ECE 113: Communication Electronics

Meeting 10: Filter Design II

February 25, 2019

Butterworth Filter Design

- All-pole filter of the form:

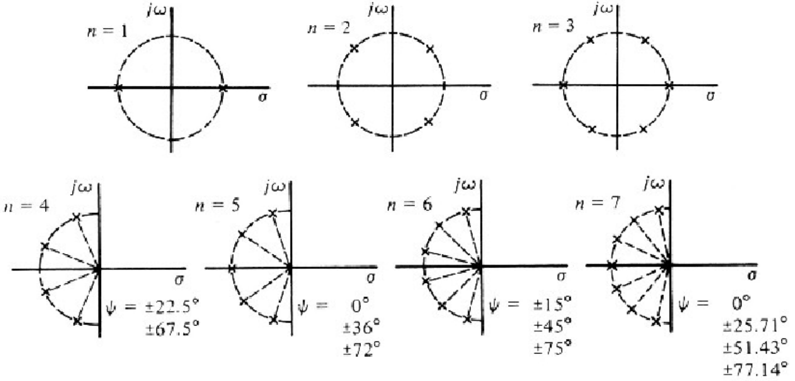
$$|T(j\omega)|^2 = T_n(j\omega)T_n(-j\omega) = \frac{1}{1+\omega^{2n}}$$

  - $|T_n(j0)| = 1$ for all n
  - $|T_n(j1) = \frac{1}{\sqrt{2}} = 0.707$, ω normalized wrt ω_0

- Roots: $1 + (-1)^n s^{2n} = 0$

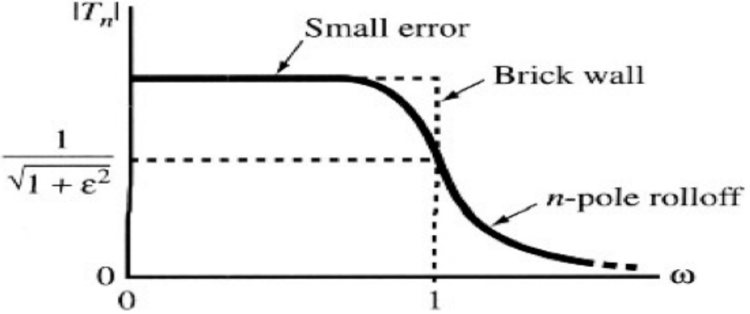
Filter order	D(s)	Expansion of D(s)	LHP Poles
n=1	$1 - s^2$	$(1+s)(1-s)$	$s = -1$
n=2	$1 + s^4$	$(1+\sqrt{2}s+s^2)(1-\sqrt{2}s+s^2)$	$s = \frac{-1 \pm j}{\sqrt{2}}$
n=3	$1 - s^6$	$(1+2s+2s^2+s^3)(1-2s+2s^2-s^3)$	$s = -1, \frac{-1 \pm j\sqrt{3}}{2}$

Pole Locations

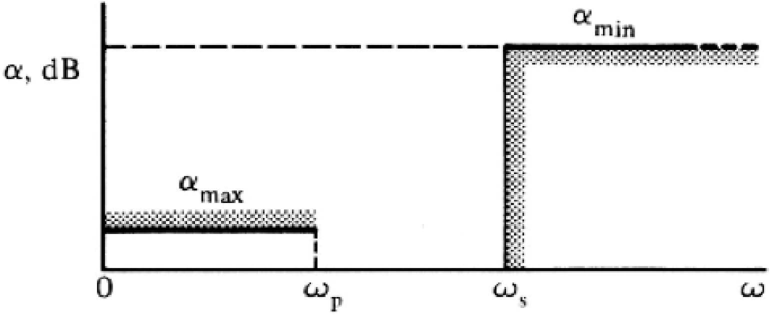


- We generalize with

$$|T_n(\omega)|^2 = \frac{1}{1+\epsilon^2\omega^{2n}}$$

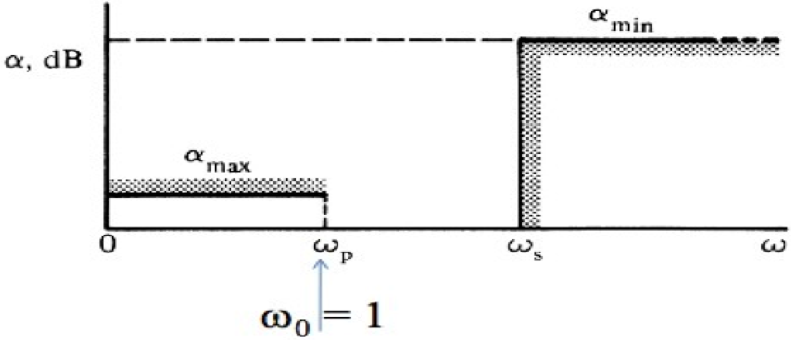


Filter Specifications



- Attenuation ($\alpha_{max}/\alpha_{min}$) in Passband/Stopband
- Passband/Stopband Cutoff Frequencies (ω_p/ω_s)
- Need to determine the filter order n

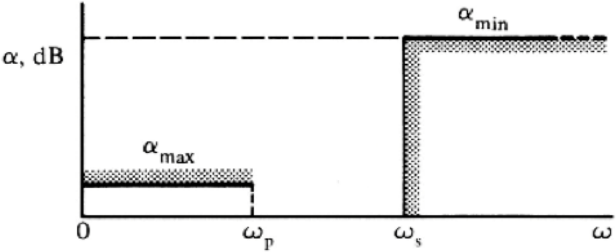
Solving for ϵ



$$\alpha_{max} = -10log|T_n(\omega = 1)|^2 = 20log\sqrt{1+\epsilon^2} = 10log(1+\epsilon^2)$$

$$\therefore \epsilon = \sqrt{10^{0.1\alpha_{max}} - 1}$$

Determining the Filter Order



$$\alpha_{min} = -10log|T_n(\omega_s)|^2 = 20log\sqrt{1+\epsilon^2\omega_s^{2n}} = 10log(1+\epsilon^2\omega_s^{2n})$$

$$\therefore \alpha_{min} = 10log(1+(10^{0.1\alpha_{max}}-1)\omega_s^{2n})$$

- It can be easily shown that:

$$n = \frac{log(10^{0.1\alpha_{min}}-1) - log(10^{0.1\alpha_{max}}-1)}{2log\omega_s}$$

New Normalizing Frequency

$$|T_n(j\omega)|^2 = \frac{1}{1+\epsilon^2(\omega/\omega_p)^{2n}}$$

- To make this equation look like that of a Butterworth filter

$$|T_n(j\omega)|^2 = \frac{1}{1+[\epsilon^{1/n}(\omega/\omega_p)]^{2n}} = \frac{1}{1+[\omega/(\epsilon^{-1/n}\omega_p)]^{2n}}$$

Example

- Design a Butterworth filter with the following specifications

$$\alpha_{max} = 0.5dB \qquad \alpha_{min} = 20dB$$
$$\omega_p = 1000rad/s \qquad \omega_s = 2000rad/s$$

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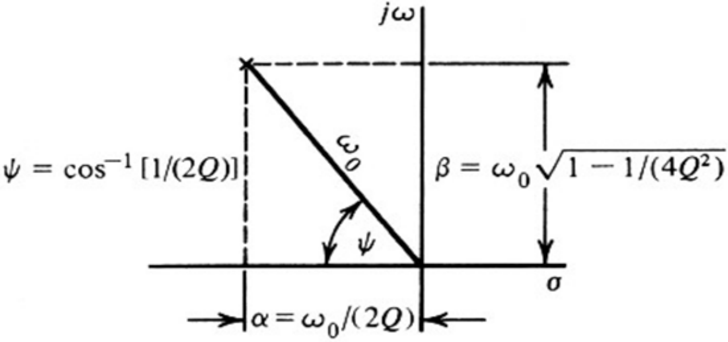
$$\alpha_{max} = 0.5dB \qquad \alpha_{min} = 20dB$$
$$\omega_p = 1000rad/s \qquad \omega_s = 2000rad/s$$

$$\epsilon^2 = 10^{0.1\alpha_{max}} - 1 = 0.122$$

$$n = \frac{log[(10^{0.1\alpha_{min}} - 1)/(10^{0.1\alpha_{max}} - 1)]}{2log\omega_s} = 4.83 \approx 5$$

$$T_5(s) = \frac{1}{(s+1)(s^2+0.618s+1)(s^2+1.618s+1)}$$

$$T_5(j\omega) = \underset{\displaystyle Q=\frac{1}{2}}{\underset{\downarrow}{\frac{1}{s+1}}} \quad \underset{\displaystyle Q=\frac{1}{1.618}}{\underset{\downarrow}{\frac{1}{s^2+1.618s+1}}} \quad \underset{\displaystyle Q=\frac{1}{0.618}}{\underset{\downarrow}{\frac{1}{s^2+0.618s+1}}}$$



$$T_5(j\omega) = \boxed{\frac{1}{s+1}} \frac{1}{s^2 + 1.618s + 1} \frac{1}{s^2 + 0.618s + 1}$$

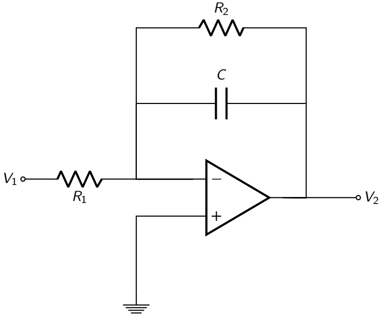
- For simplicity we set

$$R_1 = R_2 = R$$

- Transfer function

$$T(s) = -\frac{R_2}{sCR_1R_2 + R_1}$$

$$T(s) = -\frac{1}{SRC + 1}$$



Circuit Design

$$T_5(j\omega) = \boxed{\frac{1}{s+1}} \frac{1}{s^2+1.618s+1} \frac{1}{s^2+0.618s+1}$$

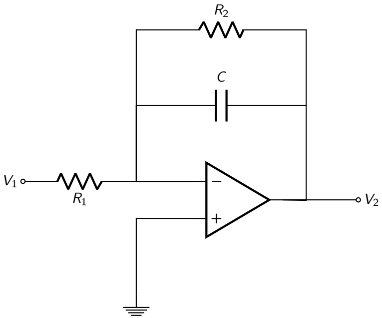
$$T(s) = -\frac{1}{sRC+1}$$

- de-normalize ω_o

$$\omega_o = \epsilon^{-1/n}\omega_p = \frac{1}{RC}$$

$$\omega_o = 1234 rad/s$$

$$C = 0.1 \mu F \qquad R = 8.1 k\Omega$$

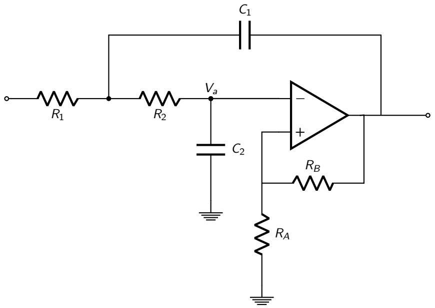


$$T_5(j\omega) = \frac{1}{s+1}\boxed{\frac{1}{s^2 + 1.618s + 1}}\frac{1}{s^2 + 0.618s + 1}$$

$$T(s) = \frac{K}{s^2 C_1 C_2 R_1 R_2 + s[C_2(R_1 + R_2) + C_1 R_1 (1 - K)] + 1}$$

- For simplicity we set

$$C_1 = C_2 = C$$
$$R_1 = R_2 = R$$



- $\omega_o = \dfrac{1}{RC}$

- $Q = \dfrac{1}{2 + 1 - K}$

Circuit Design

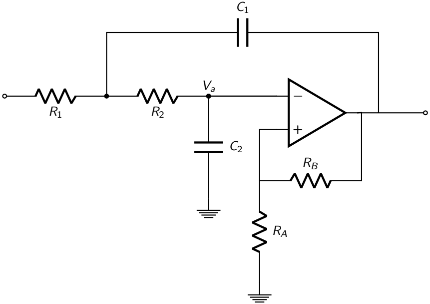
$$T_5(j\omega) = \frac{1}{s+1} \boxed{\frac{1}{s^2 + 1.618s + 1}} \frac{1}{s^2 + 0.618s + 1}$$

$$T(s) = \frac{K}{s^2 C_1 C_2 R_1 R_2 + s[C_2(R_1 + R_2) + C_1 R_1 (1 - K)] + 1}$$

$C = 0.1\mu\text{F} \quad R = 8.1\text{k}\Omega$

$K = 3 - 1.618 = 1.382$

$1/K = 0.724$

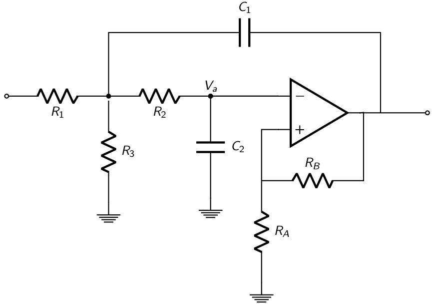


$$T_5(j\omega) = \frac{1}{s+1}\boxed{\frac{1}{s^2 + 1.618s + 1}}\frac{1}{s^2 + 0.618s + 1}$$

$$T(s) = \frac{K}{s^2 C_1 C_2 R_1 R_2 + s[C_2(R_1 + R_2) + C_1 R_1 (1 - K)] + 1}$$

$$1 + \frac{R_B}{R_A} = K$$

$$\frac{R_3}{R_1 + R_3} = \frac{1}{K}$$



$$T_5(j\omega) = \frac{1}{s+1} \boxed{\frac{1}{s^2 + 1.618s + 1}} \frac{1}{s^2 + 0.618s + 1}$$

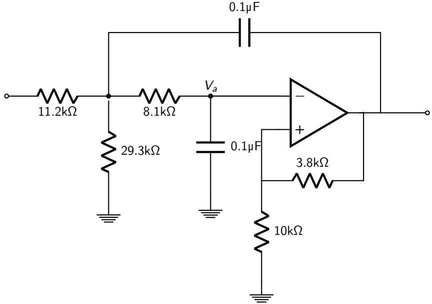
$$T(s) = \frac{K}{s^2 C_1 C_2 R_1 R_2 + s[C_2(R_1 + R_2) + C_1 R_1 (1 - K)] + 1}$$

$$R_A = 10\text{k}\Omega$$

$$R_B = 3.8\text{k}\Omega$$

$$R_1 = KR = 11.2\text{k}\Omega$$

$$R_3 = 29.3\text{k}\Omega$$



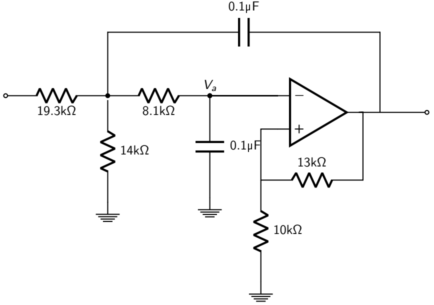
$$T_5(j\omega) = \frac{1}{s+1}\ \frac{1}{s^2+1.618s+1}\ \boxed{\frac{1}{s^2+0.618s+1}}$$

- Using the same procedure as the second transfer function, it can be derived:

$$R = 8.1\text{k}\Omega \qquad C = 0.1\mu\text{F}$$

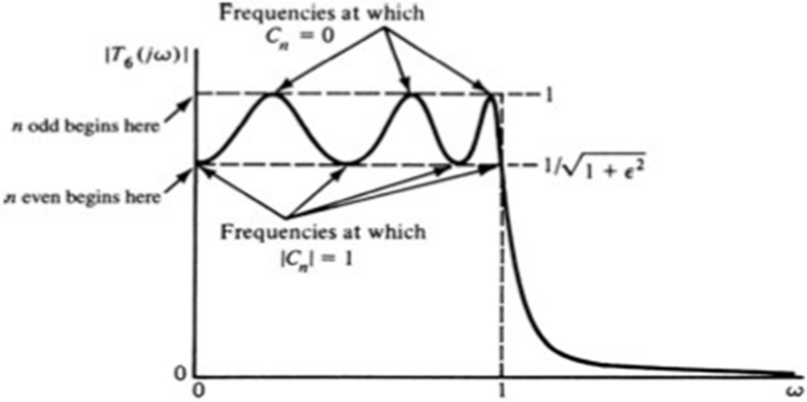
$$R_A = 10\text{k}\Omega \qquad R_B = 13\text{k}\Omega$$

$$R_1 = 19.3\text{k}\Omega \qquad R_3 = 14\text{k}\Omega$$



Chebyshev Type I Filter Design

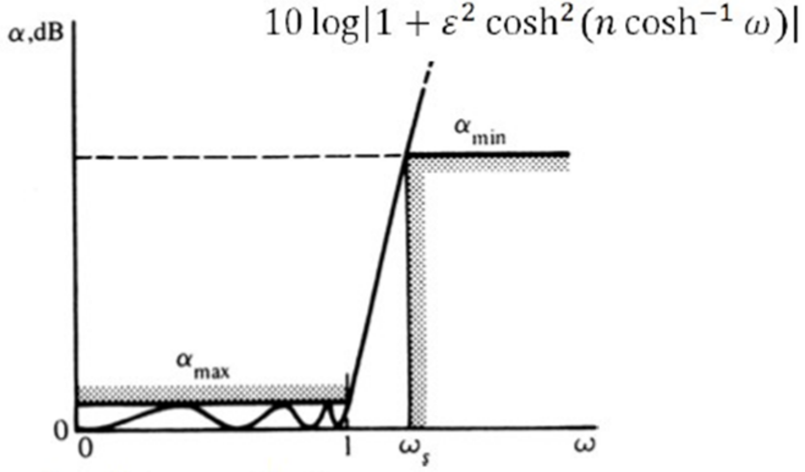
$$|T_n(j\omega)|^2 = \frac{1}{1+\epsilon^2 C_n^2(\omega)}$$



n=6

Attenuation Response

$$\alpha_n = -10log|T_n(j\omega)|^2 = 10log|1 + \epsilon^2 C_n^2(\omega)|\text{dB}$$



Pole Locations

$$T(s)T(-s) = |T_n(j\omega)|^2 \, |_{\omega=\frac{s}{j}} = \frac{1}{1+\epsilon^2 C_n^2(\frac{s}{j})}$$

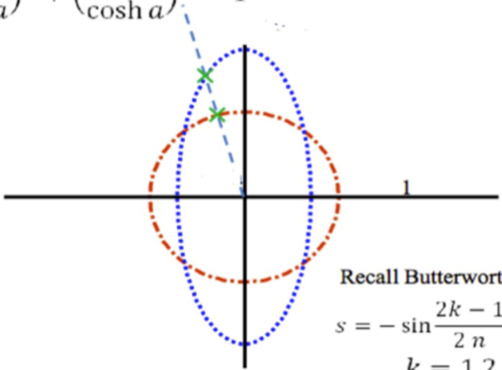
- Roots of $T(s)$ are:

$$s = -\sinh a \sin \frac{2k-1}{2n}\pi + j \cosh a \cos \frac{2k-1}{2n}\pi \qquad k = 1, 2, \ldots, n$$

$$a = \frac{1}{n} \sinh^{-1} \frac{1}{\epsilon}$$

Pole Locations

$$\left(\frac{\sigma_k}{\sinh a}\right)^2 + \left(\frac{\omega_k}{\cosh a}\right)^2 = 1$$



Recall Butterworth:

$$s = -\sin\frac{2k-1}{2n}\pi + j\cos\frac{2k-1}{2n}\pi$$

$$k = 1, 2, \dots, n$$

Normalizing Frequency

$$\alpha_n = 10log|1 + \epsilon^2 \cosh^2(n \cosh^{-1} \omega)| = 3\text{dB}$$

$$1 + \epsilon^2 \cosh^2(n \cosh^{-1} \omega) = 10^{3/10} = 2$$

$$\cosh(n \cosh^{-1} \omega) = \frac{1}{\epsilon}$$

$$\omega = \cosh(\frac{1}{n} \cosh^{-1} \frac{1}{\epsilon})$$

- De-normalizing Factor

$$\omega = \omega_p \cosh(\frac{1}{n} \cosh^{-1} \frac{1}{\epsilon})$$

Steps in Designing a Chebyshev Type I Filter

- Find n

$$n = \frac{\cosh^{-1}[(10^{0.1\alpha_{min}} - 1)/(10^{0.1\alpha_{max}} - 1)]^{1/2}}{\cosh^{-1}\omega_s}$$

- Find ϵ

$$\epsilon = \sqrt{10^{0.1\alpha_{max}} - 1}$$

- Find pole locations

$$a = \frac{1}{n}\sinh^{-1}\frac{1}{\epsilon}$$

$$s = -\sinh a \sin\frac{2k-1}{2n}\pi + j\cosh a \cos\frac{2k-1}{2n}\pi$$

- Circuit Design

Example

Design a Chebyshev filter to meet the following requirements:

$$n = 5 \qquad 0dB \text{ passband gain}$$

$$\alpha_{max} = 0.5dB \qquad \text{in } 0 \geq \omega \geq 1000 rad/s$$

- Solving ϵ: $\epsilon = \sqrt{10^{0.1\alpha_{max}} - 1} = 0.3493$
- Solving a: $a = 0.35484$
- Obtaining pole locations:

$$s = -0.3623;\ -0.2931 \pm j0.6252;\ -0.1120 \pm j1.0116$$

- Transfer function:

$$T_5(s) = \frac{K}{(s + 0.3623)(s^2 + 0.5862s + 0.4768)(s^2 + 0.2239s + 1.0358)}$$

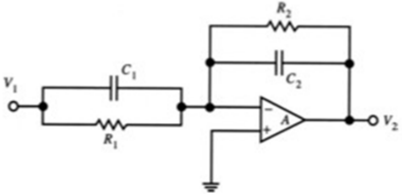
$$K = 0.1789^{-1}$$

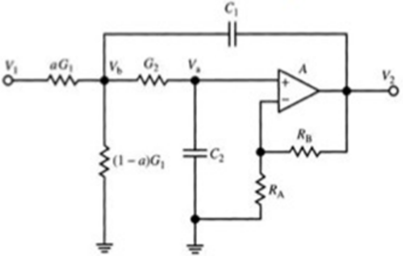
- Circuit Design

$$T_5(s) = \frac{k_1}{(s+0.3623)}\frac{k_2}{(s^2+0.5862s+0.4768)}\frac{k_3}{(s^2+0.2239s+1.0358)}$$

Sallen-key

Bilinear





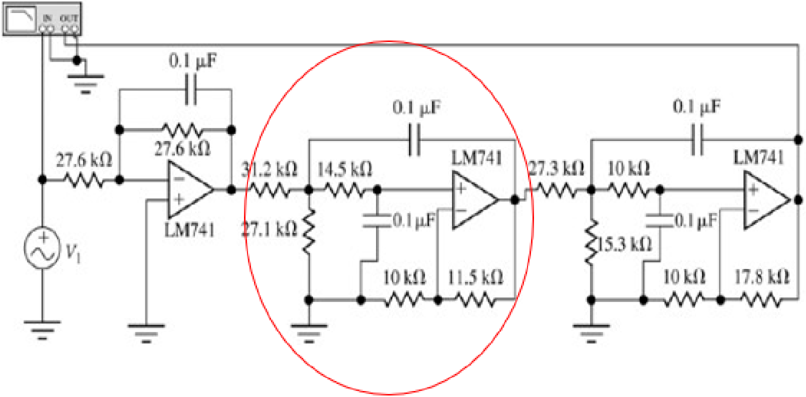
$$T(s) = \frac{H\omega_o^2}{s^2 + (\frac{\omega_o}{Q})s + \omega_o^2}$$

- For the second transfer function,
- $\omega_o = 690.5 rad/s = \dfrac{1}{RC}$

$$C = 0.1\mu\text{F} \qquad R = 14.5\text{k}\Omega$$

- $Q = 1.1778 \rightarrow K = 2.151$
- $K = 1 + \dfrac{R_B}{R_A} \rightarrow R_A = 10\text{k}\Omega \qquad R_B = 11.5\text{k}\Omega$
- $\dfrac{R_3}{R_1 + R_3} = \dfrac{1}{K}$ and $R_3 \parallel R_1 = R$

$$R_1 = 31.2\text{k}\Omega \qquad R_3 = 27.1\text{k}\Omega$$



- Component values for the other stages can be obtained using the same manner.

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