Submodular Maximization

advances in distributed/streaming computing

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March 17, 2016

Overview

Definitions of Submodularity

Definition (submodular concave)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B). \quad (1)$$

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An alternate equivalent definition is more interpretable in many situations.

Definition (diminishing returns)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A \subseteq B \subset V$, and $v \in V \backslash B$, we have that:

$$f(A + v) - f(A) \geq f(B + v) - f(B). \qquad (2)$$

Modular Functions

Definition (Modularity)

A function $f : 2^V \to \mathbb{R}$ is modular if for any $A \subseteq B \subset V$, and $v \in V \backslash B$, we have that:

$$f(A + v) - f(A) = f(B + v) - f(B). \tag{3}$$

Notably, a modular function f can always be written as

$$f(S) = f(\emptyset) + \sum_{v \in S} (f(\{v\}) - f(\emptyset))$$

for any $S \subseteq V$. If we further assume $f(\emptyset) = 0$ (in this case, we call f normalized or proper), we have a simplified expression,

$$f(S) = \sum_{v \in S} f(\{v\}).$$

Monotonitcity

Definition (Monotonitcity)

A set function $f : 2^V \to \mathbb{R}$ is said to be non-decreasing if for any $A \subseteq B \subseteq V$, $f(A) \leq f(B)$. Non-increasing set functions are defined in the similar way.

When we say a submodular function is monotone, we mean it is non-decreasing.

Properties

Submodularity is closed under addition.

Property

Let $f_1, f_2 : 2^V \to \mathbb{R}$ *be two submodular functions. Then*

$$f : 2^V \to \mathbb{R} \quad \textit{with} \quad f(A) = \alpha f_1(A) + \beta f_2(A)$$

is submodular for any fixed $\alpha, \beta \in \mathbb{R}^+$.

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is submodular for any fixed $\alpha, \beta \in \mathbb{R}^+$.

Submodularity is preserved under restriction.

Property

Let $f : 2^V \to \mathbb{R}$ be a submodular function. Let $S \subseteq V$ be a fixed set. Then

$$f' : 2^V \to \mathbb{R} \;\; \textit{with} \;\; f'(A) = f(A \cap S)$$

is submodular.

Properties cont.

The following property can be useful if we want to show that the negative of the objective function of k-median problem is submodular.

Property

Consider V as a set of indices. Let $\mathbf{c} \in \mathbb{R}^V$ be a fixed vector, c_i its ith coordinate. Then

$$f : 2^V \to \mathbb{R} \quad \textit{with} \quad f(A) = \max_{j \in A} c_i$$

is submodular.

Constraints

Submodular Maximization Problem

A submodular maximization problem usually has the following form:

$$\arg\max_{I \in \mathcal{I}} f(I), \tag{4}$$

where f is a submodular function and $\mathcal{I} \subseteq 2^V$ is the collection of all feasible solutions. We call $\mathcal{I}$ the constraint of the optimization problem.

Constraints

$\mathcal{I}$ is important!

The structure of $\mathcal{I}$ plays a crucial role in submodular optimization:

- Different constraints have different hardness results.
- Normally the difficulty increases when the constraint becomes more general.

Constraints

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Popular constraints

Some popular constraints:

- Cardinality constraint
- Knapsack constraint
- Matroid constraint
- Matching
- p-System
- ...

Constraints cont.

First we define hereditary set systems.

Definition (Hereditary)

A constraint $\mathcal{I} \subseteq 2^V$ is said to be hereditary if

$$I \in \mathcal{I} \implies J \in \mathcal{I} \text{ for any } J \subseteq I.$$

A hereditary constraint is sometimes called an independent system and each $I \in \mathcal{I}$ is called an independent set.

All constraints we will discuss are hereditary.

Constraints cont.

Cardinality

Cardinality constraint: $\mathcal{I} = \{A \subseteq V \mid |A| \leq k\}$

Constraints cont.

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Knapsack

Knapsack Constraint: each $i \in V$ is assigned a weight $w_i \geq 0$, $\mathcal{I} = \{S \subseteq V \mid \sum_{i \in S} w_i \leq W\}$.

Constraints cont.

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Matching: given a graph $G = (V, E)$, a *Matching* is a set $S \subseteq E$ such that no edges in S share common vertex.

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Matroid

Matroid is the generalization of the independence concept in linear algebra; omit details here ...

p-System

p-system is very general, it includes many other constraints as special cases.

Definition of p-System

Let $(V, \mathcal{I})$ be a set system and $\mathcal{I}$ hereditary. Let $\mathcal{B}(A)$ be the collection of all bases of A.

$$\mathcal{I} = \{A \subseteq V \mid \frac{\max_{S \in \mathcal{B}(A)} |S|}{\min_{S \in \mathcal{B}(A)} |S|} \leq p\}.$$

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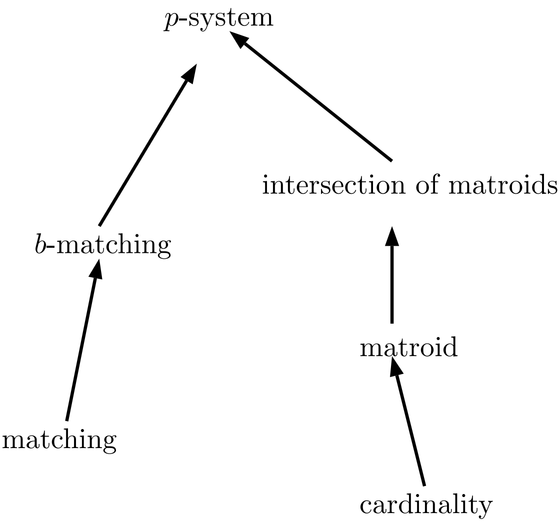
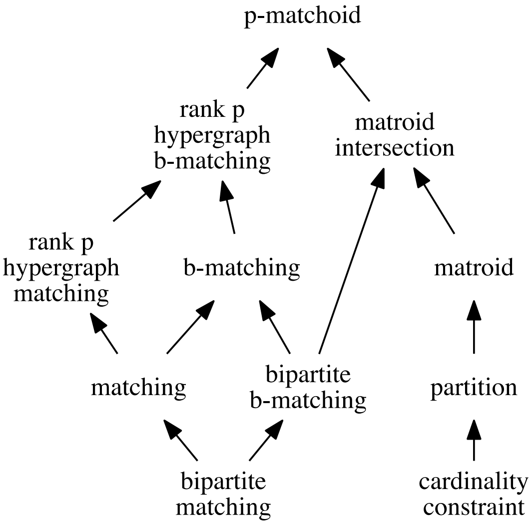
examples of p-system

- matroid is 1-system
- matching is 2-system
- intersection of p matroids is p-system
- ...

Hierarchy of constraints



Hierarchy of constraints (extended)



Notations

Some notations

- $\Delta_f(e|S) = f(S+e) - f(S)$ (or simply $\Delta(e|S)$ when f is clear from context)
- α-approximation: the returned solution S always satisfies $f(S) \geq \alpha \cdot \arg\max_{I \in \mathcal{I}} f(I)$
- When the algorithm is randomized, we normally say it guarantees α-approximation in expectation if

$$\mathbf{E}[f(S)] \geq \alpha \cdot \arg\max_{I \in \mathcal{I}} f(I).$$

The standard greedy algorithm

Algorithm 1: GREEDY algorithm for submodular maximization

Input: V the ground set, f the submodular function, $\mathcal{I}$ the constraint

Output: a set $S \subseteq V$

```
S ← ∅
while ∃ e ∈ V\S s.t. S ∪ {e} ∈ I do
    e ← arg max_{e∈V\S, S∪{e}∈I} Δ_f(e|S)
    S ← S ∪ {e}
return S
```

$S \leftarrow \emptyset$

while $\exists\ e \in V \backslash S$ s.t. $S \cup \{e\} \in \mathcal{I}$ **do**

$\quad e \leftarrow \arg\max_{e \in V \backslash S,\ S \cup \{e\} \in \mathcal{I}} \Delta_f(e|S)$

$\quad S \leftarrow S \cup \{e\}$

return S

Theorems of Algorithm 1

Theorem ([17], for cardinality constraint)

For a non-negative monotone submodular function $f : 2^V \to \mathbb{R}$*, let* $\mathcal{I}$ *be the cardinality constraint, Algorithm 1 returns a* $(1 - 1/e)$*-approximation to* $\arg\max_{I \in \mathcal{I}} f(S)$.

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Theorem ([10], modular maximization s.t. p-system)

For a non-negative monotone modular function f, given a p-system $(V, \mathcal{I})$, Algorithm 1 returns a $\frac{1}{p}$-approximation.

Speedup - GreedyLazy

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- GreedyLazy keeps an upper bound $\rho(e)$ on the marginal gain sorted in a heap.
- In each step, only update the top element in the heap and push it back, if this element remains in the top, include it into solution.
- Again gives $(1 - e^{-1})$-approximation.

Speedup - StocGreedy[16]

StocGreedy

- In each round, instead of considering all $V \backslash S$ to get

$$e \leftarrow \underset{e \in V \backslash S,\ S \cup \{e\} \in \mathcal{I}}{\arg\max} \Delta_f(e|S),$$

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- consider only $\frac{|V|}{k} \log \frac{1}{\epsilon}$ random samples from $V \backslash S$.

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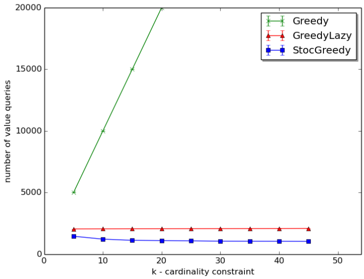
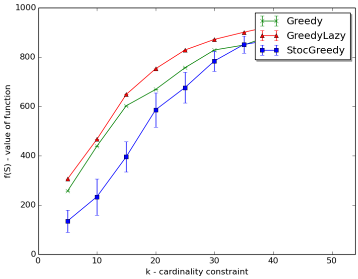
- consider only $\frac{|V|}{k} \log \frac{1}{\epsilon}$ random samples from $V \backslash S$.
- $(1 - e^{-1} - \epsilon)$-approximation in expectation.

Comparison



(a) Efficiency



(b) Quality

Figure : Experiment on SYNTHETIC dataset

Summary of state of the arts

constraint	monotone	non-negative
cardinality	$1 - 1/e$ [17]	$1/e + .004$ [3]
matroid	$1 - 1/e$ [4], R	$\frac{1-\epsilon}{e}$ [8], R
matching	$\frac{1}{2+\epsilon}$ [9]	$\frac{1}{4+\epsilon}$ [9]
intersection of p matroids	$\frac{1}{p+\epsilon}$ [13]	$\frac{p-1}{p^2+\epsilon}$ [13]
p-matchoid	$\frac{1}{p+1}$ [4, 17]	$\frac{(1-\epsilon)(2-o(1))}{e \cdot p}$ [9, 18], R

Table : Best known approximation bounds for submodular maximization in RAM model. Bounds for randomized algorithms that hold in expectation are marked (R).

Overview of Applications

- **Combinatorial Problems**: set cover, max k coverage, vertex cover, edge cover, graph cut problems etc.
- **Networks**: social networks, viral marketing, diffusion networks etc.
- **Graphical Models**: image segmentation, tree distributions, factors etc.
- **NLP**: document summarization, web search, information retrieval
- **Machine Learning**: active/semi-supervised learning etc.
- **Economics**: markets, economies of scale

Set Cover Problem

- Let E be a fixed set with finite size.
- $V = \{C_1, C_2, \ldots, C_n\}$ where each $C_i \subseteq E$.
- We define a function $f : 2^V \rightarrow \mathbb{R}$ such that $f(S) = |\cup_{C \in S} C|$.
- Goal: pick $S \subseteq V$ with $|S| \leq k$ that maximizes $f(S)$
- $f(S)$ is monotone submodular and this is a submodular maximization problem s.t. cardinality constraint!

Kernel Machines

The data set $V = \{x_1, x_2, \ldots, x_n\}$ is represented in a transformed space via a kernel matrix

$$K_V = \begin{pmatrix} \mathcal{K}(x_1, x_2) & \ldots & \mathcal{K}(x_1, x_n) \\ \vdots & \ddots & \vdots \\ \mathcal{K}(x_n, x_1) & \ldots & \mathcal{K}(x_n, x_n) \end{pmatrix},$$

where $\mathcal{K} : V \times V \to \mathbb{R}$ is the kernel function that is symmetric and positive definite.

Kernel Machines cont.

- K_V is large for large $|V|$, need to select a subset from V.
- How to measure the quality of selected subset?
- A popular way is to use *Informative Vector Machine* (IVM) introduced by Laurence et al. [12]:

$$f(S) = \frac{1}{2} \log \det \left(\mathbf{I} + \sigma^{-2} K_S \right)$$

- $f(S)$ is submodular!
- Goal:

$$\arg\max_{S \subseteq V : |S| \leq k} f(S).$$

The model

The ground set V is an ordered sequence of items $e_1, e_2, \ldots, e_n$. We process the items one by one and the maximum space being used should be sublinear (i.e. $o(n)$).

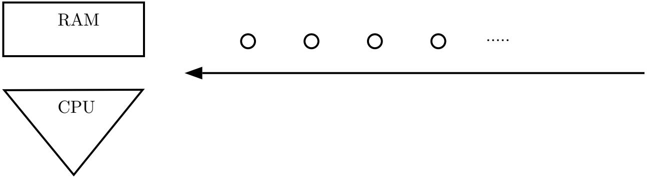


Figure : Streaming model

SIEVESTREAM assume OPT is known

Algorithm 2: SIEVESTREAMOPT for submodular maximization

Input: V as data stream, f a monotone submodular function, k the size constraint, OPT the optimal value of $f(S)$ under the constraint

Output: a set $S \subseteq V$

$S \leftarrow \emptyset$

for *each e in the data stream* **do**

  if $\Delta(e|S) \geq \frac{OPT/2 - f(S)}{k - |S|}$ *and* $|S| < k$ **then**

    $S \leftarrow S \cup \{e\}$

return S

SieveStream assume OPT is unknown

Problems with SieveStreamOPT

OPT is unknown!

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Problems with SieveStreamOPT

OPT is unknown!

So what we do?

Solution

- $m = \max_{e \in V} f(\{e\})$, for simplicity, assume $f(\emptyset) = \emptyset$
- note that $m \leq \text{OPT} \leq k \cdot m$
- if we know m, we guess OPT as $m, (1+\epsilon)m, (1+\epsilon)^2 m, \ldots \leq k \cdot m$, each guess runs an instance of SieveStreamOPT
- it runs only $O(\log_{(1+\epsilon)}) = O(\frac{k}{\epsilon})$ instances

SieveStream assume OPT is unknown, cont.

Problem again

calculating $m = \max_{e \in V} f(\{e\})$ requires an extra pass!

SieveStream assume OPT is unknown, cont.

Problem again

calculating $m = \max_{e \in V} f(\{e\})$ requires an extra pass!

Solution?

Solution

- update $m \leftarrow \max(f(e_i), m)$ on the fly!
- lazy-evaluation, create an instance of SieveStreamOPT only when necessary
- it runs only $O(\log_{(1+\epsilon)}) = O(\frac{k}{\epsilon})$ instances, using only 1 pass
- guarantee $(1/2 - \epsilon)$-approximation for monotone submodular maximization s.t. cardinality constraint

SieveStream

Algorithm 3: SieveStream for submodular maximization

Input: V as data stream, f a monotone submodular function, k the size constraint, ϵ a parameter

Output: a set $S \subseteq V$

1 $O = \{(1+\epsilon)^i \mid i \in \mathbb{Z}\}$

```
/* maintain the sets only for the necessary v's
lazily                                          */
```

2 For each $v \in O$, $S_v \leftarrow \emptyset$

3 $m \leftarrow 0$

4 **for** *each e in the data stream* **do**

5 $\quad m \leftarrow \max\{m, f(\{e\})\}$

6 $\quad O \leftarrow \{(1+\epsilon)^i \mid m \leq (1+\epsilon)^i \leq 2 \cdot k \cdot m\}$

7 $\quad$ run in parallel SieveStreamOPT with each OPT in O

8 **return** $\arg\max_{S_v : v \in O} f(S_v)$

RandomStream, assume α is known

Algorithm 4: RandomStream for submodular maximization

Input: V as data stream, f a non-negative submodular function, k the cardinality constraint, ϵ a parameter

Output: a set $S \subseteq V$

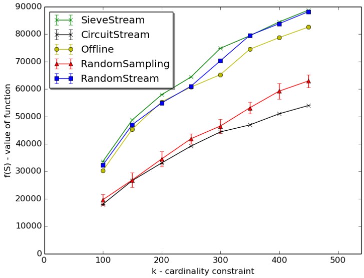
$B \leftarrow \emptyset, S \leftarrow \emptyset$
for *each e in the data stream* **do**
    if $|S| < k$ *and* $\Delta(e|S) > \alpha$ **then**
        $B \leftarrow B + e$
    if $|B| > \frac{k}{\epsilon}$ **then**
        $e \leftarrow$ uniformly random from B
        $B \leftarrow B - e, S \leftarrow S + e$
        for *all $e' \in B$ s.t.* $\Delta(e'|S) \leq \alpha$ **do**
            $B \leftarrow B - e'$
$S' \leftarrow$ offline algorithm on B
return $\arg\max_{A \in \{S, S'\}} f(A)$

RandomStream cont.

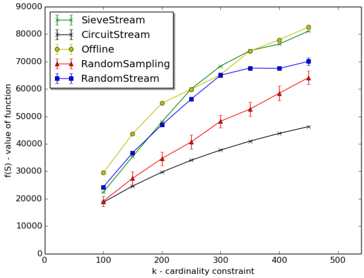
α/OPT is unknown

- In RandomStream , when $\alpha \approx \frac{\mathsf{OPT}}{k}$, then algorithm gives $\frac{1-\epsilon}{2+\epsilon}$-approximation.
- Again we can guess OPT in parallel as we did in SieveStream .

experiment



(a) Shuffled edges



(b) Edges grouped by vertices

Figure : Streaming Algorithms on FACEBOOK; ϵ is set to be 0.2 for both SIEVESTREAM and RANDOMSTREAM ; γ is set to be 1.0 for CIRCUITSTREAM .

Summary of state of the art

constraint	monotone	non-negative
cardinality	$\frac{1-\epsilon}{2}$ [1]	$\frac{1-\epsilon}{2+e}$ [6], R
matroid	$1/4$ [5]	$\frac{1-\epsilon}{4+e}$ [6], R
matching	$4/31$ [5]	$\frac{1-\epsilon}{12+\epsilon}$ [6], R
intersection of p matroids	$\frac{1}{4p}$ [5]	$\frac{(1-\epsilon)(p-1)}{5p^2-4p+\epsilon}$ [6], R
p-matchoid	$\frac{1}{4p}$ [6]	$\frac{(1-\epsilon)(2-o(1))}{(8+e)p}$ [6], R

Table : Best known approximation bounds for submodular maximization in streaming model. Bounds for randomized algorithms that hold in expectation are marked (R).

The model

Crash Introduction to MapReduce

- the data is represented as $\langle$key, value$\rangle$ pairs that are distributed across m machines
- a computation in this model proceeds in rounds. In each round, there will be two phases.
- Map phase: each pair $\langle$key, value$\rangle$ is mapped by a user-defined hash function to $\langle$hash(key), value$\rangle$, all pairs are then shuffled and sent to different machines
- Reduce phase: each machine performs computation on the pairs it received as the output or the input of the next round

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If you do not know MapReduce model ...

Think of it as a group of machines with one machine as the coordinator/center node.

GreeDi-based algorithms

framework of GreeDi-based algorithms

m - the number of machines; $C \in \mathbb{Z}^+$ is an parameter; k - the cardinality constraint. The algorithm goes as follows:

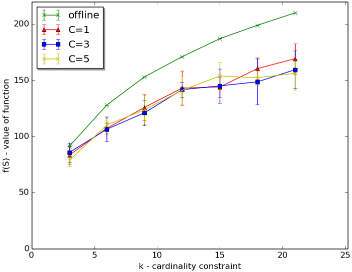
- Randomly assign each v to C out of m machines, we obtain subsets $V_1, \ldots, V_m$
- Let ALG be an offline algorithm, k' be a cardinality constraint. Run ALG on each V_i with constraint k', we obtains $U_1, U_2, \ldots, U_m$ as results.
- Let $U = \cup_i S_i$, run ALG on U with parameter k, we obtain S as the result. Also run ALG on $U_1, \ldots, U_m$ with parameter k to obtain $S_1, S_2, \ldots, S_m$.
- Return the best solution among S, $S_1, \ldots, S_m$.

Some theories about the GREEDI-Based Algorithms

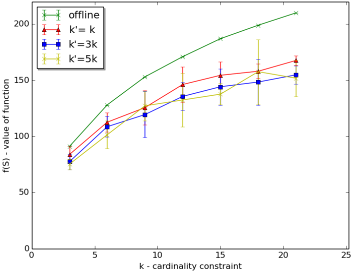
some theories (informal)

- use the standard greedy algorithm as ALG, $k' = k$, $C = 1$, the GREEDI-Based algorithm gives $\frac{1-e^{-1}}{2}$-approximation.
- increasing k' or C would **slightly** increase the approximation ratio (in worse case!), but not too much

experiment



(a) Different multiplicity C; set $k' = k$; number of machines is 20.



(b) Different k'; C is set to be 1; number of machines is set to be 20.

Figure : GreeDi-based Algorithms on Accidents dataset.

Summary of state of the art

constraint	rounds	approx.	reference
cardinality	$O(\frac{\log n}{\epsilon})$	$1 - e^{-1} - \epsilon$	[11]
	2	0.545	[15]
	$O(1/\epsilon)$	$1 - e^{-1} - \epsilon$	[2]
matroid	$O(\frac{\log n}{\epsilon})$	$1/2 - \epsilon$	[11]
	2	$1/4$	[7]
	$O(1/\epsilon)$	$1 - e^{-1} - \epsilon$	[2]
p-system	$O(\frac{\log n}{\epsilon})$	$\frac{1}{p+1} - \epsilon$	[11]
	2	$\frac{1}{2(p+1)}$	[7]
	$O(1/\epsilon)$	$\frac{1}{p+1} - \epsilon$	[2]

Table : Best known algorithms for monotone submodular maximization in the MapReduce model. All algorithms are randomized.

Question? Thank you!

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