A Survey on
Distributed/Streaming Submodular Optimization

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1   Introduction

Submodularity is a property of set functions with deep theoretical and practical consequences. Submodular functions occur in a variety of applications, including representative skyline selection [12], network structure learning [5], influence maximization [6], document summarization [8], image segmentation [2, 7] and many others.

There is a large body of research in submodular optimization which we can not cover thoroughly here. In this survey, we focus on the recent advances in optimizing (mostly, maximizing) submodular functions in distributed and streaming setting.

Through Lovász' extension, a submodular minimization problem can be solved via techniques from convex optimization. Therefore most work focuses on submodular maximization. In particular, almost all results for distributed and streaming submodular optimization are for maximization problems. In this survey, we also focus on submodular maximization, but will briefly mention minimization problems as well when necessary.

1.1   Notations

Through out this survey,

we use V to represent the ground set we consider.

2^V is the power set of V (i.e. the set of all subsets of V).

In general, A^V is the collection of maps from V to A.

For simplicity, we sometime write $A \cup \{x\}$ as $A + x$.

2   Submodularity

In this section, we first give several equivalent definitions of submodularity, and then we introduce several fundamental properties of submodular functions. We also discuss various constraints that occur frequently in submodular optimization problems. In the last part of this section, we cover algorithms that solve constrained submodular maximization problems with theoretical approximation guarantee.

2.1 Definitions

There are many equivalent definitions, and we will discuss three of them in this section.

Definition 1 (submodular concave) A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B). \tag{1}$$

An alternate equivalent definition is more interpretable in many situations,

Definition 2 (diminishing returns) A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A \subseteq B \subset V$, and $v \in V \backslash B$, we have that:

$$f(A + v) - f(A) \geq f(B + v) - f(B). \tag{2}$$

Intuitively, this definition requires that the incremental "gain" of adding a new element v decreases (diminishes) as the base set grows from A to B. We will see that this property is actually shared by many real-world phenomenons.

It turns out that a stronger but equivalent statement can also serve as the definition of a submodular function,

Definition 3 (group diminishing returns) A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A \subseteq B \subset V$, and $C \subseteq V \backslash B$, we have that:

$$f(A \cup C) - f(A) \geq f(B \cup C) - f(B). \tag{3}$$

2.2 Modularity and Supermodularity

We also briefly mention modularity and supermodularity here. These two concepts are closely related to submodularity.

A function $f : 2^V \to \mathbb{R}$ is modular if we replace inequality by equality in Definition 2 (or any of other two). Formally,

Definition 4 (Modularity) A function $f : 2^V \to \mathbb{R}$ is modular if for any $A \subseteq B \subset V$, and $v \in V \backslash B$, we have that:

$$f(A + v) - f(A) = f(B + v) - f(B). \tag{4}$$

Notably, a modular function f can always be written as

$$f(S) = f(\emptyset) + \sum_{v \in S} (f(\{v\}) - f(\emptyset))$$

for any $S \subseteq V$.

If we further assume $f(\emptyset) = 0$ (in this case, we call f normalized), we have a simplified expression,

$$f(S) = \sum_{v \in S} f(\{v\}).$$

Modularity can be useful in our discussion of submodularity, because one can use modular functions to construct submodular functions with desired properties in their applications. Examples can be found in e.g. [8, 9].

A supermodular function is defined by flipping the inequality sign in the definition of a submodular function. Formally,

Definition 5 (Supermodularity) A function $f : 2^V \to \mathbb{R}$ is modular if for any $A \subseteq B \subset V$, and $v \in V \backslash B$, we have that:

$$f(A + v) - f(A) \leq f(B + v) - f(B). \tag{5}$$

We will focus on submodular functions because a function is supermodular if and only if its negative is submodular.

2.3 Properties

Like convex and concave functions, submodular functions have many nice properties. Lovász's description of convex functions [10] can be viewed as accurate comments on submodularity:

- Convex functions occur in many mathematical models in economy, engineering, and other sciences. Convexity is a very natural property of various functions and domains occurring in such models; quite often the only non-trivial property which can be stated in general.

- Convexity is preserved under many natural operations and transformations, and thereby the effective range of results can be extended, elegant proof techniques can be developed as well as unforeseen applications of certain results can be given.

- Convex functions and domains exhibit sufficient structure so that a mathematically beautiful and practically useful theory can be developed.

- There are theoretically and practically (reasonably) efficient methods to find the minimum of a convex function.

We survey several useful properties which can be useful in our later section. Most properties can be found in [1] or [4].

Submodularity is close under addition,

Property 1 *Let $f_1, f_2 : 2^V \to \mathbb{R}$ be two submodular functions. Then*

$$f : 2^V \to \mathbb{R} \ \text{with} \ f(A) = \alpha f_1(A) + \beta f_2(A)$$

is submodular for any fixed $\alpha, \beta \in \mathbb{R}^+$.

Adding a modular function does not break submodularity,

Property 2 *Let $f_1, f_2 : 2^V \to \mathbb{R}$, f_1 is submodular and f_2 is modular. Then*

$$f : 2^V \to \mathbb{R} \ \text{with} \ f(A) = f_1(A) + \alpha f_2(A)$$

is submodular for any fixed $\alpha \in \mathbb{R}$.

Submodularity is preserved under restriction,

Property 3 *Let $f : 2^V \to \mathbb{R}$ be a submodular function. Let $S \subseteq V$ be a fixed set. Then*

$$f' : 2^V \to \mathbb{R} \ with \ f'(A) = f(A \cap S)$$

is submodular.

As a direct implication of Property 1 and Property 3, we have the following more general result,

Property 4 *Let $f : 2^V \to \mathbb{R}$ be a submodular function, $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ be a collection of subsets of V (i.e. each $C_i \subseteq V$). Then*

$$f' : 2^V \to \mathbb{R} \ with \ f'(A) = \sum_{C \in \mathcal{C}} f(A \cap C)$$

is submodular.

This property can be useful in graphical models and image processing. CHEN : *TODO: show examples*

Following property can be useful when we show that the objective function of k-medoid problem is supermodular,

Property 5 *Consider V as a set of indices. Let $\mathbf{c} \in \mathbb{R}^V$ be a fixed vector, c_i be its ith coordinate. Then*

$$f : 2^V \to \mathbb{R} \ with \ f(A) = \max_{j \in A} c_i$$

is submodular.

We can use non-negative modular function and a concave function to construct submodular functions,

Property 6 *Let $m : 2^V \to \mathbb{R}^+$ be a modular function, and f a concave function over $\mathbb{R}$. Then*

$$f : 2^V \to \mathbb{R} \ with \ f(A) = g(m(A))$$

is submodular.

Before introducing the next property, we define the monotonitcity of set function,

Definition 6 (Monotonitcity) An set function $f : 2^V \to \mathbb{R}$ is said to be monotone non-decreasing if for any $A \subseteq B \subseteq V$, $f(A) \leq f(B)$. Monotone non-increasing function can be defined similarly.

Property 7 *Let $f, g : 2^V \to \mathbb{R}$ be two submodular functions. If $(f - g)(\cdot)$ is either monotone non-decreasing or monotone non-increasing, then $f : 2^V \to \mathbb{R}$ with*

$$f(A) = \min(f(A), g(A))$$

is submodular.

2.4 Constraints

For submodular optimization problem.

2.5 Algorithms for Submodular Maximization

Greey algorithm guarantee is for normalized, monotone submodular maximization under cardinality constraint. Moreover, no polynomial time algorithm can provide a better approximation guarantee unless P = NP [3].

3 Applications

In this section, we discuss several representative applications in great detail. We will see from those examples that submodularity is such a natural property that many real-world problems can be cast in to the framework of submodular optimization.

- [11] distributed k-centers using submodular optimization.

4 Distributed Submodular Optimization

5 Streaming Submodular Maximization

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