EEE 51 Assignment 2 Answer Key

2nd Semester SY 2018-2019
Due: 5pm Tuesday, February 12, 2019 (Rm. 220)

Instructions: Write legibly. Show all solutions and state all assumptions. Write your full name, student number, and section at the upper-right corner of each page. Start each problem on a new sheet of paper. Box or encircle your final answer.

Answer sheets should be color coded according to your lecture section. The color scheme is as follows:

THQ – yellow
THU – white
WFX – pink

1. **Common Collector/Emitter Follower Sziklai Pair.**
Figure 1 shows a Sziklai pair used as an emitter follower. The capacitor C_C is just an AC coupling capacitor (open circuit at DC and shorted at AC, the value is irrelevant as of now). For both of the transistors Q_1 and Q_2, $|V_{be,on}| = 0.65\,\text{V}$, $|V_{ce,sat}| = 0.2\,\text{V}$, and $|V_A| \to \infty$. Moreover, $\beta_1 = 50$ for Q_1 and $\beta_2 = 100$ for Q_2. The transistors are biased such that the voltage at node B is $V_B = 1.85\,\text{V}$. Answer the following questions. **DO NOT ASSUME THAT $\mathbf{I_C = I_E}$**. Round all of your answers to one (1) decimal place only and use the appropriate unit of measure.

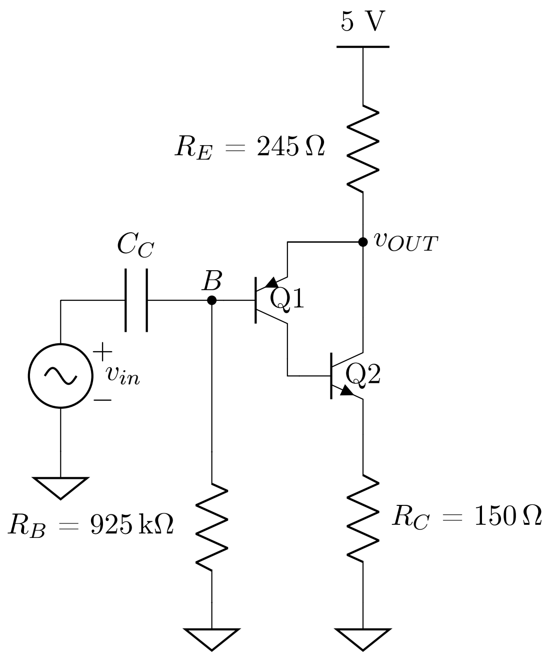


Figure 1: Emitter Follower Sziklai Pair

(a) Calculate the value of the base current of Q_1 I_{B1} and the emitter current of Q_2 I_{E2}. If I_{E2} is treated as the collector current of the transistor pair, what is the overall current gain β_{SZ}? (3 pts)

- For I_{B1}:
  Note that the coupling capacitor is open at DC.

$$\begin{aligned} I_{B1} &= \frac{V_B}{R_B} \\ &= \frac{1.85\,\mathrm{V}}{925\,\mathrm{k\Omega}} \\ &= \boxed{2\,\mathrm{\mu A}} \end{aligned}$$

- For I_{E2}:

$$\begin{aligned} I_{E2} &= (\beta_2 + 1)\,\beta_1 I_{B1} \\ &= (100 + 1)\,(50)\,(2\,\mathrm{\mu A}) \\ &= \boxed{10.1\,\mathrm{mA}} \end{aligned}$$

- For β_{SZ}:

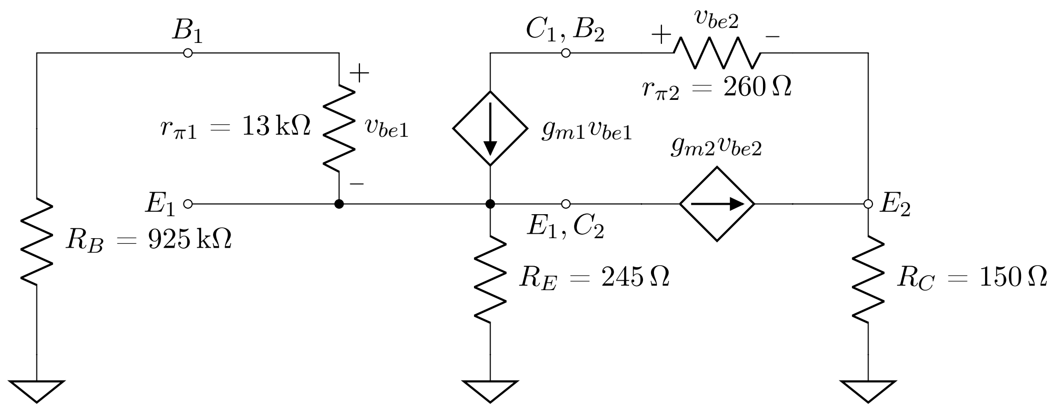
$$\begin{aligned} I_{C,SZ} &= \beta_{SZ} I_{B,SZ} \\ \beta_{SZ} &= \frac{I_{C,SZ}}{I_{B,SZ}} \\ &= \frac{I_{E2}}{I_{B1}} \\ &= \frac{10.1\,\mathrm{mA}}{2\,\mathrm{\mu A}} \\ &= \boxed{5050} \end{aligned}$$

(b) Determine the value of the DC voltage at the output node. (1 pt)

$$\begin{aligned} V_{OUT} &= (5\,\mathrm{V}) - I_{E,SZ} R_E \\ I_{E,SZ} &= I_{E1} + I_{C2} \\ &= (\beta_1 + 1)\,I_{B1} + \beta_1 \beta_2 I_{B1} \\ &= (\beta_{SZ} + 1)\,I_{B1} \\ &= (5050 + 1)\,(2\,\mathrm{\mu A}) \\ &= 10.102\,\mathrm{mA} \\ V_{OUT} &= (5\ \mathrm{V}) - I_{E,SZ} R_E \\ &= (5\ \mathrm{V}) - (10.102\,\mathrm{mA})\,(245\,\Omega) \\ &= 2.525\,01\,\mathrm{V} \\ &\approx \boxed{2.5\,\mathrm{V}} \end{aligned}$$

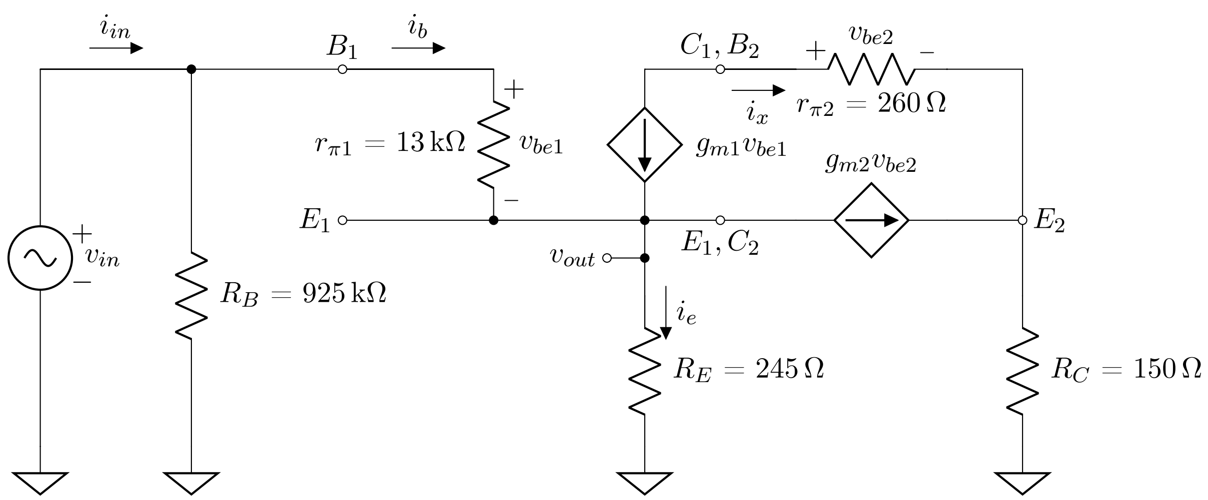
(c) Draw the small-signal model of the circuit. Label the transistor terminals, small-signal parameters, external resistors, and their values. (2 pts)

- Since $|V_A| \to \infty$, then $|r_o| \to \infty$ and is drawn as an open circuit. (Full points will still be given if the r_o's are drawn as long as the value indicated is infinity.)
- $g_{m1} \approx 3.8$ mS or mmhos or m$\frac{1}{\Omega}$; $g_{m2} \approx 384.6$ mS or mmhos or $\frac{1}{\Omega}$



(d) Find the values of the overall input resistance R_{IN}, transconductance G_M, gain A_V, and output resistance R_O of the whole emitter follower. You may draw additional schematics/diagrams to aid in your solutions. (4 pts)

- Input Resistance R_{IN}:
  We will use the following schematic to calculate R_{IN} and A_V



Perform KVL from the base to R_E to get the input resistance of the transistor:

$$\begin{aligned}
v_{in} &= i_b r_{\pi 1} + i_e R_e \\
i_e &= i_b + g_{m1} v_{be1} - g_{m2} v_{be2} \\
v_{be1} &= i_b r_{\pi 1} \\
v_{be2} &= i_x r_{\pi 2} \\
i_x &= -g_{m1} v_{be1} \\
&= -g_{m1} i_b r_{\pi 1} \\
i_e &= i_b + g_{m1} i_b r_{\pi 1} + g_{m2} g_{m1} i_b r_{\pi 1} r_{\pi 2} \\
&= i_b \left(1 + g_{m1} r_{\pi 1} + g_{m1} r_{\pi 1} g_{m2} r_{\pi 2}\right) \\
&= i_b \left(1 + \beta_1 + \beta_1 \beta_2\right) \\
&= i_b \left(\beta_{SZ} + 1\right) \\
v_{in} &= i_b r_{\pi 1} + i_b \left(\beta_{SZ} + 1\right) R_e \\
&= i_b \left[r_{\pi 1} + \left(\beta_{SZ} + 1\right) R_e\right] \\
R_b = \frac{v_{in}}{i_b} &= r_{\pi 1} + \left(\beta_{SZ} + 1\right) R_e \\
R_b &= 13\,\mathrm{k\Omega} + (5050 + 1)\,(245\,\Omega) \\
&= 1{,}250{,}495\,\Omega
\end{aligned}$$

The input resistance is thus:

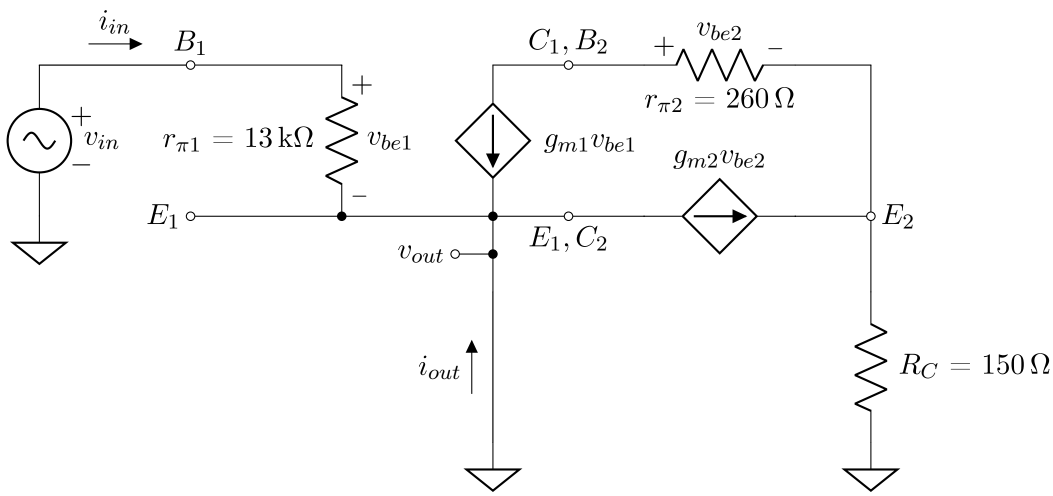
$$\begin{aligned}
R_{IN} &= R_B || R_b \\
&= (1{,}250{,}495\,\Omega)\,||\,(925\,\mathrm{k\Omega}) \\
&= 531{,}698.7054\,\Omega \\
&\approx \boxed{531{,}698.7\,\Omega} \text{ or} \\
&\approx \boxed{531.7\,\mathrm{k\Omega}}
\end{aligned}$$

Note that even if the resistor R_B is in parallel to the resistor, the input source still "sees" it and therefore should be considered in calculating the input resistance of the whole emitter follower. For the rest of the calculations, we can neglect R_B since the input voltage goes directly to the base.

- Gain A_V:
  Perform KCL at the output node (node E_1):

$$
\begin{aligned}
\frac{v_{out}}{R_E} + g_{m2}v_{be2} &= \frac{v_{in} - v_{out}}{r_{\pi 1}} + g_{m1}v_{be1} \\
\frac{v_{out}}{R_E} - g_{m2}g_{m1}v_{be1}r_{\pi 2} &= \frac{v_{in} - v_{out}}{r_{\pi 1}} + g_{m1}v_{be1} \\
\frac{v_{out}}{R_E} - g_{m2}g_{m1}r_{\pi 1}\frac{v_{in} - v_{out}}{r_{\pi 1}}r_{\pi 2} &= \frac{v_{in} - v_{out}}{r_{\pi 1}} + g_{m1}r_{\pi 1}\frac{v_{in} - v_{out}}{r_{\pi 1}} \\
\frac{v_{out}}{R_E} + g_{m1}r_{\pi 1}g_{m2}r_{\pi 2}\frac{v_{out} - v_{in}}{r_{\pi 1}} &= \frac{v_{in} - v_{out}}{r_{\pi 1}} + g_{m1}r_{\pi 1}\frac{v_{in} - v_{out}}{r_{\pi 1}} \\
\frac{v_{out}}{R_E} + \frac{g_{m1}r_{\pi 1}g_{m2}r_{\pi 2}v_{out}}{r_{\pi 1}} + \frac{v_{out}}{r_{\pi 1}} + \frac{g_{m1}r_{\pi 1}v_{out}}{r_{\pi 1}} &= \frac{v_{in}}{r_{\pi 1}} + \frac{g_{m1}r_{\pi 1}v_{in}}{r_{\pi 1}} + \frac{g_{m1}r_{\pi 1}g_{m2}r_{\pi 2}v_{in}}{r_{\pi 1}} \\
v_{out}\left[\frac{1}{R_E} + \frac{1 + g_{m1}r_{\pi 1} + g_{m1}r_{\pi 1}g_{m2}r_{\pi 2}}{r_{\pi 1}}\right] &= v_{in}\left[\frac{1 + g_{m1}r_{\pi 1} + g_{m1}r_{\pi 1}g_{m2}r_{\pi 2}}{r_{\pi 1}}\right] \\
v_{out}\left[\frac{1}{R_E} + \frac{1 + \beta_1 + \beta_1\beta_2}{r_{\pi 1}}\right] &= v_{in}\left[\frac{1 + \beta_1 + \beta_1\beta_2}{r_{\pi 1}}\right] \\
v_{out}\left[\frac{1}{R_E} + \frac{\beta_{SZ} + 1}{r_{\pi 1}}\right] &= v_{in}\left[\frac{\beta_{SZ} + 1}{r_{\pi 1}}\right] \\
A_V = \frac{v_{out}}{v_{in}} &= \frac{\frac{\beta_{SZ}+1}{r_{\pi 1}}}{\frac{1}{R_E} + \frac{\beta_{SZ}+1}{r_{\pi 1}}} \\
A_V &= \frac{\frac{5050+1}{13\,\mathrm{k}\Omega}}{\frac{1}{245\,\Omega} + \frac{5050+1}{13\,\mathrm{k}\Omega}} \\
&\approx 0.98960 \\
&\approx \boxed{1.0}
\end{aligned}
$$

- Transconductance G_M:
  To calculate G_M, we assume zero output condition so we short the output terminal:



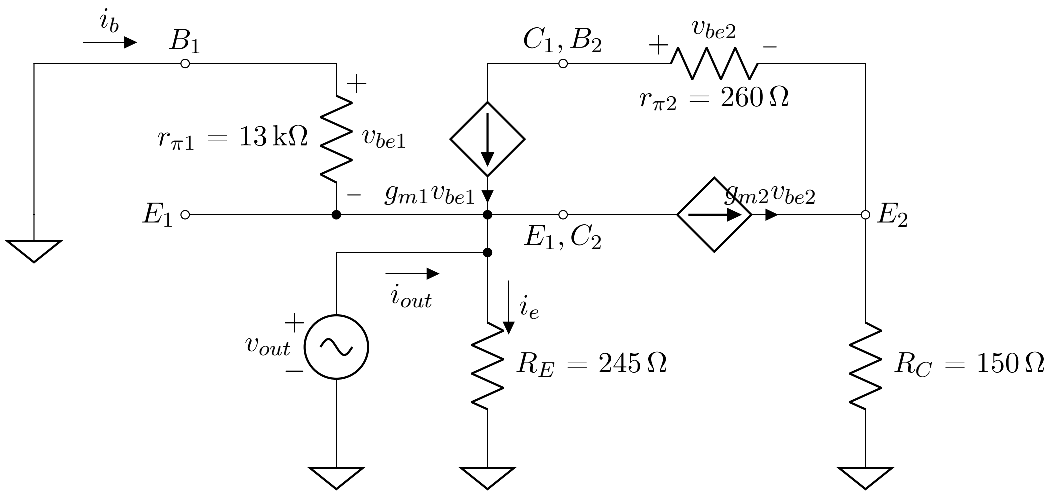
Perform KCL at the output node (node E_1):

$$
\begin{aligned}
g_{m2}v_{be2} &= i_{out} + \frac{v_{in}}{r_{\pi 1}} + g_{m1}v_{be1} \\
-i_{out} - g_{m2}g_{m1}v_{be1}r_{\pi 2} &= \frac{v_{in}}{r_{\pi 1}} + g_{m1}v_{be1} \\
-i_{out} - g_{m2}r_{\pi 2}g_{m1}v_{in} &= \frac{v_{in}}{r_{\pi 1}} + g_{m1}v_{in} \\
-i_{out} &= \frac{v_{in}}{r_{\pi 1}} + g_{m1}v_{in} + g_{m2}r_{\pi 2}g_{m1}v_{in} \\
&= v_{in}\left[\frac{1}{r_{\pi 1}} + g_{m1} + g_{m2}r_{\pi 2}g_{m1}\right] \\
&= v_{in}\left[\frac{1 + g_{m1}r_{\pi 1} + g_{m1}r_{\pi 1}g_{m2}r_{\pi 2}}{r_{\pi 1}}\right] \\
&= v_{in}\left[\frac{1 + \beta_1 + \beta_1\beta_2}{r_{\pi 1}}\right] \\
&= v_{in}\left[\frac{\beta_{SZ} + 1}{r_{\pi 1}}\right] \\
G_M = \frac{i_{out}}{v_{in}} &= -\frac{\beta_{SZ} + 1}{r_{\pi 1}} \\
G_M &= -\frac{5050 + 1}{13\,\mathrm{k}\Omega} \\
&\approx \boxed{-388.5 \text{ mS or mmhos or m}\frac{1}{\Omega} \text{ or}} \\
&\approx \boxed{-0.4 \text{ S or mhos or } \frac{1}{\Omega}}
\end{aligned}
$$

- Output Resistance R_O:
  Since A_V and G_M are already calculated, we can use $A_V = -G_M R_O$. Note that the amplifier is an emitter follower, thus the gain should be positive:

$$\begin{aligned} A_V &= -G_M R_O \\ R_O &= \frac{A_V}{-G_M} \\ &\approx \boxed{2.5\,\Omega \text{ or } 2.6\,\Omega} \end{aligned}$$

  Another solution is by assuming zero input condition, shorting the input terminal and applying a test source at the output:



  Performing KCL at the output node (node E_1):

$$\begin{aligned} i_{out} - \frac{v_{out}}{r_{\pi 1}} - g_{m1}v_{out} &= \frac{v_{out}}{R_E} + g_{m2}v_{be2} \\ i_{out} - \frac{v_{out}}{r_{\pi 1}} - g_{m1}v_{out} &= \frac{v_{out}}{R_E} + g_{m2}\left[-\left(-g_{m1}v_{out}\right)\right]r_{\pi 2} \\ i_{out} &= \frac{v_{out}}{R_E} + g_{m2}r_{\pi 2}g_{m1}v_{out} + \frac{v_{out}}{r_{\pi 1}} + g_{m1}v_{out} \\ &= v_{out}\left[\frac{1}{R_E} + g_{m2}r_{\pi 2}g_{m1} + g_{m1} + \frac{1}{r_{\pi 1}}\right] \\ &= v_{out}\left[\frac{1}{R_E} + \frac{g_{m1}r_{\pi 1}g_{m2}r_{\pi 2} + g_{m1}r_{\pi 1} + 1}{r_{\pi 1}}\right] \\ &= v_{out}\left[\frac{1}{R_E} + \frac{\beta_1\beta_2 + \beta_1 + 1}{r_{\pi 1}}\right] \\ &= v_{out}\left[\frac{1}{R_E} + \frac{\beta_{SZ} + 1}{r_{\pi 1}}\right] \\ R_O = \frac{v_{out}}{i_{out}} &= \frac{1}{\frac{1}{R_E} + \frac{\beta_{SZ}+1}{r_{\pi 1}}} \\ R_O &= \frac{1}{\frac{1}{245\,\Omega} + \frac{5050+1}{13\,\mathrm{k}\Omega}} \\ &\approx 2.546\,99\,\Omega \\ R_O &\approx \boxed{2.5\,\Omega} \end{aligned}$$

2. **Common Emitter Amplifier with Load.**

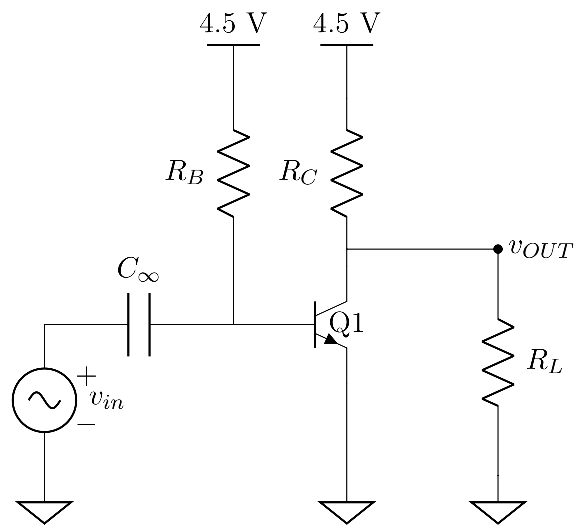


Figure 2: Common Emitter with biasing

Assume T=300K when necessary. For the CE amplifier in fig. 2 with $V_{BE,on} = 0.7V$, $V_{CE,sat} = 0.7V$, $\beta = 200$, $V_A = 100V$, $R_B = 95K\Omega$, $R_C = 204\Omega$, and $R_L = 794\Omega$:

(a) Calculate the DC biasing parameters:

i. base current (I_B) (0.5 pt)

$$\begin{aligned} I_B &= \frac{V_{DD} - V_{BE,on}}{R_B} \\ &= \frac{4.5V - 0.7V}{95K\Omega} \\ &= \boxed{40\,\mu\text{A}} \end{aligned}$$

ii. collector current (I_C) (0.5 pt)

$$\begin{aligned} I_C &= \beta I_B \\ &= (200)(40\mu A) \\ &= \boxed{8\,\text{mA}} \end{aligned}$$

iii. output voltage (V_{OUT}) (1 pt) V_{OUT} can be solved conveniently by using KCL at that node:

$$\begin{aligned} \frac{V_{DD} - V_{OUT}}{R_C} &= I_C + \frac{V_{OUT}}{R_L} \\ \frac{V_{DD}}{R_C} - I_C &= \frac{V_{OUT}}{R_L} + \frac{V_{OUT}}{R_C} \\ &= V_{OUT}(\frac{1}{R_L} + \frac{1}{R_C}) \\ V_{OUT} &= \frac{\frac{V_{DD}}{R_C} - I_C}{\frac{1}{R_L} + \frac{1}{R_C}} \\ &= \frac{\frac{4.5V}{204\Omega} - 8mA}{\frac{1}{794\Omega} + \frac{1}{204\Omega}} \\ &= \boxed{2.28V} \end{aligned}$$

(b) Draw the small-signal model of the circuit. You may omit R_B. Label the values of each small-signal parameter, and component. (3 pts)

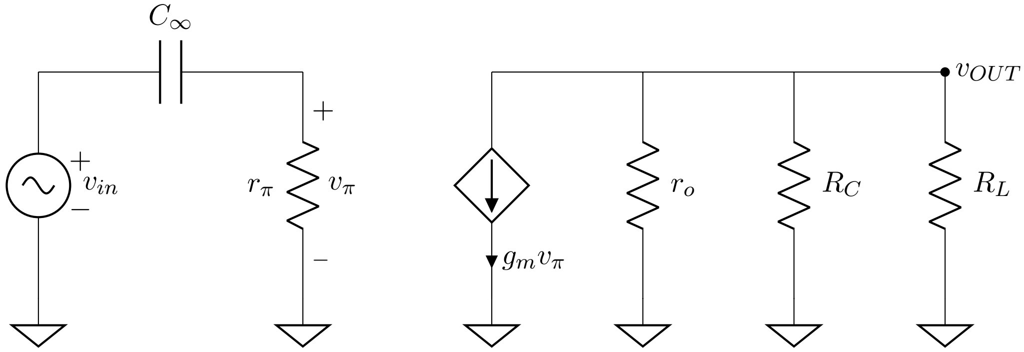


Figure 3: Common Emitter with load small-signal model

$$
\begin{aligned}
V_T &= \frac{kT}{q} = 26mV\,for T = 300K \\
g_m &= \frac{I_C}{V_T} \\
&= \frac{8mA}{26mV} \\
&= \boxed{307.692mS} \\
r_o &= \frac{V_A}{I_C} \\
&= \frac{100V}{8mA} \\
&= \boxed{12.5K\Omega}
\end{aligned}
$$

(c) Compare (a) R_{O1} = parallel combination of r_o, R_C and R_L, with (b) R_{O2} = parallel combination of R_C and R_L. Would r_o significantly affect the effective resistance? How much error relative to R_{O1} would you get, if you simply used R_{O2} rather than R_{O1} to simplify your calculations? Express this number as a percentage. (1.5 pts)

$$
\begin{aligned}
r_o &= 12.5K\Omega \\
R_C &= 204\Omega \\
R_L &= 794\Omega \\
R_{O1} &= r_o||R_C||R_L \\
&= \boxed{160.22\Omega} \\
R_{O2} &= R_C||R_L \\
&= \boxed{162.3\Omega}
\end{aligned}
$$

$$
\begin{aligned}
\%error &= |\frac{R_{O1} - R_{O2}}{R_{O1}}|x100 \\
&= |\frac{160.22\Omega - 162.3\Omega}{160.22\Omega}|x100 \\
&= \boxed{1.3\%}
\end{aligned}
$$

Usually, we'd note if $r_o >> R_C$ and $r_o >> R_L$ to simplify calculations, in which case, a parallel combination of $r_o||R_C approx R_C$. This is especially useful when only a single resistance is connected as a load, since it would be easier to calculate the amplifier's two-port parameters within a certain band of accuracy easily. In more complex circuits, this would make calculations more bearable.

(d) Calculate the small-signal gain (v_{out}/v_{in}) (1 pt)

$$G_M = g_m = 307.692mS$$
$$R_O = r_o||R_C||R_L \approx R_C||R_L = 162.3\Omega$$

$$\begin{aligned}\frac{v_{out}}{v_{in}} &= -G_M R_O \\ &= -(307.692mS)(162.3\Omega) \\ &= \boxed{49.9V/V}\end{aligned}$$

(e) Sketch v_{out} from 0 ms to 2 ms for $v_{in} = 10sin(2\pi ft)$ mV with $f = 1KHz$. Label the peak voltages, axes, and units. (1.5 pts)
Note: For this problem, any of the following answers will be considered:

i. Linear case
ii. Mathematical proof that v_{be} is not << V_T.
iii. Non-linear case

LINEAR CASE
The linear case is the simplest, and can be obtained by simply multiplying v_{in} with A_V.

$$\begin{aligned}v_{out} &= A_V v_{in} \\ &= (49.9)(10sin(2\pi ft)mV) \\ &= 499sin(2\pi ft)mV \\ &= \boxed{0.499sin(2\pi ft)V}\end{aligned}$$

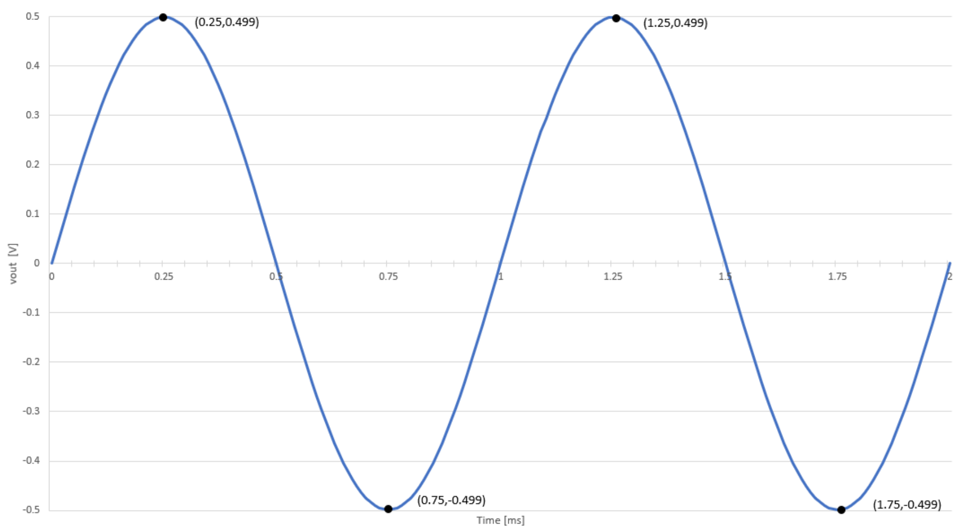


Figure 4: Output Voltage for $v_{in} = 10sin(2\pi ft)$ mV with $f = 1KHz$.

NONLINEAR CASE
But actually, with $v_{in} = 10sin(2\pi ft)$ mV, there are times when $v_{be} = 10mV$ which is not << $V_T = 26mV$. (@T=300K) Thus, $v_{be}/V_T \leq 0.3846$ which is not << 1.

There are two ways to plot this: (1) using the Shockley equation or (2) using more than one term of the Taylor series approximation. Using the Shockley equation would require knowing I_S and the dc bias V_{BE}, on top of it being more prone to approximation errors due to V_{BE}. So, we'll use the Taylor series for this case.

The Taylor series approximation is:

$$i_C = I_C[1 + (\frac{v_{be}}{V_T}) + \frac{1}{2}(\frac{v_{be}}{V_T})^2 + \frac{1}{6}(\frac{v_{be}}{V_T})^3 + ...]$$

We'll need to know how many terms we'll need so we'll compute the max value of each term until it becomes negligible. We'll use $v_{be} = 10mV$, and assume T=300K, so $V_T = 26mV$.

$$\begin{aligned}
i_{C0} &= I_C \text{DC bias} \\
&= 8mA \\
i_{C1} &= I_C(\frac{v_{be}}{V_T}) \\
&= 3.077mA \\
i_{C2} &= \frac{1}{2}I_C(\frac{v_{be}}{V_T})^2 \\
&= 0.5917mA = 19.23\% \text{ of } i_{c1} \\
i_{C3} &= \frac{1}{6}I_C(\frac{v_{be}}{V_T})^3 \\
&= 0.07586mA = 2.47\% \text{ of } i_{c1} \\
i_{C4} &= \frac{1}{24}I_C(\frac{v_{be}}{V_T})^4 \\
&= 0.007294mA = 0.237\% \text{ of } i_{c1}
\end{aligned}$$

Thus, using until the third or fourth term should be sufficient. We'll use until the fourth term:

$$i_C = I_C[1 + (\frac{v_{be}}{V_T}) + \frac{1}{2}(\frac{v_{be}}{V_T})^2 + \frac{1}{6}(\frac{v_{be}}{V_T})^3]$$

And the small-signal current i_c is the second to the fourth terms of the approximation:

$$i_c = I_C[(\frac{v_{be}}{V_T}) + \frac{1}{2}(\frac{v_{be}}{V_T})^2 + \frac{1}{6}(\frac{v_{be}}{V_T})^3]$$

The small-signal output voltage v_{out} can be obtained by multiplying i_c (small-signal), with the overall output resistance R_O.

$$\begin{aligned}
v_{out} &= i_c R_O \\
&= I_C R_O[(\frac{v_{be}}{V_T}) + \frac{1}{2}(\frac{v_{be}}{V_T})^2 + \frac{1}{6}(\frac{v_{be}}{V_T})^3]
\end{aligned}$$

Figure 5 shows the plot of this equation for $v_{be} = v_{in} = 10sin(2\pi ft)$ mV with $f = 1KHz$. The blue curve is the plot if we used only until the first term of v_{out}. The orange curve shows the plot for using until the second term (Second harmonic), while the gray curve shows the case for using until the third term (third harmonic). For this case, the second harmonic term contributes significantly and distorts v_{out} from being linear, while the third harmonic term has very little contribution.

The peaks occur at [0.25 ms, 0.75 ms, 1.25 ms, 1.75 ms]. The linear case has peaks up to 0.499 V and -0.499 V. The case with second harmonic term has peaks up to 0.595 V, and -0.403 V. While the case with third harmonic term has peaks up to 0.607 V and -0.416 V.

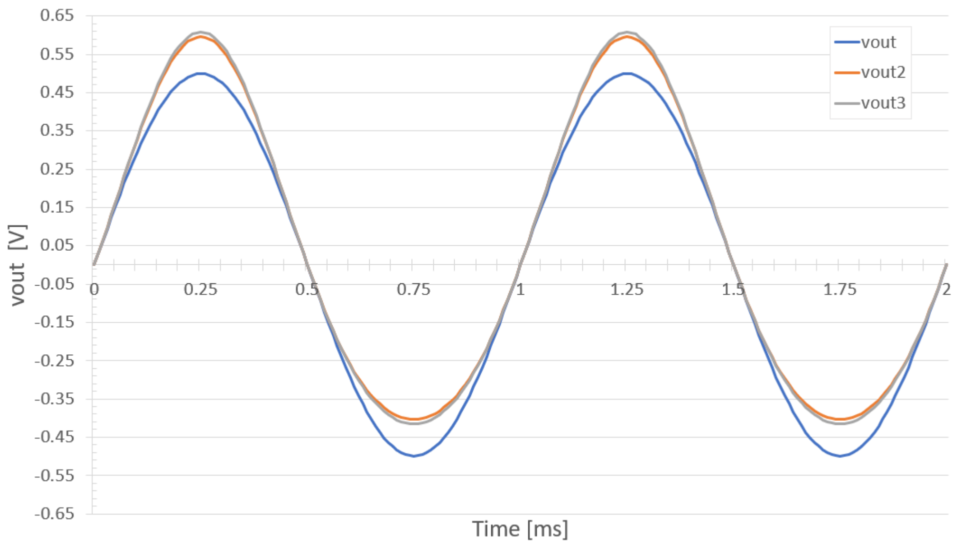


Figure 5: Output Voltage for $v_{in} = 10sin(2\pi ft)$ mV with $f = 1KHz$ for different approximations.

(f) Should you expect the amplifier to work linearly, with the same gain for $v_{in} = 52sin(2\pi ft)$ mV, with $f = 1KHz$? Why or why not? Prove mathematically. [Hint: You may revisit the Taylor series expansion of i_C.] (1 pts)

No, the amplifier shouldn't work linearly, nor will it have the same gain. The Taylor series expansion of i_C is shown below.

$$i_C = I_C[1 + (\frac{v_{be}}{V_T}) + \frac{1}{2}(\frac{v_{be}}{V_T})^2 + \frac{1}{6}(\frac{v_{be}}{V_T})^3 + ...]$$

where v_{be} is the small-signal input voltage, and $V_T = \frac{kT}{q}$. For T=300K, $V_T = 26mV$.

The accuracy of the small-signal approximation depends largely on the magnitude of the input voltage (v_{be}) in this case. With a small v_{be}, the second, third, and succeeding terms (or "harmonic terms") approximate to much less than the first $\frac{v_{be}}{V_T}$ term, improving the accuracy of the model. As v_{be} approaches V_T, the harmonic terms increase in magnitude and become significant, reducing the accuracy of the small-signal model. With $v_{be} = 52sin(2\pi ft)$ mV, there are times when $v_{be} \geq 26mV$. Therefore, there are times that the harmonic terms dominate, and the amplifier wouldn't work linearly.

3. **MOS Common Source and Common Gate Cascode.**

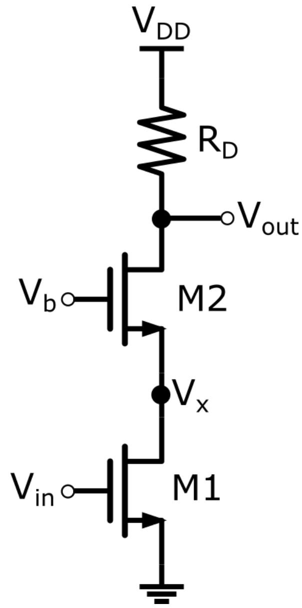


Figure 6: CS-CG Cascode

Consider the circuit shown above. For this problem, use $V_{DD} = 3V$, $I_{D1} = I_{D2} = 1mA$, $k_{M1} = 6.7m\frac{A}{V^2}$, $k_{M2} = 1.4m\frac{A}{V^2}$, $V_{TH1} = V_{TH2} = 0.7V$, $\lambda_{M1} = \lambda_{M2} = 0.1V^{-1}$ and $R_D = 1.5k\Omega$. Express your answers to four (4) decimal places and use the appropriate unit of measure.

(a) Determine the minimum voltage, V_x, needed to keep M1 in saturation. (1 pt)
The minimum V_x needed to keep transistor M1 in saturation is basically just the $V_{DS,sat}$ of M1. To ensure that M1 is in saturation, the following condition other than $V_{GS} > V_{TH}$ must be satisfied,

$$V_{DS1} \geq V_{GS1} - V_{TH1}$$

With $V_{DS1,sat}$ exactly equivalent to $V_{GS1} - V_{TH1}$ as the minimum voltage needed to keep M1 in saturation. The expression $V_{GS1} - V_{TH1}$ can also be expressed in a different manner by manipulating the MOS transfer characteristic equation,

$$I_{D1} = k(V_{GS1} - V_{TH1})^2(1 + \lambda V_{DS1})$$

We can now compute V_x as follows:

$$V_x = V_{DS1,sat} = \sqrt{\frac{I_{D1}}{k_{M1}(1 + \lambda V_{DS1,sat})}}$$

Solving this equation yields 3 possible answers which are $V_{DS1,sat} = -9.9850V$, $V_{DS1,sat} = 0.3792V$ and $V_{DS1,sat} = -0.3942V$. Of the three, the only plausible value is $V_{DS1,sat} = 0.3792V$ since the other values are negative which would imply that the transistor is not even ON even if it satisfies the condition $V_{DS1} \geq V_{GS1} - V_{TH1}$.

$$\boxed{V_x = 0.3792V \text{ (1 pt)}}$$

(b) Choose the bias voltage, V_b such that M1 is $50mV$ away from the linear region of operation. (1 pt)

If we want M1 to be $50mV$ away from the linear region, the new V_x has to be, $V_x = V_{DS1,sat} + 50mV =$

0.4292V. Doing KVL from the gate terminal of M2 down to ground, we get $V_b = V_{GS2} + V_x$. In order to get V_{GS2}, we again manipulate the MOS transfer characteristic equation as follows:

$$V_{GS2} = \sqrt{\frac{I_{D2}}{k_{M2}(1+\lambda V_{DS2})}} + V_{TH2} = 1.5032V$$

Where $V_{DS2} = V_{DD} - I_{D2}R_D - V_x = 3 - 1.5 - 0.4292 = 1.0708V$

$$\boxed{V_b = V_{GS2} + V_x = 1.5032 + 0.4292 = 1.9324V \text{ (1 pt)}}$$

(c) Draw the small-signal equivalent model of the circuit. Label the transistor terminals, the transistor small-signal parameters and their values. (2 pts)

The small-signal equivalent model of the circuit is shown in the figure below.

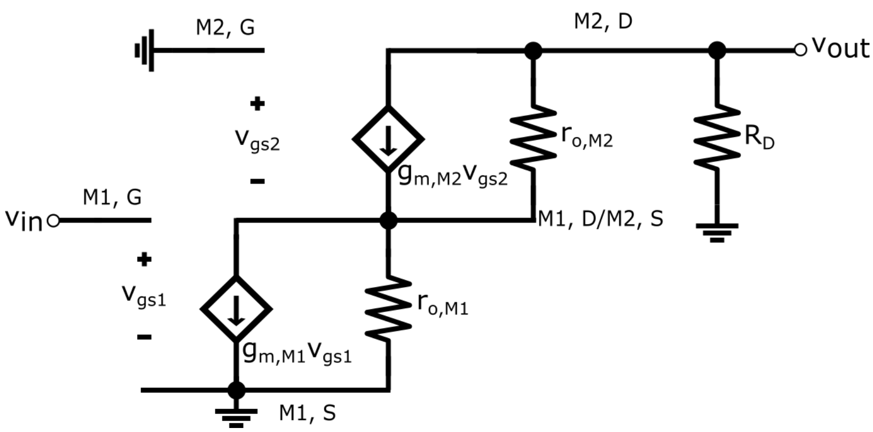


Figure 7: Small-signal circuit

The small-signal parameters are as follows:

$$r_{o,M1} = \frac{1}{\lambda I_{D1}} = 10k\Omega$$

$$r_{o,M2} = \frac{1}{\lambda I_{D2}} = 10k\Omega$$

$$g_{m,M1} = \sqrt{4k_{M1}I_{D1}} = 5.1769mS$$

$$g_{m,M2} = \sqrt{4k_{M2}I_{D2}} = 2.3664mS$$

(1 pt) will be given for the complete drawing of the circuit with proper labels. (0.5 pts) will be given for r_o and (0.5 pts) will be given for g_m.

(d) Calculate for the overall transconductance, G_m, and output impedance, R_{out}. (5 pts)

Refer to the figure below in solving for the overall transconductance, G_m. We just simplified first the notation for $r_{o,M1}$, $r_{o,M2}$, $g_{m,M1}$ and $g_{m,M2}$ to r_{o1}, r_{o2}, g_{m1} and g_{m2} respectively.

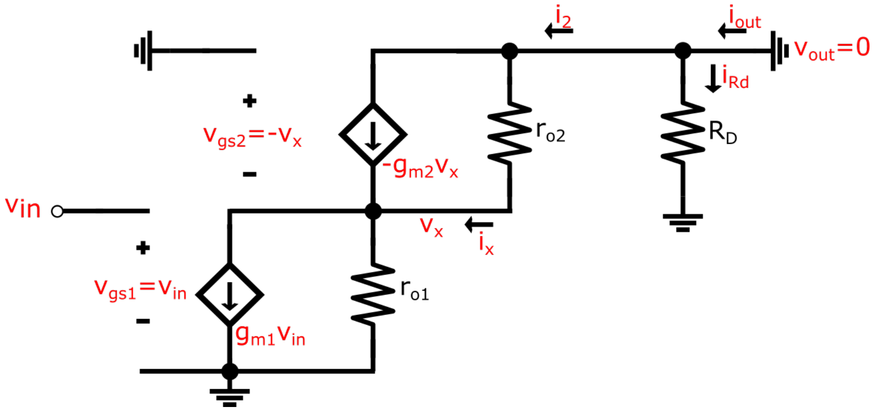


Figure 8: Small-signal circuit in solving G_m

To get G_m, we must find $\frac{i_{out}}{v_{in}}$ with $v_{out} = 0V$. Performing KCL at the output node,

$$i_{out} = i_{Rd} + i_2 = i_2 = -g_{m2}v_x - \frac{v_x}{r_{o2}}$$

Note that $i_{Rd} = 0A$. With this, i_{out} is simply i_2. Re-arranging the equation above to express v_x in terms of i_{out} we get,

$$v_x = \frac{-i_{out}}{g_{m2} + \frac{1}{r_{o2}}}$$

Note also that the value of i_2 is the same as i_x. As such,

$$i_x = i_{out} = g_{m1}v_{in} + \frac{v_x}{r_{o1}} = g_{m1}v_{in} - \frac{i_{out}}{g_{m2}r_{o1} + \frac{r_{o1}}{r_{o2}}}$$

$$i_{out}\left(1 + \frac{1}{g_{m2}r_{o1} + \frac{r_{o1}}{r_{o2}}}\right) = g_{m1}v_{in}$$

$$G_m = \frac{i_{out}}{v_{in}} = \frac{g_{m1}\left(g_{m2}r_{o1} + \frac{r_{o1}}{r_{o2}}\right)}{g_{m2}r_{o1} + \frac{r_{o1}}{r_{o2}} + 1} * \frac{r_{o2}}{r_{o2}} = \frac{g_{m1}r_{o1}(g_{m2}r_{o2} + 1)}{g_{m2}r_{o1}r_{o2} + r_{o1} + r_{o2}}$$

Substituting the small-signal parameter values,

$$\boxed{G_m = 4.9752mS \text{ (0.5 pts). (2pts) will be given for the solution.}}$$

As for R_{out}, we need to get $\frac{v_{out}}{i_{out}}$ with $v_{in} = 0V$. For this, refer to the figure below.

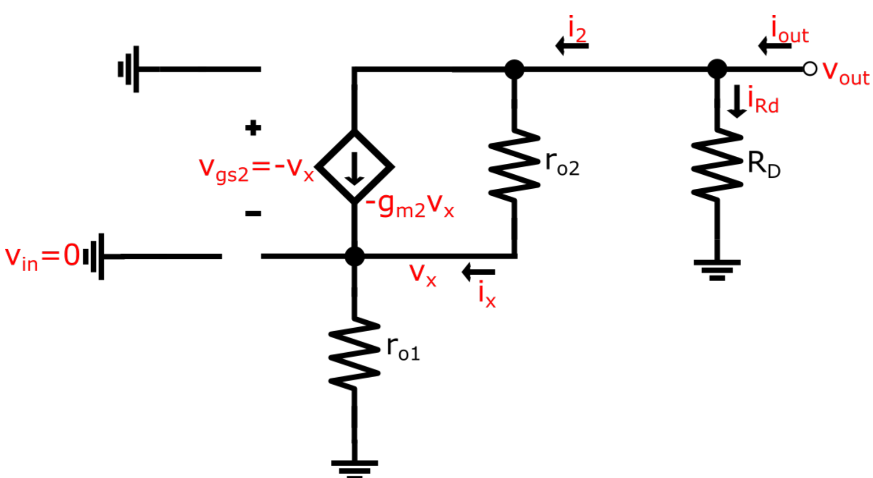


Figure 9: Small-signal circuit in solving R_{out}

We begin by performing KCL at the output node,

$$i_{out} = i_{Rd} + i_2 = \frac{v_{out}}{R_D} - g_{m2}v_x + \frac{v_{out} - v_x}{r_{o2}}$$

Note that i_2 is still the same as i_x.

$$i_2 = i_x = \frac{v_x}{r_{o1}} = i_{out} - \frac{v_{out}}{R_D}$$

We can now express v_x as

$$v_x = r_{o1}i_{out} - \frac{r_{o1}}{R_D}v_{out}$$

And substitute v_x back to the KCL equation at the output,

$$i_{out} = \frac{v_{out}}{R_D} - g_{m2}r_{o1}i_{out} + \frac{g_{m2}r_{o1}}{R_D}v_{out} + \frac{v_{out}}{r_{o2}} - \frac{r_{o1}}{r_{o2}}i_{out} + \frac{r_{o1}}{r_{o2}R_D}v_{out}$$

$$i_{out}\left(1 + g_{m2}r_{o1} + \frac{r_{o1}}{r_{o2}}\right) = v_{out}\left(\frac{1}{R_D} + \frac{g_{m2}r_{o1}}{R_D} + \frac{1}{r_{o2}} + \frac{r_{o1}}{r_{o2}R_D}\right)$$

$$\frac{v_{out}}{i_{out}} = \frac{1 + g_{m2}r_{o1} + \frac{r_{o1}}{r_{o2}}}{\frac{1}{R_D} + \frac{g_{m2}r_{o1}}{R_D} + \frac{1}{r_{o2}} + \frac{r_{o1}}{r_{o2}R_D}} * \frac{R_D r_{o2}}{R_D r_{o2}}$$

$$R_{out} = \frac{R_D(g_{m2}r_{o1}r_{o2} + r_{o1} + r_{o2})}{R_D + (g_{m2}r_{o1}r_{o2} + r_{o1} + r_{o2})} = R_D//(g_{m2}r_{o1}r_{o2} + r_{o1} + r_{o2})$$

Substituting the small-signal parameter values,

$$\boxed{R_{out} = 1.4913k\Omega \text{ (0.5 pts). (2pts) will be given for the solution.}}$$

(e) Calculate the small-signal voltage gain. (1 pt)

To get the small-signal voltage gain, A_v, we just do the following:

$$\boxed{A_v = -G_m R_{out} = 4.9752mS * 1.4913k\Omega = -7.4195 \text{ (1 pt).}}$$

TOTAL: 30 points.