SOLUTIONS FOR HOMEWORK 2

Preliminaries

Master Theorem. Use the definition from our textbook. Another more generic form of Master theorem can be found at `http://en.wikipedia.org/wiki/Master_theorem`.

n^ϵ **vs** $\log n$**.**

Claim 0.1. *For any* $\epsilon > 0$*, we have* $\log n = o(n^\epsilon) = O(n^\epsilon)$

Proof. By L'Hopital's rule,

$$\begin{aligned}\lim_{n\to\infty} \frac{\log n}{n^\epsilon} &= \lim_{n\to\infty} \frac{(\log n)'}{(n^\epsilon)'} \\ &= \lim_{n\to\infty} \frac{1/(n\ln 2)}{\epsilon n^{\epsilon-1}} \\ &= \lim_{n\to\infty} \frac{1}{\epsilon n^\epsilon \ln 2} \\ &= 0.\end{aligned}$$

Hence $\log n = o(n^\epsilon) = O(n^\epsilon)$. □

Notations. We write $\log_2 n$ as $\log n$, $\log_{10} n$ as $\lg n$, and $\log_e n$ as $\ln n$.

Question 1

Claim 0.2. $T(n) = \Theta(n^2 \log n)$.

Proof. $b = 2, a = 4$ and $f(n) = n^2$. Note $f(n) = n^2 = \Theta(n^2) = \Theta(n^{\log_2 4})$, therefore $T(n) = \Theta(n^{\log_b a} \log n) = \Theta(n^2 \log n)$ by case 2 of Master theorem. □

Question 2

Claim 0.3. $T(n) = \Theta(n^2)$.

Proof. $a = 16, b = 4$ and $f(n) = n$. Take $\epsilon = 1$ we have $f(n) = n = O(n^1) = O(n^{\log_4 16 - 1}) = O(n^{\log_b a - \epsilon})$, therefore $T(n) = \Theta(n^{\log_b a}) = \Theta(n^2)$ by case 1 of Master theorem. □

Question 3

Claim 0.4. $T(n) = \Theta(n^2 \log n)$.

Proof. $a = 6, b = 3$ and $f(n) = n^2 \log n$. Let $\epsilon = 2 - \log_3 6$, as $\log_3 6 < \log_3 3^2 = 2$, it is clear that $\epsilon > 0$. Hence $f(n) = \Omega(n^2) = \Omega(n^{\log_b a + \epsilon})$.

Furthermore, we have regularity condition: $af(n/b) = 6f(n/3) = 6(\frac{n}{3})^2 \log \frac{n}{3} \leq \frac{6}{9} \cdot n \log n \leq \frac{2}{3} f(n)$, let $c = \frac{2}{3}$, we have $0 < c < 1$.

Therefore $T(n) = \Theta(f(n)) = \Theta(n^2 \log n)$ by case 3 of Master theorem.

□

Question 4

Claim 0.5. *$T(n) = \Theta(n \log n)$.*

Proof. $a = 3, b = 4$ and $f(n) = n \log n$. Let $\epsilon = 1 - \log_4 3$, as $\log_4 3 < 1$, it is clear that $\epsilon > 0$. Hence $f(n) = n \log n = \Omega(n) = \Omega(n^{\log_b a + \epsilon})$.

Furthermore, we have regularity condition: $af(n/b) = 3f(n/4) = 3\frac{n}{4} \log \frac{n}{4} \leq \frac{3}{4} \cdot n \log n \leq \frac{3}{4} f(n)$, let $c = \frac{3}{4}$, we have $0 < c < 1$.

Therefore $T(n) = \Theta(f(n)) = \Theta(n \log n)$ by case 3 of Master theorem. □

Question 5

Claim 0.6. *$T(n) = \Theta(n^2)$.*

Proof. $a = 4, b = 2$ and $f(n) = n \log n$. Take $\epsilon = 0.5$ we have $f(n) = n \log n = O(n^{1.5}) = O(n^{\log_2 4 - 0.5}) = O(n^{\log_b a - \epsilon})$, therefore $T(n) = \Theta(n^{\log_b a}) = \Theta(n^2)$ by case 1 of Master theorem. □

Question 6

Claim 0.7. *$T(n) = \Theta(n \log^2 n)$.*

Proof. $a = 2, b = 2$ and $f(n) = n \log n$. It can be easily verified that this question can not be directly solved by applying the Master theorem in our textbook, however the Generic Form of Master Theorem does give a solution. We use substitution method here to directly solve this question.

Without loss of generality, let us assume $n = 2^k$ for some integer k.

$$\begin{aligned}
T(n) &= 2T(\frac{n}{2}) + n \log n \\
&= 2\left(2T(\frac{n}{2^2}) + \frac{n}{2} \log \frac{n}{2}\right) + n \log n \\
&= 2^2 T(\frac{n}{2^2}) + n \log \frac{n}{2} + n \log n \\
&= \cdots && \text{Further expand } T(\frac{n}{2^2}) \\
&= 2^k T(1) + n \sum_{i=1}^{k} \log \frac{n}{2^{k-i}} \\
&= nT(1) + n \sum_{i=1}^{k} \log 2^i \\
&= nT(1) + n \sum_{i=1}^{k} i \\
&= nT(1) + n \frac{k(k+1)}{2}
\end{aligned}$$

as $k = \log n$, we have $T(n) = \Theta(nk^2) = \Theta(n \log^2 n)$.

When $n \neq 2^k$. It will not affect our result when n does not have the form of 2^k. Let $k = \lceil \log n \rceil$, we have $T(2^{k-1}) \leq T(n) \leq T(2^k)$, which still gives us $T(n) = \Theta(n \log^2 n)$.

Remark. A graphical version of our solution is to use Recursion Tree which is covered in our textbook. You can also prove the result by using mathematical induction, however coming up with a correct guess in our case is non-trivial.

□

Question 7

Claim 0.8. $T(n) = \Theta(n^{\log_2 3})$

Proof. $a = 3, b = 2$ and $f(n) = n$. Since $\log_b a \approx 1.58$ we have $f(n) = O(n^{\log_b a - \epsilon})$ for $\epsilon = 0.1$. By case 1 of Master theorem, $T(n) = \Theta(n^{\log_b a}) = \Theta(n^{\log_2 3})$.

□

Question 8

Claim 0.9. $T(n) = 2^{n+1} - 1 = \Theta(2^n)$.

Proof.

$$\begin{aligned} T(n) &= T(n-1) + 2^n \\ &= T(1) + \sum_{i=2}^{n} 2^i \\ &= \sum_{i=0}^{n} 2^i \\ &= 2^{n+1} - 1. \end{aligned}$$

To see the last step, one can use the following technique:

$$\begin{aligned} \sum_{i=0}^{n} 2^i &= (2-1)(\sum_{i=0}^{n} 2^i) \\ &= \sum_{i=0}^{n} (2^{i+1} - 2^i) \\ &= (2^1 - 2^0) + (2^2 - 2^1) + \cdots + (2^i - 2^{i-1}) + (2^{i+1} - 2^i) \\ &= 2^{i+1} - 2^0. \end{aligned}$$

□