Step-by-step calculation of Black Scholes model

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1 Analytic solution of Black Scholes model

Suppose the log price process under risk neutral probability $\mathbb{Q}$ satisfies

$$dX(t) = (r - \frac{\sigma^2}{2})\,dt + \sigma dW(t). \tag{1.1}$$

Theorem (Black-Scholes)**.** *The value of a European put option at time $t \in [0, T]$ is*

$$\begin{aligned} V(x,t) &= \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}\max(\kappa - S_0 e^{X(T)}, 0)|X(t) = x] \\ &= e^{-r(T-t)}\kappa F(-d_1) - S_0 e^x F(-d_2), \end{aligned} \tag{1.2}$$

where $F(\cdot)$ is the cumulative distribution function of standard normal random variable, and

$$d_1 = \frac{x - \log(\kappa/S_0) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \tag{1.3}$$

and

$$d_2 = \frac{x - \log(\kappa/S_0) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}. \tag{1.4}$$

Proof. One can directly compute the conditional expectation by integrating functions, see Appendix A. The other way is converting the Black Scholes PDE to a heat equation and derive the solution see section 3. □

2 Black Scholes PDE

By Feynman-Kac representation (see theorem 4.8.1 in [1]), the value $V(S,t)$ of a European put option satisfies

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, &\qquad \text{for } (S,t) \in \mathbb{R}_+ \times [0,T], \\ V(0,t) = \kappa e^{-r(T-t)}, \qquad \lim_{S\to\infty} V(S,t) = 0, &\qquad \text{(boundary conditions)} \\ V(S,T) = \max(\kappa - S, 0). &\qquad \text{(final conditions)} \end{aligned} \tag{2.1}$$

By the change of variables or directly using Feynman-Kac representation to the log price process, we obtain the PDE of $V(x,t)$. Let $x = \log(S/S_0)$. The partial derivatives expressed in terms of x are

$$\begin{aligned} \frac{\partial V}{\partial S} &= \frac{1}{S}\frac{\partial V}{\partial x}, \\ \frac{\partial^2 V}{\partial S^2} &= -\frac{1}{S^2}\frac{\partial V}{\partial x} + \frac{1}{S^2}\frac{\partial^2 V}{\partial x^2}. \end{aligned} \tag{2.2}$$

Then the value $V(x,t)$ satisfies

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + (r - \frac{\sigma^2}{2})\frac{\partial V}{\partial S} - rV = 0, &\qquad \text{for } (x,t) \in \mathbb{R} \times [0,T], \\ \lim_{x\to-\infty} V(x,t) = \kappa e^{-r(T-t)}, \qquad \lim_{x\to\infty} V(x,t) = 0, &\qquad \text{(boundary conditions)} \\ V(x,T) = \max(\kappa - S_0 e^x, 0). &\qquad \text{(final conditions)} \end{aligned} \tag{2.3}$$

3 Converting Black Scholes PDE to the heat equation

By change of the variable $\tau = \frac{\sigma^2}{2}(T-t)$, we obtain

$$\begin{aligned} \frac{\partial V}{\partial \tau} &= \frac{\partial^2 V}{\partial x^2} + (\frac{2r}{\sigma^2} - 1)\frac{\partial V}{\partial x} - \frac{2r}{\sigma^2}V, \qquad &&\text{for } (x,\tau) \in \mathbb{R} \times [0, \frac{\sigma^2}{2}T], \\ V(x,0) &= \max(\kappa - S_0 e^x, 0). &&\text{(initial conditions)} \end{aligned} \tag{3.1}$$

Suppose $V(x,\tau) = e^{\alpha x + \beta \tau} u(x,\tau) = \phi u$. The partial derivatives are

$$\begin{aligned} \frac{\partial V}{\partial \tau} &= \beta \phi u + \phi \frac{\partial u}{\partial \tau}, \\ \frac{\partial V}{\partial x} &= \alpha \phi u + \phi \frac{\partial u}{\partial x}, \\ \frac{\partial^2 V}{\partial x^2} &= \alpha^2 \phi u + 2\alpha\phi \frac{\partial u}{\partial x} + \phi \frac{\partial^2 u}{\partial x^2}. \end{aligned} \tag{3.2}$$

Substitute these expressions into the PDE (3.1), we obtain

$$\alpha = \frac{\sigma^2 - 2r}{2\sigma^2}, \qquad \beta = -(\frac{\sigma^2 + 2r}{2\sigma^2})^2. \tag{3.3}$$

Then the PDE (3.1) converts to

$$\begin{aligned} \frac{\partial u}{\partial \tau} &= \frac{\partial^2 u}{\partial x^2}, \qquad &&\text{for } (x,\tau) \in \mathbb{R} \times [0, \frac{\sigma^2}{2}T], \\ u(x,0) &= e^{\alpha x} \max(\kappa - S_0 e^x, 0). &&\text{(initial condition)} \end{aligned} \tag{3.4}$$

The solution to the heat equation (3.4) is

$$u(x,\tau) = \frac{1}{\sqrt{2\pi\tau}} \int_{\mathbb{R}} u(y,0) e^{-\frac{(x-y)^2}{4\tau}}\, dy. \tag{3.5}$$

A Appendix: Proof of Black Scholes formula

Proof. The value of European put option is

$$V(x,t) = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)} \max(\kappa - S_0 e^{X(T)}, 0) | X(t) = x]. \tag{A.1}$$

We know the log price $X(t) = \log(\frac{S(t)}{S_0})$ under risk neutral measure $\mathbb{Q}$ is

$$X(t) = (r - \frac{\sigma^2}{2})t + \sigma W(t), \tag{A.2}$$

$$X(T) = (r - \frac{\sigma^2}{2})T + \sigma W(T). \tag{A.3}$$

Since Wiener process has stationary increment, then

$$X(T) = X(t) + (r - \frac{\sigma^2}{2})(T-t) + \sigma W(T-t). \tag{A.4}$$

Under the condition of $X(t) = x$, the random variable $X(T)$ has normal distribution with mean $x + (r - \frac{\sigma^2}{2})(T-t)$ and variance $\sigma\sqrt{T-t}$. So $X(T)$ has probability density function

$$f_{X(T)|X(t)=x}(y) = \frac{1}{\sigma\sqrt{2\pi(T-t)}} \exp\Big(- \frac{\big(y - x - (r - \frac{\sigma^2}{2})(T-t)\big)^2}{2\sigma^2(T-t)} \Big). \tag{A.5}$$

Therefore, the value of European put option (A.1) becomes

$$
\begin{aligned}
V(x,t) &= e^{-r(T-t)} \int_{-\infty}^{\log(\kappa/S_0)} (\kappa - S_0 e^y) f_{X(T)|X(t)=x}(y)\, dy \\
&= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \Big(\kappa \int_{-\infty}^{\log(\kappa/S_0)} \exp\big(- \frac{\big(y - x - (r - \frac{\sigma^2}{2})(T-t)\big)^2}{2\sigma^2(T-t)} \big)\, dy \\
&\qquad - S_0 \int_{-\infty}^{\log(\kappa/S_0)} \exp\big(y - \frac{\big(y - x - (r - \frac{\sigma^2}{2})(T-t)\big)^2}{2\sigma^2(T-t)} \big)\, dy \Big).
\end{aligned}
\tag{A.6}
$$

Let's compute the first integral, by change of variable $z = \frac{y-x-(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$, we obtain

$$
\begin{aligned}
\int_{-\infty}^{\log(\kappa/S_0)} \exp\big(- \frac{\big(y - x - (r - \frac{\sigma^2}{2})(T-t)\big)^2\big)}{2\sigma^2(T-t)} \big)\, dy &= \int_{-\infty}^{d_1} \exp(-\frac{z^2}{2}) \sigma\sqrt{T-t}\, dz \\
&= \sigma\sqrt{2\pi(T-t)} F(-d_1),
\end{aligned}
\tag{A.7}
$$

where

$$
-d_1 = \frac{\log(\kappa/S_0) - x - (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}. \tag{A.8}
$$

Then compute the second integral. By change of variable $w = \frac{y-x-(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$, we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\log(\kappa/S_0)} \exp\big(y - \frac{\big(y - x - (r - \frac{\sigma^2}{2})(T-t)\big)^2}{2\sigma^2(T-t)} \big)\, dy \\
&= \int_{-\infty}^{\log(\kappa/S_0)} \exp\big(- \frac{(y - x - (r + \frac{\sigma^2}{2})(T-t))^2}{2\sigma^2(T-t)} + x + (r + \frac{\sigma^2}{2})(T-t) - \frac{\sigma^2}{2}(T-t) \big)\, dy \\
&= e^{r(T-t)} e^x \int_{-\infty}^{d_2} \exp(-\frac{w^2}{2}) \sigma\sqrt{T-t}\, dw \\
&= e^{r(T-t)} e^x \sigma\sqrt{2\pi(T-t)} F(-d_2),
\end{aligned}
\tag{A.9}
$$

where

$$
-d_2 = \frac{\log(\kappa/S_0) - x - (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}. \tag{A.10}
$$

Therefore, the value of European put (A.6) becomes

$$
V(x,t) = e^{-r(T-t)} \kappa F(-d_1) - S_0 e^x F(-d_2). \tag{A.11}
$$

□

References

[1] A. Etheridge, *A course in Financial calculus*, Cambridge University Press, Cambridge, 2002.