Thermal Runaway in an Exothermic Reaction

The temperature above ambient $U(x,t)$ at position x and time t in a body undergoing an exothermic reaction satisfies the boundary value problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + Qe^{bU} \quad \text{for } x,t>0$$

Here k is the thermal diffusivity and Q, b control the rate at which the reaction is producing heat.

If it is assumed the solution of this equation either tends to infinity (thermal runaway) or reaches a steady state (equilibrium), then the existence of a steady state guarantees the temperature $U(x,t)$ stays finite. At a steady state U does not depend on t so the above equation becomes

$$kU'' + Qe^{bU} = 0$$

to be solved for $U(x)$. Multiplying by U' and integrating with respect to x gives

$$\frac{1}{2}kU'^2 + \frac{Q}{b}e^{bU} = C$$

where C is a constant. Find the solution of this first order separable equation.