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The LLV Decomposition of Hyperkähler Cohomology and Applications to the Nagai Conjecture (After Green–Kim–Laza–Robles)

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Abstract. Following work of Green, Kim, Laza, and Robles, we discuss the structure and known cases of the decomposition of the cohomology of hyperkähler varieties under the Looijenga–Lunts–Verbitsky algebra. This has applications to the Nagai conjecture concerning degenerations of hyperkähler varieties.

1. Introduction

Given a compact hyperkähler manifold X, the rational second cohomology group $H^2(X,\mathbb{Q})$ is equipped with the Beauville–Bogomolov–Fujiki form q_X. Following [7], we let

$$(V,q){:=}\big(H^2(X,\mathbb{Q})\oplus\mathbb{Q}^2, q_X\oplus\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\big)$$

denote its *Mukai completion*. Let also $h\in\operatorname{End}H^*(X,\mathbb{Q})$ be the degree operator defined by

$$h|_{H^k(X,\mathbb{Q})}=(k-\dim X)\,\mathrm{Id}$$

such that the degrees are centered at the middle cohomology.

The *Looijenga–Lunts–Verbitsky* (LLV) *algebra* $\mathfrak{g}$ is the subalgebra of $\operatorname{End}H^*(X,\mathbb{Q})$ generated by all $\mathfrak{sl}_2$-*triples* (L_a,h,Λ_a), where L_a is the operator of cup product with a class $a\in H^2(X,\mathbb{Q})$. We refer to [3] for an introduction. In particular, the fundamental theorem about this algebra is the following:

Theorem 1.1. (Looijenga–Lunts [11], Verbitsky [17])

1. $\mathfrak{g}$ *is isomorphic to* $\mathfrak{so}(V,q)$*;*
2. *Consider the adjoint action of the operator* h *on* $\mathfrak{g}$*, then we have an eigenspace decomposition* $\mathfrak{g}=\mathfrak{g}_{-2}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_2$*. In particular, the action of an element in* $\mathfrak{g}_0$ *preserves the cohomological degree.*

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3. *The reduced part* $\mathfrak{g}_0':=[\mathfrak{g}_0,\mathfrak{g}_0]$ *is isomorphic to* $\mathfrak{so}(H^2(X,\mathbb{Q}),q_X)$, *and we have the decomposition* $\mathfrak{g}_0=\mathfrak{g}_0'\oplus\mathbb{Q}h$.

The cohomology $H^*(X,\mathbb{Q})$ is a $\mathfrak{g}$-module by construction. By semisimplicity, the cohomology hence splits into a direct sum of irreducible $\mathfrak{g}$-submodules V_λ,

$$H^*(X,\mathbb{Q})\cong\bigoplus_\lambda V_\lambda^{m_\lambda},$$

called the LLV decomposition; here $m_\lambda\in\mathbf{N}$ are the multiplicities of the components. The main goal of this note is to discuss the general structure of this decomposition. One has the following basic results.

Proposition 1.2. $H^*(X,\mathbb{Q})$ *decomposes into* $H^*_{\mathrm{even}}(X,\mathbb{Q})\oplus H^*_{\mathrm{odd}}(X,\mathbb{Q})$ *as* $\mathfrak{g}$-*modules.*

Proof. This immediately follows from the fact that any element of $\mathfrak{g}$ acts by an even degree. □

Theorem 1.3. [2,17] *The subalgebra* $SH^2(X,\mathbb{Q})\subset H^*(X,\mathbb{Q})$ *generated by* $H^2(X,\mathbb{Q})$ *is an irreducible* $\mathfrak{g}$-*submodule. It is isomorphic to*

$$\mathrm{Sym}^*(H^2(X,\mathbb{Q}))/\left\langle a^{n+1}\mid q_X(a)=0\right\rangle$$

as algebra and $\mathfrak{g}_0'$-*module.*

The branching rules for $\mathfrak{g}_0'\subset\mathfrak{g}$ show that $SH^2(X,\mathbb{Q})$ is isomorphic to $V_{(n)}$ as a $\mathfrak{g}$-module (see below for notation). Hence there is always an irreducible submodule in $H^*(X,\mathbb{Q})$ that is known (and also quite big), and referred to as the *Verbitsky component*. The structure of the remaining components is however still mysterious. The so far strongest conjectural bound on their weights will be given in Remark 5.8 (*cf.* [7, Theorem 1.14]).

The plan of this note is as follows: in Sect. 2 we recall useful facts about the representation theory of Lie algebras of type B and D. In Sect. 3 we discuss the connection of the LLV algebra to the Hodge structure which will be sufficient to determine the LLV decomposition for the OG10 class. In Sect. 4 we introduce the Mumford–Tate algebra, and in Sect. 5 we almost give full details in the computation of the LLV decomposition in the K3$^{[n]}$-type case. The remaining cases of generalized Kummer varieties and OG6 are sketched in Sect. 6. Starting with Sect. 7 the last two sections will consider applications of the LLV decomposition to the Nagai conjecture, which concerns the question, how the nilpotency indices of degenerations of hyperkähler varieties are related in different degrees.

2. Representation Theory

We introduce the necessary notions for the representation theory of $\mathfrak{g}$. For this section, we let $\mathfrak{g}:=\mathfrak{so}(V,q)$ denote a Lie algebra of type B_r or D_r defined over $\mathbb{Q}$, where $\dim V=2r+1$ or $\dim V=2r$. We write Λ for the weight lattice of $\mathfrak{g}$ and $\mathfrak{W}$ for the Weyl group. Our main references are the Appendices of [7] and the book [6].

Type B

Let $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ be a Cartan subalgebra. The standard representation V decomposes as

$$V = V(0) \oplus V(\varepsilon_1) \oplus V(-\varepsilon_1) \oplus \cdots \oplus V(\varepsilon_r) \oplus V(-\varepsilon_r),$$

for some $0, \pm\varepsilon_1, \ldots, \pm\varepsilon_r \in \mathfrak{h}^\vee$ which are called the weights of V. An element $h \in \mathfrak{h}$ acts as the scalar $\varepsilon(h)$ on $V(\varepsilon)$. We choose a positive Weyl chamber generated by the fundamental weights

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \text{ for } 1 \leq i \leq r-1, \quad \varpi_r = \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r).$$

They correspond to the highest weight of $\bigwedge^i V$ for $1 \leq i \leq r-1$ and the spin module respectively. The set of dominant weights is the following

$$\Lambda^+ = \left\{ \lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r \;\middle|\; \begin{array}{c} \lambda_1 \geq \cdots \geq \lambda_r \geq 0 \\ \lambda_i \in \tfrac{1}{2}\mathbb{Z}, \lambda_i - \lambda_j \in \mathbb{Z} \text{ for all } i, j \end{array} \right\}.$$

Over $\mathbb{C}$, irreducible $\mathfrak{g}_{\mathbb{C}}$-modules are classified by their highest weight.

Over $\mathbb{Q}$, the Schur–Weyl construction for a $\mathfrak{g}$-module with integral highest weight is still available: let λ be a dominant weight with $\sum \lambda_i = d$. Then one defines (see [6, Section 19.5])

$$V_\lambda := \mathbf{S}_\lambda V \cap V^{[d]},$$

where $\mathbf{S}_\lambda$ is the Schur functor (see [6, Section 6.1]), and $V^{[d]}$ is the intersection of all the kernels $\ker(V^d \xrightarrow{q} V^{d-2})$ given by contracting any two components with the quadratic form q. On the other hand, modules with half-integer highest weight are not necessarily defined over $\mathbb{Q}$.

Example 2.1. We have $V_{(1,\ldots,1)} = \bigwedge^k V$, and $V_{(k)} = \ker(\mathrm{Sym}^k V \xrightarrow{q} \mathrm{Sym}^{k-2} V)$.

Type D

The standard representation V has weights $\pm\varepsilon_1, \ldots, \pm\varepsilon_r \in \mathfrak{h}^\vee$. The fundamental weights are given by

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \text{ for } 1 \leq i \leq r-2,$$
$$\varpi_{r-1} = \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r), \quad \varpi_r = \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r),$$

corresponding to the highest weight of $\bigwedge^i V$ for $1 \leq i \leq r-2$ and the two half-spin modules respectively. The set of dominant weights is the following

$$\Lambda^+ = \left\{ \lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r \;\middle|\; \begin{array}{c} \lambda_1 \geq \cdots \geq \lambda_{r-1} \geq |\lambda_r| \geq 0 \\ \lambda_i \in \tfrac{1}{2}\mathbb{Z}, \lambda_i - \lambda_j \in \mathbb{Z} \text{ for all } i, j \end{array} \right\}.$$

Again, all the representations with integral highest weight are defined over $\mathbb{Q}$ via the Schur–Weyl construction, which is not necessarily the case for those with half-integer highest weight.

For both type B and type D, the dimension of each V_λ can be obtained using Weyl dimension formula, which we won't state here. We will however need the following corollary of the dimension formula.

Lemma 2.2. ([7, Lemma A.9] and [6, Exercise 24.9]) *Let λ and $\mu \neq 0$ be dominant integral weights of $\mathfrak{g}$, then* $\dim V_{\lambda+\mu} > \dim V_\lambda$.

Weyl Character

We review results on the Weyl character ring for any reductive rational Lie algebra $\mathfrak{g}$, although our main interest remains in type B and D. Let $\mathrm{Rep}(\mathfrak{g})$ be the category of finite dimensional rational $\mathfrak{g}$-modules. Complexification gives a functor

$$\mathrm{Rep}(\mathfrak{g}) \longrightarrow \mathrm{Rep}(\mathfrak{g}_{\mathbb{C}})$$

to the category of $\mathfrak{g}_{\mathbb{C}}$-modules, which induces an *injective* morphism

$$K(\mathfrak{g}) \hookrightarrow K(\mathfrak{g}_{\mathbb{C}})$$

at the level of *representation rings*, that is, the Grothendieck rings of the corresponding categories.

The Weyl character of a $\mathfrak{g}_{\mathbb{C}}$-module $V = \bigoplus_\mu V(\mu)$ is given by $\mathrm{ch}\, V := \sum \dim V(\mu) e^\mu$ with value in the group ring $\mathbb{Z}[\Lambda]$, where e^μ is the element corresponding to the weight μ. The character map factors through the representation ring $K(\mathfrak{g}_{\mathbb{C}})$ and has image in $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$, the $\mathfrak{W}$-invariant subring.

Theorem 2.3. [6, Theorem 23.24] *The character map* $\mathrm{ch}\colon K(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{Z}[\Lambda]^{\mathfrak{W}}$ *is a ring isomorphism.*

We describe the Weyl character ring $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$ for $\mathfrak{g}$ of type B_r and D_r.

Proposition 2.4. [7, Section B.1.1]

1. *When* $\mathfrak{g}$ *is of type* B_r*, write* $x_i := e^{\varepsilon_i}$*. Then*
$$\mathbb{Z}[\Lambda] = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\pm \frac{1}{2}}].$$
The Weyl group $\mathfrak{W}_{2r+1}$ *is isomorphic to* $\mathfrak{S}_r \ltimes (\mathbb{Z}/2)^r$*, where* $\mathfrak{S}_r$ *acts as permutations on* $x_1, \ldots, x_r$ *and the* i*-th* $\mathbb{Z}/2$ *acts as* $x_i \mapsto x_i^{-1}$*.*
2. *When* $\mathfrak{g}$ *is of type* D_r*, the group ring* $\mathbb{Z}[\Lambda]$ *is the same as above for* B_r*, while the Weyl group* $\mathfrak{W}_{2r}$ *is the index-*2 *subgroup of* $\mathfrak{W}_{2r+1}$ *consisting of elements with an even number of non-trivial components in* $(\mathbb{Z}/2)^r$*.*

We have the following result that relates the two.

Proposition 2.5. [7, Proposition B.6] *Let* (V, q) *be a rational quadratic space of dimension* $\dim V = 2r + 1$*, and* $W \subset V$ *a non-degenerate subspace of dimension* $\dim W = 2r$*. Let* $\mathfrak{g} = \mathfrak{so}(V, q)$ *and* $\mathfrak{m} = \mathfrak{so}(W, q|_W)$*. Then the restriction functor* $\mathrm{Res}\colon \mathrm{Rep}(\mathfrak{g}) \to \mathrm{Rep}(\mathfrak{m})$ *induces an injective morphism for the character rings, and consequently, the (rational) representation rings. We have the following diagram*

$$\begin{tikzcd}
K(\mathfrak{g}) \arrow[r, hook, "\mathrm{Res}"] \arrow[d, hook] & K(\mathfrak{m}) \arrow[d, hook] \\
K(\mathfrak{g}_{\mathbb{C}}) \arrow[r, hook, "\mathrm{Res}"] \arrow[d, "\mathrm{ch}"', "\simeq"] & K(\mathfrak{m}_{\mathbb{C}}) \arrow[d, "\mathrm{ch}"', "\simeq"] \\
\mathbb{Z}[\Lambda]^{\mathfrak{W}_{2r+1}} \arrow[r, hook] & \mathbb{Z}[\Lambda]^{\mathfrak{W}_{2r}}.
\end{tikzcd}$$

In particular, for an arbitrary $\mathfrak{g}$-module, if one can obtain its decomposition as an $\mathfrak{m}$-module via restriction, then its Weyl character is uniquely determined and hence so is its $\mathfrak{g}$-module structure.

Remark 2.6. In the hyperkähler setting, the LLV algebra $\mathfrak{g}$ is of type B_{r+1} or D_{r+1}, and its reduced part $\mathfrak{g}_0'$ is of type B_r or D_r, so the proposition does not apply directly for $\mathfrak{g}_0' \subset \mathfrak{g}$. Instead, in the $\mathrm{K3}^{[n]}$-case, we will take the subalgebra $\mathfrak{m}$ to be $\mathfrak{g}(S)$, the LLV algebra of a K3 surface S.

3. Hodge Structures

From this section on, we let $r{:=}\lfloor b_2(X)/2 \rfloor$, so that $\mathfrak{g}$ is of type B_{r+1} or D_{r+1}, and $\mathfrak{g}_0'$ is of type B_r or D_r. The weights of $\mathfrak{g}$ will be denoted as $\lambda = \lambda_0\varepsilon_0 + \cdots + \lambda_r\varepsilon_r$.

The LLV decomposition is a diffeomorphism invariant, but we can obtain more information using a complex structure. Let $f \in \operatorname{End} H^*(X, \mathbb{R})$ be the Weil operator

$$f|_{H^{p,q}(X)} = \sqrt{-1}(q-p)\,\mathrm{Id}\,.$$

We will use this operator to define Hodge structures on each irreducible component V_λ, and obtain some conditions on the dominant weight λ that can appear.

Proposition 3.1. *We have $f \in (\mathfrak{g}_0')_{\mathbb{R}}$.*

Proof. Denote by I, J, K three complex structures coming from a hyperkähler metric g where I is the complex structure that we are using. We have three Kähler classes $\omega_I = g(I-,-)$, $\omega_J = g(J-,-)$, and $\omega_K = g(K-,-)$, hence three $\mathfrak{sl}_2$-triples

$$(L_I, h, \Lambda_I),\ (L_J, h, \Lambda_J),\ (L_K, h, \Lambda_K). \tag{1}$$

These are all operators on $H^*(X, \mathbb{R})$ and lie in $\mathfrak{g}_{\mathbb{R}}$ by construction.

By working pointwise on tangent spaces and then using harmonic forms, Verbitsky showed that the Weil operator $f = f_I$ for the complex structure I satisfies (cf. [7, Proposition 2.24])

$$f_I = -[L_J, \Lambda_K] = -[L_K, \Lambda_J],$$

so $f_I \in (\mathfrak{g}_0)_{\mathbb{R}}$. One may consider Weil operators f_J and f_K for the other two complex structures, and verify that

$$[f_J, f_K] = -2f_I.$$

So f_I indeed lies in $[(\mathfrak{g}_0)_{\mathbb{R}}, (\mathfrak{g}_0)_{\mathbb{R}}] = (\mathfrak{g}_0')_{\mathbb{R}}$. □

Remark 3.2. Recall that the real subalgebra $\mathfrak{g}_g$ generated by the three $\mathfrak{sl}_2$-triples (1) is isomorphic to $\mathfrak{so}(4,1)$: an explicit basis over $\mathbb{R}$ is given by

$$\Lambda_I, \Lambda_J, \Lambda_K, \quad f_I, f_J, f_K, h, \quad L_I, L_J, L_K.$$

In particular, the degree-0 part is generated by h and the three Weil operators.

Under the action of f, the standard representation V decomposes as

$$V = V^{2,0} \oplus V^{1,1} \oplus V^{0,2},$$

where f acts as $-2\sqrt{-1}$, 0, and $2\sqrt{-1}$ respectively. Similarly, we have another decomposition under the action of h

$$V = V_{-2} \oplus V_0 \oplus V_2,$$

where h acts as -2, 0, and 2 respectively. Hence if we take $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ a Cartan subalgebra that contains both h and f, then h and $\sqrt{-1}f$ are among the $\pm\varepsilon_i^\vee$. Up to the choice of a Weyl chamber, we may hence suppose that $h = \varepsilon_0^\vee$ and $\sqrt{-1}f = \varepsilon_1^\vee$. Under this choice, we can also identify $\varepsilon_1, \ldots, \varepsilon_r$ as the weights of $\mathfrak{g}_0'$.

For a $\mathfrak{g}$-module V_λ that appears in $H^*(X, \mathbb{Q})$, we take its weight decomposition with respect to the chosen Cartan subalgebra $\mathfrak{h}$: $(V_\lambda)_{\mathbb{C}} = \bigoplus_\mu V_\lambda(\mu)$, where $V_\lambda(\mu)$ is the component of weight $\mu = \mu_0\varepsilon_0 + \cdots + \mu_r\varepsilon_r$. Then h acts as $2\mu_0$ and $\sqrt{-1}f$ acts as $2\mu_1$ on $V_\lambda(\mu)$. We find

$$V_\lambda(\mu) \subset H^{p,q}(X)$$

where

$$\begin{cases} 2\mu_0 = p + q - 2n \\ 2\mu_1 = \sqrt{-1} \cdot \sqrt{-1}(q - p) = p - q \end{cases} \Rightarrow \quad \begin{cases} p = \mu_0 + \mu_1 + n \\ q = \mu_0 - \mu_1 + n \end{cases} \tag{2}$$

In other words, $V_\lambda \subset H^*(X, \mathbb{Q})$ is a sub-Hodge structure.

More generally, there is a naturally defined Hodge structure on any $\mathfrak{g}$-module V_λ determined by the actions of h and f. We simply set

$$(V_\lambda)_{\mathbb{C}}^{p,q} := \bigoplus_{\mu \text{ satisfying } 2} V_\lambda(\mu).$$

The Hodge numbers $h^{p,q}$ count the multiplicities of suitable weights.

Remark 3.3. The Hodge numbers only give information about λ_0, λ_1 in the representation V_λ and do not necessarily determine the $\mathfrak{g}$-module structure (such an example will show up in the case of OG6).

Example 3.4. The Verbitsky component $SH^2(X, \mathbb{Q})$ contains a non-trivial $H^{2n,2n}$-part. We have $p = 2n, q = 2n$ so $\mu_0 = n, \mu_1 = 0$ which must be the highest weight (n). (In fact this can be used to prove that $SH^2(X, \mathbb{Q}) \simeq V_{(n)}$: we just saw that the highest weight of $SH^2(X, \mathbb{Q})$ dominates (n); on the other hand, we have $\dim SH^2(X, \mathbb{Q}) = \dim V_{(n)}$ due to the description of Verbitsky, so by Lemma 2.2, the highest weight must be exactly (n).) In particular, since $H^{2n,2n}(X)$ is one-dimensional, the component $V_{(n)}$ appears with multiplicity 1 in $H^*(X, \mathbb{Q})$. By using the explicit description of $V_{(n)}$ one also sees that it exhausts all the outermost Hodge numbers $h^{2k,0} = 1$.

Corollary 3.5.
1. *Each component V_λ of $H^*_{\text{even}}(X, \mathbb{Q})$ has integral highest weight λ;*
2. *Each component V_λ of $H^*_{\text{odd}}(X, \mathbb{Q})$ has half-integer highest weight λ;*
3. *Each component V_λ other than the Verbitsky component satisfies $\lambda_0 + \lambda_1 \leq n - 1$ and $\lambda_0 \leq n - \frac{3}{2}$.*

Proof. For statements (1) and (2), we look at the component $V_\lambda(\lambda)$ and get

$$p + q = 2\lambda_0 + 2n,$$

which allows us to conclude that λ_0 is an integer or a half-integer in the two cases.

For statement (3), since V_λ is not the Verbitsky component, it cannot have a $H^{2n,0}$-part, so by looking at the component $V_\lambda(\lambda)$ we get

$$\lambda_0 + \lambda_1 + n = p \leq 2n - 1,$$

which gives the first inequality. By definition, the Verbitsky component exhausts the second cohomology $H^2(X, \mathbb{Q})$ and hence $H^{4n-2}(X, \mathbb{Q})$ by Hodge symmetry, so we also have

$$3 \leq p + q = 2\lambda_0 + 2n \leq 4n - 3,$$

which gives the second inequality. $\square$

Remark 3.6. 1. The two inequalities in (3) are tight: for generalized Kummer varieties Kum_n with $n \geq 2$ (whose LLV algebra is of type B_4), we can have the component $V_{(n-\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})}$. In fact, the existence of this component is equivalent to the non-vanishing of $b_3(X)$.

2. When $n = 2$, the statement (3) shows that $\lambda = (2)$ or $\lambda_0 \leq \frac{1}{2}$, so all the possible weights are $(2), (\frac{1}{2}, \ldots, \frac{1}{2}), (0)$, plus $(\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2})$ in the type D case. The half-integer weight component generates $H^*_{\mathrm{odd}}(X, \mathbb{Q})$.

   If we assume the conjecture of [7] (see Remark 5.8 below), the sum of each weight would then be bounded by $n = 2$, so the odd cohomologies must vanish entirely when $b_2(X) \geq 8$. This is indeed the case, by a theorem of Guan (see [1, Theorem 3.6] of this volume).

3. When $n = 3$ and V_λ is a component of $H^*_{\mathrm{even}}(X, \mathbb{Q})$ other than $V_{(3)}$, we get $\lambda_0 \leq 1$ so λ is a sequence of ones, and V_λ must be a wedge product $\bigwedge^k V$.

The corollary gives some constraints on the irreducible components that can appear. For O'Grady's 10-dimensional example, this is already enough to determine the full decomposition.

Proposition 3.7. *Let X be a hyperkähler manifold of dimension 10 such that $b_2(X) = 24$, $e(X) = 176904$, and $H^*_{\mathrm{odd}}(X) = 0$. Then we have the following decomposition of $\mathfrak{g}$-modules*

$$H^*(X, \mathbb{Q}) = V_{(5)} \oplus V_{(2,2)}.$$

In particular, O'Grady's example OG10 satisfies these numerical conditions, so we have obtained its LLV decomposition.

Proof. The LLV algebra $\mathfrak{g}$ is of type D_{13}. Write $H^*(X, \mathbb{Q}) = H^*_{\mathrm{even}}(X, \mathbb{Q}) = V_{(5)} \oplus V'$. We have $\dim V' = e(X) - \dim V_{(5)} = 37674$. By using the inequalities in Corollary 3.5 and by considering the dimension bound and Lemma 2.2, the only possible dominant weights that can appear are

$$\{(3), (2,2), (2,1), (2), (1,1,1,1), (1,1,1), (1,1), (1), (0)\}.$$

Each V_λ carries a Hodge structure and therefore has its own Betti numbers. Using Salamon's result on the Betti numbers of a hyperkähler manifold (see [1, Section 2] of this volume), one can verify that the only possible solution is one copy of $V_{(2,2)}$. $\square$

4. Mumford–Tate Algebra

Definition 4.1. Let W be a rational Hodge structure. Let f be the Weil operator

$$f|_{W^{p,q}} = \sqrt{-1}(q-p)\,\mathrm{Id}\,.$$

The special Mumford–Tate algebra $\mathfrak{m} = \mathfrak{m}(W)$ is the smallest rational subalgebra of $\mathrm{End}(W)$ such that $f \in \mathfrak{m}_{\mathbb{R}}$. The (full) Mumford–Tate algebra is $\mathfrak{m} \oplus \mathbb{Q}h$ where h is the degree operator $h|_{W^{p,q}} = (p+q)\,\mathrm{Id}$. It coincides with the associated Lie algebra of the Mumford–Tate group of W. (This degree operator differs from the one that we defined earlier, so we need to take a Tate twist $H^*(X,\mathbb{Q})(\dim X)$.)

When W is the cohomology $H^*(X,\mathbb{Q})$ of a hyperkähler manifold X, by Proposition 3.1 we see that $\mathfrak{m}$ is a subalgebra of $\mathfrak{g}_0'$. Conversely, we have the following result.

Proposition 4.2. *For X a very general hyperkähler manifold, the Mumford–Tate algebra $\mathfrak{m}$ is equal to $\mathfrak{g}_0'$.*

Proof. Consider the restriction map

$$\rho\colon\ \mathrm{End}\, H^*(X,\mathbb{Q}) \longrightarrow \mathrm{End}\, H^2(X,\mathbb{Q}).$$

The Weil operator f_2 on $H^2(X,\mathbb{Q})$ is the restriction of f. Since $\mathfrak{m}$ satisfies $f \in \mathfrak{m}_{\mathbb{R}}$, its restriction $\rho(\mathfrak{m})$ will satisfy $f_2 = \rho(f) \in \rho(\mathfrak{m})_{\mathbb{R}}$. Thus by definition, $\rho(\mathfrak{m})$ contains the special Mumford–Tate algebra $\mathfrak{m}(H^2(X,\mathbb{Q}))$ for the second cohomology. By the local Torelli theorem, the latter is equal to $\mathfrak{so}(H^2(X,\mathbb{Q}), q_X) \simeq \mathfrak{g}_0'$ for X very general. So $\rho(\mathfrak{m}) \simeq \mathfrak{g}_0'$, which shows that $\mathfrak{m}$ must coincide with $\mathfrak{g}_0'$. □

Consequently, for a very general X, the decomposition of $H^*(X,\mathbb{Q})$ into $\mathfrak{g}_0'$-modules is the same as decomposition into sub-Hodge structures.

Example 4.3. For X of $\mathrm{K3}^{[2]}$-type, by a dimension count we have $H^*(X,\mathbb{Q}) = V_{(2)}$ as $\mathfrak{g}$-module. Write H for the second cohomology group as a $\mathfrak{g}_0'$-module. Using the description by Verbitsky, we get an isomorphism of $\mathfrak{g}_0'$-modules

$$\begin{aligned} H^*(X,\mathbb{Q}) &= \mathbb{Q} \oplus H \oplus \quad \mathrm{Sym}^2 H \quad \oplus H \oplus \mathbb{Q} \\ &= \mathbb{Q} \oplus H \oplus (H_{(2)} \oplus \mathbb{Q}) \oplus H \oplus \mathbb{Q}\text{'} \end{aligned}$$

where $H_{(2)}$ is an irreducible $\mathfrak{g}_0'$-module obtained as $\ker(\mathrm{Sym}^2 H \xrightarrow{q_X} \mathbb{Q})$. The 1-dimensional component $\mathbb{Q} \subset H^4(X,\mathbb{Q})$ is generated by the dual of q_X, which is also proportional to $c_2(X)$.

For a Hodge special X, the Mumford–Tate algebra $\mathfrak{m}$ becomes smaller, so $H^*(X,\mathbb{Q})$ may decompose further into smaller components. This is the key idea for determining the LLV decomposition for the other three types of hyperkähler manifolds.

5. $\mathrm{K3}^{[n]}$-Type

In the $\mathrm{K3}^{[n]}$-type case, there is a natural choice of a Hodge special locus: when $X = S^{[n]}$ is actually the Hilbert scheme of a K3 surface S (not necessarily algebraic).

We have a decomposition

$$(H^2(X,\mathbb{Q}), q_X) = (H^2(S,\mathbb{Q}), q_S) \oplus \langle -2(n-1)\rangle.$$

Hence $\mathfrak{g}(S)$ is naturally realized as a subalgebra of $\mathfrak{g} = \mathfrak{g}(X)$, and $\mathfrak{m}(S) = \mathfrak{m}(H^2(S,\mathbb{Q}))$ as a subalgebra of $\mathfrak{m} = \mathfrak{m}(H^2(X,\mathbb{Q}))$. We write $W{:=}H^*(S,\mathbb{Q})$, which coincides with the Mukai completion of $H^2(S,\mathbb{Q})$ and is therefore the standard representation for $\mathfrak{g}(S)$. When S is non-algebraic and very general, $\mathfrak{m}(S)$ coincides with $\mathfrak{g}_0'(S) = \mathfrak{so}(H^2(S,\mathbb{Q}), q_S)$ and is of type D_{11}. We have the diagram

$$\begin{tikzcd}
\mathfrak{g} & \mathfrak{g}_0' \arrow[l, hook'] \\
\mathfrak{g}(S) \arrow[u, hook] & \mathfrak{g}_0'(S) = \mathfrak{m}(S) \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}$$

The Hodge structure on $H^*(S^{[n]},\mathbb{Q})$ is described by Göttsche–Soergel [8] (stated for algebraic ones only; the general case is due to de Cataldo–Migliorini [4, Theorem 5.2.1]). We recall that a partition $\alpha \vdash n$ is a sequence $\alpha = (a_1, \ldots, a_n)$ satisfying $a_1 \cdot 1 + \cdots + a_n \cdot n = n$.

Theorem 5.1. *Let S be a K3 surface, not necessarily algebraic. We have an isomorphism of Hodge structures*

$$H^*(S^{[n]},\mathbb{Q})(n) \simeq \bigoplus_{\alpha\vdash n} H^*(S^{(a_1)} \times \cdots \times S^{(a_n)},\mathbb{Q})(a_1 + \cdots + a_n).$$

The sum is taken over all partitions α of n. Here $S^{(a)}$ denotes the ath symmetric power $S^a/\mathfrak{S}_a$ of S, and we have an isomorphism of Hodge structures

$$H^*(S^{(a)},\mathbb{Q}) \simeq \operatorname{Sym}^a H^*(S,\mathbb{Q}).$$

Remark 5.2. We can omit all the Tate twists by considering the grading h on the cohomologies centered at the middle cohomology.

In other words, we have obtained the decomposition of $H^*(X,\mathbb{Q})$ as an $\mathfrak{m}(S)$-module. To deduce the $\mathfrak{g}$-module structure, we first lift this as a $\mathfrak{g}(S)$-module decomposition, and then apply Proposition 2.5.

Theorem 5.3. *We have an isomorphism*

$$H^*(S^{[n]},\mathbb{Q}) \simeq \bigoplus_{\alpha\vdash n} \bigotimes_{i=1}^{n} \operatorname{Sym}^{a_i} H^*(S,\mathbb{Q}) \tag{3}$$

of $\mathfrak{g}(S)$-modules. Consequently, the Weyl character of $H^*(S^{[n]},\mathbb{Q})$ as a $\mathfrak{g}(S)$-module is equal to

$$\operatorname{ch} H^*(S^{[n]},\mathbb{Q}) = \sum_{\alpha\vdash n} \prod_{i=1}^{n} \operatorname{ch} \operatorname{Sym}^{a_i} W.$$

In view of Proposition 2.5, this gives the Weyl character of $H^*(X,\mathbb{Q})$ as a $\mathfrak{g}$-module.

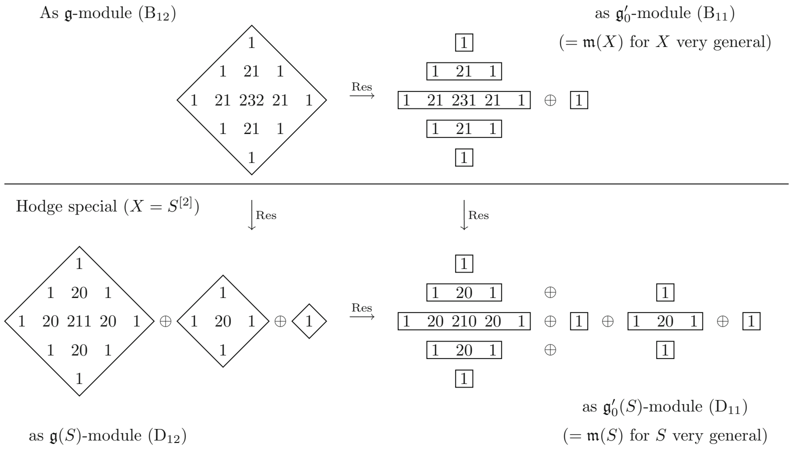

FIGURE 1 Decompositions of the Hodge diamond of $H^*(\mathrm{K3}^{[2]}, \mathbb{Q})$.

Proof. The $\mathfrak{g}(S)$-module structure is a diffeomorphism invariant, so we may assume that S is very general and non-algebraic. Recall that in this case, the special Mumford–Tate algebra $\mathfrak{m}(S)$ coincides with $\mathfrak{g}_0'(S) = \mathfrak{so}(H^2(S, \mathbb{Q}), q_S)$. So the isomorphism of Hodge structures gives an isomorphism of $\mathfrak{g}_0'(S)$-modules.

Since $\mathfrak{g}_0(S) = \mathfrak{g}_0'(S) \oplus \mathbb{Q}h$ and the decomposition respects the grading h, we can lift it to an isomorphism of $\mathfrak{g}_0(S)$-modules. Finally, the weight lattice of $\mathfrak{g}_0(S)$ is the same as that of $\mathfrak{g}(S)$, so this also is an isomorphism of $\mathfrak{g}(S)$-modules. $\square$

Remark 5.4. Alternatively, one can prove the $\mathfrak{g}(S)$-equivariance of the isomorphism (3) using Nakajima operators [13] and the explicit description of the LLV action in the Nakajima basis given in [15]. By [5, Prop.6.1.5], see also [14, Thm.2.4], the isomorphism (3) matches the Nakajima description.

Example 5.5. We consider again the $\mathrm{K3}^{[2]}$-type case. The isomorphism is given as

$$H^*(S^{[2]}, \mathbb{Q}) \simeq H^*(S^{(2)} \times S^{(0)}, \mathbb{Q}) \oplus H^*(S^{(0)} \times S^{(1)}, \mathbb{Q}) = \mathrm{Sym}^2 H^*(S, \mathbb{Q}) \oplus H^*(S, \mathbb{Q}).$$

The right hand side decomposes into 3 irreducible $\mathfrak{g}(S)$-modules, and further into 10 irreducible $\mathfrak{g}_0'(S)$-modules.

We may write the formula for the characters of $H^*(\mathrm{K3}^{[n]}, \mathbb{Q})$ in a more succinct fashion by considering all Hilbert powers at the same time (Fig. 1). Note that the LLV algebras are a priori not the same in different dimensions. But since we are considering Weyl characters, we only need the complexification $\mathfrak{g}_{\mathbb{C}}$ which is always isomorphic to $\mathfrak{so}(25)$.

Proposition 5.6. [7] *Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{so}(25)$. The generating series of the characters of the $\mathfrak{g}$-modules $H^*(\mathrm{K3}^{[n]})$ for $n \geq 2$ is given by*

$$\sum_{n=0}^{\infty} \mathrm{ch}\, H^*(\mathrm{K3}^{[n]}) q^n = \prod_{n=1}^{\infty} \prod_{i=0}^{11} \frac{1}{(1 - x_i q^n)(1 - x_i^{-1} q^n)}. \tag{4}$$

The identity lives inside the formal power series ring $A[[q]]$ where

$$A := \mathbb{Z}[\Lambda]^{\mathfrak{W}} = \mathbb{Z}[x_0^{\pm 1}, \ldots, x_{11}^{\pm 1}, (x_0 \cdots x_{11})^{\pm \frac{1}{2}}]^{\mathfrak{W}_{25}}$$

is the Weyl character ring of type B_{12}*. Note that when* $n = 1$*, the cohomology* $H^*(\mathrm{K3})$ *does not admit a structure of* $\mathfrak{g}$*-module, so we write formally*

$$\operatorname{ch} H^*(\mathrm{K3}^{[1]}) := \sum_{i=0}^{11}(x_i + x_i^{-1}).$$

Corollary 5.7. *Let* X *be a hyperkähler manifold of* $\mathrm{K3}^{[n]}$*-type. Any irreducible component* V_λ *of the LLV decomposition of* $H^*(X, \mathbb{Q})$ *with highest weight* $\lambda = \lambda_0\varepsilon_0 + \cdots + \lambda_{11}\varepsilon_{11}$ *satisfies*

$$\lambda_0 + \cdots + \lambda_{11} \le n.$$

Proof. The weight λ corresponds to the monomial $x_0^{\lambda_0}\cdots x_{11}^{\lambda_{11}}$ in the character ring. When we expand the right hand side of 4 we get

$$\prod_{n=1}^{\infty}\prod_{i=0}^{11}\left(\sum_{j\ge 0}(x_i q^n)^j\right)\left(\sum_{k\ge 0}(x_i^{-1}q^n)^k\right).$$

For each term of this product, its degree in x_i is bounded by its degree in q. So each monomial that appears in the coefficient of q^n has degree $\le n$, which gives the inequality. □

Remark 5.8. More generally, for any hyperkähler manifold X of dimension $2n$ with $r = \lfloor b_2(X)/2 \rfloor$, Green–Kim–Laza–Robles [7] conjecture the inequality

$$\lambda_0 + \cdots + \lambda_{r-1} + |\lambda_r| \le n$$

for each irreducible component V_λ of the LLV decomposition of $H^*(X, \mathbb{Q})$. This conjecture holds for all known examples of hyperkähler varieties.

Remark 5.9. Once the character of the $\mathfrak{g}$-module structure is known, one can use computer algebra to recover the actual decomposition. The results in lower dimensions can be found on the second author's webpage.

6. Generalized Kummer Varieties and OG6

After having treated the $\mathrm{K3}^{[n]}$ case in the previous section and OG10 in Proposition 3.7, we briefly remark on the remaining two cases. See [7] for details and references.

Generalized Kummer Varieties

The LLV algebra $\mathfrak{g}$ is of type B_4.

Similar to the case of $\mathrm{K3}^{[n]}$-type, we consider Hodge special members of the family: we specialize X to an actual generalized Kummer variety associated to a very general complex torus A of dimension 2. The results of Göttsche–Soergel give a complete description of the Hodge structure of $H^*(X)$ in terms of the Hodge structures on $H^*(A)$, which can be seen as a decomposition of $\mathfrak{m}(A)$-modules (of type D_3). We can similarly lift it to a $\mathfrak{g}(A)$-module decomposition (of type D_4) and apply Proposition 2.5 to obtain the character of $H^*(X)$ as a $\mathfrak{g}$-module.

OG6

This last case is more complicated. The LLV algebra $\mathfrak{g}$ is of type D_5.

Using the Hodge numbers of OG6 and the Hodge numbers of the $\mathfrak{g}$-modules, we may obtain two possible decompositions for $H^*(X, \mathbb{Q})$. To determine which case we are in, we specialize X to a Hodge special member with an explicit geometric construction given by Rapagnetta. In this situation, the Mumford–Tate algebra $\mathfrak{m}$ is of type B_2 (that of a very general abelian surface A), and the geometric construction gives a description of the Hodge structure of $H^*(X)$ in terms of $\mathfrak{m} = \mathfrak{m}(A)$-modules (Mongardi–Rapagnetta–Saccà). Then by comparing the restrictions to $\mathfrak{m}$ of the two possible $\mathfrak{g}$-module decompositions, only one agrees with the $\mathfrak{m}$-module decomposition obtained from geometry, so we may conclude.

7. Application: The Nagai Conjecture

Let $\pi : \mathcal{X} \to \Delta$ be a 1-paramer projective degeneration of hyperkähler manifolds over the disc $\Delta = \{t \in \mathbb{C} \mid |t| \leq 1\}$. We assume that the fibers $X_t = \pi^{-1}(t)$ are smooth for $t \neq 0$. Let

$$T : H^*(X_{t_0}, \mathbb{Z}) \longrightarrow H^*(X_{t_0}, \mathbb{Z})$$

be the monodromy operator for a fixed basepoint $t_0 \neq 0$. It is well-known (see e.g. [9]) that T is quasi-unipotent, i.e. there exists $m, n > 0$ such that $(T^m - 1)^n = 0$. For any such m, we hence can define the nilpotent operator

$$N = \frac{1}{m} \log(T^m).$$

Let also $N_k = N|_{H^k(X_0, \mathbb{Z})}$ be the restriction to degree k and define its nilpotency:

$$\nu_k := \min \left\{ r \in \mathbb{Z}_{\geq 1} | N_k^r \neq 0, N_k^{r+1} = 0 \right\}.$$

By a classical result of Schmid we always have $\nu_k \leq k$, and the precise value of ν_k should be viewed as measuring the change of topology under the degeneration.

For hyperkähler varieties the *type* of the degeneration is determined by ν_2 according to the following table:

Type	ν_2
I	0
II	1
III	2

For example, a quartic K3 surface S degenerating to a nodal K3 is of type I, the degeneration of S to the union of two quadrics $Q_1 \cup_E Q_2$ is type II, and breaking S into the union of 4 hyperplanes is type III. More generally, Kulikov classifies all the limits of semistable degenerations of K3 surfaces according to type.

In higher dimension a priori we need to consider all the nilpotencies, the even $\nu_2, \nu_4, \dots, \nu_{4n-2}$ and the odd ones $\nu_3, \nu_5, \dots, \nu_{4n-3}$. However, Nagai made the following prediction for the even degeneracies (the precise behaviour of the odd remains an open question).

Conjecture 7.1. (Nagai [12]) $\nu_{2k} = k\nu_2$.

Theorem 7.2. [10] *Nagai's conjecture holds if* $\nu_2 = 0$ *or* $\nu_2 = 2$.

Using the LLV decomposition one finds the following results:

Theorem 7.3. [7] *Let* $H^*(X, \mathbb{Q}) = \bigoplus_\lambda V_\lambda^{m_\lambda}$ *be the LLV decomposition. If*

$$\lambda_0 + \lambda_1 + |\lambda_2| \leq n$$

for all λ, *then Nagai's conjecture holds.*

Corollary 7.4. *Nagai's conjecture holds if* $\dim X \leq 8$. *It holds for all known examples of hyperkähler manifolds. It holds if the conjectural description of the LLV decomposition in Remark 5.8 is satisfied.*

We give a sketch of the proof below. The main geometric step is to relate the nilpotent matrices N_2 and N_k as follows. Let $\rho_k : \mathfrak{g}_0' \to \operatorname{End} H^k(X, \mathbb{Q})$ be the restriction of the LLV action to degree k. Moreover, since $T|_{H^2}$ preserves the Beauville–Bogomolov form, the nilpotent matrix $N_2 \in \operatorname{End} H^2(X, \mathbb{Q})$ lies in $\mathfrak{g}_0'$. Then one has:

Theorem 7.5. (Soldatenkov [16]) $N_k = \rho_k(N_2)$

From this, the cases $\nu_2 \in \{0, 2\}$ are fairly straightforward and follow essentially from the description of the Verbitsky component. The critical range of the Nagai conjecture is $\nu_2 = 1$.

8. Sketch of Proof for Theorems 7.3 and 7.5

We first give a sketch of the result of Soldatenkov following [7].

Sketch of Proof for Theorem 7.5

We divide the proof into two steps.

Step 1. Let $\mathcal{X}/S$ be a fixed degeneration of hyperkähler manifolds over a smooth base S, let $t_0 \in S$ be a base point, and let $\widetilde{S} \to S$ be the universal cover. Define the (extended) period domains parametrizing Hodge structures in degree $k > 2$ and 2 respectively:

$$\widehat{D}_k = \operatorname{Flag}(H^k(X_{t_0}, \mathbb{C}), f^\bullet),$$
$$\widehat{D}_2 = \mathbb{P}(H^2(X_{t_0}, \mathbb{C})).$$

Here $f^\bullet$ is the dimension vector of the Hodge filtration.

Proposition 8.1. *There exists a canonical morphism $\psi_k : \widehat{D}_2 \to \widehat{D}_k$ such that the diagram*

$$\begin{tikzcd} \widetilde{S} \arrow[rr, "\widetilde{\Phi}_2"] \arrow[dr, "\widetilde{\Phi}_k"'] & & \widehat{D}_2 \arrow[dl, "\psi_k"] \\ & \widehat{D}_k & \end{tikzcd} \tag{5}$$

commutes, where $\widetilde{\Phi}_k$ is the period mapping $t \mapsto \mathsf{pt}_{t_0,t}(F^\bullet H^k(X_t))$.

Here $\mathsf{pt}_{t_0,t}$ is obtained from the parallel transport map $H^*(X_t,\mathbb{Z}) \to H^*(X_{t_0},\mathbb{Z})$ along any path from t to t_0 by tensoring with $\mathbb{C}$, or equivalently it is the parallel transport with respect to the Gauss–Manin connection. It is well-defined since $\widetilde{S}$ is simply connected.

Proof. (Proof (Sketch)) Recall from Proposition 3.1 that the Weil operator $f \in \mathrm{End}(H^*(X,\mathbb{R}))$ defined by $f|_{H^{p,q}(X)} = \sqrt{-1}(q-p)\,\mathrm{Id}$ lies in $(\mathfrak{g}_0')_{\mathbb{R}}$ and hence satisfies

$$f|_{H^k} = \rho_k(f|_{H^2}). \tag{5}$$

This motivates the following:

Construction of ψ_k: Given $\mathfrak{o}_2 \in \widehat{D}_2$, there exists a unique semisimple $f_{\mathfrak{o}_2} \in (\mathfrak{g}_0')_{\mathbb{R}}$ such that $f_{\mathfrak{o}_2}$ induces $\mathfrak{o}_2$, i.e.

$$f_{\mathfrak{o}_2}(x) = (q-p)\sqrt{-1}x \text{ for all } x \in H^{p,q}_{\mathfrak{o}_2}(X).$$

Define $\psi_k(\mathfrak{o}_2)$ to be the filtration induced by $\rho_k(\mathfrak{o}_2)$.

Proof of Commutativity of (5): Let

$$\psi_k^t : \mathbb{P}(H^2(X_t,\mathbb{C})) \longrightarrow \widehat{D}_k^t = \mathrm{Flag}(H^k(X_{t_0},\mathbb{C}), f^\bullet)$$

be the map ψ_k above with respect to the base point t. If $\mathfrak{o}_2(t), \mathfrak{o}_k(t)$ are the elements determined by the Hodge structure of $H^*(X_t)$ then by construction of ψ_k^t and (5) we have that

$$\psi_k^t(\mathfrak{o}_2(t)) = \mathfrak{o}_k(t).$$

Parallel transport naturally intertwines the LLV algebra, that is

$$\mathrm{pt}_{t_0,t} \circ \rho_k(\alpha) = \rho_k(\mathrm{pt}_{t_0,t}(\alpha)) \circ \mathrm{pt}_{t_0,t}, \quad \text{hence} \quad \psi_k^{t_0} \circ \mathrm{pt}_{t_0,t} = \mathrm{pt}_{t_0,t} \circ \psi_k^t$$

We conclude:

$$\begin{aligned} \psi_k\left(\widetilde{\Phi}_2(t)\right) &= \psi_k^{t_0}(\mathrm{pt}_{t_0,t}(\mathfrak{o}_2(t))) \\ &= \mathrm{pt}_{t_0,t}(\psi_k^t \mathfrak{o}_2(t)) \\ &= \mathrm{pt}_{t_0,t}(\mathfrak{o}_k(t)) \\ &= \widetilde{\Phi}_k(t). \end{aligned}$$

□

Step 2. We prove Theorem 7.5. For that we require the degeneration $\mathcal{X}/S$ to be projective. We let $D_2 \subset \widehat{D}_2$ and $D_k \subset \widehat{D}_k$ be the period domains with respect to the choosen polarization, that is we define

$$D_2 = \{x \in \mathbb{P}(H^2_{\mathrm{prim}}(X_{t_0},\mathbb{C})) | x \cdot x = 0, x \cdot \overline{x}\}$$

and we let

$$D_k \subset \mathrm{Flag}(H^k_{\mathrm{prim}}(X_{t_0}, \mathbb{C}))$$

be the orbit of $\mathfrak{o}_k(t_0)$ under $\mathfrak{mt}_{k,\mathbb{R}}$, where $\mathfrak{mt}_k$ is the generic special Mumford–Tate algebra of $H^k_{\mathrm{prim}}(X_{t_0}, \mathbb{C})$ (in other words, the special Mumford–Tate algebra $\mathfrak{m}(H^k_{\mathrm{prim}}(X', \mathbb{C}))_{\mathbb{R}}$ of a generic projective deformation X' of X_{t_0}, see [7]). For the projective degeneration, the period mappings $\widetilde{\Phi}_k$ take values in D_k; we write Φ_k in this case. The key point is now:

- $\mathfrak{mt}_k = \rho_k(\mathfrak{mt}_2) = \rho_k(\mathfrak{g}'_0)$ and hence ψ_k defines a morphism $D_2 \to D_k$;
- $D_k \cong (\mathfrak{mt}_{k,\mathbb{R}} \cdot \mathfrak{o}_k(t_0))/K$ where K is a compact subgroup (see e.g. [9]).

It follows that if $T_k = T|_{H^k(X_{t_0}, \mathbb{Z})}$ is the monodromy along the loop γ, then in D_k we have

$$\begin{aligned} T_k^m \Phi_k(t_0) &= \Phi_k(\gamma^m \cdot t_0) \\ &= \psi_k(\Phi_2(\gamma^m \cdot t_0)) \\ &= \psi_k(T_2^m \Phi_2(t_0)) \\ &= \rho_k(T_2)^m \psi_k(\Phi_2(t_0)) \\ &= \rho_k(T_2)^m \Phi_k(t_0) \end{aligned}$$

for all $m \geq 1$ hence

$$T_k^{-m} \rho_k(T_2)^m \in K \cap \mathrm{GL}(H^k_{\mathrm{prim}}(X, \mathbb{Z}))$$

Since the intersection on the right is a finite subgroup, there exists m_1, m_2 such that $T_k^{-m_1} \rho_k(T_2)^{m_1} = T_k^{-m_2} \rho_k(T_2)^{m_2}$, and hence $T_k^{m_1 - m_2} = \rho_k(T_2)^{m_1 - m_2}$. Taking log on both sides, we get the desired equality $N_k = \rho_k(N_2)$.

We are ready to give an idea of the proofs of Theorems 7.2 and 7.3.

Sketch of Proof for Theorems 7.2 and 7.3

We first state the following lemma.

Lemma 8.2. *If a nilpotent operator $N : V \to V$ has nilpotency ν, then* $\mathrm{Sym}^k(N) : \mathrm{Sym}^k(V) \to \mathrm{Sym}^k(V)$ *has nilpotency $k \cdot \nu$.*

Proof. Since $N^{\nu+1}$ is zero, for all elements $x_1, \ldots, x_k \in V$ we have

$$\mathrm{Sym}^k(N)^{k\nu}(x_1 \cdots x_k) = (N^\nu x_1) \cdots (N^\nu x_k)$$

and $\mathrm{Sym}^k(N)^{k\nu+1}(x_1 \cdots x_k) = 0$. On the other hand, since N^ν is not zero, we may pick $x \in V$ such that $N^\nu x \neq 0$, and therefore $\mathrm{Sym}^k(N)^{k\nu}(x^k) = (N^\nu x)^k \neq 0$. □

For each $k \geq 1$, the group $H^{2k}(X, \mathbb{Q})$ contains the Verbitsky component $\mathrm{Sym}^k H^2 (X, \mathbb{Q})$ as a sub $\mathfrak{g}'_0$-module. Hence the nilpotency ν_{2k} of $N_k = \rho_k(N_2)$ is bounded below by the nilpotency of $\mathrm{Sym}^k N_2$, which is equal to $k \cdot \nu_2$ by the lemma. Therefore we obtain $\nu_{2k} \geq k \cdot \nu_2$.

- If $\nu_2 = 0$, then $N_2 = 0$, so $N_k = \rho_k(N_2) = 0$ is identically zero, and $\nu_{2k} = 0$.
- If $\nu_2 = 2$, then we already have the classical bound $\nu_{2k} \leq 2k$ so we may conclude that $\nu_{2k} = 2k$.

- If $\nu_2 = 1$, then we have $\nu_{2k} \geq k$. On the other hand, the nilpotency of the action of N_2 on V_λ can be bounded above by the weights, see [7]. In particular, the nilpotency of $N_k = \rho_k(N_2)$ is bounded by all the weights λ that appear in the $\mathfrak{g}_0'$-decomposition of $H^k(X_{t_0}, \mathbb{Q})$. Using the branching rules this can be expressed by the weights of the LLV decomposition of $H^*(X_{t_0}, \mathbb{Z})$. It is fortunate that this yields the rather simple weight bound of the claim.

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References

[1] Beri, P., Debarre, O.: On the Hodge and Betti numbers of hyper-Kähler manifolds **(this volume)**

[2] Bogomolov, F.A.: On the cohomology ring of a simple hyper-Kähler manifold (on the results of Verbitsky). Geom. Funct. Anal. **6**(4), 612–618 (1996)

[3] Bottini, A.: The Looijenga–Lunts–Verbitsky algebra and Verbitsky's theorem **(this volume)**

[4] de Cataldo, M.A., Migliorini, L.: The Douady space of a complex surface. Adv. Math. **151**, 283–312 (2000)

[5] de Cataldo, M.A., Migliorini, L.: The Chow groups and the motive of the Hilbert scheme of points on a surface. J. Algebra **251**(2), 824–848 (2002)

[6] Fulton, W., Harris, J.: Representation Theory, Graduate Texts in Mathematics, vol. 129. Springer, New York (1991)

[7] Green, M., Kim, Y.-J., Laza, R., Robles, C.: The LLV decomposition of hyper-Kähler cohomology (the known cases and the general conjectural behavior). Math. Ann. (2021)

[8] Göttsche, L., Soergel, W.: Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces. Math. Ann. **296**, 235–245 (1993)

[9] Kulikov, S.V., Kurchanov, P.F.: Complex algebraic varieties: periods of integrals and Hodge structures. In: Algebraic Geometry, III, Encyclopaedia Math. Sci., vol. 36, pp. 1–217, 263–270. Springer, Berlin (1998)

[10] Kollár, J., Laza, R., Saccà, G., Voisin, C.: Remarks on degenerations of hyper-Kähler manifolds. Ann. Inst. Fourier (Grenoble) **68**(7), 2837–2882 (2018)

[11] Looijenga, E., Lunts, V.: A Lie algebra attached to a projective variety. Invent. Math. **129**(2), 361–412 (1997)

[12] Nagai, Y.: On monodromies of a degeneration of irreducible symplectic Kähler manifolds. Math. Z. **258**, 407–426 (2008)

[13] Nakajima, H.: Heisenberg algebra and Hilbert schemes of points on projective surfaces. Ann. Math. **145**(2), 379–388 (1997)

[14] Neguţ, A., Oberdieck, G., Yin, Q.: Motivic decompositions for the Hilbert scheme of points of a K3 surface. J. Reine Angew. Math. **778**, 65–95 (2021)

[15] Oberdieck, G.: A Lie algebra action on the Chow ring of the Hilbert scheme of points of a K3 surface. Comment. Math. Helv. **96**(1), 65–77 (2021)

[16] Soldatenkov, A.: Limit mixed Hodge structures of Hyperkähler manifolds. Mosc. Math. J. **20**, 423–436 (2020)

[17] Verbitsky, M.: Action of the Lie algebra of SO(5) on the cohomology of a hyper-Kähler manifold. Funktsional. Anal. i Prilozhen. **24**, 70–71 (1990)

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