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Robust Output Regulation for Uncertain Discrete-Time Linear Systems under the Effect of a Sinusoidal Disturbance ⋆

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Abstract: This paper studies the robust output regulation problem of uncertain single-input single-output (SISO) discrete-time linear systems. To reject the effect of a sinusoidal disturbance that only the frequency information is known prior, a novel strategy based on adaptive feedforward control (AFC) is developed. Compared with existing regulators for uncertain discrete-time systems, neither the knowledge on the sign of the real part or the imaginary part of the transfer function at the frequency of interest (the so-called strictly positive real (SPR)-like condition), nor persistence of excitation condition is required in this approach. Stability of the closed-loop system is rigorously analyzed using small-gain theorem and Lyapunov-based stability theory. Essentially, the proposed scheme guarantees that all signals of closed-loop system are bounded while the output of system asymptotically converges to zero, which is demonstrated by a numerical example.

Keywords: Discontinuous control, output regulation, uncertain systems, adaptive control, disturbance rejection.

1. INTRODUCTION

As a fundamental topic in control theory, how to regulate the output of a system to achieve tracking of a desired trajectory (Ge et al. (2009); Jia et al. (2019)) or rejection of external disturbances (Byrnes and Isidori (2000); Wei et al. (2019)) has been extensively studied. Particularly, due to its common appearance in practical applications (Goodwin et al. (1986); Chen and Huang (2009); Chen et al. (2020)), the problem of rejecting disturbances has long been of special interest. There has been a substantial amount of research on the disturbance rejection problem for various plants, ranging from linear SISO systems (Esbrook et al. (2013)) to nonlinear multi-input multi-output (MIMO) systems (Zhong et al. (2020)), and even multi-agent systems (Su and Huang (2012)).

Back to 70s, given an accurate model of the plant and external signal, the problem is well-solved via the celebrated internal model (IM) principle (Francis and Wonham (1976)). However, once the accurate models are not available, how to incorporate a proper adaptive mechanism into the IM-based controller has drawn tremendous attention of researchers in control and signal processing community. For instance, as indicated in Chandrasekar et al. (2006), a recursive-least-squares-based adaptive control method is applied to disturbance rejection. Nevertheless, this approach assumes that the plant model is in a steady state and hence disregards the dynamic interactions between the AFC scheme and the plant dynamics. A similar result using an integral control is obtained and extended to multi-frequency disturbances in Marino and Tomei (2015), however, the design is only applicable under the SPR-like condition. In recent work Wang et al. (2020), the necessity of SPR-like condition is removed by a multiple-model adaptive controller under the condition that the frequency of disturbance is known.

Moreover, all aforementioned results are concerned with the continuous-time systems. The study for the output regulation problem for discrete-time uncertain systems is inadequate in the literature. But discrete-time (or difference) system control problem has its own interest and can find its application in various industrial fields (Bai et al. (1988); Fujii et al. (2020); Xie and Dubljevic (2021)). Applying the direct discretization of corresponding continuous-time controller to a discrete-time system in general will degrade the performance or even cause instability. As demonstrated by Yamaura and Tomizuka (2000), a continuous-time controller is discretized based on the pre-warped bilinear transformation, however, the stability of the closed-loop system is not guaranteed. Thus, many continuous-time methods can not be directly applied to discrete-time systems in general.

This motivates a number of work focusing on the discrete-time output regulation problem. In the case of known plants, Åström and Wittenmark (2013) present a standard linear control approach. Hoagg et al. (2008) approach a Markov-parameter-based adaptive controller that does not

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require direct measurements of the disturbance signals, while the plant needs to be minimal-phase and requires relative degree to be known. Doré Landau et al. (2011) extend adaptive regulators to the multivariable case. In Aranovskiy and Freidovich (2013), a further research is investigated to the case that the frequency of disturbance is unknown via a modified AFC algorithm, however, the plant must be precisely defined. In recent study Tomei (2017), parameters, order and relative degree of the plant model are unknown, yet the SPR-like condition must be satisfied for each frequency of disturbance.

Without the critical SPR-like condition, none of above approaches can address a disturbance rejection problem for a largely uncertain discrete-time system. Therefore, to relax this condition, we further study output regulation problem of uncertain discrete-time linear systems with unknown order, relative degree, and parameters, under a sinusoidal disturbance with known frequency and unknown amplitude and phase. The main challenges stem from the setting that only the frequency of disturbance is available and the plant model considered is highly uncertain. Inspired by the work Wang et al. (2020), we investigate problem in a discrete-time setting. Note that, as mentioned previously, a direct discritization of the controller in Wang et al. (2020), can not stabilize the corresponding discrete-time system. To address these difficulties, a novel AFC-based regulator is proposed and the stability of the closed-loop system is shown by employing the small-gain theorem and Lyapunov-based stability theory in this note. Overall, the contributions of this paper are summarized as follows:

i) removing the crucial SPR-like condition for output regulation of uncertain discrete-time systems;

ii) estimating the unknown parameters by proposing a non-minimal realization of closed-loop system and an adaptive law;

iii) realizing complete cancellation of the disturbance in the presence of large parametric model uncertainty.

The rest of the paper is organized as follows. In Section 2, the problem to be addressed is precisely stated. In Section 3, a novel discrete-time adaptive regulator and an adaptive law are proposed. In Section 4, the stability analysis is described. The algorithm is tested and illustrated by a numerical example in Section 5. The conclusion is drawn in Section 6.

Notation: $\mathbb{R}$ and $\mathbb{N}$ denote the set of real numbers and the set of natural number, respectively. $\|\cdot\|$ is the Euclidean norm and $\|\cdot\|_{\mathcal{L}_{2/\infty}}$ denotes the $\mathcal{L}_{2/\infty}$ norm for signals. I denotes the identity matrix with the appropriate dimension. int$(\cdot)$ is the abbreviation for interior. $\Delta x(k)$ denotes feedforward difference, i.e., $\Delta x(k) = x(k+1) - x(k)$. $\mathcal{P} \subset \mathbb{R}^p$ is a given compact set.

2. PROBLEM FORMULATION

This paper addresses the output regulation problem of a highly uncertain linear system with limited prior knowledge of the exosystem. Specifically, we consider the following uncertain discrete-time system

$$\begin{aligned} x(k+1) &= A(p)x(k) + B(p)[u_d(k) - d(k)], x(0) = x_0 \\ y(k) &= C(p)x(k), \end{aligned} \tag{1}$$

where $k \in \mathbb{N}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $u_d \in \mathbb{R}$, $d \in \mathbb{R}$ represent the state, the regulated output, the control input and the exogenous input of the system. $A(p) \in \mathbb{R}^{n\times n}$, $B(p) \in \mathbb{R}^{n\times 1}$ and $C(p) \in \mathbb{R}^{1\times n}$ are unknown matrices with uncertain parameters $p \in \mathcal{P}$. The transfer function for system (1) is defined as $H(z) = C(p)(zI - A(p))^{-1}B(p)$.

The plant is affected by a sinusoidal disturbance $d(k) = \psi_1 \cos(\omega_d k) + \psi_2 \sin(\omega_d k)$, in which ω_d is a *known* frequency and ψ_1, ψ_2 collect the unknown parameters. Let the disturbance $d(k)$ be modeled as

$$\begin{aligned} v(k+1) &= \begin{pmatrix} \cos(\omega_d) & \sin(\omega_d) \\ -\sin(\omega_d) & \cos(\omega_d) \end{pmatrix} v(k), v(0) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\ d(k) &= (1\ 0)\, v(k). \end{aligned} \tag{2}$$

Note that $\omega_d = \omega_c T$, $\omega_c > 0$ is a known frequency in continuous-time and T is the sampling time. From now on, we assume that $\omega_d \in (0, \frac{\pi}{2})$, without loss of generality.

The estimator of exosystem and the control input of plant are considered as

$$\begin{aligned} \hat{v}(k+1) &= R\hat{v}(k) + Gu(k), \hat{v}(0) = \hat{v}_0 \\ u_d(k) &= \Gamma \hat{v}(k) \end{aligned} \tag{3}$$

where

$$R = \begin{pmatrix} \cos(\omega_d) & \sin(\omega_d) \\ -\sin(\omega_d) & \cos(\omega_d) \end{pmatrix}, \Gamma = (1\ 0), G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then the dynamics of system (1)-(3) can be expressed as

$$\begin{aligned} v(k+1) &= Rv(k), d(k) = \Gamma v(k) \\ \hat{v}(k+1) &= R\hat{v}(k) + Gu(k), u_d(k) = \Gamma\hat{v}(k) \\ x(k+1) &= A(p)x(k) + B(p)[u_d(k) - d(k)] \\ y(k) &= C(p)x(k). \end{aligned} \tag{4}$$

Next, we state the following assumption that $A(p)$ is a Schur matrix (i.e., all its eigenvalues are contained in the open unit disk), robustly with respect to $p \in \mathcal{P}$.

Assumption 1. There exist constants $0 < \underline{c}_{p_0} < \bar{c}_{p_0}$ such that the unique solution $P_0(p) \in \mathbb{R}^{n\times n}$ of the discrete-time Lyapunov equation

$$A^\top(p)P_0(p)A(p) - P_0(p) = -I \tag{5}$$

satisfies $\underline{c}_{p_0} I \le P_0(p) \le \bar{c}_{p_0} I$ for all $p \in \mathcal{P}$.

Remark 1. Assumption 1 is not conservative, actually quite standard in the area of the Internal Model Principle (IMP)-based output regulation, especially for those focusing on disturbance rejection.

Then there exists a unique matrix $\Pi(p) \in \mathbb{R}^{n\times 2}$ that satisfies the Sylvester equation

$$\Pi(p)R = A(p)\Pi(p) + B(p)\Gamma. \tag{6}$$

Since all the information from (1) that is relevant for reconstructing $d(k)$ is the steady-state response

$$\begin{aligned} y_{ss}(k) &= -\,C(p)\Pi(p)v(k) \\ &= -\,(\mathrm{Re}\{H(e^{j\omega_d})\}\ \ \mathrm{Im}\{H(e^{j\omega_d})\})v(k) \end{aligned}$$

corresponding to $u_d(k) \equiv 0$, the unknown parameter vector is expressed by

$$\vartheta^\top(p) := C(p)\Pi(p) = (\mathrm{Re}\{H(e^{j\omega_d})\}\ \ \mathrm{Im}\{H(e^{j\omega_d})\}).$$

The SPR-like condition is replaced by the following weaker assumption, which is independent on the sign of the entries of ϑ.

Assumption 2. The unknown parameter vector $\vartheta(p)$ satisfies $\vartheta(p) \in \operatorname{int}\Theta$ for all $p \in \mathcal{P}$ with a compact set $\Theta := \{\vartheta \in \mathbb{R}^2 | \alpha_1^2 \leq \vartheta_1^2 + \vartheta_2^2 \leq \alpha_2^2\}$ for given boundary values $0 < \alpha_1 < \alpha_2$.

Remark 2. Assumption 2 is weaker than SPR-like condition because: First, it is independent on the sign of the entries of $\vartheta(p)$. Hence, the sign of $\mathrm{Re}\{H(e^{j\omega_d})\}$ and sign of $\mathrm{Im}\{H(e^{j\omega_d})\}$ can change over the frequency of interest, which is not permitted in the SPR-like condition. Secondly, for practical applications, one can always find sufficiently small α_1 and sufficiently large α_2 to satisfy this assumption.

The output regulation problem studied in this paper is formally described as follows:

Problem 1. Under the Assumptions 1-2, develop a dynamic output feedback controller of the form

$$\begin{aligned} \gamma(k+1) &= f(\gamma(k), y(k)), \gamma(0) = \gamma_0 \in \mathbb{R}^m \\ u_d(k) &= h(\gamma(k), y(k)) \end{aligned} \tag{7}$$

such that all solutions of the closed-loop system (1), (2) and (7) starting in all initial conditions $x_0 \in \mathbb{R}^n$, $v_0 \in \mathbb{R}^2$, $\gamma_0 \in \mathbb{R}^m$ are bounded and $\lim_{k\to\infty} y(k) = 0$ holds for all $p \in \mathcal{P}$.

Remark 3. Note that only the output y is available under our settings. Moreover, we tend to achieve global regulation of the output.

3. DESIGN OF CONTROLLER AND ADAPTIVE LAW

3.1 Controller Design and Non-minimal Realization

To decouple the exosystem from the plant, we utilize the coordinate transformation $\xi := \hat{v} - v$ and $z := x - \Pi(p)\xi$. This transformation converts the system described in (4) to an error system

$$\begin{aligned} z(k+1) &= A(p)z(k) - \Pi(p)Gu(k), z(0) = z_0 \in \mathbb{R}^n \\ \xi(k+1) &= R\xi(k) + Gu(k), \xi(0) = \xi_0 \in \mathbb{R}^2 \\ y(k) &= C(p)z(k) + \vartheta^\top(p)\xi(k). \end{aligned} \tag{8}$$

Rewrite the auxiliary system

$$\begin{aligned} \xi(k+1) &= R\xi(k) + Gu(k) \\ y_a(k) &= \vartheta^\top(p)\xi(k). \end{aligned} \tag{9}$$

This is controllable and observable for all $p \in \mathcal{P}$ under Assumption 2. Since the auxiliary output $y_a = y - C(p)z$ is unavailable for measurement, a coordinate transformation $\eta := M^{-1}\xi$ is required, where

$$M = \frac{1}{\vartheta_1^2 + \vartheta_2^2}\begin{pmatrix} \vartheta_1 & -\vartheta_2 \\ \vartheta_2 & \vartheta_1 \end{pmatrix}.$$

Consequently, the auxiliary system (9) can be expressed in new coordinates as

$$\begin{aligned} \eta(k+1) &= R\eta(k) + \theta(k)u(k) \\ y_a(k) &= \Gamma\eta(k) \end{aligned} \tag{10}$$

where $\theta = (\vartheta_1 \ \ -\vartheta_2)^\top$. Note that $\alpha_1^2 < \|\theta\|^2 < \alpha_2^2$ also holds for the new unknown parameter vector θ owing to the Assumption 2. A certainty-equivalence adaptive observer for system (10) gives as

$$\begin{aligned} \hat{\eta}(k+1) &= R\hat{\eta}(k) + \hat{\theta}(k)u(k) - \varepsilon G[\hat{y}_a(k) - y(k)] \\ \hat{y}_a(k) &= \Gamma\hat{\eta}(k) \end{aligned} \tag{11}$$

where $\hat{\theta} \in \mathbb{R}^2$ is an estimate of θ whose estimation algorithm will be designed later, and $\varepsilon > 0$ is a constant. The certainty-equivalence control law is given by

$$u(k) = -\varepsilon\hat{\theta}^\top(k)\hat{\eta}(k). \tag{12}$$

Thus, the closed-loop system in observer coordinates can be written as

$$\hat{\eta}(k+1) = F(\varepsilon, \hat{\theta})\hat{\eta}(k) - \varepsilon G[\hat{y}_a(k) - y(k)]$$

where $F(\varepsilon, \hat{\theta}) := R - \varepsilon\hat{\theta}\hat{\theta}^\top$. The dynamics of observation error $\tilde{\eta} := \hat{\eta} - \eta$ is given by the following equation

$$\begin{aligned} \tilde{\eta}(k+1) &= E(\varepsilon)\tilde{\eta}(k) + \tilde{\theta}(k)u(k) + \varepsilon GC(p)z(k) \\ \tilde{y}(k) &= \Gamma\tilde{\eta}(k) - C(p)z(k) \end{aligned} \tag{13}$$

in which $\tilde{\theta} := \hat{\theta} - \theta$, $\tilde{y} := \hat{y}_a - y$ and $E(\varepsilon) := R - \varepsilon G\Gamma$. In order to determine a parameter adaptation law for $\hat{\theta}(k)$, the dynamics (13) is rewritten in a more suitable form as specified in the lemma below, in which $\tilde{\theta}(k)$ is shifted to the output equation.

Lemma 1. The closed-loop system (8), (10), (12) and (13) can be converted to the non-minimal realization

$$\begin{aligned} \hat{\eta}(k+1) =& F\hat{\eta}(k) - \varepsilon G\tilde{\theta}^\top(k)\hat{\eta}_1(k) \\ &+ \varepsilon G[\Gamma\eta_2(k) + C(p)z(k) - \Gamma\eta_3(k)] \\ \hat{\eta}_1(k+1) =& E^\top\hat{\eta}_1(k) - \varepsilon G\hat{\theta}^\top(k)\hat{\eta}(k) \\ \eta_2(k+1) =& E\eta_2(k) + \eta_a(k+1)\,\Delta\hat{\theta}(k) \\ \eta_a(k+1) =& E\eta_a(k) - \varepsilon\hat{\theta}^\top(k)\hat{\eta}(k)I \\ \eta_3(k+1) =& E\eta_3(k) + \varepsilon GC(p)z(k) \\ z(k+1) =& A(p)z(k) + \varepsilon\Pi(p)G\hat{\theta}^\top(k)\hat{\eta}(k) \\ y(k) =& \Gamma(\hat{\eta}(k) + \eta_2(k) - \eta_3(k)) \\ &- \tilde{\theta}^\top(k)\hat{\eta}_1(k) + C(p)z(k) \end{aligned} \tag{14}$$

where $\Delta\hat{\theta}(k)$ is an adaptive law to be designed.

Proof. Invoking the Swapping lemma (Ioannou and Fidan (2006)), system (13) can be decomposed into

$$\tilde{y}_a(k) = \Gamma\sum_{j=0}^{k-1} u(j)E^{k-j-1}\tilde{\theta}(j) \tag{15}$$

$$\tilde{y}_b(k) = \varepsilon\Gamma GC(p)\sum_{j=0}^{k-1} E^{k-j-1}z(j) - C(p)z(k). \tag{16}$$

Starting with the subsystem (15), denote

$$\Psi(j) = \sum_{i=0}^{j-1} u(i)E^{-i} \in \mathbb{R}^{2\times 2}.$$

We can derive the forward difference equation $\Delta\Psi(j) = u(j)E^{-j}$, yielding

$$\tilde{y}_a(k) = \Gamma E^{k-1}\sum_{j=0}^{k-1}\Delta\Psi(j)\tilde{\theta}(j).$$

Transforming the summation of products by

$$\begin{aligned} \tilde{y}_a(k) =& \Gamma E^{k-1}\sum_{j=0}^{k-1}\Delta\Psi(j)\tilde{\theta}(k) \\ &- \Gamma E^{k-1}\sum_{j=1}^{k-1}\Psi(j+1)\,\Delta\tilde{\theta}(j). \end{aligned} \tag{17}$$

The first term of (17) can be represented by

$$\eta_1(k+1) = E^\top \eta_1(k) + Gu(k), \eta_1(0) = \eta_{1_0} \in \mathbb{R}^2$$
$$\tilde{y}_{a,1}(k) = \tilde{\theta}^\top(k)\eta_1(k).$$

Note that $\Psi(1) = 0$, then the second term of (17) gives

$$\tilde{y}_{a,2} = \Gamma \sum_{j=0}^{k-1} E^{k-j-1}\eta_a(j+1)\,\Delta\tilde{\theta}(j)$$

where $\eta_a(j+1) = \sum_{i=0}^{j} u(i)E^{j-i} \in \mathbb{R}^{2\times 2}$ admits the dynamics

$$\eta_a(k+1) = E\eta_a(k) + u(k)I, \eta_a(0) = \eta_{a_0} \in \mathbb{R}^{2\times 2}.$$

Consequently, the second term of (17) satisfies the subsequent equations

$$\eta_2(k+1) = E\eta_2(k) + \eta_a(k+1)\,\Delta\hat{\theta}(k), \eta_2(0) = \eta_{2_0} \in \mathbb{R}^2$$
$$\eta_a(k+1) = E\eta_a(k) + u(k)I$$
$$\tilde{y}_{a,2}(k) = \Gamma\eta_2(k)$$

where the estimation signal $\Delta\hat{\theta}(k)$ is a time-varying parameter vector. Noticing $\tilde{y}_a = \tilde{y}_{a,1} - \tilde{y}_{a,2}$, we completely obtain the discrete-time linear dynamics for the subsystem (15) as follows

$$\begin{aligned}\eta_1(k+1) &= E^\top \eta_1(k) + Gu(k)\\ \eta_2(k+1) &= E\eta_2(k) + \eta_a(k+1)\,\Delta\hat{\theta}(k)\\ \eta_a(k+1) &= E\eta_a(k) + u(k)I\\ \tilde{y}_a(k) &= \tilde{\theta}^\top(k)\eta_1(k) - \Gamma\eta_2(k).\end{aligned}$$

Without loss of generality, we choose an observer signal $\hat{\eta}_1$ to substitute the state η_1, which implies

$$\hat{\eta}_1(k+1) = E^\top\hat{\eta}_1(k) + Gu(k), \hat{\eta}_1(0) = \hat{\eta}_{1_0} \in \mathbb{R}^2.$$

The impulse response of subsystem (16) satisfies the discrete-time linear equation

$$\eta_3(k+1) = E\eta_3(k) + \varepsilon GC(p)z(k), \eta_3(0) = \eta_{3_0} \in \mathbb{R}^2$$
$$\tilde{y}_b(k) = \Gamma\eta_3(k) - C(p)z(k).$$

To conclude, we obtain the non-minimal realization of (13). Together with (8), (10) and (12), the non-minimal realization (14) is obtained, thus ending the proof.

3.2 Parameter Adaptation Law

After obtaining the non-minimum realization of the closed-loop system, to finally facility the AFC-based controller (12), now we focus on designing a parameter adaptation law $\Delta\hat{\theta}(k) = \varphi$.

A general option of parameter adaptation law depending on the available signals is given as

$$\begin{aligned}\Delta\hat{\theta}(k) &= \mathcal{P}(\hat{\theta}(k), \varphi(\hat{\eta}_1, \tilde{y})), \hat{\theta}(0) \in \operatorname{int}\Theta\\ \hat{\eta}_1(k+1) &= E^\top\hat{\eta}_1(k) + Gu(k)\\ \tilde{y}(k) &= \hat{y}_a(k) - y(k)\end{aligned} \tag{18}$$

in which $\hat{\theta}$ is updated with time and $\mathcal{P}(\cdot)$ is an original projection algorithm that ensures $\hat{\theta} \in \Theta$. The projection operator $\mathcal{P}(\cdot)$ is defined as

$$\mathcal{P}(\cdot) = \begin{cases}\varphi, \ \hat{\theta}(k) \in \operatorname{int}\Theta\\ \varphi, \ \hat{\theta}(k) \in \bar{\Theta} \text{ and } \hat{\theta}(k+) \in \operatorname{int}\Theta\\ 0, \ \text{otherwise.}\end{cases} \tag{19}$$

Here, $\bar{\Theta}$ denotes the boundary of the compact set Θ. The gradient-based unconstrained parameter adaptation law (Ioannou and Fidan (2006)) is of the form

$$\varphi(\hat{\eta}_1, \tilde{y}) = -\rho\varepsilon^2\frac{\hat{\eta}_1\tilde{y}}{m^2} \tag{20}$$

where $\rho > 0$ is a constant and the normalization signal $m^2 := 1 + \|\hat{\eta}_1\|^2 + |\tilde{y}|^2$ ensures that $\|\Delta\hat{\theta}(k)\| < \rho\varepsilon^2$.

To ensure the boundness of $\hat{\theta}(k)$ and prove the forthcoming stability analysis, we employ the multiple-model-based switching mechanism introduced in Wang et al. (2020), Section IV, where the non-convex set Θ is divided into several convex sets, and the standard projection is imposed on each subset. Due to the limitation of space, here we do not present the detailed proof of this switching adaptive law (has been shown in Wang et al. (2020), Section V), but one can easily verify that the estimate $\hat{\theta}$ given by such an adaptive law verifies the following two propositions:

Proposition 1. The signal $\hat{\theta}(\cdot)$ satisfies $\hat{\theta}(k) \in \Theta$ and $\|\Delta\hat{\theta}(k)\| < \rho\varepsilon^2$ for all $k \geq 0$.

Proposition 2. The normalization signal $m(\cdot)$ in the update law (20) satisfies $m(\cdot) \in \mathcal{L}_\infty$.

Remark 4. The presented parameter estimation law (18)-(20) ensures that $\hat{\theta}(k) \in \Theta$ and $\|\Delta\hat{\theta}(k)\| < \rho\varepsilon^2$ for all $k \geq 0$, which is essential to the subsequent stability analysis of the closed-loop system.

4. STABILITY ANALYSIS

In this section, we analyze the stability of the system in a closed-loop with the proposed controller and parameter adaptation law. To achieve this, we recast the closed-loop system (14) as the interconnection of the following subsystems

$$\Sigma_1 : \begin{cases}\hat{\eta}(k+1) = F\hat{\eta}(k) + \varepsilon G\nu_1(k)\\ e_1(k) = \hat{\theta}^\top(k)\hat{\eta}(k)\end{cases} \tag{21}$$

$$\Sigma_2 : \begin{cases}\eta_2(k+1) = E\eta_2(k) + \eta_a(k+1)\,\Delta\hat{\theta}(k)\\ \eta_a(k+1) = E\eta_a(k) - \varepsilon I\nu_2(k)\\ e_2(k) = \Gamma\eta_2(k)\end{cases} \tag{22}$$

$$\Sigma_3 : \begin{cases}\eta_3(k+1) = E\eta_3(k) + \varepsilon GC(p)z(k)\\ z(k+1) = A(p)z(k) + \varepsilon\Pi(p)G\nu_3(k)\\ e_3(k) = \Gamma\eta_3(k) - C(p)z(k)\end{cases} \tag{23}$$

$$\Sigma_4 : \begin{cases}\hat{\eta}_1(k+1) = E^\top\hat{\eta}_1(k) - \varepsilon G\nu_4(k)\\ e_4(k) = \hat{\eta}_1^\top(k)\tilde{\theta}(k)\end{cases} \tag{24}$$

with inputs $\nu_1 = u_1 + e_2 - e_3$, $\nu_2 = e_1$, $\nu_3 = e_1$, overall input $u_1 = \hat{\eta}_1^\top\tilde{\theta}$ and overall output $y_1 = e_3 - e_2$, depicted in Fig. 1.

Next, we will analyze the Lyapunov equations for the closed-loop system. The results obtained from this analysis will be used for subsequent stability analysis.

Lemma 2. There exist a scalar $c_{\varepsilon_1} > 0$ and constants $0 < \underline{c}_{p_e} < \bar{c}_{p_e}$ such that the symmetric and positive definite solution $P_e(\varepsilon) \in \mathbb{R}^{2\times 2}$ of the Lyapunov equation

$$E^\top(\varepsilon)P_e(\varepsilon)E(\varepsilon) - P_e(\varepsilon) = -\varepsilon I \tag{25}$$

satisfies $\underline{c}_{p_e}I \leq P_e(\varepsilon) \leq \bar{c}_{p_e}I$ for all $(\theta, \varepsilon) \in \Theta \times (0, c_{\varepsilon_1})$.

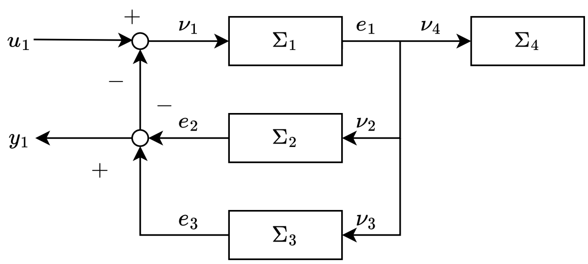

Fig. 1. Interconnection of subsystems Σ_1-Σ_4.

Proof. The matrix E can be represented as

$$E(\varepsilon) = R - \varepsilon G\Gamma = \begin{pmatrix} \cos(\omega_d) - \varepsilon & \sin(\omega_d) \\ -\sin(\omega_d) & \cos(\omega_d) \end{pmatrix}.$$

It is Schur when $0 < \varepsilon < 2$. Since $\det(E) = 1 - \varepsilon\cos(\omega_d)$ and ε is bounded, there exists an upper bound $\bar{c}_e > 0$ such that $E \le \bar{c}_e I$. Let

$$P_e = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix}.$$

After solving Lyapunov equation (25), we get

$$\begin{aligned} p_1 =& -\frac{4\sec(\omega_d)}{\varepsilon^2 - 4} \\ p_2 =& p_3 = -\frac{\varepsilon(\varepsilon\cos(\omega_d) - 2)\csc(\omega_d)\sec(\omega_d)}{\varepsilon^2 - 4} \\ p_4 =& \frac{(\varepsilon^3\cos(\omega_d) - 2 - \varepsilon^2)\csc^2(\omega_d)\sec(\omega_d)}{\varepsilon^2 - 4} \\ &+ \frac{(2 - \varepsilon^2)\cos(2\omega_d)\csc^2(\omega_d)\sec(\omega_d)}{\varepsilon^2 - 4}. \end{aligned}$$

Using the fact that

$$\det(P_e(\varepsilon)) = -\frac{\varepsilon^2\csc(\omega_d)^2 + 4\sec(\omega_d)^2}{\varepsilon^2 - 4}$$

and the spectrum of P_e satisfies

$$\text{spec } P_e(0) = \{\frac{1}{\cos(\omega_d)}, \frac{1}{\cos(\omega_d)}\}$$

we obtain that there exist a scalar $c_{\varepsilon_1} = 2$ and constants $0 < \underline{c}_{p_e} < \bar{c}_{p_e}$ such that $\underline{c}_{p_e} I \le P_e(\varepsilon) \le \bar{c}_{p_e} I$ for all $(\theta, \varepsilon) \in \Theta \times (0, c_{\varepsilon_1})$.

Lemma 3. There exist a scalar $c_{\varepsilon_2} := \min\{c_{\varepsilon_1}, \frac{2}{\alpha_2^2}\}$ and constants $0 < \underline{c}_{p_f} < \bar{c}_{p_f}$ such that the symmetric and positive definite solution $P_f(\varepsilon, \theta) \in \mathbb{R}^{2\times 2}$ of the Lyapunov equation

$$F^\top(\varepsilon, \theta)P_f(\varepsilon, \theta)F(\varepsilon, \theta) - P_f(\varepsilon, \theta) = -\varepsilon\theta^\top\theta I \qquad (26)$$

satisfies $\underline{c}_{p_f} I \le P_f(\theta, \varepsilon) \le \bar{c}_{p_f} I$ for all $(\theta, \varepsilon) \in \Theta \times (0, c_{\varepsilon_2})$, where $F(\varepsilon, \theta) = R - \varepsilon\theta\theta^\top$.

We will omit the proof of Lemma 3 as it follows a similar procedure to Lemma 2.

The following propositions will provide the stability properties of the subsystems Σ_1-Σ_3 using dissipative systems theory (Isidori (2000)).

Proposition 3. There exist $\gamma_1 > 0$ and $c_{\varepsilon_3} > 0$ such that system Σ_1 is strictly dissipative with respect to the supply rate $q_1(\nu_1, e_1) = \gamma_1^2|\nu_1|^2 - |e_1|^2$ for all $\varepsilon \in (0, c_{\varepsilon_3})$, with quadratic, positive definite and decrescent storage function $W_1(k, \hat{\eta}) = 2\varepsilon^{-1}\hat{\eta}^\top(k)P_f(\varepsilon, \hat{\theta}(k))\hat{\eta}(k)$.

Proof. First, the following preliminary result is needed.

Property 1. The gradient with respect to θ of the quadratic form $Q_f(w) := w^\top P_f(\varepsilon, \theta)w, w \in \mathbb{R}^2$, satisfies $\nabla_\theta Q_f(w) = \varepsilon^3(w^\top S_1(\varepsilon, \theta)w, w^\top S_2(\varepsilon, \theta)w)$ for some matrix-valued functions $S_i : (\varepsilon, \theta) \to \mathbb{R}^{2\times 2}, i = 1, 2$, which are continuous and bounded over $\Theta \times (0, c_{\varepsilon_2})$.

Then a quantity l gives

$$l := \max_{\hat{\theta}\in\Theta, \varepsilon\in(0, c_{\varepsilon_2})} \{\|S_1(\varepsilon, \hat{\theta})\| + \|S_2(\varepsilon, \hat{\theta})\|\}.$$

Define the Lyapunov function candidate

$$V_1(k, \hat{\eta}) := \hat{\eta}^\top(k)P_f(\varepsilon, \hat{\theta}(k))\hat{\eta}(k) \qquad (27)$$

for all $(\hat{\theta}(\cdot), \varepsilon) \in \Theta \times (0, c_{\varepsilon_2})$. Evaluating $\Delta V_1(k, \hat{\eta})$ along the trajectories of system Σ_1 yields

$$\begin{aligned} &\Delta V_1(k, \hat{\eta}) \\ =& \hat{\eta}^\top(k+1)P_f(k+1)\hat{\eta}(k+1) - \hat{\eta}^\top(k)P_f(k)\hat{\eta}(k) \\ \le& -\varepsilon(\frac{\alpha_1^2}{4} - l\bar{c}_f^2\varepsilon^2)\|\hat{\eta}\|^2 - \frac{\varepsilon}{2}|e_1|^2 \\ &+ (4\varepsilon\alpha_1^{-2}\bar{c}_{p_f}^2\bar{c}_f^2 + \varepsilon^2\bar{c}_{p_f})|\nu_1|^2 \end{aligned}$$

where we have utilized Assumption 1, Lemma 3, Property 1 and Young's inequality. Letting $c_{\varepsilon_3} := \min\{c_{\varepsilon_2}, \frac{\alpha_1}{2\bar{c}_f\sqrt{l}}\}$, it follows that

$$\begin{aligned} \Delta V_1(k, \hat{\eta}) \le& -\frac{\varepsilon}{2}a(\varepsilon)\|\hat{\eta}\|^2 - \frac{\varepsilon}{2}|e_1|^2 \\ &+ \frac{\varepsilon}{2}(8\alpha_1^{-2}\bar{c}_{p_f}^2\bar{c}_f^2 + 2\varepsilon\bar{c}_{p_f})|\nu_1|^2 \end{aligned}$$

for all $(\hat{\theta}, \varepsilon) \in \Theta \times (0, c_{\varepsilon_3})$ and $a(\varepsilon) := \frac{\alpha_1^2}{2} - 2l\bar{c}_f^2\varepsilon^2 > 0$. Define a quadratic, positive definite and decrescent storage function $W_1(k, \hat{\eta}) := 2\varepsilon^{-1}V_1(k, \hat{\eta})$, which satisfies

$$\Delta W_1(k) \le -|e_1|^2 - a(\varepsilon)\|\hat{\eta}\|^2 + \gamma_1^2|\nu_1|^2$$

where $\gamma_1^2 := 8\bar{c}_f^2\bar{c}_{p_f}^2\alpha_1^{-2} + 2\varepsilon\bar{c}_{p_f}$. Therefore, system Σ_1 is strictly dissipative with respect to the supply rate $q_1(\nu_1, e_1) = \gamma_1^2|\nu_1|^2 - |e_1|^2$. As a result, under the given conditions, system Σ_1 is exponentially stable when $\nu_1 = 0$, and has a finite $\mathcal{L}_2$-gain between ν_1 and e_1 which does not exceed γ_1, for any $\varepsilon \in (0, c_{\varepsilon_3})$.

Proposition 4. There exists $\gamma_2 > 0$ such that system Σ_2 is strictly dissipative with respect to the supply rate $q_2(\nu_2, e_2) = \varepsilon^2\gamma_2^2|\nu_2|^2 - |e_2|^2$ for all $\varepsilon \in (0, c_{\varepsilon_3})$, with quadratic and positive definite storage function $W_2(\eta_2, \eta_a)$.

Proposition 5. There exists $\gamma_3 > 0$ such that system Σ_3 is strictly dissipative with respect to the supply rate $q_3(\nu_3, e_3) = \varepsilon^2\gamma_3^2|\nu_3|^2 - |e_3|^2$ for all $\varepsilon \in (0, c_{\varepsilon_3})$ and all $p \in \mathcal{P}$, with quadratic and positive definite storage function $W_3(\eta_3, z)$.

The proof of Proposition 4 and Proposition 5 can be found in Appendix A and Appendix B, respectively.

The main result of the control performance is summarized in the following theorem.

Theorem 1. Consider the discrete-time plant model (1) and the exosystem (2) under Assumptions 1-2. There exist positive constants c_m and c_ρ such that Problem 1 can be solved using the AFC-based method, consisting of (3), (11), (12), (18) (19), and (20), for any $\varepsilon \in (0, c_m)$ and any $\rho \in (0, c_\rho)$.

Proof. We begin by applying small-gain techniques (Isidori (2000)) to analyze the interconnection properties of Σ_1-Σ_3. Let $\varepsilon \in (0, c_{\varepsilon_3})$. According to Propositions 3-5, we obtain the following inequality

$$\|e_{1_\tau}\|_{\mathcal{L}_2} \le \gamma_1 \|u_{1_\tau}\|_{\mathcal{L}_2} + \varepsilon\gamma_1(\gamma_2+\gamma_3)\|e_{1_\tau}\|_{\mathcal{L}_2}, \tau \in \mathbb{R}_{\ge 0}.$$

Here, $u_\tau(\cdot)$ represents the truncation of the signal $u(\cdot)$ over the time interval $[0,\tau]$. Additionally, we define $c_{\varepsilon_4} := \min\{c_{\varepsilon_3}, \frac{1}{\gamma_1(\gamma_2+\gamma_3)}\}$, and use it to establish that subsystems Σ_1-Σ_3 form a small-gain theorem interconnection (with respect to the $\mathcal{L}_2$-norm) for $\varepsilon \in (0, c_{\varepsilon_4})$. Applying the small-gain theorem, we obtain

$$\|y_{1_\tau}\|_{\mathcal{L}_2} \le \bar{c}_\varepsilon \|u_{1_\tau}\|_{\mathcal{L}_2}, \forall \varepsilon \in (0, c_{\varepsilon_4}), \tau \in \mathbb{R}_{\ge 0}$$

where $\bar{c}_\varepsilon := \frac{\varepsilon\gamma_1(\gamma_2+\gamma_3)}{1-\varepsilon\gamma_1(\gamma_2+\gamma_3)}$.

By using the results obtained above, we demonstrate that the signal u_1 belongs to $\mathcal{L}_2$. Define $V_4(k) = \frac{1}{2}\|\tilde{\theta}(k)\|^2$ and calculate the difference equation

$$\Delta V_4(k) = \frac{1}{2}\|\tilde{\theta}(k+1)\|^2 - \frac{1}{2}\|\tilde{\theta}(k)\|^2.$$

Note that $\tilde{y} = \hat{\eta}_1^\top \tilde{\theta} + y_1$, letting $c_\rho = \frac{2}{c_m^2}$, then for all $\rho \in (0, c_\rho)$, it holds $\rho\varepsilon^2 < 2$, which leads to

$$V_4(k) - V_4(0) \le \frac{\rho\varepsilon^2}{2\|m\|_\infty^2}\left(-\sum_{\tau=0}^{k-1}|u_1(\tau)|^2 + \sum_{\tau=0}^{k-1}|y_1(\tau)|^2\right).$$

From Proposition 2, we obtain

$$\sum_{\tau=0}^{k-1}|u_1(\tau)|^2 \le \frac{2\|m\|_\infty^2}{\rho\varepsilon^2(1-\bar{c}_\varepsilon^2)} V_4(0) < \infty.$$

This implies that the signal $u_1 \in \mathcal{L}_2$.

Finally, we derive that the trajectories of the closed-loop system (14) belong to $\mathcal{L}_\infty$ and $\lim_{k\to\infty} y(k) = 0$. This can be achieved by recalling the fact that the storage function $W_1(k, \hat{\eta})$ satisfies the dissipation inequality $\Delta W_1(k) \le \gamma_1^2|\nu_1(k)|^2 - |e_1(k)|^2$, which is based on Proposition 3. Accumulating each side of this inequality along trajectories of systems Σ_1-Σ_3 over the time interval $[0,\tau]$, and utilizing the results from Proposition 4 and Proposition 5, we can obtain the following inequality

$$\|e_{i_\tau}\|_{\mathcal{L}_2} \le \varepsilon\gamma_i \|e_{1_\tau}\|_{\mathcal{L}_2}, i = 2, 3.$$

We conclude that

$$\begin{aligned} W_1(\tau, \hat{\eta}(\tau)) <& W_1(0, \hat{\eta}(0)) + 3\gamma_1^2\|u_{1_\tau}\|^2 \\ &- (1 - 3\varepsilon^2\gamma_1^2(\gamma_2^2+\gamma_3^2))\|e_{1_\tau}\|^2. \end{aligned}$$

After utilizing Lemma 3 and the small-gain theorem, setting $c_m := \min\{c_{\varepsilon_4}, \frac{\sqrt{3}}{3\gamma_1(\gamma_2+\gamma_3)}\}$, we get

$$\|\hat{\eta}(\tau)\| \le \sqrt{\frac{\varepsilon}{2\underline{c}_{p_f}}} \max\{\sqrt{\frac{2\bar{c}_{p_f}}{\varepsilon}}\|\hat{\eta}(0)\|, \sqrt{3}\gamma_1\|u_1\|_{\mathcal{L}_2}\}$$

for all $\tau \in \mathbb{R}_{\ge 0}$ and $\varepsilon \in (0, c_m)$, then $\hat{\eta}(\cdot) \in \mathcal{L}_\infty$ can be concluded. Since $\hat{\theta}(\cdot) \in \Theta$ is a bounded signal given in Proposition 1, we can also get $e_1(\cdot) \in \mathcal{L}_\infty$. Exponentially stability of systems Σ_2, Σ_3 and Σ_4 implies that all remaining state trajectories belong to $\mathcal{L}_\infty$ as well.

Additionally, we note that the fact that $u_1(\cdot) \in \mathcal{L}_2$ implies that $\hat{\eta}_1(\cdot) \in \mathcal{L}_\infty$ and $\Delta\hat{\eta}_1(\cdot) \in \mathcal{L}_\infty$ for all $\varepsilon \in (0, c_m)$. Using the results of Proposition 1 and Proposition 2, we obtain $\hat{\theta}(\cdot) \in \mathcal{L}_\infty$ and $\Delta\hat{\theta}(\cdot) \in \mathcal{L}_\infty$. Hence, it follows that $u_1(\cdot) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\Delta u_1(\cdot) \in \mathcal{L}_\infty$ and $\lim_{k\to\infty} u_1(k) = 0$. Since systems Σ_1-Σ_3 are exponentially stable for all $\varepsilon \in (0, c_m)$, it follows that $\hat{\eta}(k)$, $\eta_2(k)$, $\eta_3(k)$, $\eta_a(k)$, $z(k)$ converge to zero asymptotically, implying that $\lim_{k\to\infty} y(k) = 0$. This completes the proof of Theorem 1.

5. NUMERICAL EXAMPLE

The effectiveness of the proposed scheme is verified via numerical examples. Consider the following linear stable and non-minimum phase system

$$H(z) = \frac{0.1704z - 0.1885}{z^2 - 1.774z + 0.8187}. \tag{28}$$

To evaluate the performance of the proposed method, a sinusoidal disturbance with a different frequency is added during algorithm implementation, it has the form

$$d(k) = \begin{cases} 2\sin(\omega_d k), \text{if } k < 1500 \\ 2\sin(0.7 \times \omega_d k), \text{if } k \ge 1500 \end{cases} \tag{29}$$

where $\omega_d = 0.1$ [rad/s]. Note that $\omega_d = \omega_c \times T$, where the sampling time T is chosen as 0.1 s. Set the controller parameters as $\varepsilon = 0.3$, $\rho = 0.5$, $\alpha_1 = 0.1$ and $\alpha_2 = 3$. All closed-loop system states have been simulated starting at zero initial conditions, except for $\hat{\theta}$, whose initial condition is $(-1, 1)$.

Applying the proposed controller, the simulation results are shown in Fig. 2-4. In the interval $k \in [0, 1500)$, the plant is affected by a sinusoidal disturbance with a frequency of $\omega_d = 0.1$ [rad/s]. Moreover, a change of disturbance occurs at $k = 1500$. Fig. 2 shows the performance of the auxiliary controller u in response to changes in the external disturbance. Fig. 3 and Fig. 4 demonstrate the ability of the control input u_d to compensate for the external disturbance d, and regulate the system output y to zero.

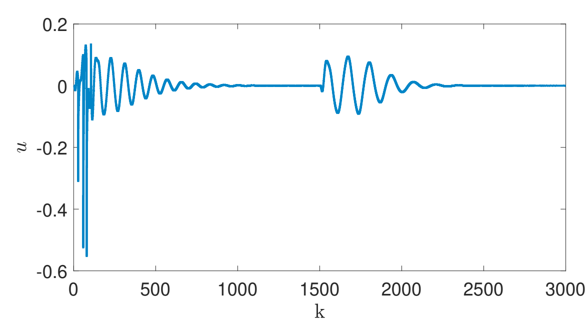

Fig. 2. The auxiliary controller u.

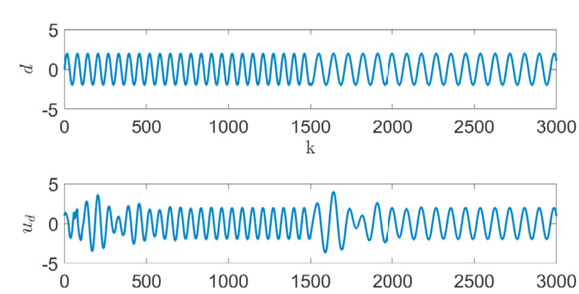

Fig. 3. Comparison of disturbance d and control input u_d.

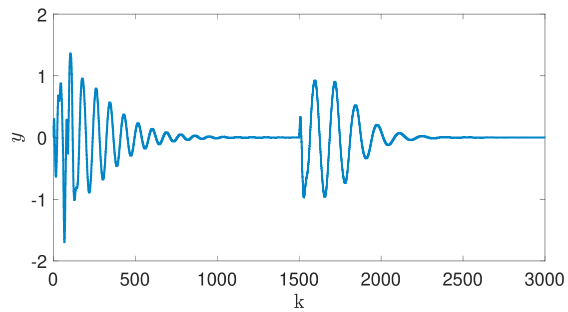

Fig. 4. The output y of plant.

Finally, we demonstrate the robustness of the proposed algorithm by analyzing the impact of a noisy output measurement (polluted by an additive Gaussian noise with zero mean and 0.1 variance). Simulation result is shown in Fig. 5. The result indicates that the proposed controller is not sensitive to output measurement noise.

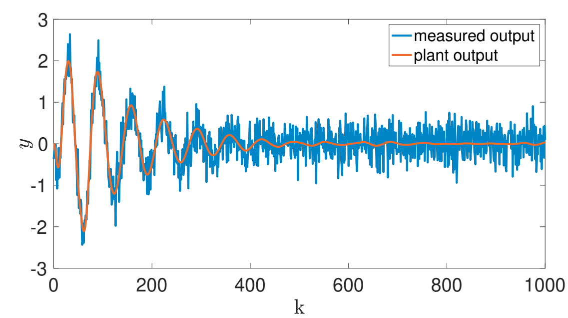

Fig. 5. The output y of plant perturbed by an additive Gaussian noise.

6. CONCLUSION

In this paper, the robust output regulation problem is addressed for uncertain discrete-time systems, under the effect of a sinusoidal signal that only the frequency information is assumed to be known. A novel AFC-based method is proposed to remove the SPR-like condition. Stability analysis shows that the trajectories of the closed-loop system are bounded and the output of system asymptotically converges to zero. Finally, a numerical example is given to illustrate the effectiveness of proposed control algorithm. Future work will be aimed to extend the results to multiple sinusoids case for MIMO uncertain systems. Meanwhile, we acknownledge that the prior information on the frequency of external disturbance should be relaxed in the future work. Currently we are able to show that the proposed algorithm is robust with respect to small frequency error, but the integration of a proper frequency estimator is under investigation.

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Appendix A. PROOF OF PROPOSITION 4

It should be noted that the system Σ_2 is Input-to-state stable if $\Delta\hat{\theta}(\cdot) \in \mathcal{L}_\infty$ for all $\varepsilon \in (0, c_{\varepsilon_3})$. Letting $V_{21}(\eta_2) = 2\varepsilon^{-1}\eta_2^\top P_e \eta_2$, along the trajectory of the η_2-subsystem, we have

$$\begin{aligned}\Delta V_{21}(k) &= \frac{2}{\varepsilon}[\eta_2^\top(k+1)P_e\eta_2(k+1) - \eta_2^\top(k)P_e\eta_2(k)] \\ &= \frac{2}{\varepsilon}[-\varepsilon\eta_2^\top(k)\Gamma\Gamma^\top\eta_2(k) + 2\eta_2^\top(k)E^\top P_e\eta_a(k+1)\,\Delta\hat{\theta}(k) \\ &+ \Delta\hat{\theta}^\top(k)\eta_a^\top(k+1)P_e\eta_a(k+1)\,\Delta\hat{\theta}(k)].\end{aligned}$$

Utilizing the Proposition 1, Lemma 2 and Young's inequality, adding and subtracting the term $|e_2|^2$, we get

$$\Delta V_{21}(k) \le -|e_2(k)|^2 - \frac{1}{2}\|\eta_2(k)\|^2 + \gamma_{21}^2\|\eta_a(k+1)\|^2.$$

One concludes that the $\mathcal{L}_2$-gain of the η_2-subsystem in system Σ_2 does not exceed $\gamma_{21} := \varepsilon\rho\sqrt{8\bar{c}_e^2\bar{c}_{p_e}^2 + 2\varepsilon\bar{c}_{p_e}}$ for all $\varepsilon \in (0, c_{\varepsilon_3})$.

Next, defining $V_{22}(\eta_a) = \mathrm{trace}[\eta_a^\top P_e \eta_a]$, along the trajectory of the η_a-subsystem, we have

$$\begin{aligned}\Delta V_{22}(k) &= \mathrm{trace}[\eta_a^\top(k+1)P_e\eta_a(k+1) - \eta_a^\top(k)P_e\eta_a(k)] \\ &\le -\frac{1}{2}\varepsilon\|\eta_a(k)\|^2 + (2\varepsilon\bar{c}_{p_e}^2\bar{c}_e^2 + \varepsilon^2\bar{c}_{p_e})|\nu_2(k)|^2.\end{aligned}$$

Finally, define the Lyapunov function candidate $V_2(\eta_2, \eta_a) = V_{21} + \lambda_1(\varepsilon)V_{22}$ for system Σ_2, where $\lambda_1(\varepsilon)$ is a positive constant. It follows that the time difference of V_2 along trajectories of System Σ_2 yields

$$\begin{aligned}\Delta V_2(k) &\le -|e_2(k)|^2 - \frac{1}{2}\|\eta_2(k)\|^2 - \frac{1}{4}\varepsilon\lambda_1(\varepsilon)\|\eta_a(k)\|^2 \\ &+ \bar{\lambda}_1(\varepsilon)\|\eta_a(k)\|^2 + \varepsilon^2\gamma_2^2|\nu_2(k)|^2\end{aligned}$$

for all $(\hat{\theta}(\cdot), \varepsilon) \in \Theta \times (0, c_{\varepsilon_3})$. Let $\bar{\lambda}_1(\varepsilon) := -\frac{1}{4}\varepsilon\lambda_1(\varepsilon) + 8\varepsilon^2\bar{c}_e^4\bar{c}_{p_e}^2\rho^2 + 2\varepsilon^3\rho^2\bar{c}_{p_e}\bar{c}_e^2 + 8\varepsilon^3\bar{c}_e^3\bar{c}_{p_e}^2\rho^2 + 2\rho^2\varepsilon^4\bar{c}_e\bar{c}_{p_e} = 0$, we have

$$\begin{aligned}\Delta V_2(k) &\le -\lambda_2(\varepsilon)(\|\eta_2(k)\|^2 + \|\eta_a(k)\|^2) - |e_2(k)|^2 \\ &+ \varepsilon^2\gamma_2^2|\nu_2(k)|^2\end{aligned}$$

in which $\gamma_2^2 = 64\varepsilon\bar{c}_e^5\bar{c}_{p_e}^4\rho^2 + 48\varepsilon\bar{c}_e^4\bar{c}_{p_e}^3\rho^2 + 16\varepsilon^2\rho^2\bar{c}_{p_e}^2\bar{c}_e^2 + 2\varepsilon^3\rho^2\bar{c}_{p_e} + 64\bar{c}_e^4\bar{c}_{p_e}^6\rho^2 + 48\varepsilon^2\bar{c}_{p_e}^3\bar{c}_e^3\rho^2 + 8\varepsilon^3\bar{c}_e\bar{c}_{p_e}^2\rho^2 + 2\varepsilon^2\rho^2\bar{c}_e\bar{c}_{p_e} + 8\varepsilon\rho^2\bar{c}_e^3\bar{c}_{p_e}^2$ and $\lambda_2(\varepsilon) := \min\{\frac{1}{2}, \frac{1}{4}\varepsilon\lambda_1(\varepsilon)\}$. Define a quadratic, positive definite and decrescent storage function

$$W_2(\eta_2, \eta_a) := V_2(\eta_2, \eta_a)$$

which satisfies

$$\begin{aligned}\Delta W_2(k) \le &- \lambda_2(\varepsilon)\|\eta_2(k)\|^2 - \lambda_2(\varepsilon)\|\eta_a(k)\|^2 \\ &- |e_2(k)|^2 + \varepsilon^2\gamma_2^2|\nu_2(k)|^2.\end{aligned}$$

As a result, the system Σ_2 is strictly dissipative with respect to the supply rate $q_2(\nu_2, e_2) = \varepsilon^2\gamma_2^2|\nu_2|^2 - |e_2|^2$ for any $\varepsilon \in (0, c_{\varepsilon_3})$. Hence, Σ_2 is exponentially stable when $\nu_2 = 0$, and has a finite $\mathcal{L}_2$-gain between ν_2 and e_2 which does not exceed $\varepsilon\gamma_2$ for any $\varepsilon \in (0, c_{\varepsilon_3})$.

Appendix B. PROOF OF PROPOSITION 5

The system Σ_3 is the cascade of two exponentially stable linear systems. Define the Lyapunov function candidate $V_3(\eta_3, z) = \eta_3^\top P_e\eta_3 + \lambda_3(\varepsilon)z^\top P_0 z$ for Σ_3, in which $\lambda_3(\varepsilon) > 0$ is a constant will be designed later. For any $p \in \mathcal{P}$, we have

$$\begin{aligned}&\Delta V_3(k) = \eta_3^\top(k+1)P_e\eta_3(k+1) + \lambda_3 z^\top(k+1)P_0 z(k+1) \\ &- \eta_3^\top(k)P_e\eta_3(k) - \lambda_3 z^\top(k)P_0 z(k) \\ &\le -\frac{\varepsilon}{2}\|\eta_3(k)\|^2 - \frac{\lambda_3}{2}\|z(k)\|^2 \\ &+ (2\varepsilon\bar{c}_{p_e}^2\bar{c}_e^2\|C(p)\|^2 + \varepsilon^2\bar{c}_{p_e}\|C(p)\|^2)\|z(k)\|^2 \\ &+ (2\lambda_3\varepsilon^2\bar{c}_{p_0}^2\|A(p)\|^2\|\Pi(p)\|^2 + \lambda_3\varepsilon^2\bar{c}_{p_0}\|\Pi(p)\|^2)|\nu_3(k)|^2\end{aligned}$$

where we used Assumption 1, Lemma 2 and Young's inequality. Letting

$$c_c := \max_{p\in\mathcal{P}}\|C(p)\|, c_\pi := \max_{p\in\mathcal{P}}\|\Pi(p)\|, c_a := \max_{p\in\mathcal{P}}\|A(p)\|$$

we obtain

$$\begin{aligned}\Delta V_3(k) \le &- \frac{\varepsilon}{4}\|\eta_3(k)\|^2 - \frac{\lambda_3}{4}\|z(k)\|^2 - \frac{\varepsilon}{8}|e_3(k)|^2 \\ &- (\frac{\lambda_3}{4} - 2\varepsilon\bar{c}_e^2\bar{c}_{p_e}^2c_c^2 - \varepsilon^2\bar{c}_{p_e}c_c^2 - \frac{\varepsilon c_c^2}{4})\|z(k)\|^2 \\ &+ (2\lambda_3\varepsilon^2c_a^2\bar{c}_{p_0}^2c_\pi^2 + \lambda_3\varepsilon^2\bar{c}_{p_0}c_\pi^2)|\nu_3(k)|^2.\end{aligned}$$

Set $\lambda_4(\varepsilon) := 8\bar{c}_e^2\bar{c}_{p_e}^2c_c^2 + 4\varepsilon\bar{c}_{p_e}c_c^2 + c_c^2$, $\lambda_3(\varepsilon) := \varepsilon\lambda_4(\varepsilon)$, then

$$\begin{aligned}\Delta V_3(k) \le &- \frac{\varepsilon}{8}\lambda_5(\varepsilon)\|\eta_3(k)\|^2 - \frac{\varepsilon}{8}\lambda_5(\varepsilon)\|z(k)\|^2 - \frac{\varepsilon}{8}|e_3(k)|^2 \\ &+ \frac{\varepsilon}{8}(16\lambda_4(\varepsilon)\varepsilon^2c_a^2\bar{c}_{p_0}^2c_\pi^2 + 8\lambda_4(\varepsilon)\varepsilon^2\bar{c}_{p_0}c_\pi^2)|\nu_3(k)|^2\end{aligned}$$

where $\lambda_5(\varepsilon) := \min\{2, 2\lambda_4(\varepsilon)\}$. Define the quadratic, positive definite and decrescent storage function $W_3(\eta_3, z) = 8\varepsilon^{-1}V_3(\eta_3, z)$, which satisfies

$$\begin{aligned}\Delta W_3(\eta_3, z) \le &- \lambda_5(\varepsilon)\|\eta_3(k)\|^2 - \lambda_5(\varepsilon)\|z(k)\|^2 \\ &- |e_3(k)|^2 + \varepsilon^2\gamma_3^2|\nu_3(k)|^2.\end{aligned}$$

The system Σ_3 can be shown to be strictly dissipative with respect to the supply rate $q_3(\nu_3, e_3) = \varepsilon^2\gamma_3^2|\nu_3|^2 - |e_3|^2$ for any $\varepsilon \in (0, c_{\varepsilon_3})$, where $\gamma_3^2 := 16\lambda_4(\varepsilon)c_a^2\bar{c}_{p_0}^2c_\pi^2 + 8\lambda_4(\varepsilon)\bar{c}_{p_0}c_\pi^2$. As a result, system Σ_3 is exponentially stable when $\nu_3 = 0$, and has a finite $\mathcal{L}_2$-gain between ν_3 and e_3 which does not exceed $\varepsilon\gamma_3$ for any $\varepsilon \in (0, c_{\varepsilon_3})$.