Boosting

CSci 5525: Machine Learning

Instructor: Arindam Banerjee

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- We assume the existence of such a learner

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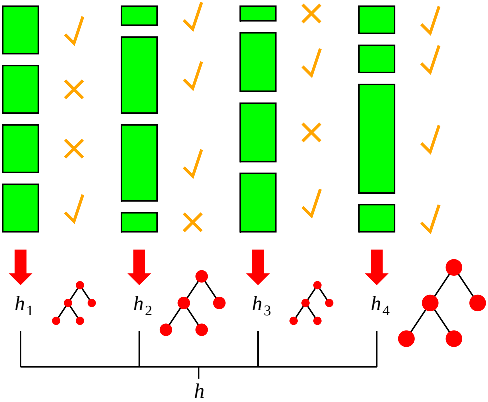
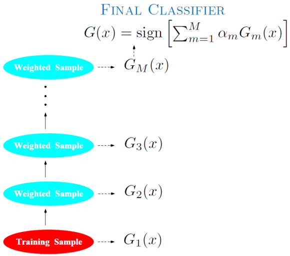
Boosting Algorithms



- Weight decreased on correct samples
- Weight increased on incorrect samples

Adaboost Training



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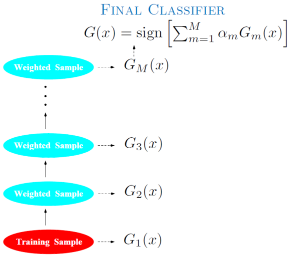

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$$\epsilon_t = P_{\mathbf{x} \sim w_t}[G_t(\mathbf{x}) \neq y] = \frac{\sum_{i=1}^{N} w_t(i) \mathbb{I}(y_i \neq G_t(\mathbf{x}_i))}{\sum_{i=1}^{N} w_t(i)}$$

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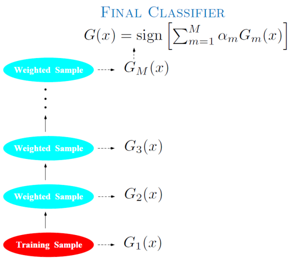

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- The combined classifier

$$g(\mathbf{x}) = \text{sign}\left[\sum_{t=1}^{T} \alpha_t G_t(\mathbf{x})\right]$$

Adaboost Algorithm

Input: Training set $(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_N, y_N)$

Algorithm: Initialize $w_1(i) = 1/N$
For $t = 1, \ldots, T$

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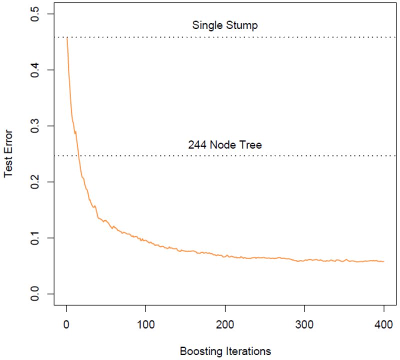
Adaboost Example

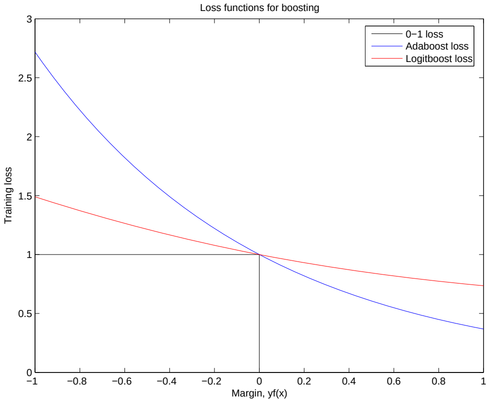


- $X_1, \ldots, X_{10}$ are univariate independent Gaussians

$$Y = \begin{cases} 1 & \text{if } \sum_j X_j^2 > \chi_{10}^2(0.5) \\ -1 & \text{otherwise} \end{cases}$$

- $\chi_{10}^2(0.5) = 9.34$, median of chi-squared r.v. with 10 degrees of freedom

Training Error, Margin Error



The 0-1 training set loss with convex upper bounds: exponential loss and logistic loss

The Training Error

- The training error of the final classifier is bounded

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{1}(g(\mathbf{x}_i) \neq y_i) \leq \frac{1}{N}\sum_{i=1}^{N}\exp(-y_i G(\mathbf{x}_i)) = \prod_{t=1}^{T} Z_t$$

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- Other boosting algorithms minimize other upper bounds

Obtaining α_t

- For a given $G_t(\mathbf{x})$, goal is to minimize

$$f(\alpha) = \sum_{i=1}^{N} w_t(i) \exp(-\alpha y_i G_t(\mathbf{x}_i)) = e^{-\alpha} \sum_{y_i = G_t(\mathbf{x}_i)} w_t(i) + e^{\alpha} \sum_{y_i \neq G(\mathbf{x}_i)} w_t(i)$$

$$= (e^{\alpha} - e^{-\alpha}) \sum_{i=1}^{N} w_t(i) \mathbb{1}(y_i \neq G_t(\mathbf{x}_i)) + e^{-\alpha} \sum_{i=1}^{n} w_t(i)$$

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- For round t, with $\epsilon_t = 1/2 - \gamma_t/2$

$$Z_t = 2\sqrt{\epsilon_t(1-\epsilon_t)} = \sqrt{1-\gamma_t^2}$$

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- A more careful argument gives the margin error

$$\frac{1}{m}\sum_i \mathbb{I}_{[y_i h(\mathbf{x}_i) \leq \theta]} \leq \prod_{t=1}^{T} (1-\gamma_t)^{\frac{1-\theta}{2}} (1+\gamma_t)^{\frac{1+\theta}{2}}$$

Boosting as Entropy Projection

For a general α we have

$$D_{t+1}^{\alpha}(i) = \frac{D_t(i)\exp(-\alpha y_i h_t(\mathbf{x}_i))}{Z_t(\alpha)}$$

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Then

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Further

$$\operatorname{argmin}_{D_{t+1}, E_{D_{t+1}}[yh(\mathbf{x})]=0} KL(D_{t+1}, D_t) = D_{t+1}^{\alpha_t} ,$$

which is the update Adaboost uses.

Additive Models

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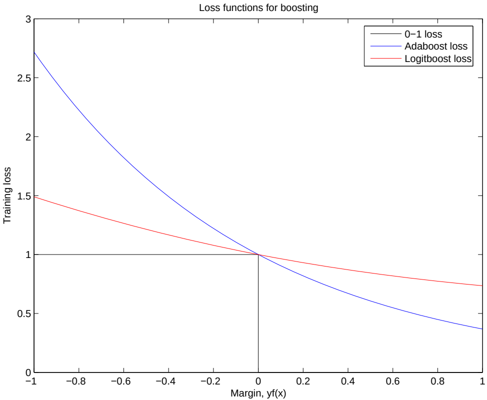
- For Logitboost,

$$C(yh(\mathbf{x})) = \log(1 + \exp(-yh(\mathbf{x})))$$

Upper Bounds on Training Errors



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$$\sum_{i=1}^{N} C(y_i[H_{t-1}(\mathbf{x}_i) + \alpha_t h_t(\mathbf{x}_i)]) = \sum_{i=1}^{N} u_t(i) \exp(-y_i \alpha_t h_t(\mathbf{x}_i))$$

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- The optimum solution

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Gradient Descent in Function Space

- Choose any appropriate margin cost function C
- Goal is to minimize margin cost on the training set
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- Margin bounds on error rate based on n samples

$$\Pr[yf(\mathbf{x}) \leq 0] \;\leq\; \Pr_S[yf(\mathbf{x}) \leq \theta] + O\left(\frac{C_n(F)}{\theta}\right) + O\left(\sqrt{\frac{\log \frac{1}{\delta}}{n}}\right)$$

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- Complexity does not change with convex combination
- With probability at least $(1-\delta)$ over the draw of train set S of size N, $\forall f \in \text{co}_k(\mathcal{F}), \theta > 0$ we have

$$\Pr[yf(\mathbf{x}) \leq 0] \;\leq\; \Pr_S[yf(\mathbf{x}) \leq \theta] + \frac{4R_n(F)}{\theta} + O\left(\sqrt{\frac{\log \frac{1}{\delta}}{n}}\right)$$

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