

A Survey on Distributed/Streaming Submodular Optimization

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1 Introduction

Submodularity is a property of set functions with deep theoretical and practical consequences. Submodular functions occur in a variety of applications, including representative skyline selection [16], network structure learning [6], influence maximization [7], document summarization [9], image segmentation [3, 8] and many others.

There is a large body of research in submodular optimization which we can not cover thoroughly here. In this survey, we focus on the recent advances in optimizing (mostly, maximizing) submodular functions in distributed and streaming setting.

Through Lovász' extension, a submodular minimization problem can be solved via techniques from convex optimization. Therefore most work focuses on submodular maximization. In particular, almost all results for distributed and streaming submodular optimization are for maximization problems. In this survey, we also focus on submodular maximization, but will briefly mention minimization problems as well when necessary.

1.1 Notations

Through out this survey,

we use V to represent the ground set we consider.

2^V is the power set of V (i.e. the set of all subsets of V).

In general, A^V is the collection of maps from V to A .

For simplicity, we sometime write $A \cup \{x\}$ as $A + x$.

2 Submodularity

In this section, we first give several equivalent definitions of submodularity, and then we introduce several fundamental properties of submodular functions. We also discuss various constraints that occur frequently in submodular optimization problems. In the last part of this section, we cover algorithms that solve constrained submodular maximization problems with theoretical approximation guarantee.

2.1 Definitions

There are many equivalent definitions, and we will discuss three of them in this section.

Definition 1 (submodular concave) A function $f : 2^V \rightarrow \mathbb{R}$ is **submodular** if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B). \quad (1)$$

An alternate equivalent definition is more interpretable in many situations,

Definition 2 (diminishing returns) A function $f : 2^V \rightarrow \mathbb{R}$ is **submodular** if for any $A \subseteq B \subset V$, and $v \in V \setminus B$, we have that:

$$f(A + v) - f(A) \geq f(B + v) - f(B). \quad (2)$$

Intuitively, this definition requires that the incremental “gain” of adding a new element v decreases (diminishes) as the base set grows from A to B . We will see that this property is actually shared by many real-world phenomena.

It turns out that a stronger but equivalent statement can also serve as the definition of a submodular function,

Definition 3 (group diminishing returns) A function $f : 2^V \rightarrow \mathbb{R}$ is **submodular** if for any $A \subseteq B \subset V$, and $C \subseteq V \setminus B$, we have that:

$$f(A \cup C) - f(A) \geq f(B \cup C) - f(B). \quad (3)$$

2.2 Modularity and Supermodularity

We also briefly mention modularity and supermodularity here. These two concepts are closely related to submodularity.

A function $f : 2^V \rightarrow \mathbb{R}$ is modular if we replace inequality by equality in Definition 2 (or any of other two). Formally,

Definition 4 (Modularity) A function $f : 2^V \rightarrow \mathbb{R}$ is **modular** if for any $A \subseteq B \subset V$, and $v \in V \setminus B$, we have that:

$$f(A + v) - f(A) = f(B + v) - f(B). \quad (4)$$

Notably, a modular function f can always be written as

$$f(S) = f(\emptyset) + \sum_{v \in S} (f(\{v\}) - f(\emptyset))$$

for any $S \subseteq V$.

If we further assume $f(\emptyset) = 0$ (in this case, we call f **normalized**), we have a simplified expression,

$$f(S) = \sum_{v \in S} f(\{v\}).$$

Modularity can be useful in our discussion of submodularity, because one can use modular functions to construct submodular functions with desired properties in their applications. Examples can be found in e.g. [9, 10].

A supermodular function is defined by flipping the inequality sign in the definition of a submodular function. Formally,

Definition 5 (Supermodularity) A function $f : 2^V \rightarrow \mathbb{R}$ is **modular** if for any $A \subseteq B \subset V$, and $v \in V \setminus B$, we have that:

$$f(A + v) - f(A) \leq f(B + v) - f(B). \quad (5)$$

We will focus on submodular functions because a function is supermodular if and only if its negative is submodular.

2.3 Properties

Like convex and concave functions, submodular functions have many nice properties. Lovász's description of convex functions [11] can be viewed as accurate comments on submodularity:

- Convex functions occur in many mathematical models in economy, engineering, and other sciences. Convexity is a very natural property of various functions and domains occurring in such models; quite often the only non-trivial property which can be stated in general.
- Convexity is preserved under many natural operations and transformations, and thereby the effective range of results can be extended, elegant proof techniques can be developed as well as unforeseen applications of certain results can be given.
- Convex functions and domains exhibit sufficient structure so that a mathematically beautiful and practically useful theory can be developed.
- There are theoretically and practically (reasonably) efficient methods to find the minimum of a convex function.

We survey several useful properties which can be useful in our later section. More properties of submodularity can be found in e.g. [2, 5].

Submodularity is close under addition,

Property 1 Let $f_1, f_2 : 2^V \rightarrow \mathbb{R}$ be two submodular functions. Then

$$f : 2^V \rightarrow \mathbb{R} \text{ with } f(A) = \alpha f_1(A) + \beta f_2(A)$$

is submodular for any fixed $\alpha, \beta \in \mathbb{R}^+$.

Adding a modular function does not break submodularity,

Property 2 Let $f_1, f_2 : 2^V \rightarrow \mathbb{R}$, f_1 is submodular and f_2 is modular. Then

$$f : 2^V \rightarrow \mathbb{R} \text{ with } f(A) = f_1(A) + \alpha f_2(A)$$

is submodular for any fixed $\alpha \in \mathbb{R}$.

Submodularity is preserved under restriction,

Property 3 Let $f : 2^V \rightarrow \mathbb{R}$ be a submodular function. Let $S \subseteq V$ be a fixed set. Then

$$f' : 2^V \rightarrow \mathbb{R} \text{ with } f'(A) = f(A \cap S)$$

is submodular.

As a direct implication of Property 1 and Property 3, we have the following more general result,

Property 4 Let $f : 2^V \rightarrow \mathbb{R}$ be a submodular function, $\mathcal{C} = \{C_1, C_2, \dots, C_k\}$ be a collection of subsets of V (i.e. each $C_i \subseteq V$). Then

$$f' : 2^V \rightarrow \mathbb{R} \text{ with } f'(A) = \sum_{C \in \mathcal{C}} f(A \cap C)$$

is submodular.

This property can be useful in graphical models and image processing. **CHEN : TODO: show examples**

Following property can be useful when we show that the objective function of k-medoid problem is supermodular,

Property 5 Consider V as a set of indices. Let $\mathbf{c} \in \mathbb{R}^V$ be a fixed vector, c_i be its i th coordinate. Then

$$f : 2^V \rightarrow \mathbb{R} \text{ with } f(A) = \max_{j \in A} c_j$$

is submodular.

We can use non-negative modular function and a concave function to construct submodular functions,

Property 6 Let $m : 2^V \rightarrow \mathbb{R}^+$ be a modular function, and g a concave function over \mathbb{R} . Then

$$f : 2^V \rightarrow \mathbb{R} \text{ with } f(A) = g(m(A))$$

is submodular.

Before introducing the next property, we define the monotonicity of set function,

Definition 6 (Monotonicity) An set function $f : 2^V \rightarrow \mathbb{R}$ is said to be monotone non-decreasing if for any $A \subseteq B \subseteq V$, $f(A) \leq f(B)$. Monotone non-increasing function can be defined similarly.

Property 7 Let $f, g : 2^V \rightarrow \mathbb{R}$ be two submodular functions. If $(f - g)(\cdot)$ is either monotone non-decreasing or monotone non-increasing, then $f : 2^V \rightarrow \mathbb{R}$ with

$$f(A) = \min(f(A), g(A))$$

is submodular.

2.4 Constraints

Now we discuss the constraints in submodular optimization problems. A submodular maximization problem usually has the following form,

$$\arg \max_{I \in \mathcal{I}} f(I) \quad (6)$$

where f is a submodular function and $\mathcal{I} \subseteq 2^V$ is the collection of all feasible solutions. We call \mathcal{I} the **constraint** of the optimization problem. The structure of \mathcal{I} plays a crucial role in submodular optimization. Different constraints have different hardness results, normally the difficulty increases when the constraint becomes more general. We will introduce cardinality constraint, knapsack constraint, matroid, and p-matchoid.

2.4.1 Cardinality Constraint

Cardinality constraint is perhaps the most straightforward constraint we would discuss in this survey. Efficient algorithms have been developed for finding or approximating the optimal solution of (6). There are also a lot of discussions on optimization subject to cardinality constraint, in both streaming and distributed setting.

A **cardinality constraint** is parameterized with a fixed constant $k \in \mathbb{Z}^+$. It is simply defined as $\mathcal{I} = \{A \subseteq V \mid |V| \leq k\}$, i.e. all subsets of V with size no larger than k . Cardinality constraint is arguably the most popular constraint, and it occurs everywhere. For example, in k-medoid clustering, we want to find a set S of *at most* k points, that minimizes the total distance of all points to S .

2.4.2 Knapsack Constraint

Knapsack constraint generalizes cardinality constraint by assigning each element in V a weight. Given a budget $W > 0$ and assume that each $i \in V$ is assigned a weight $w_i \geq 0$, a **knapsack constraint** can be defined as $\mathcal{I} = \{S \subseteq V \mid \sum_{i \in S} w_i \leq W\}$.

2.4.3 Matroid

Informally, a **Matroid** is the abstraction of the *independence* concept in linear algebra. In fact there are so many results around Matroid and the Matroid itself becomes a subfield of algebra. We cover some basics of Matroid theory and from which readers can easily see how powerful this concept is.

Before discussing the concept of a matroid, we briefly review the independence concept from linear algebra. For simplicity, let us just consider \mathbb{R}^d instead of a general linear space. A subset S of \mathbb{R}^d is said to be *independent* if there does not exist any $\mathbf{x} \in S$ such that \mathbf{x} can be represented by linear combination of vectors in $S \setminus \{\mathbf{x}\}$.

Let $\mathcal{I} = \{S \subseteq \mathbb{R}^d \mid S \text{ is independent}\}$, i.e. only a independent set can be considered feasible. From what we learn in college linear algebra course, we know \mathcal{I} has the following properties: 1) $\emptyset \in \mathcal{I}$; 2) if $I \in \mathcal{I}$, any of I 's subsets is also in \mathcal{I} ; 3) if $J, I \in \mathcal{I}$ and J has smaller size than I , we must be able to find an element $\mathbf{x} \in I \setminus J$ such that $J \cup \{\mathbf{x}\} \in \mathcal{I}$.

Even for the "trivial" size function $f : 2^V \rightarrow \mathbb{Z}$ with $f(A) = |A|$, optimizing $\arg \max_{I \in \mathcal{I}} f(I)$ would have tremendous applications because its optimal solution is a base of the vector space.

We will see shortly how this optimization problem has direct connection with *Maximum Spanning Tree* problem. If we somehow generalize the definition of independence, we may be able to model a much more larger class of problems into the form (6).

It turns out that the properties of \mathcal{I} we just described are sufficient to give a meaningful definition for Matroid. Formally,

Definition 7 (Matroid) A set system (V, \mathcal{I}) is a **Matroid** if it has the following properties,

1. $\emptyset \in \mathcal{I}$
2. $\forall I \in \mathcal{I}, J \subseteq I \implies J \in \mathcal{I}$
3. $\forall I, J \in \mathcal{I}$, with $|I| = |J| + 1$, then $\exists x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$

Note that, unlike in the \mathbb{R}^d case, we restrict on a finite set V . **CHEN : is it necessary?**

2.4.4 Matchoids

Let $\mathcal{M}_1 = (V_1, \mathcal{I}_1), \dots, \mathcal{M}_q = (V_q, \mathcal{I}_q)$ be q matroids where $V = V_1 \cup \dots \cup V_q$. Let $\mathcal{I} = \{S \subseteq V \mid S \cap V_i \in \mathcal{I}_i \text{ for all } i\}$. The finite set system (V, \mathcal{I}) is a **p-matchoid** if for every $e \in V$, e is a member of V_i for at most p indices $i \in [q]$. The concept of p-matchoid generalizes the intersection of matroids (taking $p = q$ and $V_i = V$ for all i).

2.5 Algorithms for Submodular Maximization

There are a lot of results for submodular maximization in the centralized setting where the data can fit into the RAM. Those results normally assume the **oracle model**: one is given a value oracle and a membership oracle. Given $S \subseteq V$, the membership oracle answers if $S \in \mathcal{I}$ and the value oracle returns $f(S)$. We cover several classical results which serve as the building blocks for distributed/streaming algorithms for submodular maximization.

We introduce the notation for **marginal gain**: $\Delta_f(e|S) = f(S + e) - f(S)$. The following algorithm shows a popular greedy strategy for submodular optimization.

Algorithm 1: GREEDY algorithm for submodular maximization

Input: V the ground set, f the submodular function, \mathcal{I} the constraint

Output: a set $S \subseteq V$

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1  $S \leftarrow \emptyset$ 
2 while  $\exists e \in V \setminus S$  s.t.  $S \cup \{e\} \in \mathcal{I}$  do
3    $e \leftarrow \arg \max_{e \in V \setminus S, S \cup \{e\} \in \mathcal{I}} \Delta_f(e|S)$ 
4    $S \leftarrow S \cup \{e\}$ 
5 return  $S$ 
```

2.5.1 Algorithms for Cardinality Constraint

A celebrated result of [15] shows that,

Theorem 1 ([15]) For a non-negative monotone non-decreasing submodular function $f : 2^V \rightarrow \mathbb{R}$, let \mathcal{I} be the cardinality constraint, Algorithm 1 returns a $(1 - 1/e)$ -approximation to $\arg \max_{I \in \mathcal{I}} f(I)$.

For several classes of submodular functions, this result is actually the best one can expect for any efficient algorithm. In fact the hardness in [4, 15] shows that any algorithm that is only allowed sub-exponential number of value queries can not achieve better than $(1 - 1/e)$ -approximation (for a large class of submodular functions).

There are several papers improving the running time of Algorithm 1 under cardinality constraint of size k (under which the membership oracle is trivial). Minoux [13] proposed LAZY-GREEDY as a fast implementation for Algorithm 1. Instead of computing $\Delta_f(e|S)$ for each $e \in V \setminus S$ in Line 3, LAZY-GREEDY keeps an upper bound $\rho(e)$ (initially $+\infty$) on the marginal gain sorted in decreasing order (or kept in a heap). In each iteration, the LAZY-GREEDY algorithm evaluates the element on top of the heap and updates its upper bound $\rho(e) \leftarrow \Delta(e|S)$. If the updated $\rho(e) \geq \rho(e')$ for all other e' , submodularity guarantees that e is the element with the largest marginal gain. The exact number of value queries consumed by LAZY-GREEDY is unknown because it heavily relies on both f and V , experimental study however shows that the LAZY-GREEDY algorithm is in order of magnitude faster than the naive implementation of Algorithm 1.

Wei et al. [17] improved the running time of LAZY-GREEDY by approximating the underlying submodular function with a set of (sub)modular functions. Badanidiyuru et al. [1] proposed a different approach that uses only $O(\frac{|V|}{\epsilon} \log \frac{1}{\epsilon})$ number of value queries and guarantees $(1 - 1/e - \epsilon)$ -approximation. This result was improved by Mirzasoleiman et al. [14] recently where they proposed a randomized algorithm (STOCHASTIC-GREEDY) that reduce the number of queries to $O(|V| \log \frac{1}{\epsilon})$ and the approximation guarantee is $(1 - 1/e - \epsilon)$, in expectation. The key observation made in [14] is that, instead of considering all $e \in V \setminus S$, one can only consider $O(\frac{|V|}{k} \log \frac{1}{\epsilon})$ random samples from $V \setminus S$.

2.5.2 Algorithms for Matroid Constraint

Let $f(S) = \sum_{i \in S} w_i$, where $S \subseteq V$ and each element $i \in V$ is assigned a non-negative weight w_i . A notable result shows the deep connection between Algorithm 1 and the concept of matroid,

Theorem 2 (see e.g. [2, 5]) *Let (V, \mathcal{I}) be a set system we consider, \mathcal{I} is a matroid if and only if for any modular function f defined in above way, Algorithm 1 leads to a optimal solution for $\arg \max_{I \in \mathcal{I}} f(I)$.*

Note that Algorithm 1 actually includes many greedy algorithms as special cases (e.g. maximum weighted spanning tree algorithm). The statement of Theorem 2 is so strong that it provides a complete characterization of a large class of problems.

3 Applications

In this section, we discuss several representative applications in great detail. We will see from those examples that submodularity is such a natural property that many real-world problems can be cast in to the framework of submodular optimization.

- [12] distributed k-centers using submodular optimization.

4 Distributed Submodular Optimization

5 Streaming Submodular Maximization

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