called a *Jacobi-style* algorithm, after Jacobi's classical use of this method. It is also sometimes called a *synchronous* algorithm because it can be performed in parallel, with separate processors simultaneously updating the values of individual states using input from other processors. The second array is needed to simulate this parallel computation sequentially. The in-place version of the algorithm is often called a *Gauss–Seidel-style* algorithm after the classical Gauss–Seidel algorithm for solving systems of linear equations. In addition to iterative policy evaluation, other DP algorithms can be implemented in these different versions. Bertsekas and Tsitsiklis (1989) provide excellent coverage of these variations and their performance differences.

4.5 Asynchronous DP algorithms are due to Bertsekas (1982, 1983), who also called them distributed DP algorithms. The original motivation for asynchronous DP was its implementation on a multiprocessor system with communication delays between processors and no global synchronizing clock. These algorithms are extensively discussed by Bertsekas and Tsitsiklis (1989). Jacobi-style and Gauss–Seidel-style DP algorithms are special cases of the asynchronous version. Williams and Baird (1990) presented DP algorithms that are asynchronous at a finer grain than the ones we have discussed: the backup operations themselves are broken into steps that can be performed asynchronously.

4.7 This section, written with the help of Michael Littman, is based on Littman, Dean, and Kaelbling (1995). The phrase "curse of dimensionality" is due to Bellman (1957).

Reinforcement Learning: An Introduction

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****Draft****

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Chapter 5

Monte Carlo Methods

In this chapter we consider our first learning methods for estimating value functions and discovering optimal policies. Unlike the previous chapter, here we do not assume complete knowledge of the environment. Monte Carlo methods require only *experience*—sample sequences of states, actions, and rewards from actual or simulated interaction with an environment. Learning from *actual* experience is striking because it requires no prior knowledge of the environment's dynamics, yet can still attain optimal behavior. Learning from *simulated* experience is also powerful. Although a model is required, the model need only generate sample transitions, not the complete probability distributions of all possible transitions that is required for dynamic programming (DP). In surprisingly many cases it is easy to generate experience sampled according to the desired probability distributions, but infeasible to obtain the distributions in explicit form.

Monte Carlo methods are ways of solving the reinforcement learning problem based on averaging sample returns. To ensure that well-defined returns are available, here we define Monte Carlo methods only for episodic tasks. That is, we assume experience is divided into episodes, and that all episodes eventually terminate no matter what actions are selected. Only on the completion of an episode are value estimates and policies changed. Monte Carlo methods can thus be incremental in an episode-by-episode sense, but not in a step-by-step (online) sense. The term "Monte Carlo" is often used more broadly for any estimation method whose operation involves a significant random component. Here we use it specifically for methods based on averaging complete returns (as opposed to methods that learn from partial returns, considered in the next chapter).

Monte Carlo methods sample and average *returns* for each state–action pair much like the bandit methods we explored in Chapter 2 sample and average *rewards* for each action. The main difference is that now there are multiple states, each acting like a different bandit problem (like an associative-search or contextual bandit) and that the different bandit problems are interrelated. That is, the return after taking an action in one state depends on the actions taken in later states in the same episode. Because all the action selections are undergoing learning, the problem becomes nonstationary from the point of view of the earlier state.

To handle the nonstationarity, we adapt the idea of general policy iteration (GPI) developed in Chapter 4 for DP. Whereas there we *computed* value functions from knowledge of the MDP, here we *learn* value functions from sample returns with the MDP. The value functions and corresponding policies still interact to attain optimality in essentially the same way (GPI). As in the DP chapter, first we consider the prediction problem (the computation of v_π and q_π for a fixed arbitrary policy π) then policy improvement, and, finally, the control problem and its solution by GPI. Each of these ideas taken from DP is extended to the Monte Carlo case in which only sample experience is available.

5.1 Monte Carlo Prediction

We begin by considering Monte Carlo methods for learning the state-value function for a given policy. Recall that the value of a state is the expected return—expected cumulative future discounted reward—starting from that state. An obvious way to estimate it from experience, then, is simply to average the returns observed after visits to that state. As more returns are observed, the average should converge to the expected value. This idea underlies all Monte Carlo methods.

In particular, suppose we wish to estimate $v_\pi(s)$, the value of a state s under policy π, given a set of episodes obtained by following π and passing through s. Each occurrence of state s in an episode is called a *visit* to s. Of course, s may be visited multiple times in the same episode; let us call the first time it is visited in an episode the *first visit* to s. The *first-visit MC method* estimates $v_\pi(s)$ as the average of the returns following first visits to s, whereas the *every-visit MC method* averages the returns following all visits to s. These two Monte Carlo (MC) methods are very similar but have slightly different theoretical properties. First-visit MC has been most widely studied, dating back to the 1940s, and is the one we focus on in this chapter. Every-visit MC extends more naturally to function approximation and eligibility traces, as discussed in Chapters 9 and 7. First-visit MC is shown in procedural form in Figure 5.1.

```
Initialize:
    $\pi \leftarrow$ policy to be evaluated
    $V \leftarrow$ an arbitrary state-value function
    $Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Repeat forever:
    Generate an episode using $\pi$
    For each state $s$ appearing in the episode:
        $G \leftarrow$ return following the first occurrence of $s$
        Append $G$ to $Returns(s)$
        $V(s) \leftarrow$ average($Returns(s)$)
```

Figure 5.1: The first-visit MC method for estimating v_π.

Both first-visit MC and every-visit MC converge to $v_\pi(s)$ as the number of visits (or first visits) to s goes to infinity. This is easy to see for the case of first-visit MC. In this case each return is an independent, identically distributed estimate of $v_\pi(s)$ with finite variance. By the law of large numbers the sequence of averages of these estimates converges to their expected value. Each average is itself an unbiased estimate, and the standard deviation of its error falls as $1/\sqrt{n}$, where n is the number of returns averaged. Every-visit MC is less straightforward, but its estimates also converge asymptotically to $v_\pi(s)$ (Singh and Sutton, 1996).

The use of Monte Carlo methods is best illustrated through an example.

Example 5.1: Blackjack The object of the popular casino card game of *blackjack* is to obtain cards the sum of whose numerical values is as great as possible without exceeding 21. All face cards count as 10, and an ace can count either as 1 or as 11. We consider the version in which each player competes independently against the dealer. The game begins with two cards dealt to both dealer and player. One of the dealer's cards is face up and the other is face down. If the player has 21 immediately (an ace and a 10-card), it is called a *natural*. He then wins unless the dealer also has a natural, in which case the game is a draw. If the player does not have a natural, then he can request additional cards, one by one (*hits*), until he either stops (*sticks*) or exceeds 21 (*goes bust*). If he goes bust, he loses; if he sticks, then it becomes the dealer's turn. The dealer hits or sticks according to a fixed strategy without choice: he sticks on any sum of 17 or greater, and hits otherwise. If the dealer goes bust, then the player wins; otherwise, the outcome—win, lose, or draw—is determined by whose final sum is closer to 21.

Playing blackjack is naturally formulated as an episodic finite MDP. Each game of blackjack is an episode. Rewards of $+1$, -1, and 0 are given for winning, losing, and drawing, respectively. All rewards within a game are zero, and we do not discount ($\gamma = 1$); therefore these terminal rewards are also the returns. The player's actions are to hit or to stick. The states depend on the player's cards and the dealer's showing card. We assume that cards are dealt from an infinite deck (i.e., with replacement) so that there is no advantage to keeping track of the cards already dealt. If the player holds an ace that he could count as 11 without going bust, then the ace is said to be *usable*. In this case it is always counted as 11 because counting it as 1 would make the sum 11 or less, in which case there is no decision to be made because, obviously, the player should always hit. Thus, the player makes decisions on the basis of three variables: his current sum (12–21), the dealer's one showing card (ace–10), and whether or not he holds a usable ace. This makes for a total of 200 states.

Consider the policy that sticks if the player's sum is 20 or 21, and otherwise hits. To find the state-value function for this policy by a Monte Carlo approach, one simulates many blackjack games using the policy and averages the returns following each state. Note that in this task the same state never recurs within one episode, so there is no difference between first-visit and every-visit MC methods. In this way, we obtained the estimates of the state-value function shown in Figure 5.2. The estimates for states with a usable ace are less certain and less regular because these states are less common. In any event, after 500,000 games the value function is very

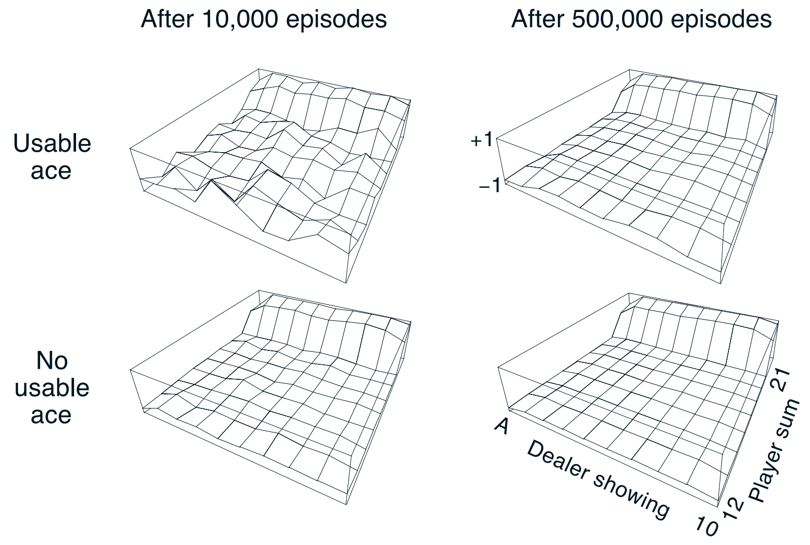


Figure 5.2: Approximate state-value functions for the blackjack policy that sticks only on 20 or 21, computed by Monte Carlo policy evaluation.

well approximated.

Although we have complete knowledge of the environment in this task, it would not be easy to apply DP methods to compute the value function. DP methods require the distribution of next events—in particular, they require the quantities $p(s', r|s, a)$—and it is not easy to determine these for blackjack. For example, suppose the player's sum is 14 and he chooses to stick. What is his expected reward as a function of the dealer's showing card? All of these expected rewards and transition probabilities must be computed *before* DP can be applied, and such computations are often complex and error-prone. In contrast, generating the sample games required by Monte Carlo methods is easy. This is the case surprisingly often; the ability of Monte Carlo methods to work with sample episodes alone can be a significant advantage even when one has complete knowledge of the environment's dynamics.

■

Can we generalize the idea of backup diagrams to Monte Carlo algorithms? The general idea of a backup diagram is to show at the top the root node to be updated and to show below all the transitions and leaf nodes whose rewards and estimated values contribute to the update. For Monte Carlo estimation of v_π, the root is a state node, and below it is the entire trajectory of transitions along a particular single episode, ending at the terminal state, as in Figure 5.3. Whereas the DP diagram (Figure 3.4a) shows all possible transitions, the Monte Carlo diagram shows only those sampled on the one episode. Whereas the DP diagram includes only one-step transitions, the Monte Carlo diagram goes all the way to the end of the episode. These differences in the diagrams accurately reflect the fundamental differences between the algorithms.

An important fact about Monte Carlo methods is that the estimates for each state

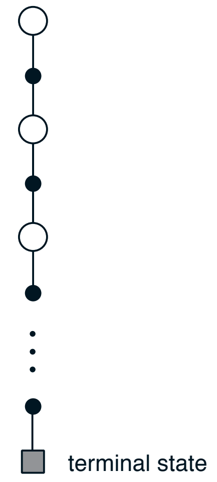
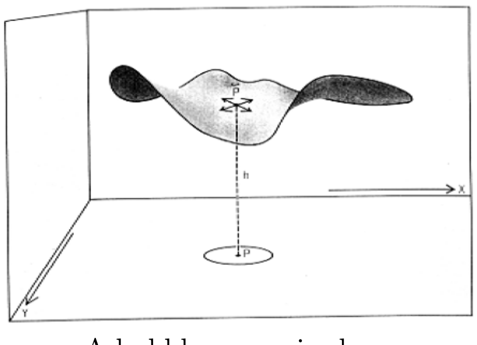


Figure 5.3: The backup diagram for Monte Carlo estimation of v_π.

are independent. The estimate for one state does not build upon the estimate of any other state, as is the case in DP. In other words, Monte Carlo methods do not *bootstrap* as we defined it in the previous chapter.

In particular, note that the computational expense of estimating the value of a single state is independent of the number of states. This can make Monte Carlo methods particularly attractive when one requires the value of only one or a subset of states. One can generate many sample episodes starting from the states of interest, averaging returns from only these states ignoring all others. This is a third advantage Monte Carlo methods can have over DP methods (after the ability to learn from actual experience and from simulated experience).

Example 5.2: Soap Bubble



A bubble on a wire loop

Suppose a wire frame forming a closed loop is dunked in soapy water to form a soap surface or bubble conforming at its edges to the wire frame. If the geometry of the wire frame is irregular but known, how can you compute the shape of the surface? The shape has the property that the total force on each point exerted by neighboring points is zero (or else the shape would change). This means that the surface's height at any point is the average of its heights at points in a small circle around that point. In addition, the surface must meet at its boundaries with the wire frame. The usual approach to problems of this kind is to put a grid over the area covered by the surface and solve for its height at the grid points by an iterative computation. Grid points at the boundary are forced to the wire frame, and all others are adjusted toward the average of the heights of their four nearest neighbors. This process then iterates, much like DP's iterative policy evaluation, and ultimately converges to a close approximation to the desired surface.

This is similar to the kind of problem for which Monte Carlo methods were originally designed. Instead of the iterative computation described above, imagine standing on the surface and taking a random walk, stepping randomly from grid point to neighboring grid point, with equal probability, until you reach the boundary. It turns out that the expected value of the height at the boundary is a close approximation to the height of the desired surface at the starting point (in fact, it is exactly the value computed by the iterative method described above). Thus, one can closely approximate the height of the surface at a point by simply averaging the boundary heights of many walks started at the point. If one is interested in only the value at one point, or any fixed small set of points, then this Monte Carlo method can be far more efficient than the iterative method based on local consistency. ■

Exercise 5.1 Consider the diagrams on the right in Figure 5.2. Why does the estimated value function jump up for the last two rows in the rear? Why does it drop off for the whole last row on the left? Why are the frontmost values higher in the upper diagrams than in the lower?

5.2 Monte Carlo Estimation of Action Values

If a model is not available, then it is particularly useful to estimate *action* values (the values of state–action pairs) rather than *state* values. With a model, state values alone are sufficient to determine a policy; one simply looks ahead one step and chooses whichever action leads to the best combination of reward and next state, as we did in the chapter on DP. Without a model, however, state values alone are not sufficient. One must explicitly estimate the value of each action in order for the values to be useful in suggesting a policy. Thus, one of our primary goals for Monte Carlo methods is to estimate q_*. To achieve this, we first consider the policy evaluation problem for action values.

The policy evaluation problem for action values is to estimate $q_\pi(s, a)$, the expected return when starting in state s, taking action a, and thereafter following policy π. The Monte Carlo methods for this are essentially the same as just presented for state values, except now we talk about visits to a state–action pair rather than to a state. A state–action pair s, a is said to be visited in an episode if ever the state s is visited and action a is taken in it. The every-visit MC method estimates the value of a state–action pair as the average of the returns that have followed visits all the visits to it. The first-visit MC method averages the returns following the first time in each episode that the state was visited and the action was selected. These methods converge quadratically, as before, to the true expected values as the number of visits to each state–action pair approaches infinity.

The only complication is that many state–action pairs may never be visited. If π is a deterministic policy, then in following π one will observe returns only for one of the actions from each state. With no returns to average, the Monte Carlo estimates of the other actions will not improve with experience. This is a serious problem because the purpose of learning action values is to help in choosing among the actions available in each state. To compare alternatives we need to estimate the value of *all* the actions from each state, not just the one we currently favor.

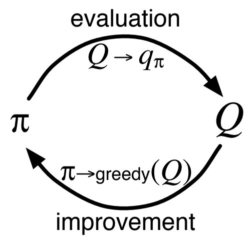
This is the general problem of *maintaining exploration*, as discussed in the context of the k-armed bandit problem in Chapter 2. For policy evaluation to work for action values, we must assure continual exploration. One way to do this is by specifying that the episodes *start in a state–action pair*, and that every pair has a nonzero probability of being selected as the start. This guarantees that all state–action pairs will be visited an infinite number of times in the limit of an infinite number of episodes. We call this the assumption of *exploring starts*.

The assumption of exploring starts is sometimes useful, but of course it cannot be relied upon in general, particularly when learning directly from actual interaction with an environment. In that case the starting conditions are unlikely to be so helpful. The most common alternative approach to assuring that all state–action pairs are encountered is to consider only policies that are stochastic with a nonzero probability of selecting all actions in each state. We discuss two important variants of this approach in later sections. For now, we retain the assumption of exploring starts and complete the presentation of a full Monte Carlo control method.

Exercise 5.2 What is the backup diagram for Monte Carlo estimation of q_π?

5.3 Monte Carlo Control

We are now ready to consider how Monte Carlo estimation can be used in control, that is, to approximate optimal policies. The overall idea is to proceed according to the same pattern as in the DP chapter, that is, according to the idea of generalized policy iteration (GPI). In GPI one maintains both an approximate policy and an approximate value function. The value function is repeatedly altered to more closely approximate the value function for the current policy, and the policy is repeatedly improved with respect to the current value function, as suggested by the diagram to the right. These two kinds of changes work against each other to some extent, as each creates a moving target for the other, but together they cause both policy and value function to approach optimality.



To begin, let us consider a Monte Carlo version of classical policy iteration. In this method, we perform alternating complete steps of policy evaluation and policy improvement, beginning with an arbitrary policy π_0 and ending with the optimal policy and optimal action-value function:

$$\pi_0 \xrightarrow{\text{E}} q_{\pi_0} \xrightarrow{\text{I}} \pi_1 \xrightarrow{\text{E}} q_{\pi_1} \xrightarrow{\text{I}} \pi_2 \xrightarrow{\text{E}} \cdots \xrightarrow{\text{I}} \pi_* \xrightarrow{\text{E}} q_*,$$

where $\xrightarrow{\text{E}}$ denotes a complete policy evaluation and $\xrightarrow{\text{I}}$ denotes a complete policy improvement. Policy evaluation is done exactly as described in the preceding section. Many episodes are experienced, with the approximate action-value function approaching the true function asymptotically. For the moment, let us assume that we do indeed observe an infinite number of episodes and that, in addition, the episodes are generated with exploring starts. Under these assumptions, the Monte Carlo methods will compute each q_{π_k} exactly, for arbitrary π_k.

Policy improvement is done by making the policy greedy with respect to the current value function. In this case we have an *action*-value function, and therefore no model is needed to construct the greedy policy. For any action-value function q, the corresponding greedy policy is the one that, for each $s \in \mathcal{S}$, deterministically chooses an action with maximal action-value:

$$\pi(s) \doteq \arg\max_a q(s,a). \tag{5.1}$$

Policy improvement then can be done by constructing each π_{k+1} as the greedy policy with respect to q_{π_k}. The policy improvement theorem (Section 4.2) then applies to π_k and π_{k+1} because, for all $s \in \mathcal{S}$,

$$\begin{aligned} q_{\pi_k}(s, \pi_{k+1}(s)) &= q_{\pi_k}(s, \arg\max_a q_{\pi_k}(s,a)) \\ &= \max_a q_{\pi_k}(s,a) \\ &\geq q_{\pi_k}(s, \pi_k(s)) \\ &\geq v_{\pi_k}(s). \end{aligned}$$

As we discussed in the previous chapter, the theorem assures us that each π_{k+1} is uniformly better than π_k, or just as good as π_k, in which case they are both optimal policies. This in turn assures us that the overall process converges to the optimal policy and optimal value function. In this way Monte Carlo methods can be used to find optimal policies given only sample episodes and no other knowledge of the environment's dynamics.

We made two unlikely assumptions above in order to easily obtain this guarantee of convergence for the Monte Carlo method. One was that the episodes have exploring starts, and the other was that policy evaluation could be done with an infinite number of episodes. To obtain a practical algorithm we will have to remove both assumptions. We postpone consideration of the first assumption until later in this chapter.

For now we focus on the assumption that policy evaluation operates on an infinite number of episodes. This assumption is relatively easy to remove. In fact, the same issue arises even in classical DP methods such as iterative policy evaluation, which also converge only asymptotically to the true value function. In both DP and Monte Carlo cases there are two ways to solve the problem. One is to hold firm to the idea of approximating q_{π_k} in each policy evaluation. Measurements and assumptions are made to obtain bounds on the magnitude and probability of error in the estimates, and then sufficient steps are taken during each policy evaluation to assure that these bounds are sufficiently small. This approach can probably be made completely satisfactory in the sense of guaranteeing correct convergence up to some level of approximation. However, it is also likely to require far too many episodes to be useful in practice on any but the smallest problems.

The second approach to avoiding the infinite number of episodes nominally required for policy evaluation is to forgo trying to complete policy evaluation before returning to policy improvement. On each evaluation step we move the value function *toward* q_{π_k}, but we do not expect to actually get close except over many steps. We used this idea when we first introduced the idea of GPI in Section 4.6. One extreme form of the idea is value iteration, in which only one iteration of iterative policy evaluation is performed between each step of policy improvement. The in-place version of value iteration is even more extreme; there we alternate between improvement and evaluation steps for single states.

For Monte Carlo policy evaluation it is natural to alternate between evaluation and improvement on an episode-by-episode basis. After each episode, the observed returns are used for policy evaluation, and then the policy is improved at all the states visited in the episode. A complete simple algorithm along these lines is given in Figure 5.4. We call this algorithm *Monte Carlo ES*, for Monte Carlo with Exploring Starts.

In Monte Carlo ES, all the returns for each state–action pair are accumulated and averaged, irrespective of what policy was in force when they were observed. It is easy to see that Monte Carlo ES cannot converge to any suboptimal policy. If it did, then the value function would eventually converge to the value function for that policy, and that in turn would cause the policy to change. Stability is achieved only when both the policy and the value function are optimal. Convergence to this optimal fixed point seems inevitable as the changes to the action-value function decrease over time, but has not yet been formally proved. In our opinion, this is one of the most fundamental open theoretical questions in reinforcement learning (for a partial solution, see Tsitsiklis, 2002).

Example 5.3: Solving Blackjack It is straightforward to apply Monte Carlo ES to blackjack. Since the episodes are all simulated games, it is easy to arrange

```
Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
    $Q(s,a) \leftarrow$ arbitrary
    $\pi(s) \leftarrow$ arbitrary
    $Returns(s,a) \leftarrow$ empty list

Repeat forever:
    Choose $S_0 \in \mathcal{S}$ and $A_0 \in \mathcal{A}(S_0)$ s.t. all pairs have probability $> 0$
    Generate an episode starting from $S_0, A_0$, following $\pi$
    For each pair $s, a$ appearing in the episode:
        $G \leftarrow$ return following the first occurrence of $s, a$
        Append $G$ to $Returns(s,a)$
        $Q(s,a) \leftarrow$ average($Returns(s,a)$)
    For each $s$ in the episode:
        $\pi(s) \leftarrow \arg\max_a Q(s,a)$
```

Figure 5.4: Monte Carlo ES: A Monte Carlo control algorithm assuming exploring starts and that episodes always terminate for all policies.

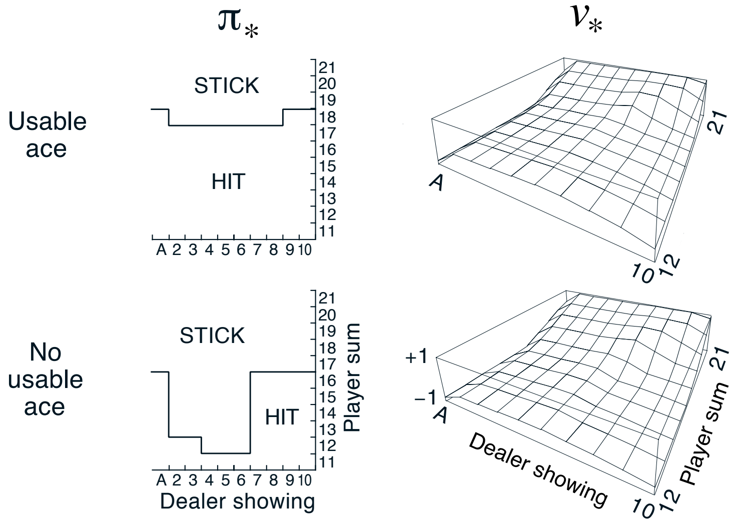


Figure 5.5: The optimal policy and state-value function for blackjack, found by Monte Carlo ES (Figure 5.4). The state-value function shown was computed from the action-value function found by Monte Carlo ES.

for exploring starts that include all possibilities. In this case one simply picks the dealer's cards, the player's sum, and whether or not the player has a usable ace, all at random with equal probability. As the initial policy we use the policy evaluated in the previous blackjack example, that which sticks only on 20 or 21. The initial action-value function can be zero for all state–action pairs. Figure 5.5 shows the optimal policy for blackjack found by Monte Carlo ES. This policy is the same as the "basic" strategy of Thorp (1966) with the sole exception of the leftmost notch in the policy for a usable ace, which is not present in Thorp's strategy. We are uncertain of the reason for this discrepancy, but confident that what is shown here is indeed the optimal policy for the version of blackjack we have described. ■

5.4 Monte Carlo Control without Exploring Starts

How can we avoid the unlikely assumption of exploring starts? The only general way to ensure that all actions are selected infinitely often is for the agent to continue to select them. There are two approaches to ensuring this, resulting in what we call *on-policy* methods and *off-policy* methods. On-policy methods attempt to evaluate or improve the policy that is used to make decisions, whereas off-policy methods evaluate or improve a policy different from that used to generate the data. The Monte Carlo ES method developed above is an example of an on-policy method. In this section we show how an on-policy Monte Carlo control method can be designed that does not use the unrealistic assumption of exploring starts. Off-policy methods are considered in the next section.

In on-policy control methods the policy is generally *soft*, meaning that $\pi(a|s) > 0$ for all $s \in \mathcal{S}$ and all $a \in \mathcal{A}(s)$, but gradually shifted closer and closer to a deterministic optimal policy. Many of the methods discussed in Chapter 2 provide mechanisms for this. The on-policy method we present in this section uses *ε-greedy* policies, meaning that most of the time they choose an action that has maximal estimated action value, but with probability ε they instead select an action at random. That is, all nongreedy actions are given the minimal probability of selection, $\frac{\epsilon}{|\mathcal{A}(s)|}$, and the remaining bulk of the probability, $1 - \varepsilon + \frac{\epsilon}{|\mathcal{A}(s)|}$, is given to the greedy action. The ε-greedy policies are examples of *ε-soft* policies, defined as policies for which $\pi(a|s) \geq \frac{\epsilon}{|\mathcal{A}(s)|}$ for all states and actions, for some $\varepsilon > 0$. Among ε-soft policies, ε-greedy policies are in some sense those that are closest to greedy.

The overall idea of on-policy Monte Carlo control is still that of GPI. As in Monte Carlo ES, we use first-visit MC methods to estimate the action-value function for the current policy. Without the assumption of exploring starts, however, we cannot simply improve the policy by making it greedy with respect to the current value function, because that would prevent further exploration of nongreedy actions. Fortunately, GPI does not require that the policy be taken all the way to a greedy policy, only that it be moved *toward* a greedy policy. In our on-policy method we will move it only to an ε-greedy policy. For any ε-soft policy, π, any ε-greedy policy with respect to q_π is guaranteed to be better than or equal to π.

That any ε-greedy policy with respect to q_π is an improvement over any ε-soft policy π is assured by the policy improvement theorem. Let π' be the ε-greedy policy. The conditions of the policy improvement theorem apply because for any $s \in \mathcal{S}$:

$$\begin{aligned} q_\pi(s, \pi'(s)) &= \sum_a \pi'(a|s) q_\pi(s,a) \\ &= \frac{\epsilon}{|\mathcal{A}(s)|} \sum_a q_\pi(s,a) \;+\; (1-\varepsilon) \max_a q_\pi(s,a) \qquad (5.2) \\ &\geq \frac{\epsilon}{|\mathcal{A}(s)|} \sum_a q_\pi(s,a) \;+\; (1-\varepsilon) \sum_a \frac{\pi(a|s) - \frac{\epsilon}{|\mathcal{A}(s)|}}{1-\varepsilon} q_\pi(s,a) \end{aligned}$$

(the sum is a weighted average with nonnegative weights summing to 1, and as such it must be less than or equal to the largest number averaged)

$$\begin{aligned} &= \frac{\epsilon}{|\mathcal{A}(s)|} \sum_a q_\pi(s,a) \;-\; \frac{\epsilon}{|\mathcal{A}(s)|} \sum_a q_\pi(s,a) \;+\; \sum_a \pi(a|s) q_\pi(s,a) \\ &= v_\pi(s). \end{aligned}$$

Thus, by the policy improvement theorem, $\pi' \geq \pi$ (i.e., $v_{\pi'}(s) \geq v_\pi(s)$, for all $s \in \mathcal{S}$). We now prove that equality can hold only when both π' and π are optimal among the ε-soft policies, that is, when they are better than or equal to all other ε-soft policies.

Consider a new environment that is just like the original environment, except with the requirement that policies be ε-soft "moved inside" the environment. The new environment has the same action and state set as the original and behaves as follows. If in state s and taking action a, then with probability $1 - \varepsilon$ the new environment behaves exactly like the old environment. With probability ε it repicks the action at random, with equal probabilities, and then behaves like the old environment with the new, random action. The best one can do in this new environment with general policies is the same as the best one could do in the original environment with ε-soft policies. Let $\widetilde{v}_*$ and $\widetilde{q}_*$ denote the optimal value functions for the new environment. Then a policy π is optimal among ε-soft policies if and only if $v_\pi = \widetilde{v}_*$. From the definition of $\widetilde{v}_*$ we know that it is the unique solution to

$$\begin{aligned}
\widetilde{v}_*(s) &= (1-\varepsilon)\max_a \widetilde{q}_*(s,a) + \frac{\epsilon}{|\mathcal{A}(s)|}\sum_a \widetilde{q}_*(s,a) \\
&= (1-\varepsilon)\max_a \sum_{s',r} p(s',r|s,a)\Big[r + \gamma\widetilde{v}_*(s')\Big] \\
&\quad + \frac{\epsilon}{|\mathcal{A}(s)|}\sum_a\sum_{s',r} p(s',r|s,a)\Big[r + \gamma\widetilde{v}_*(s')\Big].
\end{aligned}$$

When equality holds and the ε-soft policy π is no longer improved, then we also know, from (5.2), that

$$\begin{aligned}
v_\pi(s) &= (1-\varepsilon)\max_a q_\pi(s,a) + \frac{\epsilon}{|\mathcal{A}(s)|}\sum_a q_\pi(s,a) \\
&= (1-\varepsilon)\max_a \sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_\pi(s')\Big] \\
&\quad + \frac{\epsilon}{|\mathcal{A}(s)|}\sum_a\sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_\pi(s')\Big].
\end{aligned}$$

However, this equation is the same as the previous one, except for the substitution of v_π for $\widetilde{v}_*$. Since $\widetilde{v}_*$ is the unique solution, it must be that $v_\pi = \widetilde{v}_*$.

```
Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
    $Q(s,a) \leftarrow$ arbitrary
    $Returns(s,a) \leftarrow$ empty list
    $\pi(a|s) \leftarrow$ an arbitrary $\varepsilon$-soft policy

Repeat forever:
    (a) Generate an episode using $\pi$
    (b) For each pair $s, a$ appearing in the episode:
            $G \leftarrow$ return following the first occurrence of $s, a$
            Append $G$ to $Returns(s,a)$
            $Q(s,a) \leftarrow$ average($Returns(s,a)$)
    (c) For each $s$ in the episode:
            $A^* \leftarrow \arg\max_a Q(s,a)$
            For all $a \in \mathcal{A}(s)$:
                $\pi(a|s) \leftarrow \begin{cases} 1-\varepsilon+\varepsilon/|\mathcal{A}(s)| & \text{if } a = A^* \\ \varepsilon/|\mathcal{A}(s)| & \text{if } a \neq A^* \end{cases}$
```

Figure 5.6: An on-policy first-visit MC control algorithm for ε-soft policies.

In essence, we have shown in the last few pages that policy iteration works for ε-soft policies. Using the natural notion of greedy policy for ε-soft policies, one is assured of improvement on every step, except when the best policy has been found among the ε-soft policies. This analysis is independent of how the action-value functions are determined at each stage, but it does assume that they are computed exactly. This brings us to roughly the same point as in the previous section. Now we only achieve the best policy among the ε-soft policies, but on the other hand, we have eliminated the assumption of exploring starts. The complete algorithm is given in Figure 5.6.

5.5 Off-policy Prediction via Importance Sampling

All learning control methods face a dilemma: They seek to learn action values conditional on subsequent *optimal* behavior, but they need to behave non-optimally in order to explore all actions (to *find* the optimal actions). How can they learn about the optimal policy while behaving according to an exploratory policy? The on-policy approach in the preceding section is actually a compromise—it learns action values not for the optimal policy, but for a near-optimal policy that still explores. A more straightforward approach is to use two policies, one that is learned about and that becomes the optimal policy, and one that is more exploratory and is used to generate behavior. The policy being learned about is called the *target policy*, and the policy used to generate behavior is called the *behavior policy*. In this case we say that learning is from data "off" the target policy, and the overall process is termed *off-policy learning*.

Throughout the rest of this book we consider both on-policy and off-policy methods. On-policy methods are generally simpler and are considered first. Off-policy methods require additional concepts and notation, and because the data is due to a different policy, off-policy methods are often of greater variance and are slower to converge. On the other hand, off-policy methods are more powerful and general. They include on-policy methods as the special case in which the target and behavior policies are the same. Off-policy methods also have a variety of additional uses in applications. For example, they can often be applied to learn from data generated by a conventional non-learning controller, or from a human expert. Off-policy learning is also seen by some as key to learning multi-step predictive models of the world's dynamics (Sutton, 2009, 2011).

In this section we begin the study of off-policy methods by considering the *prediction* problem, in which both target and behavior policies are fixed. That is, suppose we wish to estimate v_π or q_π, but all we have are episodes following another policy μ, where $\mu \neq \pi$. In this case, π is the target policy, μ is the behavior policy, and both policies are considered fixed and given.

In order to use episodes from μ to estimate values for π, we require that every action taken under π is also taken, at least occasionally, under μ. That is, we require that $\pi(a|s) > 0$ implies $\mu(a|s) > 0$. This is called the assumption of *coverage*. It follows from coverage that μ must be stochastic in states where it is not identical to π. The target policy π, on the other hand, may be deterministic, and, in fact, this is a case of particular interest in control problems. In control, the target policy is typically the deterministic greedy policy with respect to the current action-value function estimate. This policy becomes a deterministic optimal policy while the behavior policy remains stochastic and more exploratory, for example, an ε-greedy policy. In this section, however, we consider the prediction problem, in which π is unchanging and given.

Almost all off-policy methods utilize *importance sampling*, a general technique for estimating expected values under one distribution given samples from another. We apply importance sampling to off-policy learning by weighting returns according to the relative probability of their trajectories occurring under the target and behavior policies, called the *importance-sampling ratio.* Given a starting state S_t, the probability of the subsequent state–action trajectory, $A_t, S_{t+1}, A_{t+1}, \dots, S_T$, occurring under any policy π is

$$\prod_{k=t}^{T-1} \pi(A_k|S_k)p(S_{k+1}|S_k, A_k),$$

where p here is the state-transition probability function defined by (3.8). Thus, the relative probability of the trajectory under the target and behavior policies (the importance-sampling ratio) is

$$\rho_t^T \doteq \frac{\prod_{k=t}^{T-1} \pi(A_k|S_k)p(S_{k+1}|S_k, A_k)}{\prod_{k=t}^{T-1} \mu(A_k|S_k)p(S_{k+1}|S_k, A_k)} = \prod_{k=t}^{T-1} \frac{\pi(A_k|S_k)}{\mu(A_k|S_k)}. \tag{5.3}$$

Note that although the trajectory probabilities depend on the MDP's transition probabilities, which are generally unknown, all the transition probabilities cancel. The importance sampling ratio ends up depending only on the two policies and not at all on the MDP.

Now we are ready to give a Monte Carto algorithm that uses a batch of observed episodes following policy μ to estimate $v_\pi(s)$. It is convenient here to number time steps in a way that increases across episode boundaries. That is, if the first episode of the batch ends in a terminal state at time 100, then the next episode begins at time $t = 101$. This enables us to use time-step numbers to refer to particular steps in particular episodes. In particular, we can define the set of all time steps in which state s is visited, denoted $\mathcal{T}(s)$. This is for an every-visit method; for a first-visit method, $\mathcal{T}(s)$ would only include time steps that were first visits to s within their episodes. Also, let $T(t)$ denote the first time of termination following time t, and G_t denote the return after t up through $T(t)$. Then $\{G_t\}_{t\in\mathcal{T}(s)}$ are the returns that pertain to state s, and $\{\rho_t^{T(t)}\}_{t\in\mathcal{T}(s)}$ are the corresponding importance-sampling ratios. To estimate $v_\pi(s)$, we simply scale the returns by the ratios and average the results:

$$V(s) \doteq \frac{\sum_{t\in\mathcal{T}(s)} \rho_t^{T(t)} G_t}{|\mathcal{T}(s)|}. \tag{5.4}$$

When importance sampling is done as a simple average in this way it is called *ordinary importance sampling.*

An important alternative is *weighted importance sampling*, which uses a *weighted* average, defined as

$$V(s) \doteq \frac{\sum_{t\in\mathfrak{T}(s)} \rho_t^{T(t)} G_t}{\sum_{t\in\mathfrak{T}(s)} \rho_t^{T(t)}}, \tag{5.5}$$

or zero if the denominator is zero. To understand these two varieties of importance sampling, consider their estimates after observing a single return. In the weighted-average estimate, the ratio $\rho_t^{T(t)}$ for the single return cancels in the numerator and denominator, so that the estimate is equal to the observed return independent of the ratio (assuming the ratio is nonzero). Given that this return was the only one observed, this is a reasonable estimate, but of course its expectation is $v_\mu(s)$ rather than $v_\pi(s)$, and in this statistical sense it is biased. In contrast, the simple average (5.4) is always $v_\pi(s)$ in expectation (it is unbiased), but it can be extreme. Suppose the ratio were ten, indicating that the trajectory observed is ten times as likely under the target policy as under the behavior policy. In this case the ordinary importance-sampling estimate would be *ten times* the observed return. That is, it would be quite far from the observed return even though the episode's trajectory is considered very representative of the target policy.

Formally, the difference between the two kinds of importance sampling is expressed in their biases and variances. The ordinary importance-sampling estimator is unbiased whereas the weighted importance-sampling estimator is biased (the bias converges asymptotically to zero). On the other hand, the variance of the ordinary importance-sampling estimator is in general unbounded because the variance of the ratios can be unbounded, whereas in the weighted estimator the largest weight on any single return is one. In fact, assuming bounded returns, the variance of the weighted importance-sampling estimator converges to zero even if the variance of the ratios themselves is infinite (Precup, Sutton, and Dasgupta 2001). In practice, the weighted estimator usually has dramatically lower variance and is strongly preferred. Nevertheless, we will not totally abandon ordinary importance sampling as it is easier to extend to the approximate methods using function approximation that we explore in the second part of this book.

A complete every-visit MC algorithm for off-policy policy evaluation using weighted importance sampling is given in the next section in Figure 5.9.

Example 5.4: Off-policy Estimation of a Blackjack State Value
We applied both ordinary and weighted importance-sampling methods to estimate the value of a single blackjack state from off-policy data. Recall that one of the advantages of Monte Carlo methods is that they can be used to evaluate a single state without forming estimates for any other states. In this example, we evaluated the state in which the dealer is showing a deuce, the sum of the player's cards is 13, and the player has a usable ace (that is, the player holds an ace and a deuce, or equivalently three aces). The data was generated by starting in this state then

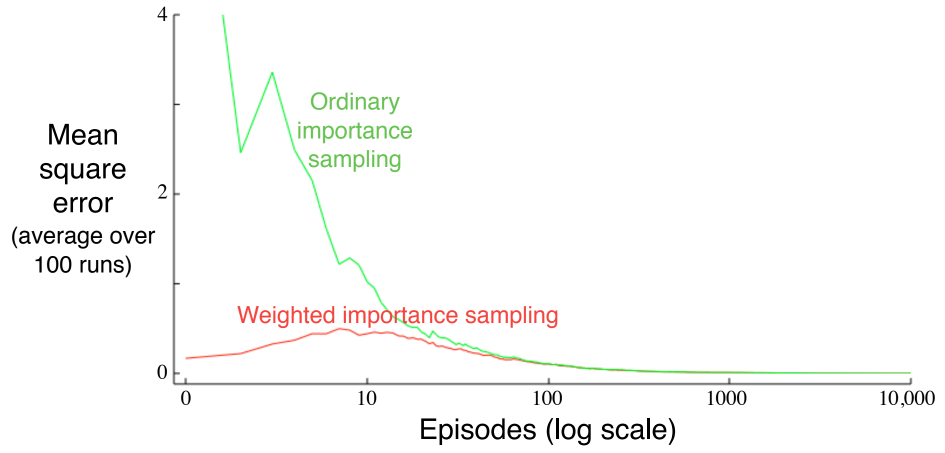


Figure 5.7: Weighted importance sampling produces lower error estimates of the value of a single blackjack state from off-policy episodes (see Example 5.4).

choosing to hit or stick at random with equal probability (the behavior policy). The target policy was to stick only on a sum of 20 or 21, as in Example 5.1. The value of this state under the target policy is approximately -0.27726 (this was determined by separately generating one-hundred million episodes using the target policy and averaging their returns). Both off-policy methods closely approximated this value after 1000 off-policy episodes using the random policy. Figure 5.7 shows the mean squared error (estimated from 100 independent runs) for each method as a function of number of episodes. The weighted importance-sampling method has much lower overall error in this example, as is typical in practice. ■

Example 5.5: Infinite Variance
The estimates of ordinary importance sampling will typically have infinite variance, and thus unsatisfactory convergence properties, whenever the scaled returns have infinite variance—and this can easily happen in off-policy learning when trajectories contain loops. A simple example is shown inset in Figure 5.8. There is only one nonterminal state s and two actions, end and back. The end action causes a deterministic transition to termination, whereas the back action transitions, with probability 0.9, back to s or, with probability 0.1, on to termination. The rewards are +1 on the latter transition and otherwise zero. Consider the target policy that always selects back. All episodes under this policy consist of some number (possibly zero) of transitions back to s followed by termination with a reward and return of +1. Thus the value of s under the target policy is 1. Suppose we are estimating this value from off-policy data using the behavior policy that selects end and back with equal probability.

The lower part of Figure 5.8 shows ten independent runs of the first-visit MC algorithm using ordinary importance sampling. Even after millions of episodes, the estimates fail to converge to the correct value of 1. In contrast, the weighted importance-sampling algorithm would give an estimate of exactly 1 everafter the

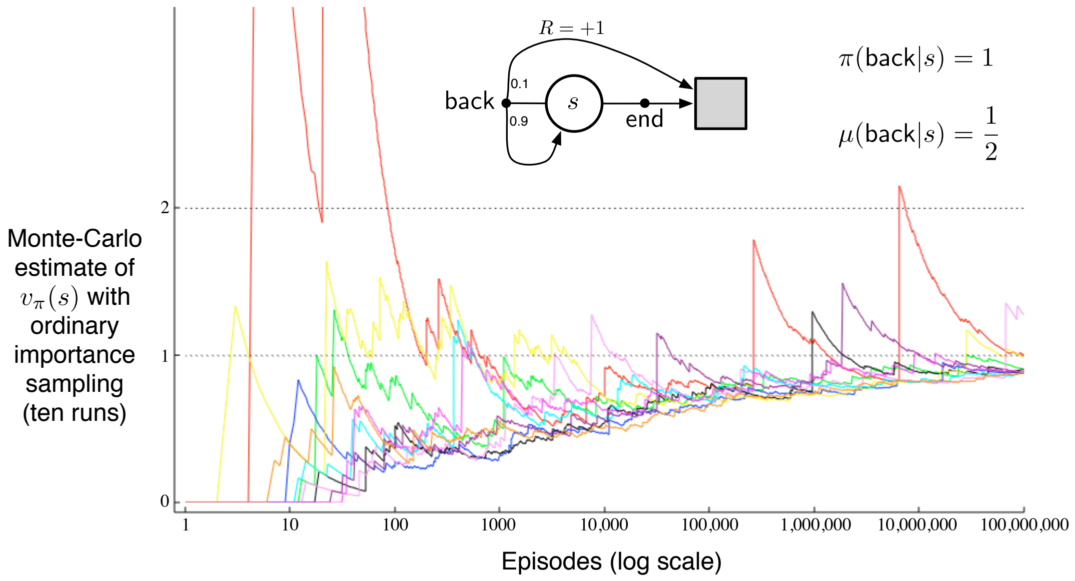


Figure 5.8: Ordinary importance sampling produces surprisingly unstable estimates on the one-state MDP shown inset (Example 5.5). The correct estimate here is 1, and, even though this is the expected value of a sample return (after importance sampling), the variance of the samples is infinite, and the estimates do not convergence to this value. These results are for off-policy first-visit MC.

first episode that was consistent with the target policy (i.e., that ended with the back action). This is clear because that algorithm produces a weighted average of the returns consistent with the target policy, all of which would be exactly 1.

We can verify that the variance of the importance-sampling-scaled returns is infinite in this example by a simple calculation. The variance of any random variable X is the expected value of the deviation from its mean $\bar{X}$, which can be written

$$\mathrm{Var}[X] \doteq \mathbb{E}\Big[\big(X - \bar{X}\big)^2\Big] = \mathbb{E}\big[X^2 - 2X\bar{X} + \bar{X}^2\big] = \mathbb{E}\big[X^2\big] - \bar{X}^2.$$

Thus, if the mean is finite, as it is in our case, the variance is infinite if and only if the expectation of the square of the random variable is infinite. Thus, we need only show that the expected square of the importance-sampling-scaled return is infinite:

$$\mathbb{E}\left[\left(\prod_{t=0}^{T-1} \frac{\pi(A_t|S_t)}{\mu(A_t|S_t)} G_0\right)^2\right].$$

To compute this expectation, we break it down into cases based on episode length and termination. First note that, for any episode ending with the end action, the importance sampling ratio is zero, because the target policy would never take this action; these episodes thus contribute nothing to the expectation (the quantity in parenthesis will be zero) and can be ignored. We need only consider episodes that involve some number (possibly zero) of back actions that transition back to the nonterminal state, followed by a back action transitioning to termination. All of these episodes have a return of 1, so the G_0 factor can be ignored. To get the expected square we need only consider each length of episode, multiplying the probability of the episode's occurrence by the square of its importance-sampling ratio, and add these up:

$$\begin{aligned}
&= \frac{1}{2}\cdot 0.1\left(\frac{1}{0.5}\right)^2 && \text{(the length 1 episode)}\\
&\quad + \frac{1}{2}\cdot 0.9\cdot\frac{1}{2}\cdot 0.1\left(\frac{1}{0.5}\frac{1}{0.5}\right)^2 && \text{(the length 2 episode)}\\
&\quad + \frac{1}{2}\cdot 0.9\cdot\frac{1}{2}\cdot 0.9\cdot\frac{1}{2}\cdot 0.1\left(\frac{1}{0.5}\frac{1}{0.5}\frac{1}{0.5}\right)^2 && \text{(the length 3 episode)}\\
&\quad + \cdots\\
&= 0.1\sum_{k=0}^{\infty} 0.9^k\cdot 2^k\cdot 2\\
&= 0.2\sum_{k=0}^{\infty} 1.8^k\\
&= \infty.
\end{aligned}$$

■

Exercise 5.3 What is the equation analogous to (5.5) for *action* values $Q(s,a)$ instead of state values $V(s)$, again given returns generated using μ?

Exercise 5.4 In learning curves such as those shown in Figure 5.7 error generally decreases with training, as indeed happened for the ordinary importance-sampling method. But for the weighted importance-sampling method error first increased and then decreased. Why do you think this happened?

Exercise 5.5 The results with Example 5.5 and shown in Figure 5.8 used a first-visit MC method. Suppose that instead an every-visit MC method was used on the same problem. Would the variance of the estimator still be infinite? Why or why not?

5.6 Incremental Implementation

Monte Carlo prediction methods can be implemented incrementally, on an episode-by-episode basis, using extensions of the techniques described in Chapter 2 (Section 2.3). Whereas in Chapter 2 we averaged *rewards*, in Monte Carlo methods we average *returns*. In all other respects exactly the same methods as used in Chapter 2 can be used for *on-policy* Monte Carlo methods. For *off-policy* Monte Carlo methods, we need to separately consider those that use *ordinary* importance sampling and those that use *weighted* importance sampling.

In ordinary importance sampling, the returns are scaled by the importance sampling ratio $\rho_t^{T(t)}$ (5.3), then simply averaged. For these methods we can again use the incremental methods of Chapter 2, but using the scaled returns in place of the rewards of that chapter. This leaves the case of off-policy methods using *weighted* importance sampling. Here we have to form a weighted average of the returns, and a slightly different incremental algorithm is required.

Suppose we have a sequence of returns $G_1, G_2, \dots, G_{n-1}$, all starting in the same state and each with a corresponding random weight W_i (e.g., $W_i = \rho_t^{T(t)}$). We wish to form the estimate

$$V_n \doteq \frac{\sum_{k=1}^{n-1} W_k G_k}{\sum_{k=1}^{n-1} W_k}, \qquad n \geq 2, \tag{5.6}$$

and keep it up-to-date as we obtain a single additional return G_n. In addition to keeping track of V_n, we must maintain for each state the cumulative sum C_n of the weights given to the first n returns. The update rule for V_n is

$$V_{n+1} \doteq V_n + \frac{W_n}{C_n}\Big[G_n - V_n\Big], \qquad n \geq 1, \tag{5.7}$$

and

$$C_{n+1} \doteq C_n + W_{n+1},$$

where $C_0 \doteq 0$ (and V_1 is arbitrary and thus need not be specified). Figure 5.9 gives a complete episode-by-episode incremental algorithm for Monte Carlo policy evaluation. The algorithm is nominally for the off-policy case, using weighted importance

```
Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
    $Q(s,a) \leftarrow$ arbitrary
    $C(s,a) \leftarrow 0$
    $\mu(a|s) \leftarrow$ an arbitrary soft behavior policy
    $\pi(a|s) \leftarrow$ an arbitrary target policy

Repeat forever:
    Generate an episode using $\mu$:
        $S_0, A_0, R_1, \dots, S_{T-1}, A_{T-1}, R_T, S_T$
    $G \leftarrow 0$
    $W \leftarrow 1$
    For $t = T-1, T-2, \dots$ downto 0:
        $G \leftarrow \gamma G + R_{t+1}$
        $C(S_t, A_t) \leftarrow C(S_t, A_t) + W$
        $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{W}{C(S_t,A_t)}\left[G - Q(S_t, A_t)\right]$
        $W \leftarrow W \frac{\pi(A_t|S_t)}{\mu(A_t|S_t)}$
        If $W = 0$ then ExitForLoop
```

Figure 5.9: An incremental every-visit MC policy-evaluation algorithm, using weighted importance sampling. The approximation Q converges to q_π (for all encountered state–action pairs) even though all actions are selected according to a potentially different policy, μ. In the on-policy case ($\pi = \mu$), W is always 1.

sampling, but applies as well to the on-policy case just by choosing the target and behavior policies as the same.

Exercise 5.6 Modify the algorithm for first-visit MC policy evaluation (Figure 5.1) to use the incremental implementation for sample averages described in Section 2.3.

Exercise 5.7 Derive the weighted-average update rule (5.7) from (5.6). Follow the pattern of the derivation of the unweighted rule (2.3).

5.7 Off-Policy Monte Carlo Control

We are now ready to present an example of the second class of learning control methods we consider in this book: off-policy methods. Recall that the distinguishing feature of on-policy methods is that they estimate the value of a policy while using it for control. In off-policy methods these two functions are separated. The policy used to generate behavior, called the *behavior* policy, may in fact be unrelated to the policy that is evaluated and improved, called the *target* policy. An advantage of this separation is that the target policy may be deterministic (e.g., greedy), while the behavior policy can continue to sample all possible actions.

Off-policy Monte Carlo control methods use one of the techniques presented in the preceding two sections. They follow the behavior policy while learning about and improving the target policy. These techniques requires that the behavior policy has a nonzero probability of selecting all actions that might be selected by the target policy (coverage). To explore all possibilities, we require that the behavior policy be

```
Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
    $Q(s,a) \leftarrow$ arbitrary
    $C(s,a) \leftarrow 0$
    $\pi(s) \leftarrow$ a deterministic policy that is greedy with respect to $Q$

Repeat forever:
    Generate an episode using any soft policy $\mu$:
        $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T, S_T$
    $G \leftarrow 0$
    $W \leftarrow 1$
    For $t = T-1, T-2, \ldots$ downto 0:
        $G \leftarrow \gamma G + R_{t+1}$
        $C(S_t, A_t) \leftarrow C(S_t, A_t) + W$
        $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{W}{C(S_t,A_t)} [G - Q(S_t, A_t)]$
        $\pi(S_t) \leftarrow \arg\max_a Q(S_t, a)$    (with ties broken consistently)
        If $A_t \neq \pi(S_t)$ then ExitForLoop
        $W \leftarrow W \frac{1}{\mu(A_t|S_t)}$
```

Figure 5.10: An off-policy every-visit MC control algorithm, using weighted importance sampling. The policy π converges to optimal at all encountered states even though actions are selected according to a different soft policy μ, which may change between or even within episodes.

soft (i.e., that it select all actions in all states with nonzero probability).

Figure 5.10 shows an off-policy Monte Carlo method, based on GPI and weighted importance sampling, for estimating q_*. The target policy π is the greedy policy with respect to Q, which is an estimate of q_π. The behavior policy μ can be anything, but in order to assure convergence of π to the optimal policy, an infinite number of returns must be obtained for each pair of state and action. This can be assured by choosing μ to be ε-soft.

A potential problem is that this method learns only from the *tails* of episodes, after the last nongreedy action. If nongreedy actions are frequent, then learning will be slow, particularly for states appearing in the early portions of long episodes. Potentially, this could greatly slow learning. There has been insufficient experience with off-policy Monte Carlo methods to assess how serious this problem is. If it is serious, the most important way to address it is probably by incorporating temporal-difference learning, the algorithmic idea developed in the next chapter. Alternatively, if γ is less than 1, then the idea developed in the next section may also help significantly.

Exercise 5.8: Racetrack (programming) Consider driving a race car around a turn like those shown in Figure 5.11. You want to go as fast as possible, but not so fast as to run off the track. In our simplified racetrack, the car is at one of a discrete set of grid positions, the cells in the diagram. The velocity is also discrete, a number of grid cells moved horizontally and vertically per time step. The actions are increments to the velocity components. Each may be changed by $+1$, -1, or 0 in one step, for a total of nine actions. Both velocity components are restricted to be nonnegative and less than 5, and they cannot both be zero. Each episode begins in one of the randomly selected start states and ends when the car crosses the finish line. The rewards are -1 for each step that stays on the track, and -5 if the agent tries to drive off the track. Actually leaving the track is not allowed, but the position is always advanced by at least one cell along either the horizontal or vertical axes.

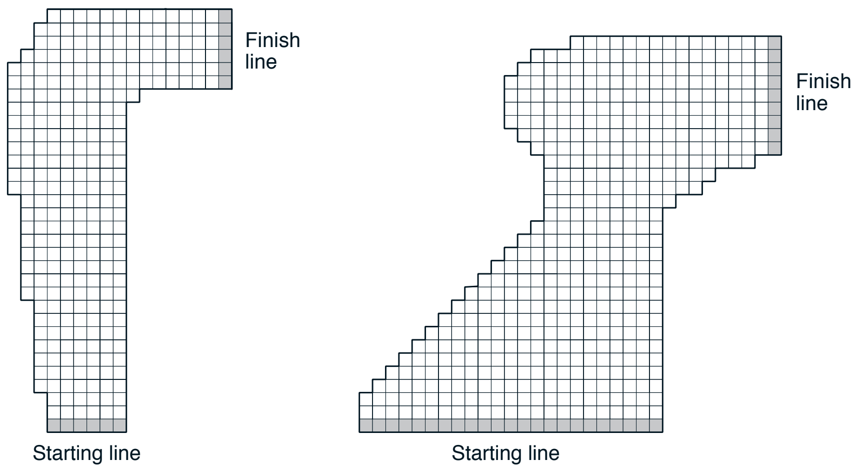


Figure 5.11: A couple of right turns for the racetrack task.

With these restrictions and considering only right turns, such as shown in the figure, all episodes are guaranteed to terminate, yet the optimal policy is unlikely to be excluded. To make the task more challenging, we assume that on half of the time steps the position is displaced forward or to the right by one additional cell beyond that specified by the velocity. Apply a Monte Carlo control method to this task to compute the optimal policy from each starting state. Exhibit several trajectories following the optimal policy.

*5.8 Return-Specific Importance Sampling

The off-policy methods that we have considered so far are based on forming importance-sampling weights for returns considered as unitary wholes, without taking into account the returns' internal structures as sums of discounted rewards. In this section we briefly consider cutting-edge research ideas for using this structure to significantly reduce the variance of off-policy estimators.

For example, consider the case where episodes are long and γ is significantly less than 1. For concreteness, say that episodes last 100 steps and that $\gamma = 0$. The return from time 0 will then be just $G_0 = R_1$, but its importance sampling ratio will be a product of 100 factors, $\frac{\pi(A_0|S_0)}{\mu(A_0|S_0)}\frac{\pi(A_1|S_1)}{\mu(A_1|S_1)}\cdots\frac{\pi(A_{99}|S_{99})}{\mu(A_{99}|S_{99})}$. In ordinary importance sampling, the return will be scaled by the entire product, but it is really only necessary to scale by the first factor, by $\frac{\pi(A_0|S_0)}{\mu(A_0|S_0)}$. The other 99 factors $\frac{\pi(A_1|S_1)}{\mu(A_1|S_1)}\cdots\frac{\pi(A_{99}|S_{99})}{\mu(A_{99}|S_{99})}$ are irrelevant because after the first reward the return has already been determined. These later factors are all independent of the return and of expected value 1; they do not change the expected update, but they add enormously to its variance. In some cases they could even make the variance infinite. Let us now consider an idea for avoiding this large extraneous variance.

The essence of the idea is to think of discounting as determining a probability of termination or, equivalently, a *degree* of partial termination. For any $\gamma \in [0, 1)$, we can think of the return G_0 as partly terminating in one step, to the degree $1 - \gamma$, producing a return of just the first reward, R_1, and as partly terminating after two steps, to the degree $(1 - \gamma)\gamma$, producing a return of $R_1 + R_2$, and so on. The latter degree corresponds to terminating on the second step, $1 - \gamma$, and not having already terminated on the first step, γ. The degree of termination on the third step is thus $(1 - \gamma)\gamma^2$, with the γ^2 reflecting that termination did not occur on either of the first two steps. The partial returns here are called *flat partial returns*:

$$\bar{G}_t^h \doteq R_{t+1} + R_{t+2} + \cdots + R_h, \qquad 0 \leq t < h \leq T,$$

where "flat" denotes the absence of discounting, and "partial" denotes that these returns do not extend all the way to termination but instead stop at h, called the *horizon* (and T is the time of termination of the episode). The conventional full return G_t can be viewed as a sum of flat partial returns as suggested above as follows:

$$\begin{aligned}
G_t &\doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots + \gamma^{T-t-1} R_T \\
&= (1-\gamma) R_{t+1} \\
&\quad + (1-\gamma)\gamma \left(R_{t+1} + R_{t+2}\right) \\
&\quad + (1-\gamma)\gamma^2 \left(R_{t+1} + R_{t+2} + R_{t+3}\right) \\
&\quad \vdots \\
&\quad + (1-\gamma)\gamma^{T-t-2} \left(R_{t+1} + R_{t+2} + \cdots + R_{T-1}\right) \\
&\quad + \gamma^{T-t-1} \left(R_{t+1} + R_{t+2} + \cdots + R_T\right) \\
&= (1-\gamma) \sum_{h=t+1}^{T-1} \gamma^{h-t-1} \bar{G}_t^h \quad + \quad \gamma^{T-t-1} \bar{G}_t^T
\end{aligned}$$

Now we need to scale the flat partial returns by an importance sampling ratio that is similarly truncated. As G_t^h only involves rewards up to a horizon h, we only need the ratio of the probabilities up to h. We define an ordinary importance-sampling estimator, analogous to (5.4), as

$$V(s) \doteq \frac{\sum_{t \in \mathfrak{T}(s)} \left((1-\gamma) \sum_{h=t+1}^{T(t)-1} \gamma^{h-t-1} \rho_t^h \bar{G}_t^h \quad + \quad \gamma^{T(t)-t-1} \rho_t^{T(t)} \bar{G}_t^{T(t)} \right)}{|\mathfrak{T}(s)|}, \tag{5.8}$$

and a weighted importance-sampling estimator, analogous to (5.5), as

$$V(s) \doteq \frac{\sum_{t \in \mathfrak{T}(s)} \left((1-\gamma) \sum_{h=t+1}^{T(t)-1} \gamma^{h-t-1} \rho_t^h \bar{G}_t^h \quad + \quad \gamma^{T(t)-t-1} \rho_t^{T(t)} \bar{G}_t^{T(t)} \right)}{\sum_{t \in \mathfrak{T}(s)} \left((1-\gamma) \sum_{h=t+1}^{T(t)-1} \gamma^{h-t-1} \rho_t^h \quad + \quad \gamma^{T(t)-t-1} \rho_t^{T(t)} \right)}. \tag{5.9}$$

We call these two estimators *discounting-aware* importance sampling estimators. They take into account the discount rate but have no affect (are the same as the off-policy estimators from Section 5.5) if $\gamma = 1$.

There is one more way in which the structure of the return as a sum of rewards can be taken into account in off-policy importance sampling, a way that may be able to reduce variance even in the absence of discounting (that is, even if $\gamma = 1$). In the off-policy estimators (5.4) and (5.5), each term of the sum in the numerator is itself a sum:

$$\begin{aligned}
\rho_t^T G_t &= \rho_t^T \left(R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{T-t-1} R_T \right) \\
&= \rho_t^T R_{t+1} + \gamma \rho_t^T R_{t+2} + \cdots + \gamma^{T-t-1} \rho_t^T R_T. \qquad (5.10)
\end{aligned}$$

The off-policy estimators rely on the expected values of these terms; let us see if we can write them in a simpler way. Note that each sub-term of (5.10) is a product of a random reward and a random importance-sampling ratio. For example, the first sub-term can be written, using (5.3), as

$$\rho_t^T R_{t+1} = \frac{\pi(A_t|S_t)}{\mu(A_t|S_t)} \frac{\pi(A_{t+1}|S_{t+1})}{\mu(A_{t+1}|S_{t+1})} \frac{\pi(A_{t+2}|S_{t+2})}{\mu(A_{t+2}|S_{t+2})} \cdots \frac{\pi(A_{T-1}|S_{T-1})}{\mu(A_{T-1}|S_{T-1})} R_{t+1}.$$

Now notice that, of all these factors, only the first and the last (the reward) are correlated; all the other ratios are independent random variables whose expected value is one:

$$\mathbb{E}_{A_k \sim \mu}\left[\frac{\pi(A_k|S_k)}{\mu(A_k|S_k)}\right] = \sum_a \mu(a|S_k)\frac{\pi(a|S_k)}{\mu(a|S_k)} = \sum_a \pi(a|S_k) = 1.$$

Thus, because the expectation of the product of independent random variables is the product of their expectations, all the ratios except the first drop out in expectation, leaving just

$$\mathbb{E}\left[\rho_t^T R_{t+1}\right] = \mathbb{E}\left[\rho_t^{t+1} R_{t+1}\right].$$

If we repeat this analysis for the kth term of (5.10), we get

$$\mathbb{E}\left[\rho_t^T R_{t+k}\right] = \mathbb{E}\left[\rho_t^{t+k} R_{t+k}\right].$$

It follows then that the expectation of our original term (5.10) can be written

$$\mathbb{E}\left[\rho_t^T G_t\right] = \mathbb{E}\left[\tilde{G}_t\right],$$

where

$$\tilde{G}_t = \rho_t^{t+1} R_{t+1} + \gamma \rho_t^{t+2} R_{t+2} + \gamma^2 \rho_t^{t+3} R_{t+3} + \cdots + \gamma^{T-t-1} \rho_t^T R_T.$$

We call this idea *per-reward* importance sampling. It follows immediately that there is an alternate importance-sampling estimator, with the same unbiased expectation as the OIS estimator (5.4), using $\tilde{G}_t$:

$$V(s) \doteq \frac{\sum_{t \in \mathcal{T}(s)} \tilde{G}_t}{|\mathcal{T}(s)|}, \tag{5.11}$$

which we might expect to sometimes be of lower variance.

Is there a per-reward version of *weighted* importance sampling? This is less clear. So far, all the estimators that have been proposed for this that we know of are not consistent.

***Exercise 5.9** Modify the algorithm for off-policy Monte Carlo control (Figure 5.10) to use the idea of the truncated weighted-average estimator (5.9). Note that you will first need to convert this equation to action values.

5.9 Summary

The Monte Carlo methods presented in this chapter learn value functions and optimal policies from experience in the form of *sample episodes.* This gives them at least three kinds of advantages over DP methods. First, they can be used to learn optimal behavior directly from interaction with the environment, with no model of the environment's dynamics. Second, they can be used with simulation or *sample models*. For surprisingly many applications it is easy to simulate sample episodes even though it is difficult to construct the kind of explicit model of transition probabilities required by DP methods. Third, it is easy and efficient to *focus* Monte Carlo methods on a small subset of the states. A region of special interest can be accurately evaluated without going to the expense of accurately evaluating the rest of the state set (we explore this further in Chapter 8).

A fourth advantage of Monte Carlo methods, which we discuss later in the book, is that they may be less harmed by violations of the Markov property. This is because they do not update their value estimates on the basis of the value estimates of successor states. In other words, it is because they do not bootstrap.

In designing Monte Carlo control methods we have followed the overall schema of *generalized policy iteration* (GPI) introduced in Chapter 4. GPI involves interacting processes of policy evaluation and policy improvement. Monte Carlo methods provide an alternative policy evaluation process. Rather than use a model to compute the value of each state, they simply average many returns that start in the state. Because a state's value is the expected return, this average can become a good approximation to the value. In control methods we are particularly interested in approximating action-value functions, because these can be used to improve the policy without requiring a model of the environment's transition dynamics. Monte Carlo methods intermix policy evaluation and policy improvement steps on an episode-by-episode basis, and can be incrementally implemented on an episode-by-episode basis.

Maintaining *sufficient exploration* is an issue in Monte Carlo control methods. It is not enough just to select the actions currently estimated to be best, because then no returns will be obtained for alternative actions, and it may never be learned that they are actually better. One approach is to ignore this problem by assuming that episodes begin with state–action pairs randomly selected to cover all possibilities. Such *exploring starts* can sometimes be arranged in applications with simulated episodes, but are unlikely in learning from real experience. In *on-policy* methods, the agent commits to always exploring and tries to find the best policy that still explores. In *off-policy* methods, the agent also explores, but learns a deterministic optimal policy that may be unrelated to the policy followed.

Off-policy prediction refers to learning the value function of a *target policy* from data generated by a different *behavior policy*. Such learning methods are based on some form of *importance sampling*, that is, on weighting returns by the ratio of the probabilities of taking the observed actions under the two policies. *Ordinary importance sampling* uses a simple average of the weighted returns, whereas *weighted importance sampling* uses a weighted average. Ordinary importance sampling produces unbiased estimates, but has larger, possibly infinite, variance, whereas weighted importance sampling always has finite variance and are preferred in practice. Despite their conceptual simplicity, off-policy Monte Carlo methods for both prediction and control remain unsettled and are a subject of ongoing research.

The Monte Carlo methods treated in this chapter differ from the DP methods treated in the previous chapter in two major ways. First, they operate on sample experience, and thus can be used for direct learning without a model. Second, they do not bootstrap. That is, they do not update their value estimates on the basis of other value estimates. These two differences are not tightly linked, and can be separated. In the next chapter we consider methods that learn from experience, like Monte Carlo methods, but also bootstrap, like DP methods.

Bibliographical and Historical Remarks

The term "Monte Carlo" dates from the 1940s, when physicists at Los Alamos devised games of chance that they could study to help understand complex physical phenomena relating to the atom bomb. Coverage of Monte Carlo methods in this sense can be found in several textbooks (e.g., Kalos and Whitlock, 1986; Rubinstein, 1981).

An early use of Monte Carlo methods to estimate action values in a reinforcement learning context was by Michie and Chambers (1968). In pole balancing (Example 3.4), they used averages of episode durations to assess the worth (expected balancing "life") of each possible action in each state, and then used these assessments to control action selections. Their method is similar in spirit to Monte Carlo ES with every-visit MC estimates. Narendra and Wheeler (1986) studied a Monte Carlo method for ergodic finite Markov chains that used the return accumulated from one visit to a state to the next as a reward for adjusting a learning automaton's action probabilities.

Barto and Duff (1994) discussed policy evaluation in the context of classical Monte Carlo algorithms for solving systems of linear equations. They used the analysis of Curtiss (1954) to point out the computational advantages of Monte Carlo policy evaluation for large problems. Singh and Sutton (1996) distinguished between every-visit and first-visit MC methods and proved results relating these methods to reinforcement learning algorithms.

The blackjack example is based on an example used by Widrow, Gupta, and Maitra (1973). The soap bubble example is a classical Dirichlet problem whose Monte Carlo solution was first proposed by Kakutani (1945; see Hersh and Griego, 1969; Doyle and Snell, 1984). The racetrack exercise is adapted from Barto, Bradtke, and Singh (1995), and from Gardner (1973).

Monte Carlo ES was introduced in the 1998 edition of this book. That may have been the first explicit connection between Monte Carlo estimation and control methods based on policy iteration.

Efficient off-policy learning has become recognized as an important challenge that arises in several fields. For example, it is closely related to the idea of "interventions" and "counterfactuals" in probabalistic graphical (Bayesian) models (e.g., Pearl, 1995; Balke and Pearl, 1994). Off-policy methods using importance sampling have a long history and yet still are not well understood. Weighted importance sampling, which is also sometimes called normalized importance sampling (e.g., Koller and Friedman, 2009), is discussed by Rubinstein (1981), Hesterberg (1988), Shelton (2001), and Liu (2001) among others. Combining off-policy learning with temporal-difference learning and approximation methods introduces subtle issues that we consider in later chapters.

The target policy in off-policy learning is sometimes referred to in the literature as the “estimation” policy, as it was in the first edition of this book.

Our treatment of the idea of discounting-aware importance sampling is based on the analysis and “forward view” of Sutton, Mahmood, Precup, and van Hasselt (2014). Per-reward importance sampling was introduced by Precup, Sutton, and Singh (2000), who called it “per-decision” importance sampling.