

SUMMARY :
MATHEMATICAL ASPECTS OF MIXING TIMES IN MARKOV CHAIN

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ABSTRACT. In this paper, important concepts and basic idea from *Mathematical Aspects of Mixing Times in Markov Chains* will be covered.

1. BASIC BOUND ON MIXING TIMES

1.1. Preliminaries: Distance and mixing times. Let (Ω, P, π) denote Markov kernel of a finite Markov chain on a finite state space Ω with a unique invariant measure π . That is

$$P(x, y) \geq 0, \forall x, y \in \Omega, \text{ and } \sum_{y \in \Omega} P(x, y) = 1, \forall x \in \Omega$$

$$\sum_{x \in \Omega} \pi(x) P(x, y) = \pi(y), \forall y \in \Omega.$$

In this summary, we assume P is irreducible and π has full support (Ω).

Minimal holding probability $\alpha \in [0, 1]$ satisfies $\forall x \in \Omega : P(x, x) \geq \alpha$

Let $k_n^x(y) = P^n(x, y)/\pi(y)$ denote the density with respect to π at time $n \geq 0$, or simply $k_n(y)$ when start rate or start distribution is unimportant or clear from the context. Then $\lim_{n \rightarrow \infty} k_n^x(y) = 1$

If μ is a probability distribution on Ω , then

- Total Variation Distance: $\|\mu - \pi\|_{TV} = \frac{1}{2} \|\frac{\mu}{\pi} - 1\| = \frac{1}{2} \sum_{y \in \Omega} |\mu(y) - \pi(y)|$
- Variance: $Var_{\pi}(\mu/\pi) = \|\frac{\mu}{\pi} - 1\|_{2, \pi}^2 = \sum_{y \in \Omega} \pi(y) (\frac{\mu(y)}{\pi(y)} - 1)^2$
- Informational divergence: $D(P^n(x, \cdot) \|\pi) = Ent_{\pi}(k_n^x) = \sum_{y \in \Omega} P^n(x, y) \log \frac{P^n(x, y)}{\pi(y)}$

Where the entropy $Ent_{\pi}(f) = E_{\pi} f \log \frac{f}{E_{\pi} f}$.

Each of these distance are convex, which means given $s \in [0, 1]$, $dist((1-s)\mu + s\nu, \pi) \leq (1-s)dist(\mu, \pi) + s dist(\nu, \pi)$. A convex distance satisfies the condition

$$dist(\sigma P^n, \pi) = dist(\sum_{x \in \Omega} \sigma(x) P^n(x, \cdot), \pi) \leq \sum_{x \in \Omega} \sigma(x) dist(P^n(x, \cdot), \pi) \leq \max_{x \in \Omega} dist(P^n(x, \cdot), \pi) \quad (1)$$

so the distance is maximized when the initial distribution concentrate at a point.

Definition 1.1. The total variation, relative entropy and L^2 mixing times are defined as follows.

- $\tau(\epsilon) = \min\{n : \forall x \in \Omega, \|P^n(x, \cdot) - \pi\|_{TV} \leq \epsilon\}$
- $\tau_D(\epsilon) = \min\{n : \forall x \in \Omega, D(P^n(x, \cdot) \|\pi) \leq \epsilon\}$
- $\tau_2(\epsilon) = \min\{n : \forall x \in \Omega, \|k_n^x - 1\|_{2, \pi} \leq \epsilon\}$

The time-reversal P^* is defined by $\pi(x)P^*(x, y) = \pi(y)P(y, x)$, $x, y \in \Omega$, and we have $\langle f, Pg \rangle_\pi = \langle P^*f, g \rangle_\pi$

For $f, g : \Omega \rightarrow R$, let $\mathcal{E}(f, g) = \mathcal{E}_P(f, g)$ denote the Dirichlet form,

$$\mathcal{E}(f, g) = \langle f, (I - P)g \rangle_\pi \quad (2)$$

if $f = g$ then

$$\mathcal{E}(f, f) = \frac{1}{2} \sum_{x, y} (f(x) - f(y))^2 P(x, y) \pi(x) \quad (3)$$

and

$$\mathcal{E}_P(f, f) = \mathcal{E}_{P^*}(f, f) = \mathcal{E}_{\frac{P+P^*}{2}}(f, f) \quad (4)$$

A useful property of the reversal is that $k_n = P^*k_{n-1}$. If $P^* = P$, then P is said to be the time-reversible, or to satisfy the detailed balance condition. It's almost trivial to see that $\frac{P+P^*}{2}$ and PP^* are time-reversible. If P is reversible then $\mathcal{E}(f, g) = \mathcal{E}(g, f)$

Also, we borrow some notation from complexity theory, including O, Θ, Ω .

1.2. Continuous Time. Discrete Laplacian $\mathcal{L} = -(I - P)$. Then for $t \geq 0$, $H_t = e^{t\mathcal{L}}$ represents the continuized chain, corresponding to the discrete Markov kernel P . The contuized chain simply represents a Markov process $\{X_t\}_{t \geq 0}$. Also, let $h_t^x(y) = H_t(x, y)/\pi(y)$.

Lemma 1.2. For any h_0 and all $t \geq 0$, $h_t = H_t^* h_0$. Consequently, for any $x \in \Omega$,

$$\frac{dh_t(x)}{dt} = \mathcal{L}h_t(x)$$

Lemma 1.3.

$$\frac{d}{dt} \text{Var}(h_t) = -2\mathcal{E}(h_t, h_t) \quad (5)$$

$$\frac{d}{dt} \text{Ent}(h_t) = -\mathcal{E}(h_t, \log h_t) \quad (6)$$

Proof. Trivial to show. □

Spectral gap λ and entropy constant ρ_0

Definition 1.4. Let $\lambda > 0$ and $\rho_0 > 0$ be the optimal constants in the inequalities:

$$\lambda \text{Var}_\pi f \leq \mathcal{E}(f, f), \forall f : \Omega \rightarrow R. \quad (7)$$

$$\rho_0 \text{Ent}_\pi f \leq \mathcal{E}(f, \log f), \forall f : \Omega \rightarrow R_+. \quad (8)$$

Corollary 1.5. Let $\pi_* = \min_{x \in \Omega} \pi(x)$. Then in the continuous time,

$$\tau_2(\epsilon) \leq \frac{1}{\lambda} \left(\frac{1}{2} \log \frac{1 - \pi_*}{\pi_*} + \log \frac{1}{\epsilon} \right) \quad (9)$$

$$\tau_D(\epsilon) \leq \frac{1}{\rho_0} \left(\log \log \frac{1}{\pi_*} + \frac{1}{\epsilon} \right) \quad (10)$$

Proof. Simply solve the differential equations,

$$\frac{d}{dt} \text{Var}(h_t^x) = -2\mathcal{E}(h_t^x, h_t^x) \leq -2\lambda \text{Var}(h_t^x) \quad (11)$$

$$\frac{d}{dt} \text{Ent}(h_t^x) = -\mathcal{E}(h_t^x, \log h_t^x) \leq \text{Ent}(h_t^x) \quad (12)$$

and note that $\text{Var}(h_0) \leq \frac{1-\pi_*}{\pi_*}$ and $\text{Ent}(h_0) \leq \log \frac{1}{\pi_*}$ (e.q. by equation (1)) \square

Proposition 1.6. *If $c > 0$ then*

- (1) $\text{Var}_\pi(H_t f) \leq e^{-ct} \text{Var}_\pi f, \forall f$ and $t > 0$, *if and only if* $\lambda \geq c$.
- (2) $\text{Ent}_\pi(H_t f) \leq e^{-ct} \text{Ent}_\pi f, \forall f > 0$ and $t > 0$, *if and only if* $\rho_0 \geq c$

ρ_0 is rather challenging to estimate while there have been several techniques (linear-algebraic and functional-analytic) to help bound the spectral gap. The following inequality relating the two Dirichlet forms introduced above also motivates the study of classical logarithmic Sobolev inequality.

Lemma 1.7. *If $f \geq 0$ then*

$$2\mathcal{E}(\sqrt{f}, \sqrt{f}) \leq \mathcal{E}(f, \log f) \quad (13)$$

Let $\rho_P > 0$ denote the logarithmic Sobolev constant of P defined as follows.

Definition 1.8.

$$\rho = \rho_P = \inf_{\text{Ent} f^2 \neq 0} \frac{\mathcal{E}(f, f)}{\text{Ent} f^2}$$

Proposition 1.9. *For every irreducible chain P ,*

$$2\rho \leq \rho_0 \leq 2\lambda$$

Proof. Using (13), the first inequality is immediate. The second follows from applying (8) to function $f = 1 + \epsilon g$, where $g \in L^2(\pi)$ and $E_\pi g = 0$. Assume $\epsilon \ll 1$, so that $f \geq 0$. Then using the Taylor approximation, $\log(1 + \epsilon g) = \epsilon g - 1/2(\epsilon)^2 g^2 + o(\epsilon^2)$, we have

$$\text{Ent}_\pi(f) = \frac{1}{2}\epsilon^2 \pi(g)^2 + o(\epsilon^2)$$

and

$$\mathcal{E}(f, \log f) = -\epsilon E_\pi((\mathcal{L}g) \log(1 + \epsilon g)) = \epsilon^2 \mathcal{E}(g, g) + o(\epsilon^2).$$

Thus starting from (8), and applying to f as above, we get

$$\epsilon^2 \mathcal{E}(g, g) \geq \frac{\rho_0}{2} \epsilon^2 E_\pi g^2 + o(\epsilon^2).$$

Canceling ϵ^2 and letting $\epsilon \downarrow 0$, yields the second inequality of the proposition, since $E_\pi g = 0$ \square

1.3. Discrete Time. A mixing bound in terms of the spectral gap will be shown in a fashion similar to that in continuous time. There seems to be no discrete-time analog of the modified log-Sobolev bound on relative entropy.

In discrete time we consider two approaches to mixing time, both of which are equivalent. the first approach involves operator norms, and is perhaps the more intuitive of two methods.

Proposition 1.10.

$$\tau_2(\epsilon) \leq \lceil \frac{1}{1 - \|P^*\|} \log \frac{1}{\epsilon \sqrt{\pi_*}} \rceil \quad (14)$$

where $\pi_* = \min_{x \in \Omega} \pi(x)$ and

$$\|P^*\| = \sup_{f: \Omega \rightarrow R, Ef=0} \frac{\|P^*f\|_2}{\|f\|_2}$$

This results has appeared in mixing time literature in many equivalent forms.

Proof. Since $k_{i+1} - 1 = P^*(k_i - 1)$ and $E(k_i - 1) = 0$ for all i then

$$\|k_n - 1\|_2 = \|P^*\|^n \|k_0 - 1\|_2$$

Solving for when this expression drops to ϵ and using the approximations $\log x \leq -(1 - x)$ and $\|k_0 - 1\|_2 \leq \sqrt{\frac{1 - \pi_*}{\pi_*}}$ \square

In above proposition, the mixing bound followed almost immediately from the definition. However, there is an alternate approach to this problem which bear more of a resemblance to the continuous time result and is more convenient for showing refined bounds.

The discrete time analog of differentiating $Var(h_t)$ is take the difference $Var(k_n) - Var(k_{n-1})$, or more generally, $Var(P^*f) - Var(f)$.

Lemma 1.11. *Given Markov chain P and function $f : \Omega \rightarrow R$, then*

$$Var(P^*f) - Var(f) = -\mathcal{E}_{PP^*}(f, f) \leq -Var(f)\lambda_{PP^*}$$

.

Lemma 1.12. *Note, for any transition probability matrix K , $E_\pi f = E_\pi(Kf)$. Then*

$$Var(P^*f) - Var(f) = \langle P^*f, P^*f \rangle_\pi - \langle f, f \rangle_\pi = -\langle f, (I - PP^*)f \rangle_\pi$$

gives what we need.

Corollary 1.13. *A discrete time Markov chain P satisfies*

$$\tau_2(\epsilon) \leq \lceil \frac{1}{\lambda_{PP^*}} \log \frac{1}{\epsilon \sqrt{\pi_*}} \rceil \quad (15)$$

Proof.

$$Var(k_n) \leq Var(k_0)(1 - \lambda_{PP^*})^n \quad (16)$$

The result follows by solving for when variance drops to ϵ^2 and using the approximation $\log(1 - \lambda_{PP^*}) \leq -\lambda_{PP^*}$ \square

It is preferable to work with P instead of PP^* . Here are several simplifications make this possible.

Corollary 1.14. *In discrete time, a Markov chain with holding probability α satisfies*

$$\tau_2(\epsilon) \leq \lceil \frac{1}{\alpha\lambda} \log \frac{1}{\epsilon\sqrt{\pi_*}} \rceil \quad (17)$$

For a reversible Markov chain,

$$\tau_2(\epsilon) \leq \lceil \frac{1}{1 - \lambda_{max}} \log \frac{1}{\epsilon\sqrt{\pi_*}} \rceil \leq \lceil \frac{1}{\min\{2\alpha, \lambda\}} \log \frac{1}{\epsilon\sqrt{\pi_*}} \rceil \quad (18)$$

Where $\lambda_{max} = \max\{\lambda_1, |\lambda_{n-1}|\}$ when $\lambda_1 = 1 - \lambda$ is the largest non-trivial eigenvalue of P and $\lambda_{n-1} \geq -1$ is the smallest eigenvalue.

Now we will show that those two approaches to bounding discrete mixing in this section are equivalent.

Proposition 1.15.

$$1 - \lambda_{PP^*} = \|P^*\| \quad (19)$$

Proof. See the original paper, page 20. □

1.4. Does Reversibility matter? Many mixing result were originally shown only in the context of a reversible Markov chain. We can actually avoid this requirement in most cases. However, there are still some cases that this requirement is necessary. In this section, some difference would be covered, and some classical results of reversible Markov chain will be illustrated.

The difference between reversible and non-reversible results is most apparent when upper and lower bounds on distances are given. Let

$$d(n) = \max_x \|P^n(x, \cdot) - \pi(\cdot)\|_{TV}$$

denotes the worst variation distance after n steps. Then combining the above work, we have

Proposition 1.16. *if P is reversible:*

$$\frac{1}{2}\lambda_{max}^n \leq d(n) \leq \frac{1}{2}\lambda_{max}^n \sqrt{\frac{1 - \pi_*}{\pi_*}} \quad (20)$$

if P is non-reversible:

$$x \quad (21)$$

Lemma 1.17. *If P is reversible and irreducible on state space of size $|\Omega| = n$, then it has a complete spectrum of real eigenvalues with magnitudes at most one, that is*

$$1 = \lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{n-1} \geq -1. \quad (22)$$

The Courant-Fischer theorem show the connection between eigenvalues and Dirichlet forms for a reversible Markov chain.

Lemma 1.18. *In a reversible Markov chain the second largest eigenvalue λ_1 and the smallest eigenvalue λ_{n-1} satisfy*

$$1 - \lambda_1 = \inf_{Var(f) \neq 0} \frac{\mathcal{E}(f, f)}{Var(f)} = \lambda \quad (23)$$

$$1 + \lambda_{n-1} = \inf_{Var(f) \neq 0} \frac{\mathcal{F}(f, f)}{Var(f)} = \lambda \quad (24)$$

Where

$$\mathcal{F}(f, f) = \langle f, (I + P)f \rangle_\pi$$

Proof. See the original paper, page 25. □

2. ADVANCED FUNCTIONAL TECHNIQUES

In this section, it is shown that information on function of large variance, or on functions with small support, can be exploited to show better mixing time bounds.

The argument is simple. Recall that we have $\frac{d}{dt} \text{Var}(h_t) = -2\mathcal{E}(h_t, h_t)$. If $\mathcal{E}(f, f) \geq G(\text{Var}(f))$ for some $G : R_+ \rightarrow R_+$ and $f : \Omega \rightarrow R_+$ with $Ef = 1$, then it follows that $\frac{d}{dt} \text{Var}(h_t) = -2\mathcal{E}(h_t, h_t) \leq -2G(\text{Var}(f))$, so we have the following inequality:

$$\tau_2(\epsilon) = \int_0^{\tau_2(\epsilon)} 1 dt \leq \int_{\text{Var}(h_0)}^{\epsilon^2} \frac{dI}{-2G(I)} \quad (25)$$

When $G(r) = \lambda r$, it simply gives the bound we got in previous chapter. More generally, in this chapter we derive such functions G in terms of the log-Sobolev constant, Nash inequalities, spectral profile, or via comparison to another Markov chain.

In the discrete time, replace $\frac{d}{dt} \text{Var}(h_t) = -2\mathcal{E}(h_t, h_t)$ with $\text{Var}(k_n) - \text{Var}(k_{n-1}) = -\mathcal{E}_{PP^*}(k_n, k_n)$ (is it correct? shouldn't it be k_{n-1}). Then if $\mathcal{E}_{PP^*}(f, f) \geq G_{PP^*}(\text{Var}(f))$, and $I(n) = \text{Var}(k_n)$ and $G_{PP^*}(r)$ are non-decreasing, the piecewise linear extension of $I(n)$ to $t \in R_+$ will satisfy

$$\frac{dI}{dt} \leq -G_{PP^*}(I). \quad (26)$$

Then

$$\tau_2(\epsilon) = \int_0^{\tau_2(\epsilon)} 1 dt \leq \int_{\text{Var}(h_0)}^{\epsilon^2} \frac{dI}{-G_{PP^*}(I)} \quad (27)$$

Since we have

$$\mathcal{E}_{PP^*}(f, f) \leq 2\alpha \mathcal{E}(f, f) \leq 2\alpha G(\text{Var}(f))$$

then we may take $G_{PP^*}(r) = 2\alpha G(r)$

2.1. Log-Sobolev and Nash Inequalities. Some of the best bounds on L^2 mixing times were shown by use of the log-Sobolev constant. Following is a example. (But I can't talk about what's the bound is in this case).

Example 2.1. A matroid \mathcal{M} is given by a ground set $E(\mathcal{M})$ with $|E(\mathcal{M})| = n$ and a collection of bases $\mathcal{B}(\mathcal{M}) \subseteq 2^{E(\mathcal{M})}$. The bases $\mathcal{B}(\mathcal{M})$ must all have the same cardinality r , and $\forall X, Y \in \mathcal{B}(\mathcal{M}), \forall e \in X, \exists f \in Y : X \cup \{f\} \setminus \{e\} \in \mathcal{B}(\mathcal{M})$. One choice of a Markov chain on matroids is, given state $X \in \mathcal{B}(\mathcal{M})$ half the time do nothing, and otherwise choose $e \in X, f \in \mathcal{B}(\mathcal{M})$ and transition to state $X - e + f$ if it is also a basis.

Equation (25) and (26) will bound mixing time in terms of log-Sobolev constant if we can show a relation between the Dirichlet form and a function of the variance. The following lemma establishes this connection.

Lemma 2.2. If f is non-negative then

$$\text{Ent}(f^2) \geq Ef^2 \log \frac{Ef^2}{(Ef)^2}$$

and in particular, if $Ef = 1$ then

$$\mathcal{E}(f, f) \geq \rho \text{Ent}(f^2) \geq \rho(1 + \text{Var}(f)) \log(1 + \text{Var}(f)).$$

Proof. See the original paper on page 29. \square

In some case, Nash inequality can be used to supplement the log-Sobolev result and improve the bound. Unfortunately, however, Nash inequalities are notoriously difficult to establish. We now show that the Dirichlet form can also be lower bounded in terms of variance by using Nash inequality.

Lemma 2.3. *Given a Nash Inequality*

$$\|f\|_2^{2+1/D} \leq C \left[\mathcal{E}(f, f) + \frac{1}{T} \|f\|_2^2 \right] \|f\|_1^{1/D}$$

which holds for every function $f : \Omega \rightarrow \mathbb{R}$ and some constants $C, D, T \in \mathbb{R}_+$, then whenever $f \geq 0$ and $Ef = 1$ then

$$\mathcal{E}(f, f) \geq (1 + \text{Var}(f)) \left(\frac{(1 + \text{Var}(f))^{1/D}}{C} - \frac{1}{T} \right)$$

Proof. Simply rewrite the inequality and use the facts that $\|f\|_1 = 1$ and $\text{Var}(f) = \|f\|_2^2 - 1$. \square

2.2. Spectral profile. In previous section it was found that log-Sobolev bounds on mixing time can on spectral gap results, by replacing the $\log(1/\pi_*)$ term with $\log \log(1/\pi_*)$. However, the log-Sobolev is much more difficult to bound than the spectral gap and, to date, bounds on it are known for only a handful of problems. Moreover, sometimes even log-Sobolev is not strong enough. In this section, the method of Spectral Profile will be introduced. The main idea is to improve the basic relation $\mathcal{E}(f, f) \geq \lambda \text{Var}(f)$ by considering the support of f .

Definition 2.4. *For a non-empty subset $S \subset \Omega$ the first Dirichlet eigenvalue on S is given by*

$$\lambda_1(S) = \inf_{f \in c_0^+(S)} \frac{\mathcal{E}(f, f)}{\text{Var}(f)} \quad (28)$$

where $c_0^+(S) = \{f \geq 0 : \text{supp}(f) \subset S\}$ is the set of non-negative function supported on S . The spectral profile $\Lambda : [\pi_*, \infty) \rightarrow \mathbb{R}$ is given by $\Lambda(r) = \inf_{\pi_* \leq \pi(S) \leq r} \lambda_1(S)$

The spectral profile is a natural extension of spectral gap λ , and will we now see that it can improve on the basic bound $\mathcal{E}(f, f) \geq \lambda \text{Var}(f)$ used earlier.

Lemma 2.5. *For every non-constant function $f : \Omega \rightarrow \mathbb{R}_+$*

$$\mathcal{E}(f, f) \leq \frac{1}{2} \Lambda \left(\frac{4(Ef)^2}{\text{Var}f} \right) \text{Var}(f) \quad (29)$$

Proof. Proof can be founded in the original paper, page 35. I can also talk about that. \square

Applying this formula together with (25), we obtain following inequality.

$$\tau_2(\epsilon) \leq \int_{\text{Var}(h_0)}^8 \frac{dI}{-I\Lambda(4/I)} + \int_8^{\epsilon^2} \frac{dI}{-2\lambda I}$$

Let σ be the initial distribution, and let $h_0(x) = \frac{\sigma(x)}{\pi(x)}$, a change of variables to $r = 4/I$ gives the mixing time bound (in continuous time).

$$\tau_2(\epsilon) \leq \int_{4/\text{Var}(\sigma/\pi)}^{1/2} \frac{dr}{r\Lambda(r)} + \frac{1}{\lambda} \log \frac{2\sqrt{2}}{\epsilon} \quad (30)$$

Similar things can be done in discrete-time case.

The theorem, with the trivial bound $\Lambda(r) \geq \lambda$ sometimes produces worse bound than we got in previous chapter, however, when $\Lambda(r) \gg \lambda$ when r are small, we could expect the obtain sharper bound.

Spectral profile is a fairly new tool, and it has not been widely studied yet. However, some researchers still got some results to bound spectral profile, for example, it can be shown that log-Sobolev constant and a Nash inequality induce the following lower bounds:

$$\Lambda(r) \geq \rho \frac{\log(1/r)}{1-r}$$

and

$$\Lambda(r) \geq \frac{1}{Cr^{1/2D}} - \frac{1}{T}.$$

2.3. Comparison methods. It sometimes happened that a Markov chain is difficult to study, but a related chain is more manageable. In this situation, the comparison method has been widely used to bound spectral gap, log-Sobolev constant and Nash inequalities.

Before deriving comparison results for the quantities in this chapter, a preliminary result is needed.

Theorem 2.6. *Consider two Markov chains P and \hat{P} on the same state space Ω , and for every $x \neq y \in \Omega$ with $\hat{P}(x, y) \geq 0$ define a directed path γ_{xy} from x to y along edges in P . Let Γ denote the set of all such paths. Then*

$$\mathcal{E}_P(f, f) \geq \frac{1}{A} \mathcal{E}_{\hat{P}}(f, f), \quad (31)$$

$$\text{Var}_{\pi}(f) \leq M \text{Var}_{\hat{\pi}}(f), \quad \text{Ent}_{\pi}(f^2) \leq M \text{Ent}_{\hat{\pi}}(f^2) \quad (32)$$

Where $M = \max_x \frac{\pi(x)}{\hat{\pi}(x)}$, and

$$A = A(\Gamma) = \max_{a \neq b; P(a, b) \neq 0} \frac{1}{\pi(a)P(a, b)} \sum_{x \neq y; (a, b) \in \gamma_{xy}} \hat{\pi}(x) \hat{P}(x, y) |\gamma_{xy}|$$

Proof. see page 39 in original paper. □

Corollary 2.7.

$$\lambda_P \geq \frac{1}{MA} \lambda_{\hat{P}}, \quad \rho_P \geq \frac{1}{MA} \rho_{\hat{P}}, \quad \Lambda_P(r) \geq \frac{1}{MA} \Lambda_{\hat{P}}(r)$$

In the case of a reversible chain, it is also possible to compare λ_{n-1} if the paths are of odd length. First, consider the preliminary result.

Theorem 2.8. *Consider two Markov chains P and \hat{P} on the same state space Ω , and for every $x \neq y \in \Omega$ with $\hat{P}(x, y) \geq 0$ define a directed path γ_{xy} of odd length $|\gamma_{xy}|$ from x to y along edges in P . Let Γ^* denote the set of all such paths. Then*

$$\mathcal{F}_P(f, f) \geq \frac{1}{A^*} \mathcal{F}_{\hat{P}}(f, f), \quad (33)$$

Where $M = \max_x \frac{\pi(x)}{\hat{\pi}(x)}$, and

$$A^* = A^*(\Gamma^*) = \max_{a \neq b; P(a, b) \neq 0} \frac{1}{\pi(a)P(a, b)} \sum_{x \neq y; (a, b) \in \gamma_{xy}} \hat{\pi}(x) \hat{P}(x, y) |\gamma_{xy}| r_{xy}(a, b)$$

where $r_{xy}(a, b)$ is the number of times the edge (a, b) appears in path γ_{xy} .

If P and \hat{P} are reversible then

$$1 - \lambda_{\max}(P) \geq \frac{1}{MA^*}(1 - \lambda_{\max}(\hat{P}))$$

The most widely used example of these comparison results is the canonical path theorem

Corollary 2.9. *Given a Markov chain P on state space Ω , and directed paths λ_{xy} between every pair of vertices $x \neq y \in \Omega$, then*

$$\lambda \geq \left(\max_{a \neq b; P(a,b) \neq 0} \frac{1}{\pi(a)P(a,b)} \sum_{x \neq y; (a,b) \in \gamma_{xy}} \pi(x)\pi(y)|\gamma_{xy}| \right)^{-1}$$

Proof. Let $\hat{P}(x, y) = \pi(y)$, $\hat{\pi} = \pi$, then $M = 1$, note that

$$\mathcal{E}_{\hat{P}}(f, f) = \frac{1}{2} \sum_{x, y \in \Omega} (f(x) - f(y))^2 \pi(x)\pi(y) = \text{Var}_{\pi}(f).$$

□

Corollary 2.10. *Consider two Markov chains P on the state space Ω , and a set of cycles γ_x of odd length from each vertex $x \in \Omega$ to itself. Then the smallest eigenvalue λ_{n-1} of P satisfies the relation*

$$1 + \lambda_{n-1} \geq 2 \left(\max_{a \neq b; P(a,b) \neq 0} \frac{1}{\pi(a)P(a,b)} \sum_{x \neq y; (a,b) \in \gamma_{xy}} \pi(x)|\gamma_{xy}|r_{xy}(a,b) \right)^{-1}$$

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