Linear Regression

Professor Jie Peng, PhD

Department of Statistics

University of California, Davis

Overview: Part I

Regression Analysis: What for?

Regression analysis is a statistical technique to:

(i) **Describe** the relationship between a response variable and a set of predictor variables;

(ii) **Predict** the value of the response variable based on values of the predictor variables;

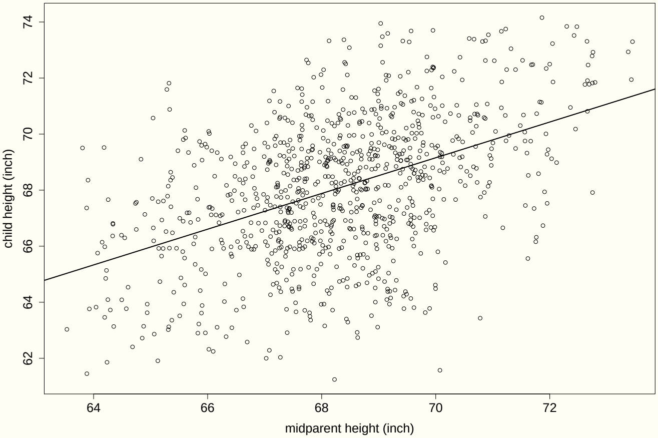
A Bit of History

- Francis Galton: Study of family resemblances, 1885
- 928 child-parent pairs: Height of the adult child and the "midparent height" (average height of the father and the mother)
- "**Regression to mean**": Children's heights tend to be more "moderate" than their parents

	Child(in)	Midparent(in)
1	61.57220	70.07404
2	61.24382	68.22505
3	61.90968	65.12639
4	61.85769	64.23529
5	61.44986	63.88177
6	62.00005	67.02702

......

Figure: Scatter plot: Child's height versus midparent height



- Foot-ball shaped data cloud $\rightarrow$ linear relationship
- Fitted regression line:

$$Y = 24.54 + 0.637X$$

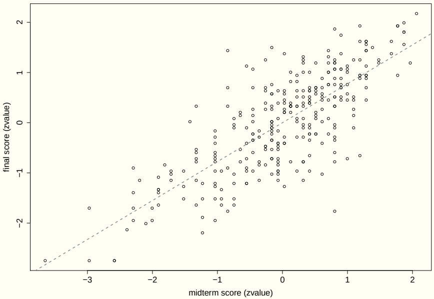
- If the midparent's height is 72in, then the child's height is predicted to be:

$$24.54 + 0.637 \times 72in = 70.4in.$$

Overview: Part II

Exam Scores

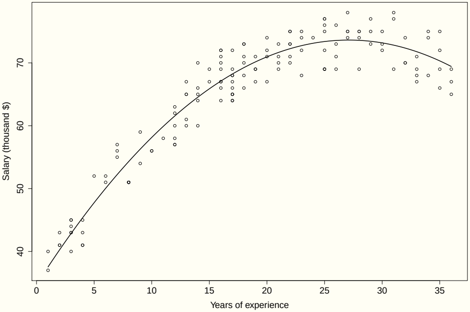
Figure: Standardized final score versus standardized midterm score



If a student's midterm score is 2 standard deviations above the class mean, then their predicted final score would be ...?

Salary

Figure: Salary versus years of experience



Would a straight line fit the data well?

Body Fat

Accurate measure of body fat is costly. It is desirable to use a set of easily obtainable measurements to predict the body fat. E.g., Age (years), Weight (lbs), Height (inches), Neck circumference (cm), Chest (cm), Abdomen 2 (cm), Hip (cm), Thigh (cm), Knee(cm), Ankle (cm), Biceps (cm), Forearm (cm), Wrist (cm).

Are all these needed for predicting body fat? Are their effects linear?

Questions to Be Studied

- How to estimate the regression relationship?
- How reliable are the regression estimates?
- How reliable are the predictions?
- How to interpret estimated regression coefficients?
- Does the model fit the data? Do model assumptions hold?
- How to choose X variables? How to choose between competing models? How to validate a model?

Review of Some Basics

Summation Operator: Σ

- $\sum_{i=1}^{n} Y_i = Y_1 + Y_2 + \cdots + Y_n$
- Variants:

$$\sum_{i=1}^{n} Y_i, \quad \sum_{1}^{n} Y_i, \quad \sum Y_i$$

- Useful facts:

$$\begin{aligned}
\sum_{i=1}^{n} c &= n \cdot c \\
\sum_{i=1}^{n} (Y_i + Z_i) &= \sum_{i=1}^{n} Y_i + \sum_{i=1}^{n} Z_i \\
\sum_{i=1}^{n} (c \cdot Y_i) &= c \cdot \sum_{i=1}^{n} Y_i
\end{aligned}$$

Notes: in general $\sum_{i=1}^{n} (c_i \cdot Y_i) \neq (\sum_{i=1}^{n} c_i) \cdot (\sum_{i=1}^{n} Y_i)$*!*

Expectation Operator: $E(\cdot)$

- Discrete random variable: $E(Y) = \sum_y y \cdot P(Y = y)$
- Continuous random variable: $E(Y) = \int y \cdot f(y)dy$
- Variants: expectation, mean, expected value
- Useful facts: c is a (non-random) constant and Y, Z are random variables

$$
\begin{aligned}
E(c) &= c \\
E(Y + Z) &= E(Y) + E(Z) \\
E(c + Y) &= c + E(Y) \\
E(c \cdot Y) &= c \cdot E(Y)
\end{aligned}
$$

Variance Operator: $Var(\cdot)$

- $Var(Y) = E\Big((Y - E(Y))^2\Big)$
- Variants: $\sigma^2\{Y\}$
- Useful facts: c is a (non-random) constant and Y, Z are random variables

$$
\begin{aligned}
Var(Y) &= E\big(Y^2\big) - \big(E(Y)\big)^2 \geq 0 \\
Var(c) &= 0 \\
Var(Y + Z) &= Var(Y) + Var(Z), \quad \text{if } Y \text{ and } Z \text{ are uncorrelated} \\
Var(c + Y) &= Var(Y) \\
Var(c \cdot Y) &= c^2 \cdot Var(Y)
\end{aligned}
$$

Covariance Operator: $Cov(\cdot, \cdot)$

- $Cov(Y, Z) = E\Big(\big(Y - E(Y)\big) \cdot \big(Z - E(Z)\big)\Big)$
- Variants: $\sigma\{Y, Z\}$
- Useful facts: c_1, c_2 are (non-random) constants and Y, Z are random variables

$$
\begin{aligned}
Cov(Y, Z) &= E(Y \cdot Z) - E(Y) \cdot E(Z) \\
Cov(Y, Y) &= Var(Y) \\
Cov(c_1 + Y, c_2 + Z) &= Cov(Y, Z) \\
Cov(c_1 \cdot Y, c_2 \cdot Z) &= c_1 \cdot c_2 \cdot Cov(Y, Z) \\
Var(Y + Z) &= Var(Y) + Var(Z) + 2 \cdot Cov(Y, Z)
\end{aligned}
$$

Hypothesis Testing: Components

- **Null hypothesis** H_0: "status quo". E.g., H_0 : not guilty
  - Null hypothesis is assumed to be true unless there is strong evidence against it. E.g., A defendant is assumed not guilty unless proved otherwise by evidence that is beyond reasonable doubts.
- **Alternative hypothesis** H_a: "new theory". E.g., H_a : guilty
  - Alternative hypothesis is being accepted only when one can reject the null hypothesis with strong evidence.
- A testing procedure is to ascertain if the evidence (in data) against the null hypothesis is strong enough to reject it.

Type I and Type II Errors

Decision \ Actual Fact	H_0 true	H_0 false
Accept H_0	Correct	**Type II Error**
Reject H_0	**Type I Error**	Correct

- **Significance level** α: (prespecified) maximum allowable **type I error rate**. E.g., $\alpha = 0.05$, $\alpha = 0.1$, $\alpha = 0.01$. **Type I error rate** is not to exceed the significance level α.
- **Type II error rate (=1-power)** is affected by the testing procedure, signal-to-noise ratio, sample size, significance level α, etc.

Point Estimator

A procedure to calculate a numerical quantity based on a (random) sample.

- E.g., Sample mean as an estimator for the population mean; sample proportion as an estimator for the population proportion.
- An estimator is a random variable.
- Error in the estimation: a numerical measure of the reliability of the estimation. E.g., standard error (SE) of an estimator.
  What is the standard error of the sample mean?

Confidence Interval

A (random) interval that covers the parameter of interest with a (pre-specified) high probability.

- Confidence coefficient (level/coverage) : the pre-specified probability of coverage, denoted by $1-\alpha$. E.g., $1-\alpha = 90\%$, $1-\alpha = 95\%$, $1-\alpha = 99\%$.
- The corresponding interval is called a $(1-\alpha)100\%$-confidence interval.
- One common form: $\textit{Estimator} \pm \textit{multiplier}_\alpha \times \textit{SE}(\textit{Estimator})$

What is a 95% confidence interval for the population mean?