EEE 51 Assignment 10 Solution

2nd Semester SY 2017-2018

1. **BJT common-collector amplifier.** Given the amplifier below with $I_{C,Q1} = 50mA$, $V_A = 200V$, $\beta = 150$, $V_{CC} = 5V$, $R_E = R_S = 100\Omega$, and $C_1 = 10pF$, solve for the following:

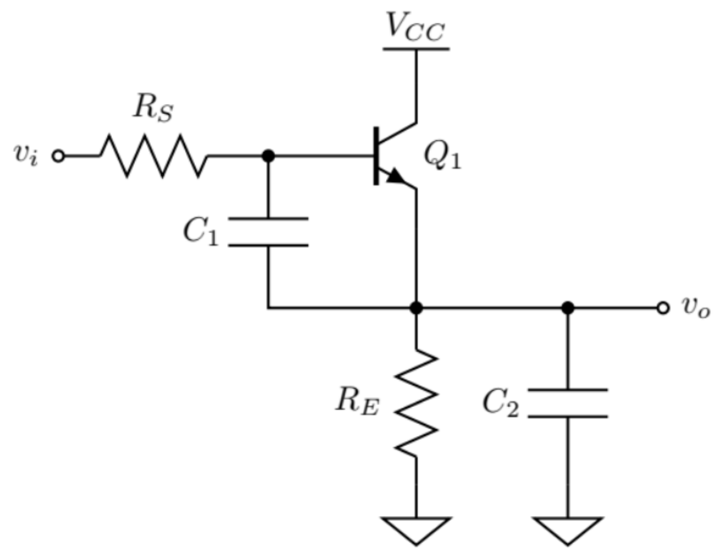

Figure 1

(a) Draw the small-signal model and redraw the circuit for feedback. Find the feedback factor F. [2 pts]

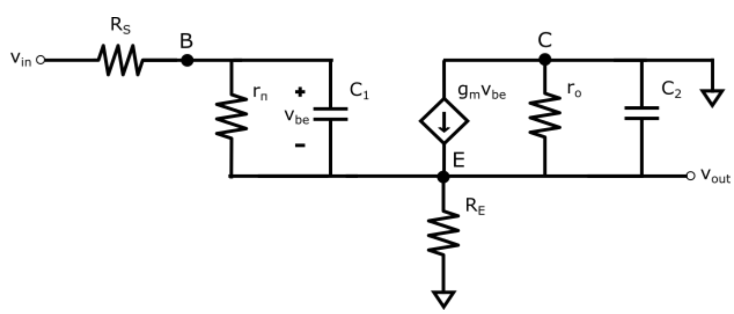

Figure 2: Small-signal equivalent circuit.

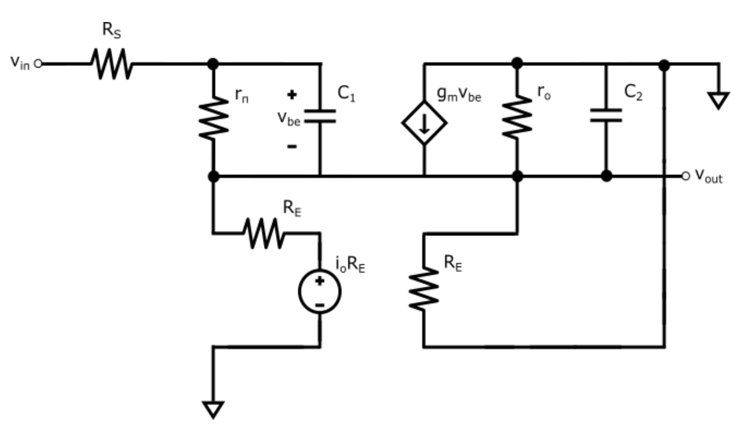

Figure 3: Small-signal equivalent circuit for feedback.

$$V_{fb} = i_o R_E \tag{1}$$

$$F = \frac{V_{fb}}{i_o} = R_E \tag{2}$$

$$\boxed{F = R_E = 100\Omega \text{ [2 pts]}}$$

(b) Find the value of C_2 such that the phase margin is 60°. [2 pts]

Starting by obtaining the small-signal parameters g_m, r_o and r_π,

$$g_m = \frac{I_{C,Q}}{V_T} = 1.92S \tag{3}$$

$$r_o = \frac{V_A}{I_{C,Q}} = 4k\Omega \tag{4}$$

$$r_\pi = \frac{\beta V_T}{I_{C,Q}} = 78\Omega \tag{5}$$

Computing for the closed loop gain transconductance, we can express $R_A = r_o \parallel 1/sC_1$, $R_B = r_o \parallel 1/sC_2$ and $R_C = R_S + R_E$ temporarily. Solving for i_o,

$$i_o = g_m v_{be} \frac{R_B}{R_B + R_E} = g_m(v_i) \frac{R_A}{R_A + R_C} \frac{R_B}{R_B + R_E} \tag{6}$$

$$G_{m,CL} = \frac{i_o}{v_i} = g_m \frac{R_A}{R_A + R_C} \frac{R_B}{R_B + R_E} \tag{7}$$

Substituting the values for R_A, R_B, and R_C and simplifying the expression,

$$G_{m,CL} = g_m \frac{\frac{r_\pi}{r_\pi + R_E + R_S}}{1 + sC_1 \frac{r_\pi(R_E+R_S)}{r_\pi + R_E + R_S}} \frac{\frac{r_o}{r_o + R_E}}{1 + sC_2 \frac{r_o R_E}{r_o + R_E}} \tag{8}$$

Obtaining the expressions for the poles,

$$\omega_{p1} = \frac{r_\pi + R_E + R_S}{r_\pi(R_E + R_S)(C_1)} = 1.782 \times 10^9 \tag{9}$$

$$\omega_{p2} = \frac{r_o + R_E}{r_o R_E (C_2)} = \frac{10.25 \times 10^{-3}}{C_2} \tag{10}$$

Solving for T_0,

$$T_0 = G_{m,CL}(s = 0) = g_m \frac{r_\pi}{r_\pi + R_E + R_S} \frac{r_o}{r_o + R_E} = 0.525566 \tag{11}$$

For the phase margin to be equal to 60^o,

$$180^o - tan^{-1}\left(\frac{\omega_u}{\omega_{p1}}\right) - tan^{-1}\left(\frac{\omega_u}{\omega_{p2}}\right) = 60^o \tag{12}$$

where $\omega_u = KT_0\omega_{p1}$. Substituting ω_u with $K = 1$ and solving for the value of ω_{p2},

$$tan^{-1}\left(\frac{T_0\omega_{p1}}{\omega_{p2}}\right) = 120^o - tan^{-1}(T_0) \tag{13}$$

$$\omega_{p2} = \frac{T_0\omega_{p1}}{tan(120^o - tan^{-1}(T_0))} \tag{14}$$

Solving for the value of K when $\omega_u = \omega_{p2}$,

$$K = \frac{1}{tan(120^o - tan^{-1}(T_0))} = -0.04 \tag{15}$$

Obtaining the value of C_2 using $K = -0.04$,

$$tan^{-1}\left(\frac{KT_0\omega_{p1}}{\omega_{p2}}\right) = 120^o - tan^{-1}(KT_0) \tag{16}$$

$$\omega_{p2} = \frac{KT_0\omega_{p1}}{tan(120^o - tan^{-1}(KT_0))} = 22.55 \times 10^6 = \frac{0.01025}{C_2} \tag{17}$$

Solving for C_2,

$$C_2 = 454.37pF \tag{18}$$

$$\boxed{C_2 = 454.37pF \text{ [2 pts]}}$$

(c) Find the value of C_2 such that the phase margin is 45°. [2 pts]

For the phase margin to be equal to 45^o,

$$180^o - tan^{-1}\left(\frac{\omega_u}{\omega_{p1}}\right) - tan^{-1}\left(\frac{\omega_u}{\omega_{p2}}\right) = 45^o \tag{19}$$

where $\omega_u = T_0\omega_{p1}$. Following the same process as (b), the solution is as follows:

$$tan^{-1}\left(\frac{T_0\omega_{p1}}{\omega_{p2}}\right) = 135^o - tan^{-1}(T_0) \tag{20}$$

$$\omega_{p2} = \frac{T_0\omega_{p1}}{tan(135^o - tan^{-1}(T_0))} \tag{21}$$

$$K = \frac{1}{tan(135^o - tan^{-1}(T_0))} = -0.311 \tag{22}$$

$$tan^{-1}\left(\frac{KT_0\omega_{p1}}{\omega_{p2}}\right) = 135^o - tan^{-1}(KT_0) \tag{23}$$

$$\omega_{p2} = \frac{KT_0\omega_{p1}}{tan(135^o - tan^{-1}(KT_0))} = 405.102 \times 10^6 = \frac{0.01025}{C_2} \tag{24}$$

Solving for C_2,

$$C_2 = 25.3pF \tag{25}$$

$$\boxed{C_2 = 25.3pF \text{ [2 pts]}}$$

(d) For a phase margin of 45°, find the loop gain $T(s)$ and closed loop gain $A_{CL}(s) = \frac{v_o}{v_i}$. What are the values of T_0 and A_{v0}? [2 pts]

To obtain open loop gain $T(s)$ with a phase margin of 45°, we use the expression,

$$T(s) = G_m F \tag{26}$$

Solving for G_m, expressions for v_{be} and i_o are obtained first. $R_A = r_\pi \parallel 1/sC_1$ is used for v_be.

$$v_{be} = \frac{R_A}{R_A + R_S} v_i \tag{27}$$

$$i_o = g_m v_{be} \tag{28}$$

Substituting v_{be} into i_o and solving for $G_m = i_o/v_i$,

$$G_m = g_m \frac{R_A}{R_S + R_A} \tag{29}$$

Substituting this to the expression for $T(s)$ along with $F = R_E$ and R_A,

$$T(s) = g_m R_E \frac{\frac{r_\pi}{r_\pi + R_S}}{1 + sC_1 \frac{r_\pi R_S}{r_\pi + R_S}} \tag{30}$$

$$T_0 = g_m R_E \frac{r_\pi}{r_\pi + R_S} = 84.135 \tag{31}$$

For the expression of $A_{CL}(s)$, we can use the closed loop circuit transconductance $G_{m,CL}$ obtained earlier given the topology of the small-signal model.

$$A_{CL}(s) = g_m \cdot \frac{\frac{r_\pi}{r_\pi + R_E + R_S}}{1 + sC_1 \frac{r_\pi (R_E + R_S)}{r_\pi + R_E + R_S}} \cdot \frac{\frac{r_o}{r_o + R_E}}{1 + sC_2 \frac{r_o R_E}{r_o + R_E}} \tag{32}$$

$$A_{v0} = g_m \cdot \frac{r_\pi}{r_\pi + R_E + R_S} \cdot \frac{r_o}{r_o + R_E} = 0.525566 \tag{33}$$

$$\boxed{\begin{gathered} T(s) = g_m R_E \frac{\frac{r_\pi}{r_\pi + R_S}}{1 + sC_1 \frac{r_\pi R_S}{r_\pi + R_S}},\ T_0 = 84.135 \text{ [1 pt]} \\ A_{CL}(s) = g_m \cdot \frac{\frac{r_\pi}{r_\pi + R_E + R_S}}{1 + sC_1 \frac{r_\pi (R_E + R_S)}{r_\pi + R_E + R_S}} \cdot \frac{\frac{r_o}{r_o + R_E}}{1 + sC_2 \frac{r_o R_E}{r_o + R_E}},\ A_{v0} = 0.525566 \text{ [1 pt]} \end{gathered}}$$

(e) Plot the magnitude and phase response of the circuit as a function of frequency. [2 pts]

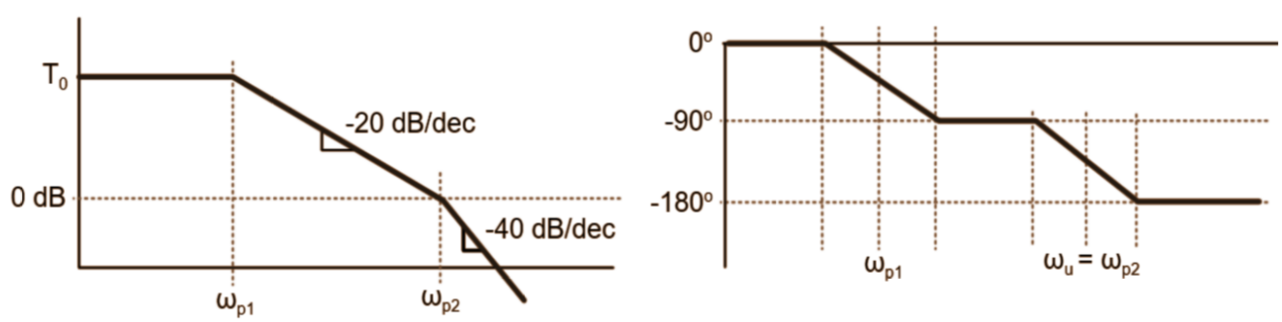

Figure 4: Magnitude and phase response.

2. **Unity-Gain Feedback.** In order to design a larger, multi-stage amplifier, the common BJT to be used for the stages is placed under a simple test to characterize some performance basics.

   The single NPN Q_1 under test is placed in common-emitter (as seen in Figure 5a) and is loaded at the input and output with capacitors C_i and C_L to simulate expected loading in between stages. The common-emitter is constructed with an ideal current source to bias it. Unity-gain feedback is achieved by using an ideal summing amplifier, which also takes care of the quiescent input current to match the bias created by the ideal current source. This setup is as shown in Figure 5b. For all intents and purposes there is no other output load than the load capacitor C_L and the input should be treated as ideal, trusting in this summing amplifier and not loading this system any further.

   Q_1 has known and finite values of V_A, V_T at the operating temperature, as well as β. The bias current I_C can be set and is chosen to be some starting value that ensures that Q_1 is in forward-active.

   For the following questions, express all values asked for in terms of I_C, V_A, V_T, β, and C_i. You may assume that parasitic capacitances are negligible for this setup when calculating for the values asked for.

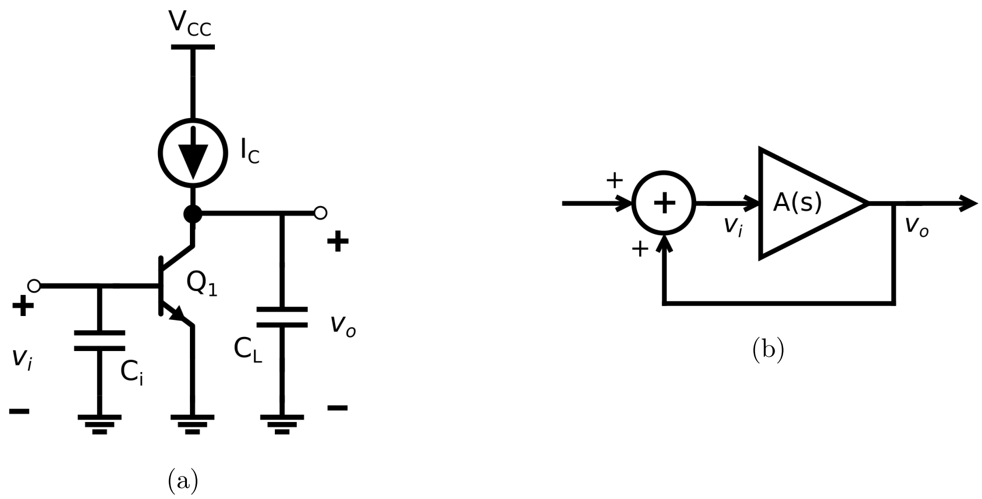

Figure 5

   (a) What is the gain of the amplifier at DC? [1 pt]

   The amplifier is part of a larger system, but what is being asked for is only its own gain and not that of the system. Therefore, all that is needed is A(s) at DC, which is when ω is zero.

   At DC, the capacitances don't contribute anything and the amplifier is simply a common-emitter with no effect from the poles. With an ideal current source as its only load, its gain is ideal and equal to intrinsic gain.

$$\boxed{A_0 = a_0 = -g_m r_o = -\tfrac{V_A}{V_T} \text{ [1 pt]}}$$

   (b) This amplifier is intended to be loaded such that the phase margin is at 90°. C_i is already at the minimum and cannot be changed; C_L is already at the same capacitance as C_i but can be increased. What should the value of C_L be in order to meet this phase margin? Assume that the output pole created by C_L should have a frequency smaller than the input pole. [2 pts]

   The input pole is simply the input resistance of the CE, r_π, in parallel with C_i. That makes the pole frequency

$$\omega_i = \frac{1}{r_\pi C_i} \tag{34}$$

   which when converted to the terms asked for

$$\omega_i = \frac{I_C}{\beta V_T C_i} \tag{35}$$

Similarly the output pole is just the output resistance r_o in parallel with the load capacitance C_L.

$$\omega_o = \frac{1}{r_o C_L} = \frac{I_C}{V_A C_L} \tag{36}$$

We can then say that the relative location of ω_o is

$$\omega_o = \omega_i \frac{\beta V_T C_i}{V_A C_L} \tag{37}$$

$$\frac{\omega_o}{\omega_i} = \frac{\beta}{a_o} \frac{C_i}{C_L} \tag{38}$$

In order to reach the 90° PM, the phase lag of the open-loop gain has to be 90° when the magnitude of that gain reaches unity. However, a single pole will create a total phase lag of 90° by itself, and so that implies that the second pole must be after the gain reaches unity, else it would cause the phase lag to go lower. In Bode plot estimation, the phase contribution of the second pole would start a decade before the pole frequency. Therefore, the second pole should be one decade after the unity-gain frequency. This is illustrated in Figure 6.

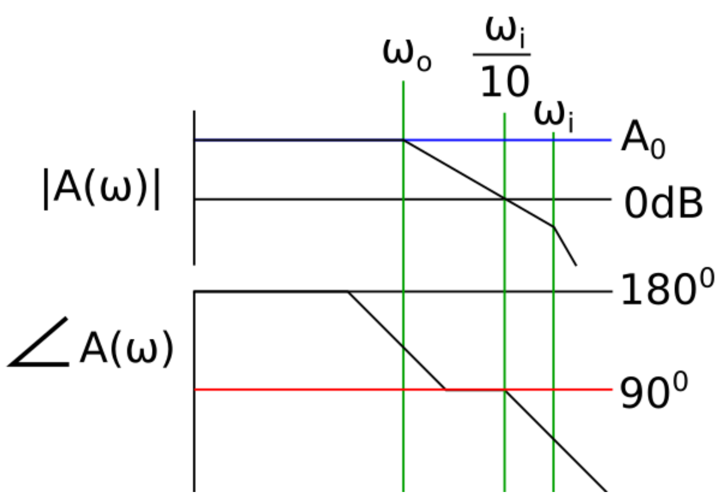

Figure 6: Bode Plot for expected Phase Margin

When does unity-gain frequency occur then? In the magnitude response, the first pole causes the magnitude to drop one decade per decade (20 dB/dec.) starting at the pole frequency, and initial value is the gain at DC. Therefore, unity gain is achieved at a frequency equivalent to the DC gain multiplied to the first pole frequency.

$$\omega_{0\,\mathrm{dB}} = \omega_o \times a_o \tag{39}$$

We know also that the second pole should be placed one decade after that.

$$\omega_{0\,\mathrm{dB}} = \frac{\omega_i}{10} \tag{40}$$

$$\frac{\omega_o}{\omega_i} = \frac{1}{10 a_o} \tag{41}$$

Combining it with (38),

$$\frac{\beta}{a_o}\frac{C_i}{C_L} = \frac{1}{10a_o} \tag{42}$$

$$\frac{\beta C_i}{C_L} = \frac{1}{10} \tag{43}$$

$$\boxed{C_L = 10\beta C_i \text{ [2 pts]}}$$

Which is a very interesting conclusion. Not only is the phase margin (and therefore stability) independent of bias current I_C, it is also independent of intrinsic gain. The only factor inherent to the BJT that should affect this outcome is the β current gain, which should make some sense as a difference in current flow between the input and output would determine how those capacitors are allowed to change in voltage at rates relative to each other.

(c) The bandwidth is determined by the location of the first pole with the smallest frequency, in order to approximate the $-3\,\text{dB}$ point. What is the bandwidth? [1 pt]

The first pole location should be the output pole, and already we know that

$$\omega_o = \frac{I_C}{V_A C_L} \tag{44}$$

Knowing the value of C_L,

$$\boxed{BW = \frac{I_C}{10\beta V_A C_i} \text{ [1 pt]}}$$

(d) In this setup, what is the trade-off between the gain-bandwidth product and power consumption? [2 pts]

Knowing the bandwidth, we can calculate GBP to be

$$GBP = A_0 \times BW = \frac{I_C}{10\beta V_T C_i} \tag{45}$$

$$\boxed{GBP = \frac{I_C}{10\beta V_T C_i} \text{ [1 pt]}}$$

> We can easily see then that the GBP scales linearly to the bias current I_C. Assuming that the voltage remains constant and just the CE is biased differently to accept more or less collector current at DC, that means that power scales linearly to the GBP. [1 pt]

(e) Since C_L is now at the proper value for a 90° phase margin to be achieved for some bias point generated by I_C, if I_C had to be changed later on, how would the phase margin change? [1 pt]

Knowing that

$$C_L = 10\beta C_i \tag{46}$$

fulfills the 90° PM and that it is independent of bias current,

> The phase margin will not change for a reasonable range of I_C around the initial value, so long as the BJT remains in forward-active. [1 pt]

(f) In constructing a multi-stage amplifier in a manner similar to this, would it make sense to follow this setup such that poles nearer to the output have a lower pole frequency than the ones at the input? Or is the opposite true, or does it not matter? Explain. [2 pts]

In an ideal case being modeled, it should be remembered that when talking about linear, time-invariant systems in a cascade the order does not matter. Whether one pole or another takes effect first in the system does not affect the output. If the ideal case is being considered and the system is being modeled as linear, then the arrangement of the poles do not matter whether they are closer to the output or not.

However, once non-idealities are considered, for example, noise being introduced at some point or non-linear distortion in the amplifier, then the effects of the poles change. If for example noise is introduced between the input and output pole (that is, this amplifying stage is itself the source of it), then the signal has been filtered by the input pole but the noise has not. Having the more dominant pole closer to the output means that the most filtering can occur near the output, after any noise or distortion from within this stage has been added. This allows the noise to be filtered about as much as the signal, and so the output is closer to being ideal. Therefore, lower-frequency poles should be placed closer to the output. [2 pts]

(g) Suppose the concept of a multi-stage amplifier now upsets you, and you attempt instead to construct a Cascode amplifier in an attempt to bypass the problem of having loading between stages. What are the trade-offs, frequency response-wise, in doing so? [1 pt]

We know that roughly, a Cascode amplifier has a transconductance of g_{m1} but a scaled output resistance of $r_{o1}\left(g_{m2}r_{o2}\right)$, creating a higher gain but also a closer output pole, essentially having the same GBP as a single-stage CE. The difference is, it achieves this high gain without introducing multiple stages, and so involves less poles. If we only consider output capacitances, then a two-stage CE-CE has about as much gain as a single CE-CB Cascode, but with one more pole. And the more poles are involved, the harder it is to maintain bandwidth as one pole must become dominant be set well before all the others (at a much lower pole frequency), this distance across frequency scaling with gain.

This difference is best seen between a single Cascode and a two-stage CE-CE, because if the single Cascode only has the one output pole, then stability is pretty much always assured with a fixed PM of 90°. The two-stage will always have to have one pole have a much lower frequency than the other.

However, once you consider the intrinsic capacitances in BJTs, then a normally negligible pole created by the junction capacitance of the CB BJT of the Cascode, or really any capacitance introduced at the node connecting the CE and CB BJTs, can become a much larger pole as that node sees high resistance towards either ground or the output. So not only is the output pole scaled by $g_{m2}r_{o2}$, so is this once-negligible pole. A Cascode has, when these non-idealities are taken into account, just as many poles as a multi-stage equivalent, and since the Cascode runs along a single branch, manipulating these poles such that stability is achieved can become harder than in a multi-stage, where each stage can be controlled relatively separately. [1 pt]

3. Given the circuit with the parameters below, solve for the following:

$V_{DD} = 5V$, I_{M8}=$5pA$, $k_{1,2} = 2.5\frac{pA}{V^2}$, $k_{3,4} = 3.75\frac{pA}{V^2}$, $k_{5,6,7,9} = 2.5\frac{nA}{V^2}$, $k_8 = 5\frac{pA}{V^2}$, $\lambda = 0.01V^{-1}$, $C_L = 5*10^{-21}F$, $C_{L2} = 500*10^{-24}F$, and $|V_{TH_{n|p}}| = 1V$

Assume that the compensating capacitances have no effect on the analysis of the first stage and is purely for the 2nd stage and that $C_M >> C_{L2}$

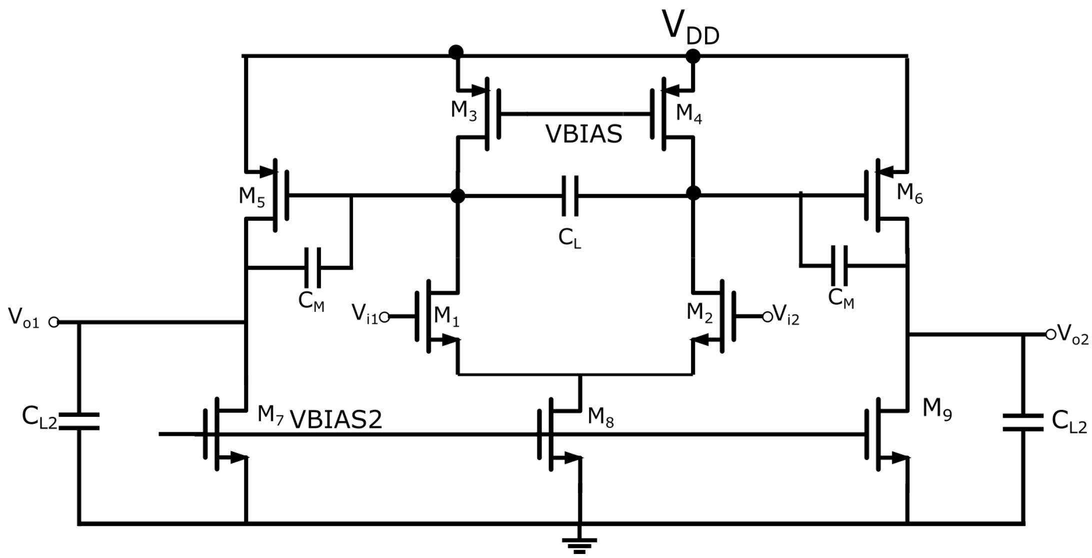

Figure 7

(a) Plot the magnitude and phase response of the amplifier. When $C_M = 0$ [2.5 pts]

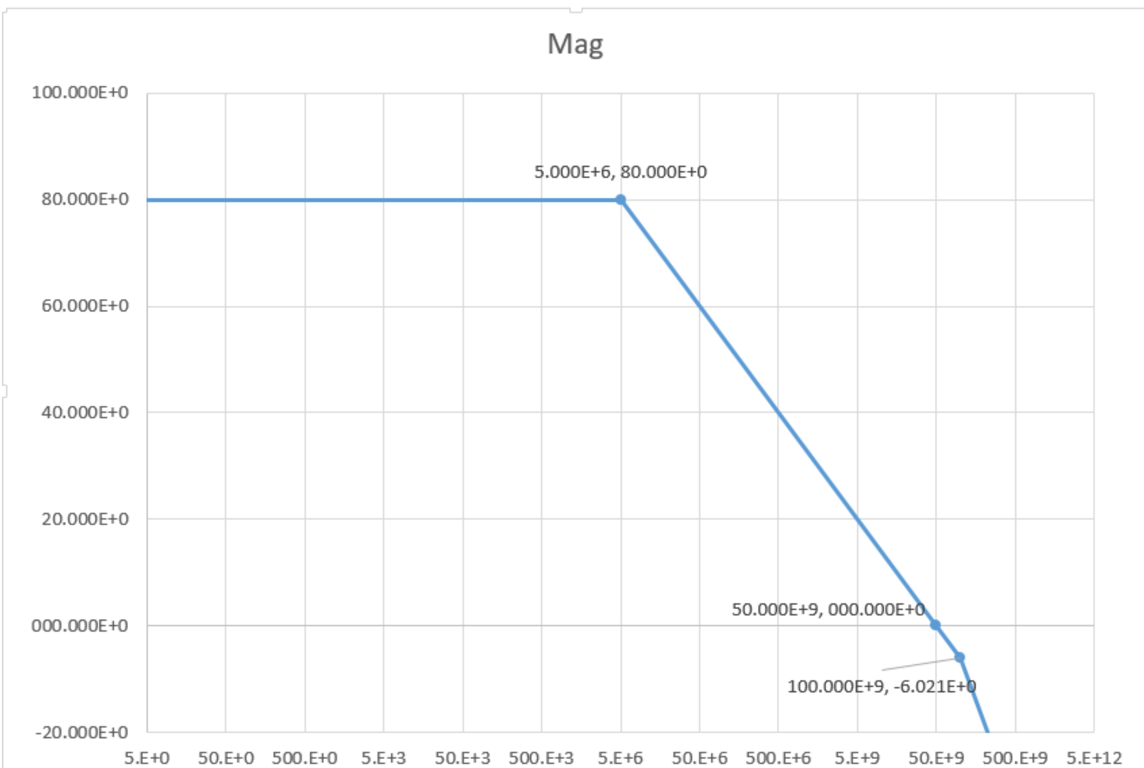

Figure 8

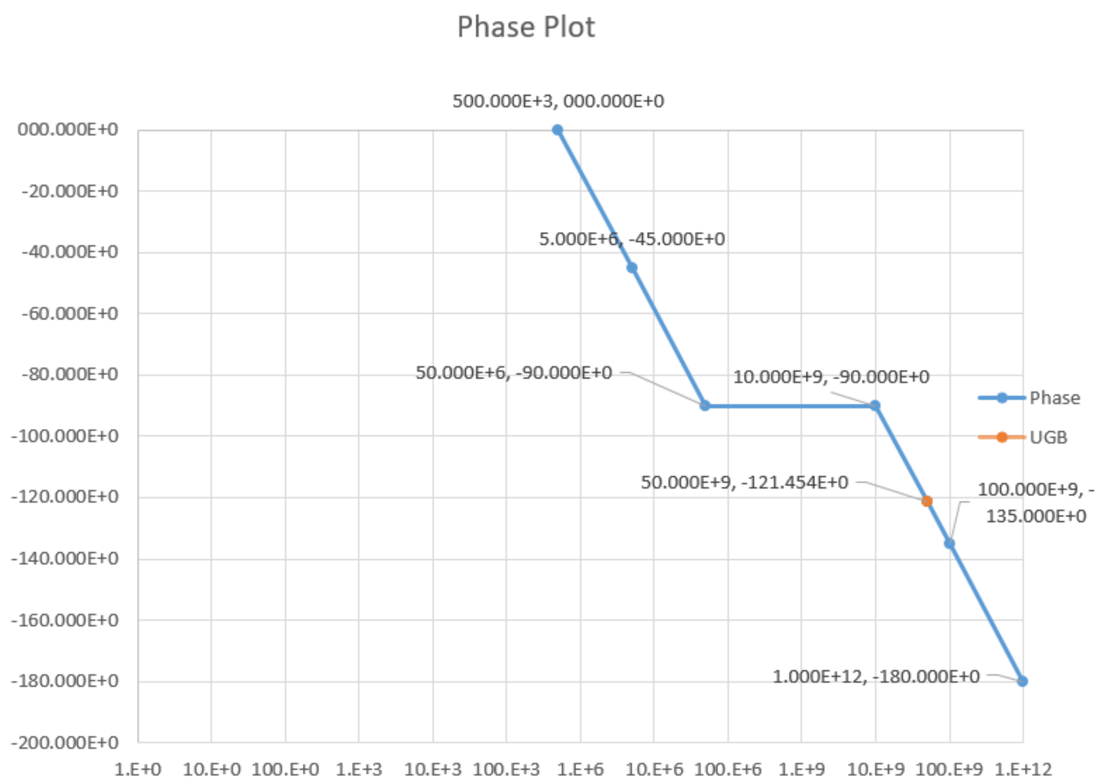

Figure 9

$$Av = \frac{Av_1}{1+\frac{s}{\omega_{p1}}} * \frac{Av_2}{1+\frac{s}{\omega_{p2}}} \tag{47}$$

$$Av_1 = -g_{m2}(r_{o2}//r_{o4}) = -g_{m2}R_{o1} = -100 \tag{48}$$

$$Av_2 = -g_{m6}(r_{o6}//r_{o9}) = -g_{m6}R_{o2} = -100 \tag{49}$$

$$\omega_{p1} = \frac{1}{2C_L R_{o1}} = 5MHz \tag{50}$$

$$\omega_{p2} = \frac{1}{C_{L2} R_{o2}} = 100GHz \tag{51}$$

(b) Solve for the poles, zero and $A_{total}(s)$ equations.[2.5 pts]
Assuming the addition of C_M has no effect on stage 1.

$$A_{stage1} = \frac{Av_1}{1+\frac{s}{\omega_{p1}}} \tag{52}$$

with the same gain and pole in equation A_{v1}, ω_{p1}.
Solving for the general equation for stage 2. $R_{od} = R_{o1}$

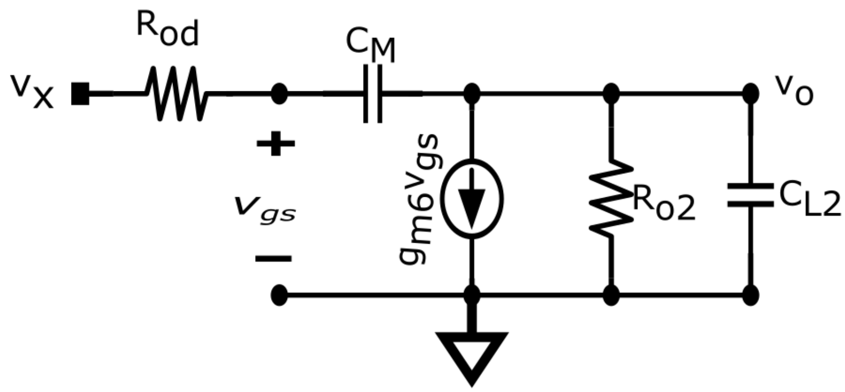

Figure 10

Using KCL at the gate.

$$(1) => \frac{v_{gs} - v_x}{R_{od}} + (vgs - v_o)SC_M = 0 \tag{53}$$

$$(1^*) => v_{gs} = \frac{v_o SC_M + \frac{v_x}{R_{od}}}{SC_M + \frac{1}{R_{od}}} \tag{54}$$

Using KCL at the drain.

$$(2) => v_o(SC_M + SC_{L2} + \frac{1}{R_{o2}}) = v_{gs}(SC_M - g_{m6}) \tag{55}$$

Combining (1*) and (2).

$$v_o(SC_M + SC_{L2} + \frac{1}{R_{o2}})(SC_M + \frac{1}{R_{od}}) = v_o SC_M(SC_M - g_{m6}) + \frac{v_x}{R_{od}}(SC_M - g_{m6}) \tag{56}$$

Simplifying

$$\frac{v_o}{v_x} = \frac{-g_{m6}(1 - \frac{SC_M}{g_{m6}})}{R_{od}[s^2 C_{L2}C_M + s(\frac{C_M}{R_{od}} + \frac{C_{L2}}{R_{od}} + \frac{C_M}{R_{o2}} + g_{m6}C_M) + \frac{1}{R_{o2}R_{od}}]} \tag{57}$$

Further simplification and using the assumption that $C_M >> C_{L2}$

$$A_{stage2} = \frac{-g_{m6}R_{o2}(1 - \frac{SC_M}{g_{m6}})}{s^2 C_{L2}C_M R_{o2}R_{od} + s(C_M(R_{o2} + R_{od}) + g_{m6}C_M R_{o2}R_{od}) + 1} \tag{58}$$

The denominator can be further simplified using the following equation format.

$$(1 + \frac{s}{\omega_{p2}})(1 + \frac{s}{\omega_{p3}}) = s^2(\frac{1}{\omega_{p2}\omega_{p3}}) + s(\frac{1}{\omega_{p2}} + \frac{1}{\omega_{p3}}) + 1 \tag{59}$$

if $\omega_{p2} = \frac{1}{C_M(R_{o2}+R_{od})}$

$$\frac{C_M(R_{o2} + R_{od})}{\omega_{p3}} = C_{L2}C_M R_{o2}R_{od} \tag{60}$$

therefore

$$\omega_{p3} = \frac{1}{C_{L2}(R_{o2}//R_{od})} \tag{61}$$

if $\omega_{p2} = \frac{1}{g_{m6}C_M R_{o2}R_{od}}$

$$\omega_{p3} = \frac{g_{m6}}{C_{L2}} \tag{62}$$

(c) Compute the necessary C_M to achieve a 45° phase margin at the w_p of the first stage. [2.5 pts].

In order to place the UGB at the pole of first stage and achieve a 45 degrees PM, we may move the dominant pole of the 2nd stage using the compensating capacitance.

$$\omega_{p2} = \frac{\omega_{p1}}{|A_{total}|} = \frac{5MHz}{10k} = 500 \tag{63}$$

We may then use either of the 2 equations for ω_{p2} to solve for C_M.

$$C_M = \frac{1}{\omega_{p2}(R_{o2} + R_{od})} = 99.9x10^{-18}F \tag{64}$$

or

$$C_M = \frac{1}{\omega_{p2} g_{m6} R_{o2} R_{od}} = 1x10^{-18}F \tag{65}$$

(d) Plot the new magnitude and phase response. [2.5 pts]

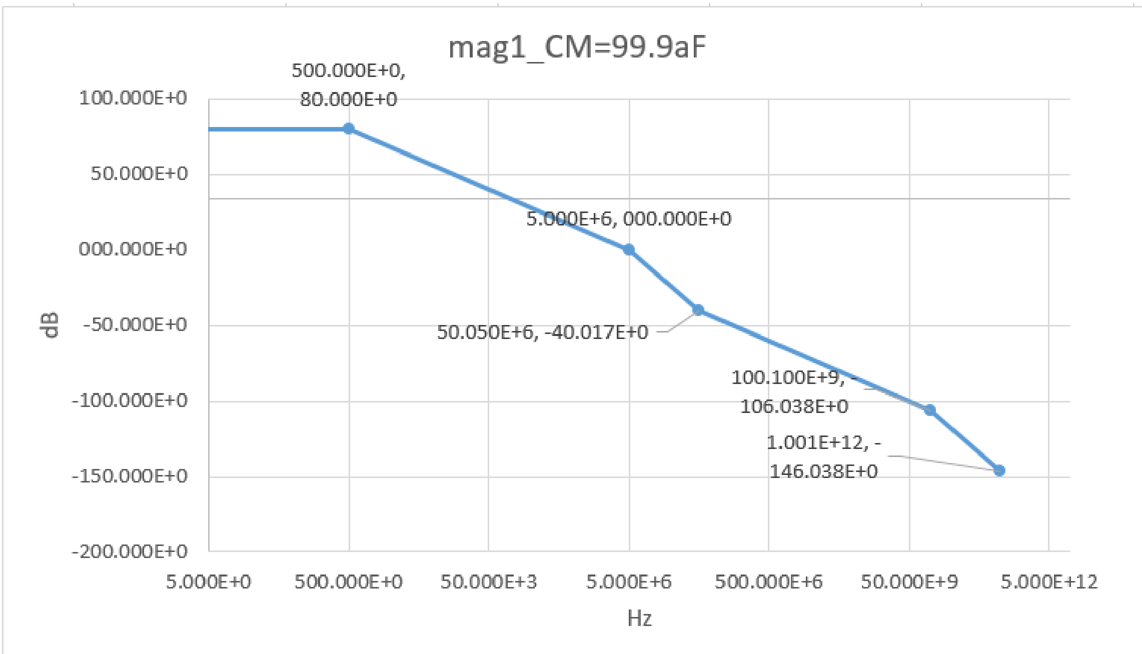

Figure 11

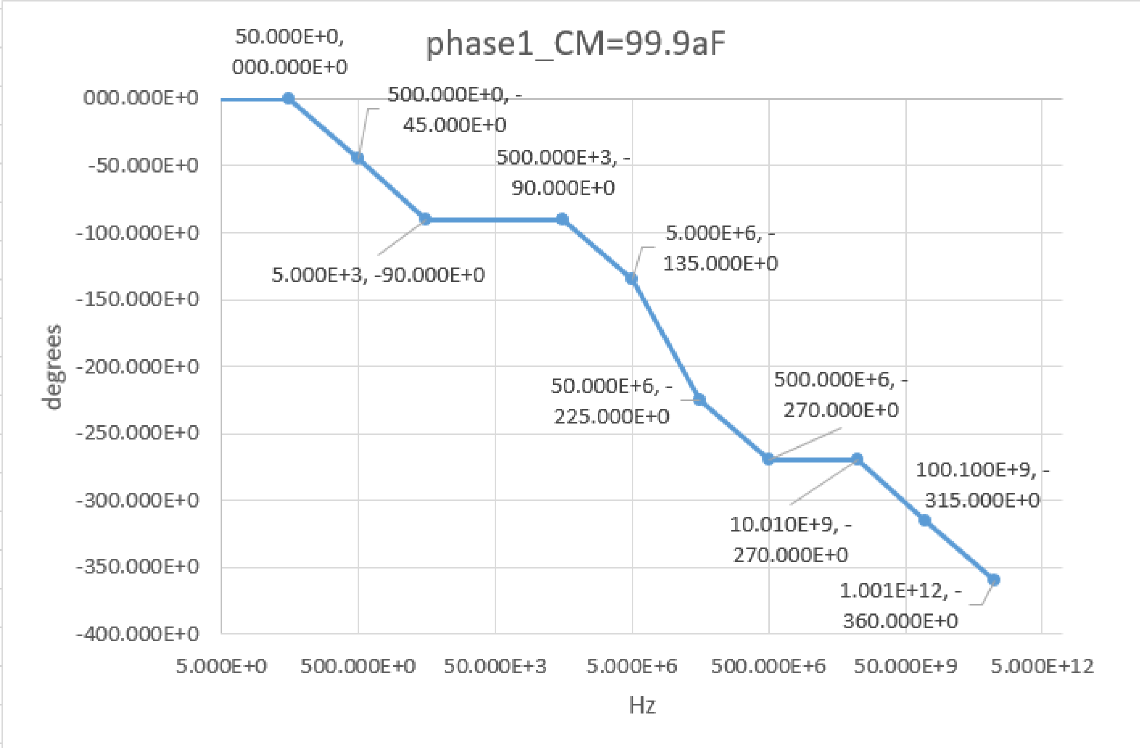

Figure 12

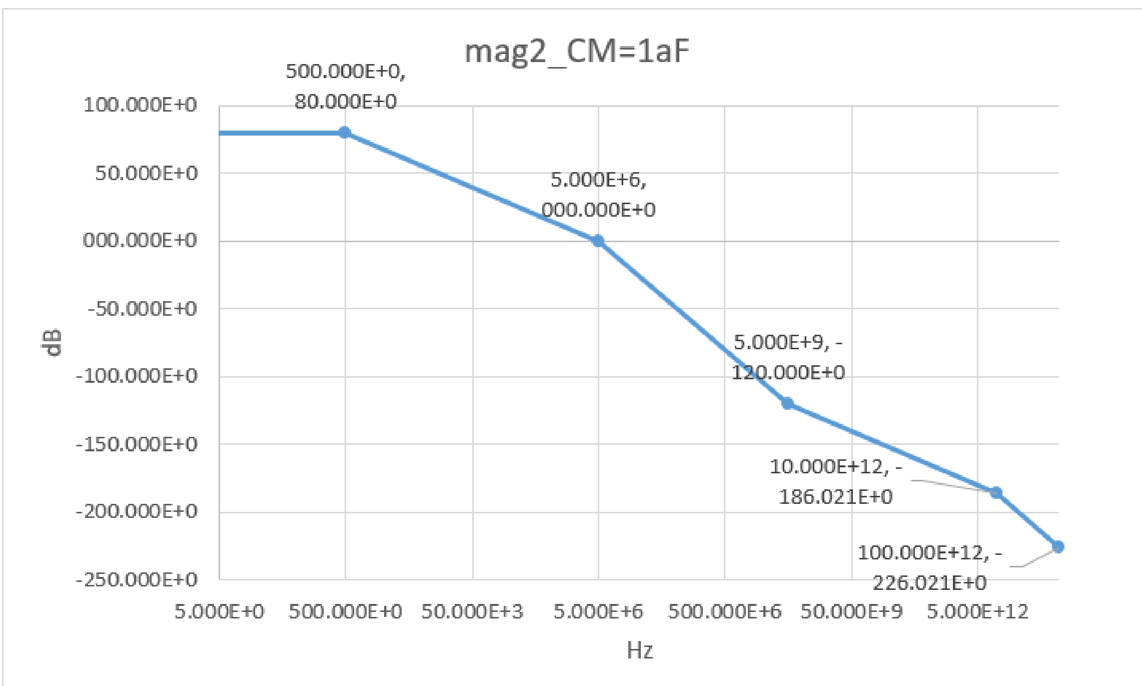

Figure 13

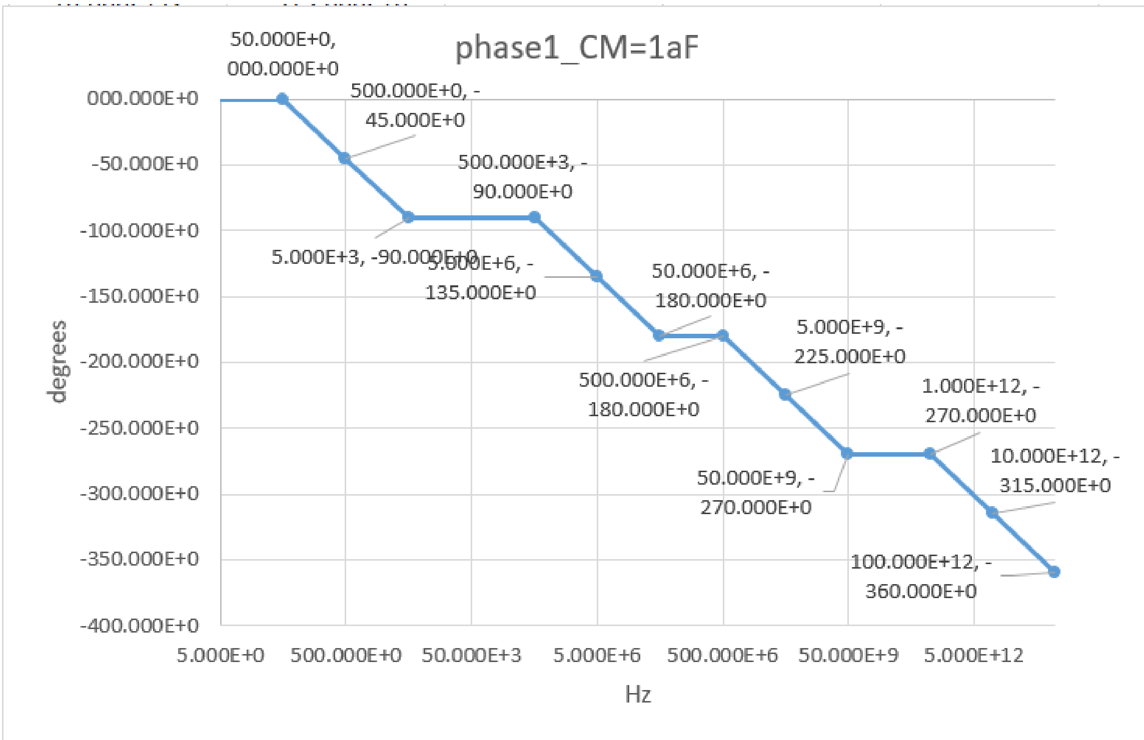
phase1_CM=1aF
50.000E+0, 000.000E+0
500.000E+0, -45.000E+0
500.000E+3, -90.000E+0
5.000E+3, -90.000E+0
5.000E+6, -135.000E+0
50.000E+6, -180.000E+0
500.000E+6, -180.000E+0
5.000E+9, -225.000E+0
50.000E+9, -270.000E+0
1.000E+12, -270.000E+0
10.000E+12, -315.000E+0
100.000E+12, -360.000E+0
000.000E+0
-50.000E+0
-100.000E+0
-150.000E+0
-200.000E+0
-250.000E+0
-300.000E+0
-350.000E+0
-400.000E+0
degrees
5.000E+0
500.000E+0
50.000E+3
5.000E+6
500.000E+6
50.000E+9
5.000E+12
Hz

Figure 14