Submodular Maximization

advances in distributed/streaming computing

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Overview

Definitions of Submodularity

Definition (submodular concave)

A function $f: 2^V \rightarrow \mathbb{R}$ is submodular if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B). \tag{1}$$

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An alternate equivalent definition is more interpretable in many situations.

Definition (diminishing returns)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A \subseteq B \subset V$, and $v \in V \backslash B$, we have that:

$$f(A + v) - f(A) \geq f(B + v) - f(B). \tag{2}$$

Properties

Submodularity is closed under addition.

Property

Let $f_1, f_2 : 2^V \to \mathbb{R}$ be two submodular functions. Then

$$f : 2^V \to \mathbb{R} \quad \textit{with} \quad f(A) = \alpha f_1(A) + \beta f_2(A)$$

is submodular for any fixed $\alpha, \beta \in \mathbb{R}^+$.

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is submodular for any fixed $\alpha, \beta \in \mathbb{R}^+$.

Submodularity is preserved under restriction.

Property

Let $f : 2^V \to \mathbb{R}$ *be a submodular function. Let* $S \subseteq V$ *be a fixed set. Then*

$$f' : 2^V \to \mathbb{R} \quad \textit{with} \quad f'(A) = f(A \cap S)$$

is submodular.

Properties cont.

The following property can be useful if we want to show that the negative of the objective function of k-median problem is submodular.

Property

Let V be the ground set we consider, each element in V is a real number. Then

$$f : 2^V \to \mathbb{R} \quad \textit{with} \quad f(A) = \max_{c \in A} c$$

is submodular.

Constraints

Submodular Maximization Problem

A submodular maximization problem usually has the following form:

$$\arg\max_{I \in \mathcal{I}} f(I), \tag{3}$$

where f is a submodular function and $\mathcal{I} \subseteq 2^V$ is the collection of all feasible solutions. We call $\mathcal{I}$ the constraint of the optimization problem.

Example

Cardinality constraint: $\mathcal{I} = \{A \subseteq V \mid |A| \leq k\}$

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The structure of $\mathcal{I}$ plays a crucial role in submodular optimization:

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Popular constraints

Some popular constraints:

- Cardinality constraint
- Knapsack constraint
- Matroid constraint
- Matching
- p-System
- ...

Constraints cont.

First we define hereditary set systems.

Definition (Hereditary)

A constraint $\mathcal{I} \subseteq 2^V$ is said to be hereditary if

$$I \in \mathcal{I} \implies J \in \mathcal{I} \text{ for any } J \subseteq I.$$

A hereditary constraint is sometimes called an independent system and each $I \in \mathcal{I}$ is called an independent set.

All constraints we will discuss are hereditary.

Constraints cont.

Cardinality

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Knapsack

Knapsack Constraint: each $i \in V$ is assigned a weight $w_i \geq 0$, $\mathcal{I} = \{S \subseteq V \mid \sum_{i \in S} w_i \leq W\}$.

Constraints cont.

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Matching: given a graph $G = (V, E)$, a *Matching* is a set $S \subseteq E$ such that no edges in S share common vertex.

Constraints cont.

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Matroid

Matroid is the generalization of the independence concept in linear algebra; omit details here ...

p-System

p-system is very general, it includes many other constraints as special cases.

Definition of p-System

Let $(V, \mathcal{I})$ be a set system and $\mathcal{I}$ hereditary. Let $\mathcal{B}(A)$ be the collection of all bases of A.

$$\mathcal{I} = \{A \subseteq V \mid \frac{\max_{S \in \mathcal{B}(A)} |S|}{\min_{S \in \mathcal{B}(A)} |S|} \leq p\}.$$

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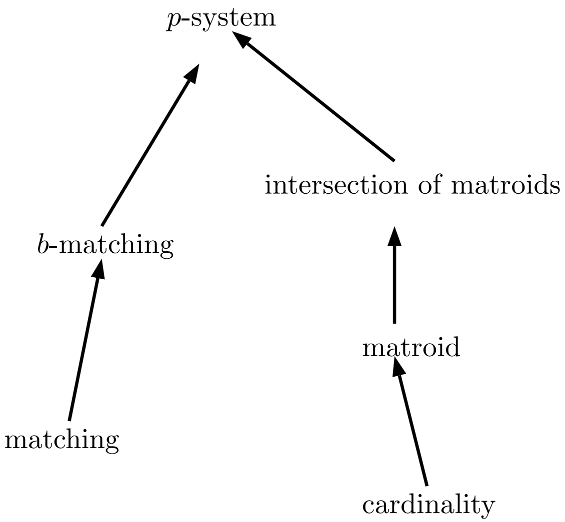
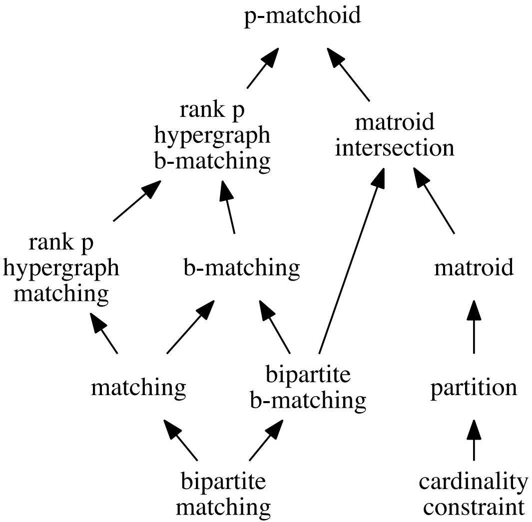
examples of p-system

- matroid is 1-system
- matching is 2-system
- intersection of p matroids is p-system
- ...

Hierarchy of constraints



Hierarchy of constraints (extended)



Notations

Some notations

- $\Delta_f(e|S) = f(S+e) - f(S)$ (or simply $\Delta(e|S)$ when f is clear from context), call it marginal gain.
- α-approximation: the returned solution S always satisfies $f(S) \geq \alpha \cdot \arg\max_{I \in \mathcal{I}} f(I)$
- When the algorithm is randomized, we normally say it guarantees α-approximation in expectation if

$$\mathbf{E}[f(S)] \geq \alpha \cdot \arg\max_{I \in \mathcal{I}} f(I).$$

The standard greedy algorithm

Algorithm 1: GREEDY algorithm for submodular maximization

Input: V the ground set, f the submodular function, $\mathcal{I}$ the constraint

Output: a set $S \subseteq V$

$S \leftarrow \emptyset$

while $\exists\ e \in V \backslash S$ *s.t.* $S \cup \{e\} \in \mathcal{I}$ **do**

  $e \leftarrow \arg\max_{e \in V \backslash S,\ S \cup \{e\} \in \mathcal{I}} \Delta_f(e|S)$

  $S \leftarrow S \cup \{e\}$

return S

Theorems of Algorithm 1

Theorem ([17], for cardinality constraint)

For a non-negative monotone submodular function $f : 2^V \to \mathbb{R}$, let $\mathcal{I}$ be the cardinality constraint, Algorithm 1 returns a $(1 - 1/e)$-approximation to $\arg\max_{I \in \mathcal{I}} f(S)$.

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Theorem ([17, 4], for p-system)

For a non-negative monotone submodular function f, given a p-system $(V, \mathcal{I})$, Algorithm 1 returns a $\frac{1}{p+1}$-approximation.

Speedup - GreedyLazy

GreedyLazy

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- GreedyLazy keeps an upper bound $\rho(e)$ on the marginal gain sorted in a heap.
- In each step, only update the top element in the heap and push it back, if this element remains in the top, include it into solution.
- Again gives $(1 - e^{-1})$-approximation.

Speedup - StocGreedy[16]

StocGreedy

- In each round, instead of considering all $V \backslash S$ to get

$$e \leftarrow \underset{e \in V \backslash S,\ S \cup \{e\} \in \mathcal{I}}{\arg\max} \Delta_f(e|S),$$

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$$e \leftarrow \underset{e \in V \backslash S, \; S \cup \{e\} \in \mathcal{I}}{\arg\max} \Delta_f(e|S),$$

- consider only $\frac{|V|}{k} \log \frac{1}{\epsilon}$ random samples from $V \backslash S$.

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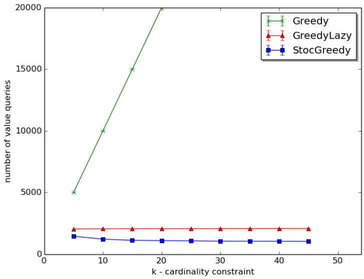
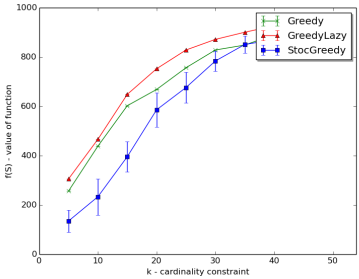
- consider only $\frac{|V|}{k} \log \frac{1}{\epsilon}$ random samples from $V \backslash S$.
- $(1 - e^{-1} - \epsilon)$-approximation in expectation.

Comparison



(a) Efficiency



(b) Quality

Figure : Experiment on SYNTHETIC dataset

Summary of state of the arts

constraint	monotone	non-negative
cardinality	$1 - 1/e$ [17]	$1/e + .004$ [3]
matroid	$1 - 1/e$ [4], R	$\frac{1-\epsilon}{e}$ [8], R
matching	$\frac{1}{2+\epsilon}$ [9]	$\frac{1}{4+\epsilon}$ [9]
intersection of p matroids	$\frac{1}{p+\epsilon}$ [13]	$\frac{p-1}{p^2+\epsilon}$ [13]
p-matchoid	$\frac{1}{p+1}$ [4, 17]	$\frac{(1-\epsilon)(2-o(1))}{e \cdot p}$ [9, 18], R

Table : Best known approximation bounds for submodular maximization in RAM model. Bounds for randomized algorithms that hold in expectation are marked (R).

Overview of Applications

- **Combinatorial Problems**: set cover, max k coverage, vertex cover, edge cover, graph cut problems etc.
- **Networks**: social networks, viral marketing, diffusion networks etc.
- **Graphical Models**: image segmentation, tree distributions, factors etc.
- **NLP**: document summarization, web search, information retrieval
- **Machine Learning**: active/semi-supervised learning etc.
- **Economics**: markets, economies of scale

Set Cover Problem

- Let E be a fixed set with finite size.
- $V = \{C_1, C_2, \ldots, C_n\}$ where each $C_i \subseteq E$.
- We define a function $f : 2^V \to \mathbb{R}$ such that $f(S) = |\cup_{C \in S} C|$.
- Goal: pick $S \subseteq V$ with $|S| \leq k$ that maximizes $f(S)$
- $f(S)$ is monotone submodular and this is a submodular maximization problem s.t. cardinality constraint!

Kernel Machines

The data set $V = \{x_1, x_2, \ldots, x_n\}$ is represented in a transformed space via a kernel matrix

$$K_V = \begin{pmatrix} \mathcal{K}(x_1, x_2) & \ldots & \mathcal{K}(x_1, x_n) \\ \vdots & \ddots & \vdots \\ \mathcal{K}(x_n, x_1) & \ldots & \mathcal{K}(x_n, x_n) \end{pmatrix},$$

where $\mathcal{K} : V \times V \to \mathbb{R}$ is the kernel function that is symmetric and positive definite.

Kernel Machines cont.

- K_V is large for large $|V|$, need to select a subset from V.
- How to measure the quality of selected subset?
- A popular way is to use *Informative Vector Machine* (IVM) introduced by Laurence et al. [12]:

$$f(S) = \frac{1}{2} \log \det \left(\mathbf{I} + \sigma^{-2} K_S \right)$$

- $f(S)$ is submodular!
- Goal:

$$\underset{S \subseteq V : |S| \leq k}{\arg\max} \; f(S).$$

The model

The ground set V is an ordered sequence of items $e_1, e_2, \ldots, e_n$. We process the items one by one and the maximum space being used should be sublinear (i.e. $o(n)$).

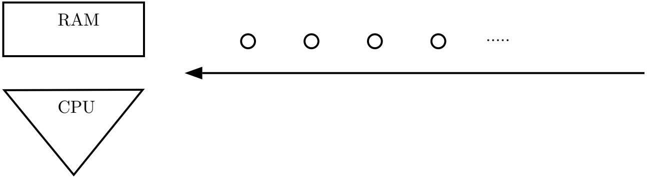


Figure : Streaming model

SieveStream assume OPT is known

Algorithm 2: SieveStreamOPT for submodular maximization

Input: V as data stream, f a monotone submodular function, k the size constraint, OPT the optimal value of $f(S)$ under the constraint

Output: a set $S \subseteq V$

1. $S \leftarrow \emptyset$
2. **for** *each* e *in the data stream* **do**
3. **if** $\Delta(e|S) \geq \frac{OPT/2 - f(S)}{k - |S|}$ *and* $|S| < k$ **then**
4. $S \leftarrow S \cup \{e\}$
5. **return** S

SieveStream assume OPT is unknown

Problems with SieveStreamOPT

OPT is unknown!

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So what we do?

SieveStream assume OPT is unknown

Problems with SieveStreamOPT

OPT is unknown!

So what we do?

Solution

- $m = \max_{e \in V} f(\{e\})$, for simplicity, assume $f(\emptyset) = \emptyset$
- note that $m \leq \text{OPT} \leq k \cdot m$
- if we know m, we guess OPT as $m, (1+\epsilon)m, (1+\epsilon)^2 m, \ldots \leq k \cdot m$, each guess runs an instance of SieveStreamOPT
- it runs only $O(\log_{(1+\epsilon)}) = O(\frac{k}{\epsilon})$ instances

SieveStream assume OPT is unknown, cont.

Problem again

calculating $m = \max_{e \in V} f(\{e\})$ requires an extra pass!

SieveStream assume OPT is unknown, cont.

Problem again

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Solution?

SieveStream assume OPT is unknown, cont.

Problem again

calculating $m = \max_{e \in V} f(\{e\})$ requires an extra pass!

Solution?

Solution

- update $m \leftarrow \max(f(e_i), m)$ on the fly!
- lazy-evaluation, create an instance of SieveStreamOPT only when necessary
- it runs only $O(\log_{(1+\epsilon)}) = O(\frac{k}{\epsilon})$ instances, using only 1 pass
- guarantee $(1/2 - \epsilon)$-approximation for monotone submodular maximization s.t. cardinality constraint

SieveStream

Algorithm 3: SieveStream for submodular maximization

Input: V as data stream, f a monotone submodular function, k the size constraint, ϵ a parameter

Output: a set $S \subseteq V$

```
O = {(1 + ε)^i | i ∈ Z}
/* maintain the sets only for the necessary v's lazily */
For each v ∈ O, S_v ← ∅
m ← 0
for each e in the data stream do
    m ← max{m, f({e})}
    O ← {(1 + ε)^i | m ≤ (1 + ε)^i ≤ 2 · k · m}
    run in parallel SieveStreamOPT with each OPT in O
return argmax_{S_v : v ∈ O} f(S_v)
```

RandomStream , assume α is known

Algorithm 4: RandomStream for submodular maximization

Input: V as data stream, f a non-negative submodular function, k the cardinality constraint, ϵ a parameter

Output: a set $S \subseteq V$

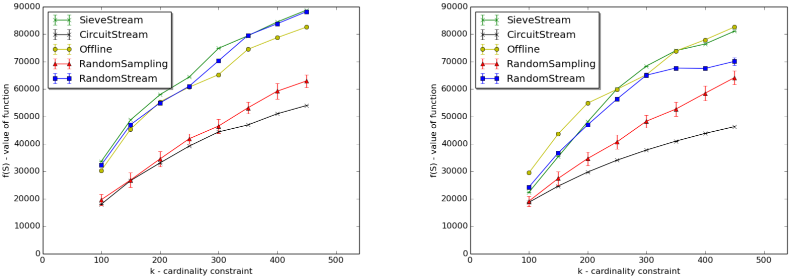
```
B ← ∅, S ← ∅
for each e in the data stream do
    if |S| < k and Δ(e|S) > α then
        B ← B + e
    if |B| > k/ε then
        e ← uniformly random from B
        B ← B − e, S ← S + e
        for all e′ ∈ B s.t. Δ(e′|S) ≤ α do
            B ← B − e′
S′ ← offline algorithm on B
return arg max_{A∈{S,S′}} f(A)
```

RandomStream cont.

α/OPT is unknown

- In RandomStream , when $\alpha \approx \frac{\text{OPT}}{k}$, then algorithm gives $\frac{1-\epsilon}{2+\epsilon}$-approximation.
- Again we can guess OPT in parallel as we did in SieveStream .

experiment



(a) Shuffled edges (b) Edges grouped by vertices

Figure : Streaming Algorithms on FACEBOOK; ϵ is set to be 0.2 for both SIEVESTREAM and RANDOMSTREAM ; γ is set to be 1.0 for CIRCUITSTREAM .

Summary of state of the art

constraint	monotone	non-negative
cardinality	$\frac{1-\epsilon}{2}$ [1]	$\frac{1-\epsilon}{2+e}$ [6], R
matroid	$1/4$ [5]	$\frac{1-\epsilon}{4+e}$ [6], R
matching	$4/31$ [5]	$\frac{1-\epsilon}{12+\epsilon}$ [6], R
intersection of p matroids	$\frac{1}{4p}$ [5]	$\frac{(1-\epsilon)(p-1)}{5p^2-4p+\epsilon}$ [6], R
p-matchoid	$\frac{1}{4p}$ [6]	$\frac{(1-\epsilon)(2-o(1))}{(8+e)p}$ [6], R

Table : Best known approximation bounds for submodular maximization in streaming model. Bounds for randomized algorithms that hold in expectation are marked (R).

The model

Crash Introduction to MapReduce

- the data is represented as $\langle$key, value$\rangle$ pairs that are distributed across m machines
- a computation in this model proceeds in rounds. In each round, there will be two phases.
- Map phase: each pair $\langle$key, value$\rangle$ is mapped by a user-defined hash function to $\langle$hash(key), value$\rangle$, all pairs are then shuffled and sent to different machines
- Reduce phase: each machine performs computation on the pairs it received as the output or the input of the next round

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- Reduce phase: each machine performs computation on the pairs it received as the output or the input of the next round

If you do not know MapReduce model ...

Think of it as a group of machines with one machine as the coordinator/center node.

GreeDi-based algorithms

framework of GreeDi-based algorithms

m - the number of machines; $C \in \mathbb{Z}^+$ is an parameter; k - the cardinality constraint. The algorithm goes as follows:

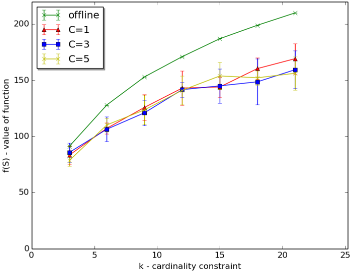
- Randomly assign each v to C out of m machines, we obtain subsets $V_1, \ldots, V_m$
- Let ALG be an offline algorithm, k' be a cardinality constraint. Run ALG on each V_i with constraint k', we obtains $U_1, U_2, \ldots, U_m$ as results.
- Let $U = \cup_i S_i$, run ALG on U with parameter k, we obtain S as the result. Also run ALG on $U_1, \ldots, U_m$ with parameter k to obtain $S_1, S_2, \ldots, S_m$.
- Return the best solution among S, $S_1, \ldots, S_m$.

Some theories about the GREEDI-Based Algorithms

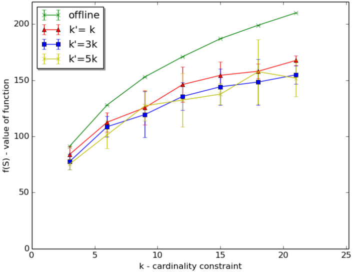
some theories (informal)

- use the standard greedy algorithm as ALG, $k' = k$, $C = 1$, the GREEDI-Based algorithm gives $\frac{1-e^{-1}}{2}$-approximation.
- increasing k' or C would **slightly** increase the approximation ratio (in worse case!), but not too much

experiment



(a) Different multiplicity C; set $k' = k$; number of machines is 20.



(b) Different k'; C is set to be 1; number of machines is set to be 20.

Figure : GREEDI-based Algorithms on ACCIDENTS dataset.

Summary of state of the art

constraint	rounds	approx.	reference
cardinality	$O(\frac{\log n}{\epsilon})$	$1 - e^{-1} - \epsilon$	[11]
	2	0.545	[15]
	$O(1/\epsilon)$	$1 - e^{-1} - \epsilon$	[2]
matroid	$O(\frac{\log n}{\epsilon})$	$1/2 - \epsilon$	[11]
	2	$1/4$	[7]
	$O(1/\epsilon)$	$1 - e^{-1} - \epsilon$	[2]
p-system	$O(\frac{\log n}{\epsilon})$	$\frac{1}{p+1} - \epsilon$	[11]
	2	$\frac{1}{2(p+1)}$	[7]
	$O(1/\epsilon)$	$\frac{1}{p+1} - \epsilon$	[2]

Table : Best known algorithms for monotone submodular maximization in the MapReduce model. All algorithms are randomized.

Question? Thank you!

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