EEE 51 Assignment 7 Solution

2nd Semester SY 2017-2018
Due: 5pm Tuesday, April 10, 2018 (Rm. 220)

Instructions: Write legibly. Show all solutions and state all assumptions. Write your full name, student number, and section at the upper-right corner of each page. Start each problem on a new sheet of paper. Box or encircle your final answer.

Answer sheets should be colored according to your lecture section. The color scheme is as follows:

THQ – yellow
THR – blue
THU – white
THX – green
WFX – pink

1. **Frequency response of RC circuits.** You are Special Agent Jeongyeon, a new member of the SWAT team. As part of your training, your boss, Officer Nayeon, hands you the following exercises on frequency response.

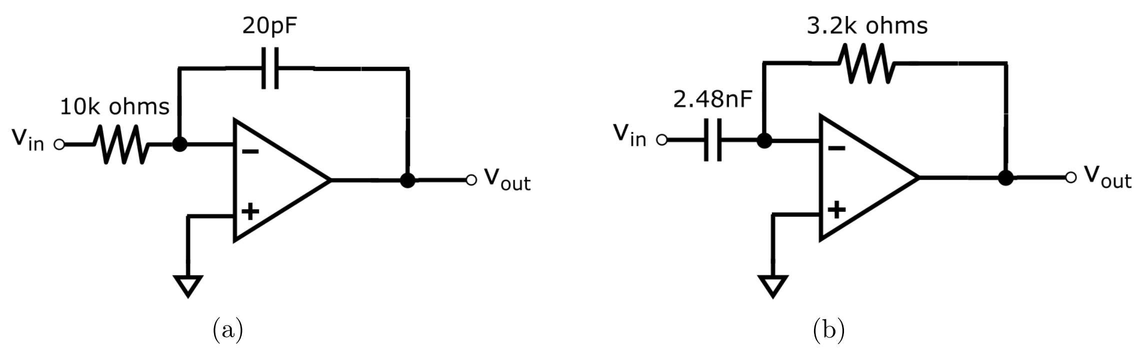


Figure 1: RC circuits

(a) Determine the transfer function $H(s)$ of (a) and (b) in Figure 1. Assume the op amps are ideal. [2 pts]

Since both op amps are ideal, $v_+ = v_- = 0$ for both configurations, and $i_+ = i_- = 0$.
For circuit (a) :

$$\frac{v_{in} - 0}{R} = \frac{0 - v_{out}}{\frac{1}{sC}} \tag{1}$$

$$\frac{v_{in}}{R} = -v_{out}(sC) \tag{2}$$

$$\boxed{\frac{v_{out}}{v_{in}} = -\frac{1}{sRC} \text{ [1 pt]}}$$

For circuit (b) :

$$\frac{v_{in} - 0}{\frac{1}{sC}} = \frac{0 - v_{out}}{R} \tag{3}$$

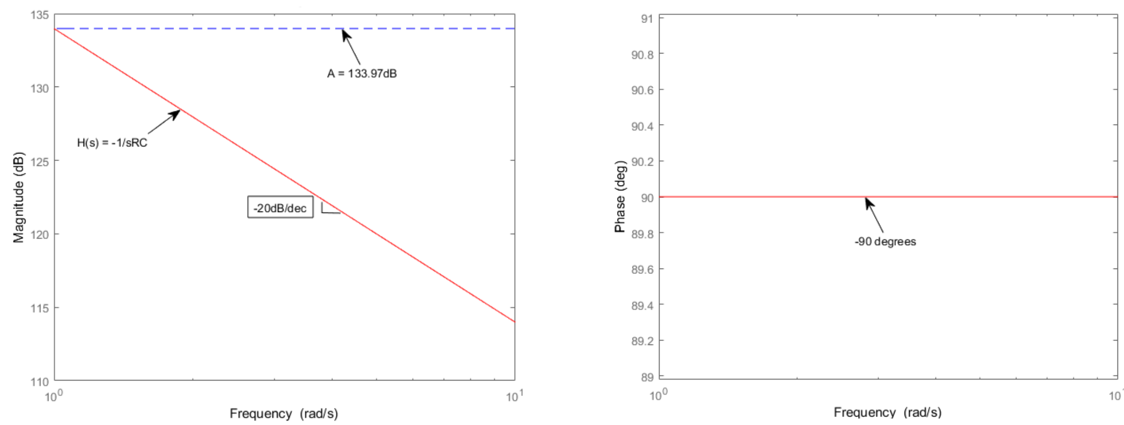
$$v_{in}(sC) = -\frac{v_{out}}{R} \tag{4}$$

$$\boxed{\frac{v_{out}}{v_{in}} = -sRC \text{ [1 pt]}}$$

(b) Draw the frequency response of both circuits. Assume that the magnitude and phase response are piecewise linear functions. Label the graph properly. Also label the slope, DC gain, phase, 3-dB point and other relevant frequencies accordingly. (*Note* : You may use Matlab for this part, but in this case, you should not assume that the response is piecewise.) [4 pts]

For circuit (a) :
The magnitude and phase responses are shown in Figure 2. To get the magnitude at $\omega = 0$:

$$20log\left(\frac{1}{RC}\right) = 20log\left[\frac{1}{(10k)(20p)}\right] = 133.9794dB \tag{5}$$

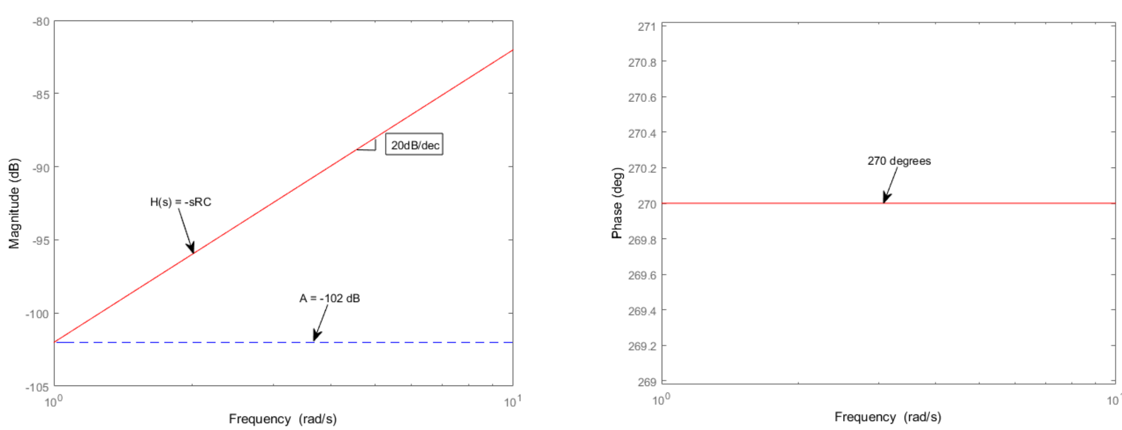


(a) (b)

Figure 2: Frequency response of circuit (a)

For circuit (b) :
The magnitude and phase responses are shown in Figure 3. To get the magnitude at $\omega = 0$:

$$20log(RC) = 20log[(2.48n)(3.2k)] = -102.008dB \tag{6}$$



(a) (b)

Figure 3: Frequency response of circuit (b)

(c) What do circuits (a) and (b) do? (i.e. What is the functionality of each circuit?) [1 pt]

(a) Integrator [0.5 pt] (b) Differentiator [0.5 pt]

Emergency! Officer Nayeon is kidnapped by a killer who specializes in making bombs that emit a certain frequency. Your team has a measuring device, an active high pass filter which can be found in Figure 4.

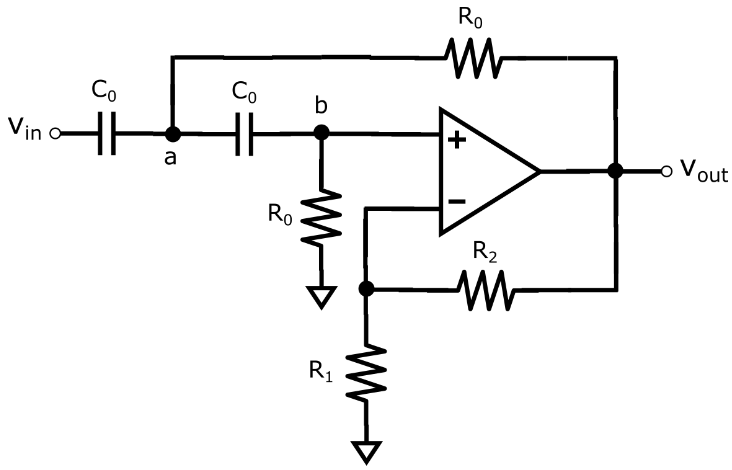


Figure 4: Active high pass filter

(d) Your team knows that the killer's signature frequency is $4.2kHz$. From what you have gathered, you know that the signals emitted by the special bombs are small enough that they need a voltage amplification of $5dB$. Now, it's your job to configure the circuit and choose the resistor and capacitor values according to the information given and save Nayeon. Make sure that these are standard values. Good luck, Agent. [3 pts]

First let us define nodes a and b, which have been labeled accordingly in Figure 4. Remember the assumptions for an ideal op amp. The goal is to obtain the transfer function of this circuit. To do this, we will use KCL analysis, first in node a:

$$\frac{v_{in} - v_a}{\frac{1}{sC_0}} = \frac{v_a - v_{out}}{R_0} + \frac{v_a - v_b}{\frac{1}{sC_0}} \tag{7}$$

KCL analysis in the negative input of the op amp:

$$v_+ = v_- = v_b \tag{8}$$

$$\frac{v_b}{R_1} = \frac{v_{out} - v_b}{R_2} \tag{9}$$

$$v_b\left(\frac{1}{R_2} + \frac{1}{R_1}\right) = \frac{v_{out}}{R_2} \tag{10}$$

$$v_b = \frac{v_{out}}{1 + \frac{R_2}{R_1}} \tag{11}$$

KCL analysis at node b:

$$\frac{v_a - v_b}{\frac{1}{sC_0}} = \frac{v_b}{R_0} \tag{12}$$

$$v_a = v_b\left(\frac{1}{sC_0}\right)\left(\frac{1}{R_0} + sC_0\right) \tag{13}$$

Substituting (11) into (13),

$$v_a = \frac{v_{out}}{1 + \frac{R_2}{R_1}}\left(\frac{1 + sC_0R_0}{sC_0R_0}\right) \tag{14}$$

Now let us define G as:

$$G = 1 + \frac{R_2}{R_1} \tag{15}$$

Replacing v_a and v_b in (7) using (14) and (11),

$$sC_0 v_{in} - \frac{sC_0 v_{out}}{G}\left(\frac{1+sC_0R_0}{sC_0R_0}\right) = \frac{v_{out}}{GR_0}\left(\frac{1+sC_0R_0}{sC_0R_0}\right) - \frac{v_{out}}{R_0} + \frac{sC_0 v_{out}}{G}\left(\frac{1+sC_0R_0}{sC_0R_0}\right) - \frac{sC_0 v_{out}}{G} \tag{16}$$

Simplifying further we should arrive at the transfer function:

$$\frac{v_{out}}{v_{in}} = G\frac{s^2C_0^2R_0^2}{s^2C_0^2R_0^2 + 3sC_0R_0 + 1} \tag{17}$$

The resistor and capacitor values should be able to satisfy the following equations to be given credit:

$$20logG = 20log\left(1+\frac{R_2}{R_1}\right) = 5dB \text{ [1.5 pts]}$$
$$f_c = \frac{1}{2\pi R_0 C_0} < 4.2kHz \text{ [1.5 pts]}$$

(e) (BONUS) What is the order of the filter? [0.5 pt]

It can be seen in the transfer function in (17) that the circuit is a **second-order** high pass filter. [0.5 pt]

2. **Frequency response of a common emitter amplifier.** A common emitter amplifier is shown in Figure 5. Given that $V_{IN} = 5\,\text{V}$, $V_A = 100\,\text{V}$, $I_S = 1\,\text{fA}$, $R_B = 10\,\text{k}\Omega$, $R_C = 500\,\Omega$, $C_L = 1\,\text{pF}$, $C_\pi = C_\mu = 10\,\text{fF}$, and $\beta = 50$, answer the following questions.

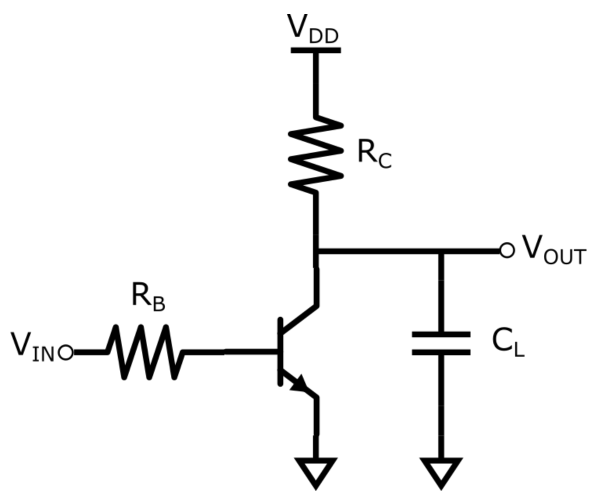


Figure 5: Common Emitter Amplifier with Capacitive Load

(a) What is the quiescent collector current of the common emitter amplifier (ignoring the effects of early voltage)?

$$\beta I_B = \frac{\beta}{\beta+1} I_E \tag{18}$$

$$\beta\frac{V_{IN} - V_{BE}}{R_B} = \frac{\beta}{\beta+1} I_S(e^{\frac{V_{BE}}{V_T}} - 1) \tag{19}$$

$$50\frac{5 - V_{BE}}{10k} = \frac{50}{50+1}(1\,\text{fA})(e^{\frac{V_{BE}}{0.026}} - 1) \tag{20}$$

Finding V_{BE},

$$V_{BE} = 0.79809V \tag{21}$$

Therefore,

$$I_C = \frac{50}{50+1}(1\,\text{fA})(e^{\frac{0.79809}{0.026}} - 1) = 21\,\text{mA [2 pts]}$$

(b) Draw the small signal equivalent circuit including C_π and C_μ. Calculate all small-signal parameter values.

First, we calculate for the small-signal parameter values.

$$g_m = \frac{I_C}{V_T} = 0.808\,\text{S [0.5 pts]} \tag{22}$$

$$r_O = \frac{V_A}{I_C} = 4759\,\Omega\ \text{[0.5 pts]} \tag{23}$$

$$r_\pi = \frac{\beta V_T}{I_C} = 61.87\,\Omega\ \text{[0.5 pts]} \tag{24}$$

Given $C_\pi = C_\mu = 10\,\text{fF}$, the small signal circuit is

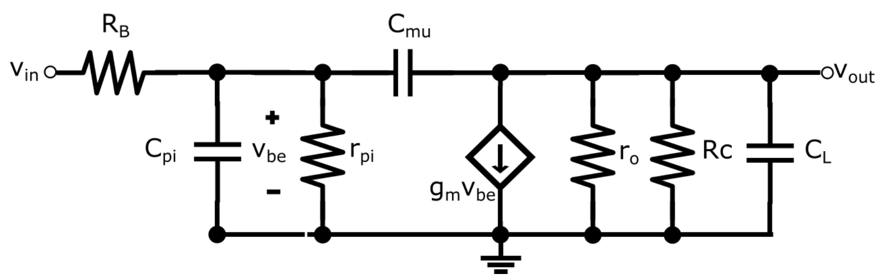


Figure 6: Small Signal Equivalent circuit of CE Amplifier with capacitive load. [1.5 pts]

(c) Find the small-signal transfer function of the circuit.

KCL at the output node:

$$sC_L v_{out} + \frac{v_{out}}{r_o} + g_m v_{be} + sC_\mu(v_{out} - v_{be}) = 0 \tag{25}$$

KCL at the base node:

$$\frac{v_{be} - v_{in}}{R_B} + \frac{v_{be}}{r_\pi} + sC_\pi v_{be} + sC_\mu(v_{be} - v_{out}) = 0 \tag{26}$$

Manipulating equation 26,

$$v_{be} = \frac{\frac{v_{in}}{R_B} - sC_\mu v_{out}}{\frac{1}{R_B} + \frac{1}{r_\pi} + s(C_\pi + C_\mu)} \tag{27}$$

Manipulating equation 25,

$$v_{be} = \frac{sC_L + \frac{1}{r_o} + sC_\mu}{sC_\mu g_m} v_{out} \tag{28}$$

Combining equations 27 and 28,

$$\frac{s(C_L + C_\mu) + \frac{1}{R_o}}{sC_\mu - g_m} v_{out} = \frac{\frac{v_{in}}{R_B} - sC_\mu v_{out}}{\frac{1}{R_B} + \frac{1}{r_\pi} + s(C_\pi + C_\mu)} \tag{29}$$

Let,

$$C' = C_\pi + C_\mu \tag{30}$$

$$Y_x = \frac{1}{R_B} + \frac{1}{r_\pi} \tag{31}$$

Therefore

$$\frac{v_{out}}{v_{in}} = \frac{\frac{1}{R_B(Y_x + sC')}}{\frac{s(C_L + C_\mu) + \frac{1}{R_o}}{sC_\mu - g_m} + \frac{sC_\mu}{Y_x + sC'}} \tag{32}$$

Simplifying,

$$\frac{v_{out}}{v_{in}} = \frac{sC_\mu - g_m}{R_B(s^2(C_\mu^2 + (C_L + C_\mu)C') + s(\frac{C'}{R_o} + Y_x(C_L + C_\mu) + g_m C_\mu) + \frac{Y_x}{R_o})} \tag{33}$$

Substituting the derived values,

$$\boxed{\frac{v_{out}}{v_{in}} = \frac{10^{-14}s - 0.808}{2.033\times10^{-22}s^2 + 8.389\times10^{-11}s + 0.35843} \text{ [3 pts]}}$$

(d) Draw the magnitude and phase response of the circuit. [BONUS]

3. **Frequency analysis of basic circuits.** Capacitive and inductive elements are frequency-dependent components of our circuits whose impedances can vary and even dominate the circuit at certain operating frequencies.

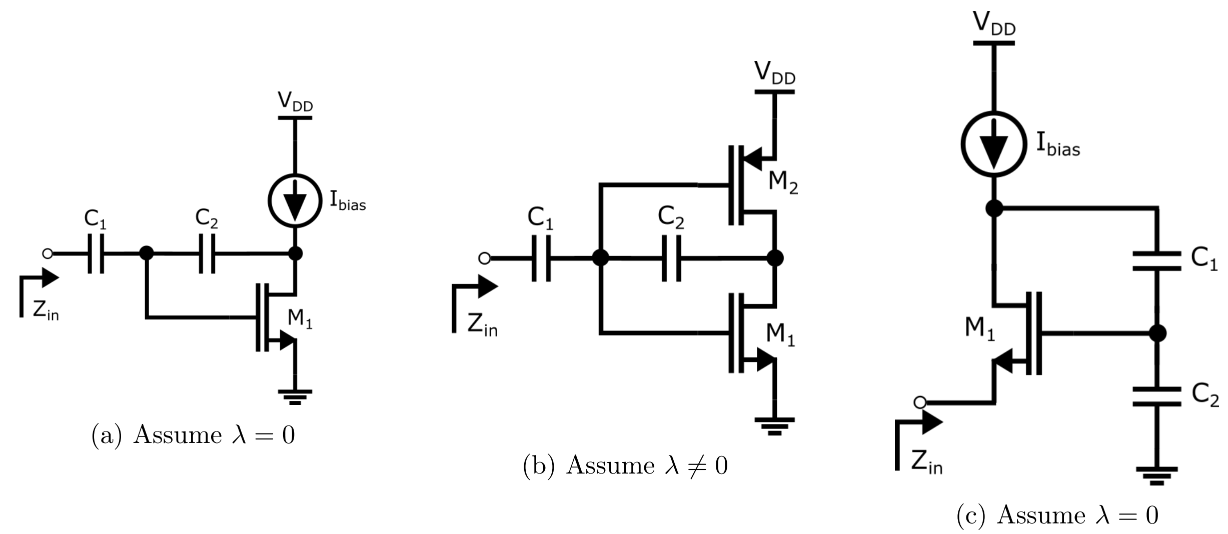


Figure 7: Input impedance analysis for single stage circuits

For the circuits shown in Fig. 7 and neglecting all other intrinsic capacitances,

(a) Determine the input impedances, Z_{in} of each circuit. [8 pts]

For Fig. 7a, the equivalent small signal circuit is shown below:

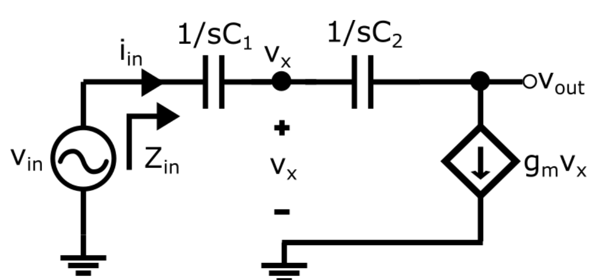


Figure 8: Small signal circuit of Fig. 7a

From the figure,

$$i_{in} = (v_{in} - v_x)sC_1 = g_m v_x \tag{34}$$

$$v_{in}sC_1 = (g_m + sC_1)v_x \tag{35}$$

$$v_{in} = \frac{(g_m + sC_1)v_x}{sC_1} \tag{36}$$

$$Z_{in} = \frac{v_{in}}{i_{in}} = \frac{g_m + sC_1}{g_m sC_1} \tag{37}$$

$$\boxed{Z_{in} = \tfrac{g_m+sC_1}{g_m sC_1} \text{ [1 pt]}}$$

[1 pt] will be given for the solution.

For Fig. 7b, the equivalent small signal circuit is shown below:

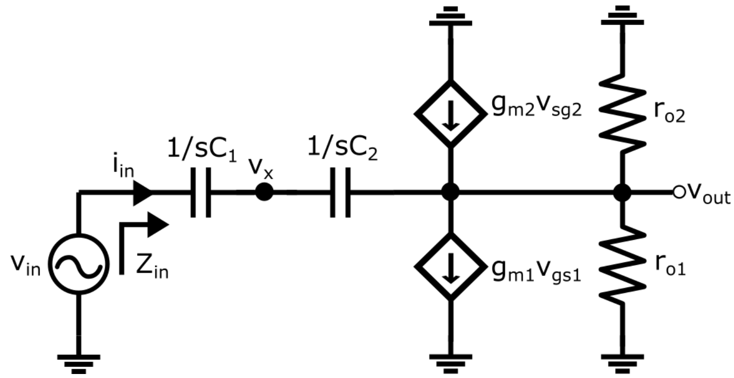


Figure 9: Small signal circuit of Fig. 7b

Simplifying Fig. 9,

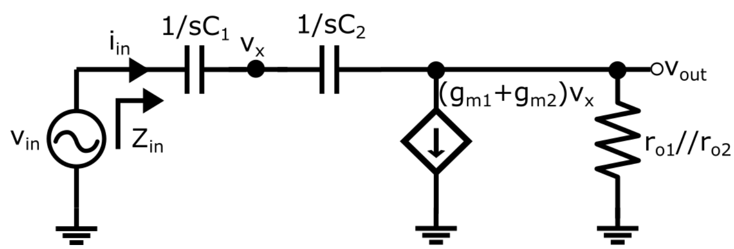


Figure 10: Simplified small signal circuit of Fig. 7b

$$i_{in} = (v_{in} - v_x)sC_1 = (v_x - v_{out})sC_2 = (g_{m1} + g_{m2})v_x + \frac{v_{out}}{r_{o1}//r_{o2}} \tag{38}$$

For this part of the solution, let us simply denote $g_m = g_{m1} + g_{m2}$ and $r_o = r_{o1}//r_{o2}$. From Eq. 38,

$$\frac{v_{out}}{v_x} = \frac{sC_2 - g_m}{sC_2 + \frac{1}{r_o}} \tag{39}$$

Also from Eq. 38,

$$s(C_1 + C_2)v_x = sC_1 v_{in} + sC_2 v_{out} \tag{40}$$

$$s(C_1 + C_2)v_x = sC_1 v_{in} + \frac{(sC_2)^2 - g_m sC_2}{sC_2 + \frac{1}{r_o}} v_x \tag{41}$$

Simplifying Eq. 41,

$$v_x = \left(\frac{sC_1C_2 + \frac{C_1}{r_o}}{sC_1C_2 + \frac{C_1+C_2}{r_o} + g_m C_2}\right) v_{in} \tag{42}$$

Again, using Eq. 38,

$$i_{in} = sC_1(v_{in} - v_x) = sC_1 \left(1 - \frac{sC_1C_2 + \frac{C_1}{r_o}}{sC_1C_2 + \frac{C_1+C_2}{r_o} + g_m C_2}\right) v_{in} \tag{43}$$

$$i_{in} = \left(\frac{sC_1C_2(\frac{1}{r_o} + g_m)}{sC_1C_2 + \frac{C_1+C_2}{r_o} + g_mC_2} \right) v_{in} \tag{44}$$

$$Z_{in} = \left(\frac{i_{in}}{v_{in}} \right)^{-1} = \frac{sC_1C_2 + \frac{C_1+C_2}{r_o} + g_mC_2}{sC_1C_2(\frac{1}{r_o} + g_m)} \tag{45}$$

$$\boxed{Z_{in} = \frac{sC_1C_2+\frac{C_1+C_2}{r_o}+g_mC_2}{sC_1C_2(\frac{1}{r_o}+g_m)} \text{ [1 pt]}}$$

[2 pts] will be given for the solution.

For Fig. 7c, the equivalent small signal circuit is shown below:

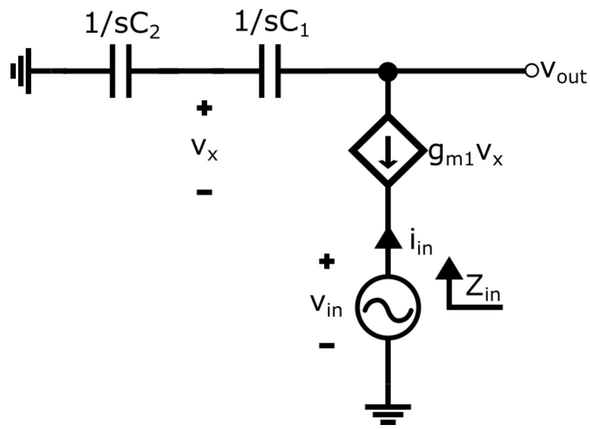


Figure 11: Small signal circuit of Fig. 7c

First we express i_{in} as,

$$i_{in} = -g_{m1}v_x \tag{46}$$

Also we can say that,

$$g_{m1}v_x = -sC_2(v_x + v_{in}) \tag{47}$$

$$v_{in} = -\frac{(g_{m1} + sC_2)}{sC_2}v_x \tag{48}$$

$$Z_{in} = \frac{v_{in}}{i_{in}} = \frac{-\frac{(g_{m1}+sC_2)}{sC_2}v_x}{-g_{m1}v_x} = \frac{g_{m1} + sC_2}{g_{m1}sC_2} \tag{49}$$

$$\boxed{Z_{in} = \frac{g_{m1}+sC_2}{g_{m1}sC_2} \text{ [1 pt]}}$$

[2 pts] will be given for the solution.

(b) Find the transfer function $H(s) = \frac{v_{out}(s)}{v_{in}(s)}$ of Fig. 7a. [2 pts]

In solving this, refer to the small signal circuit shown in Fig. 8. We express the current $g_{m1}v_x$ as,

$$g_{m1}v_x = (v_{in} - v_x)sC_1 \tag{50}$$

As well as,

$$g_{m1}v_x = (v_x - v_{out})sC_2 \tag{51}$$

Re-writing Eq. 50,

$$v_{in} = \frac{g_{m1} + sC_1}{sC_1}v_x \tag{52}$$

As well as Eq. 51,

$$v_{out} = \frac{sC_2 - g_{m1}}{sC_2} v_x \tag{53}$$

Combining Eq. 52 and Eq. 53 to get the transfer function,

$$H(s) = \frac{v_{out}(s)}{v_{in}(s)} = \frac{\frac{sC_2 - g_{m1}}{sC_2} v_x}{\frac{g_{m1} + sC_1}{sC_1} v_x} = \frac{s^2 C_1 C_2 - s g_m C_1}{s^2 C_1 C_2 + s g_m C_2} \tag{54}$$

$$\boxed{H(s) = \tfrac{sC_1C_2 - g_m C_1}{sC_1C_2 + g_m C_2} \text{ [1 pt]}}$$

[1 pt] will be given for the solution.

TOTAL: 30.5/30 points.