Graph Theory Theorems in random graphs

CGT 2024, Qingdao

Jie Han
han.jie@bit.edu.cn

School of Mathematics and Statistics
Beijing Institute of Technology

September 20, 2024

Subgraphs

北京理工大学
BEIJING INSTITUTE OF TECHNOLOGY

Question 1

When does a graph G contain F as a subgraph?

- Small F: Turán, Ramsey
- Large F: Dirac, Hajnal–Szemerédi

Question 2

When does a random graph $G(n, p)$ contain F as a subgraph?

- Threshold function
- Kahn–Kalai, Johansson–Kahn–Vu, Montgomery

Question 3

Can we combine these two lines of research?

Subgraphs

北京理工大学
BEIJING INSTITUTE OF TECHNOLOGY

Question 1

When does a graph G contain F as a subgraph?

- Small F: Turán, Ramsey
- Large F: Dirac, Hajnal–Szemerédi

Question 2

When does a random graph $G(n, p)$ contain F as a subgraph?

- Threshold function
- Kahn–Kalai, Johansson–Kahn–Vu, Montgomery

Question 3

Can we combine these two lines of research?

Subgraphs

北京理工大学 BEIJING INSTITUTE OF TECHNOLOGY

Question 1

When does a graph G contain F as a subgraph?

- Small F: Turán, Ramsey
- Large F: Dirac, Hajnal–Szemerédi

Question 2

When does a random graph $G(n, p)$ contain F as a subgraph?

- Threshold function
- Kahn–Kalai, Johansson–Kahn–Vu, Montgomery

Question 3

Can we combine these two lines of research?

Subgraphs

Question 1

When does a graph G contain F as a subgraph?

- Small F: Turán, Ramsey
- Large F: Dirac, Hajnal–Szemerédi

Question 2

When does a random graph $G(n, p)$ contain F as a subgraph?

- Threshold function
- Kahn–Kalai, Johansson–Kahn–Vu, Montgomery

Question 3

Can we combine these two lines of research?

Question 1

When does a graph G contain F as a subgraph?

- Small F: Turán, Ramsey
- Large F: Dirac, Hajnal–Szemerédi

Question 2

When does a random graph $G(n,p)$ contain F as a subgraph?

- Threshold function
- Kahn–Kalai, Johansson–Kahn–Vu, Montgomery

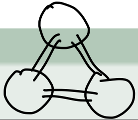
Question 3

Can we combine these two lines of research?

Graph Theory Theorems

Turán's Theorem

If a graph G has no copy of K_r, then its density is at most $1 - \frac{1}{r-1}$.



Ramsey's Theorem

In every 2-edge coloring of K_n one can find a monochromatic clique of size $\log n/2$.

Dirac's Theorem

Every graph G with at least $n \geq 3$ vertices and $\delta(G) \geq n/2$ has a Hamilton cycle.

In random graphs?

$G(n,p)$: n vertices, every pair of vertices forms an edge with probability p independent of others.

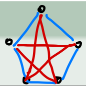
Graph Theory Theorems

Turán's Theorem

If a graph G has no copy of K_r, then its density is at most $1 - \frac{1}{r-1}$.

Ramsey's Theorem

In every 2-edge coloring of K_n one can find a monochromatic clique of size $\log n/2$.



Dirac's Theorem

Every graph G with at least $n \geq 3$ vertices and $\delta(G) \geq n/2$ has a Hamilton cycle.

In random graphs?

$G(n,p)$: n vertices, every pair of vertices forms an edge with probability p independent of others.

Graph Theory Theorems

Turán's Theorem

If a graph G has no copy of K_r, then its density is at most $1 - \frac{1}{r-1}$.

Ramsey's Theorem

In every 2-edge coloring of K_n one can find a monochromatic clique of size $\log n/2$.

Dirac's Theorem

Every graph G with at least $n \geq 3$ vertices and $\delta(G) \geq n/2$ has a Hamilton cycle.

In random graphs?

$G(n, p)$: n vertices, every pair of vertices forms an edge with probability p independent of others.

Graph Theory Theorems

Turán's Theorem

If a graph G has no copy of K_r, then its density is at most $1 - \frac{1}{r-1}$.

Ramsey's Theorem

In every 2-edge coloring of K_n one can find a monochromatic clique of size $\log n/2$.

Dirac's Theorem

Every graph G with at least $n \geq 3$ vertices and $\delta(G) \geq n/2$ has a Hamilton cycle.

In random graphs?
$G(n,p)$: n vertices, every pair of vertices forms an edge with probability p independent of others.

2-density

Definition. (2-density)

Given a graph F, let

$$m_2(F) := \max_{F' \subseteq F, v(F') \geq 3} \frac{e(F') - 1}{v(F') - 2}.$$

- If $p \ll n^{-\frac{1}{m_2(F)}}$, then in $G(n,p)$ aas the number of copies of F is much smaller than its number of edges.
- The above threshold is called the "deletion threshold".

2-density

Definition. (2-density)

Given a graph F, let

$$m_2(F) := \max_{F' \subseteq F, v(F') \geq 3} \frac{e(F') - 1}{v(F') - 2}.$$

- If $p \ll n^{-\frac{1}{m_2(F)}}$, then in $G(n,p)$ aas the number of copies of F is much smaller than its number of edges.
- The above threshold is called the "deletion threshold".

2-density

BEIJING INSTITUTE OF TECHNOLOGY 北京理工大學

Definition. (2-density)

Given a graph F, let
$$m_2(F) := \max_{F' \subseteq F, v(F') \geq 3} \frac{e(F') - 1}{v(F') - 2}.$$

- If $p \ll n^{-\frac{1}{m_2(F)}}$, then in $G(n, p)$ aas the number of copies of F is much smaller than its number of edges.
- The above threshold is called the "deletion threshold".

- Turán density $\pi(F)$: the maximum edge density an F-free graph can have.
- Asymptotically almost surely (aas): the probability of the event goes to 1 as $n \to \infty$.

Theorem. [Conlon–Gowers, Schacht, Ann. Math., 2016, Conjectured by Kohayakawa–Łuczak–Rödl]

Let $p \gg n^{-\frac{1}{m_2(F)}}$. Then as $n \to \infty$, aas the largest F-free subgraph of $G(n,p)$ has $p(\pi(F) + o(1))\binom{n}{2}$ edges.

1. If $p \ll n^{-\frac{1}{m_2(F)}}$, then in $G(n,p)$ aas the largest F-free subgraph of $G(n,p)$ has $p(1 - o(1))\binom{n}{2}$ edges.
2. Its general form works for all r-uniform hypergraphs
3. Optimal value still open for bipartite F.

- Turán density $\pi(F)$: the maximum edge density an F-free graph can have.
- Asymptotically almost surely (aas): the probability of the event goes to 1 as $n \to \infty$.

Theorem. [Conlon–Gowers, Schacht, Ann. Math., 2016, Conjectured by Kohayakawa–Łuczak–Rödl]

Let $p \gg n^{-\frac{1}{m_2(F)}}$. Then as $n \to \infty$, aas the largest F-free subgraph of $G(n,p)$ has $p(\pi(F)+o(1))\binom{n}{2}$ edges.

1. If $p \ll n^{-\frac{1}{m_2(F)}}$, then in $G(n,p)$ aas the largest F-free subgraph of $G(n,p)$ has $p(1-o(1))\binom{n}{2}$ edges.
2. Its general form works for all r-uniform hypergraphs
3. Optimal value still open for bipartite F.

Random Turán Theorem – The KŁR Conjecture

- Turán density $\pi(F)$: the maximum edge density an F-free graph can have.
- Asymptotically almost surely (aas): the probability of the event goes to 1 as $n \to \infty$.

Theorem. [Conlon–Gowers, Schacht, Ann. Math., 2016, Conjectured by Kohayakawa–Łuczak–Rödl]

Let $p \gg n^{-\frac{1}{m_2(F)}}$. Then as $n \to \infty$, aas the largest F-free subgraph of $G(n,p)$ has $p(\pi(F) + o(1))\binom{n}{2}$ edges.

1. If $p \ll n^{-\frac{1}{m_2(F)}}$, then in $G(n,p)$ aas the largest F-free subgraph of $G(n,p)$ has $p(1 - o(1))\binom{n}{2}$ edges.
2. Its general form works for all r-uniform hypergraphs
3. Optimal value still open for bipartite F.

Random Turán Theorem – The KŁR Conjecture

- Turán density $\pi(F)$: the maximum edge density an F-free graph can have.
- Asymptotically almost surely (aas): the probability of the event goes to 1 as $n \to \infty$.

Theorem. [Conlon–Gowers, Schacht, Ann. Math., 2016, Conjectured by Kohayakawa–Łuczak–Rödl]

Let $p \gg n^{-\frac{1}{m_2(F)}}$. Then as $n \to \infty$, aas the largest F-free subgraph of $G(n,p)$ has $p(\pi(F)+o(1))\binom{n}{2}$ edges.

1. If $p \ll n^{-\frac{1}{m_2(F)}}$, then in $G(n,p)$ aas the largest F-free subgraph of $G(n,p)$ has $p(1-o(1))\binom{n}{2}$ edges.
2. Its general form works for all r-uniform hypergraphs
3. Optimal value still open for bipartite F.

Random Ramsey Theorem

Ramsey's Theorem

In every 2-edge coloring of K_n one can find a monochromatic clique of size $\log n/2$.

Theorem. [Rödl–Ruciński, J.AMS, 1995]

Let F be a graph with a cycle. Let $p \gg n^{-\frac{1}{m_2(F)}}$. Then aas every 2-edge coloring of $G(n,p)$ contains a monochromatic copy of F.
If $p \ll n^{-\frac{1}{m_2(F)}}$, then aas this doesn't happen.

Random Dirac Theorem

Dirac's Theorem

Every graph G with at least $n \geq 3$ vertices and $\delta(G) \geq n/2$ has a Hamilton cycle.

Threshold for Hamiltonicity, Pósa–Korshunov

Let $p \geq C \log n/n$. Then aas $G(n,p)$ has a Hamilton cycle.

Theorem. [Lee–Sudakov, RS&A, 2012]

For every ε, there exists $C = C(\varepsilon)$ s.t. TFH. Let $p \geq C \log n/n$. Then aas every subgraph of $G(n,p)$ with minimum degree at least $(1/2 + \varepsilon)np$ has a Hamilton cycle.

- $1/2$ is best possible: removing edges of a bisection makes a graph disconnected.
- This line of research is called the "local resilience".

Random Dirac Theorem

Dirac's Theorem

Every graph G with at least $n \geq 3$ vertices and $\delta(G) \geq n/2$ has a Hamilton cycle.

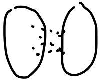
Threshold for Hamiltonicity, Pósa–Korshunov

Let $p \geq C \log n/n$. Then aas $G(n,p)$ has a Hamilton cycle.

Theorem. [Lee–Sudakov, RS&A, 2012]

For every ε, there exists $C = C(\varepsilon)$ s.t. TFH. Let $p \geq C \log n/n$. Then aas every subgraph of $G(n,p)$ with minimum degree at least $(1/2 + \varepsilon)np$ has a Hamilton cycle.

- $1/2$ is best possible: removing edges of a bisection makes a graph disconnected.
- This line of research is called the "local resilience".



Random Dirac Theorem

Dirac's Theorem

Every graph G with at least $n \geq 3$ vertices and $\delta(G) \geq n/2$ has a Hamilton cycle.

Threshold for Hamiltonicity, Pósa–Korshunov

Let $p \geq C \log n/n$. Then aas $G(n,p)$ has a Hamilton cycle.

Theorem. [Lee–Sudakov, RS&A, 2012]

For every ε, there exists $C = C(\varepsilon)$ s.t. TFH. Let $p \geq C \log n/n$. Then aas every subgraph of $G(n,p)$ with minimum degree at least $(1/2+\varepsilon)np$ has a Hamilton cycle.

- $1/2$ is best possible: removing edges of a bisection makes a graph disconnected.
- This line of research is called the "local resilience".

Robust Dirac Theorem

Theorem. [Lee–Sudakov, RS&A, 2012]

For every ε, there exists $C = C(\varepsilon)$ s.t. TFH. Let $p \geq C \log n/n$. Then aas every subgraph of $G(n, p)$ with minimum degree at least $(1/2 + \varepsilon)np$ has a Hamilton cycle.

Later, they proved a closely related version.

Theorem. [Krivelevich–Lee–Sudakov, T.AMS, 2014]

There exists $C > 0$ s.t. TFH. Let G be an n-vertex graph with $\delta(G) \geq n/2$. Let $p \geq C \log n/n$. Then aas $G \cap G(n, p)$ has a Hamilton cycle.

- We call $G_p := G \cap G(n, p)$ the p-random sparsification of G.

Robust Dirac Theorem

Theorem. [Lee–Sudakov, RS&A, 2012]

For every ε, there exists $C = C(\varepsilon)$ s.t. TFH. Let $p \geq C \log n/n$. Then aas every subgraph of $G(n,p)$ with minimum degree at least $(1/2+\varepsilon)np$ has a Hamilton cycle.

Later, they proved a closely related version.

Theorem. [Krivelevich–Lee–Sudakov, T.AMS, 2014]

There exists $C > 0$ s.t. TFH. Let G be an n-vertex graph with $\delta(G) \geq n/2$. Let $p \geq C \log n/n$. Then aas $G \cap G(n,p)$ has a Hamilton cycle.

- We call $G_p := G \cap G(n,p)$ the p-random sparsification of G.

Robust Dirac Theorem

Theorem. [Lee–Sudakov, RS&A, 2012]

For every ε, there exists $C = C(\varepsilon)$ s.t. TFH. Let $p \geq C \log n/n$. Then aas every subgraph of $G(n, p)$ with minimum degree at least $(1/2 + \varepsilon)np$ has a Hamilton cycle.

Later, they proved a closely related version.

Theorem. [Krivelevich–Lee–Sudakov, T.AMS, 2014]

There exists $C > 0$ s.t. TFH. Let G be an n-vertex graph with $\delta(G) \geq n/2$. Let $p \geq C \log n/n$. Then aas $G \cap G(n, p)$ has a Hamilton cycle.

- We call $G_p := G \cap G(n, p)$ the p-random sparsification of G.

Results: Robustness

Robust Hajnal–Szemerédi Theorem. [Allen et al., Pham et al. 2024+]

Let G be an n-vertex graph with $\delta(G) \geq (1-1/r)n$. Let $p \geq Cn^{-2/r}(\log n)^{1/\binom{r}{2}}$. Then aas G_p has a K_r-factor.

Generalized to F-factors for all strictly balanced graphs F by Kelly–Müyesser–Pokrovskiy.

$th(F)$: the threshold for F-containment property of $G(n,p)$.

Meta Theorem: Robustness.

Suppose that every n-vertex graph G with $\delta(G) \geq \alpha n$ contains a subgraph F. Let $p \gg th(F)$. Let G be an n-vertex graph G with $\delta(G) \geq (\alpha+\varepsilon)n$. Then aas G_p contains a copy of F.

Known for hypergraph perfect matchings, (certain) hypergraph Hamilton cycles under minimum degree conditions.

Results: Robustness

Robust Hajnal–Szemerédi Theorem. [Allen et al., Pham et al. 2024+]

Let G be an n-vertex graph with $\delta(G) \geq (1 - 1/r)n$. Let $p \geq Cn^{-2/r}(\log n)^{1/\binom{r}{2}}$. Then aas G_p has a K_r-factor.

Generalized to F-factors for all strictly balanced graphs F by Kelly–Müyesser–Pokrovskiy.

$th(F)$: the threshold for F-containment property of $G(n, p)$.

Meta Theorem: Robustness.

Suppose that every n-vertex graph G with $\delta(G) \geq \alpha n$ contains a subgraph F. Let $p \gg th(F)$. Let G be an n-vertex graph G with $\delta(G) \geq (\alpha + \varepsilon)n$. Then aas G_p contains a copy of F.

Known for hypergraph perfect matchings, (certain) hypergraph Hamilton cycles under minimum degree conditions.

Results: Robustness

北京理工大學 BEIJING INSTITUTE OF TECHNOLOGY

Robust Hajnal–Szemerédi Theorem. [Allen et al., Pham et al. 2024+]

Let G be an n-vertex graph with $\delta(G) \geq (1 - 1/r)n$. Let $p \geq Cn^{-2/r}(\log n)^{1/\binom{r}{2}}$. Then aas G_p has a K_r-factor.

Generalized to F-factors for all strictly balanced graphs F by Kelly–Müyesser–Pokrovskiy.

$th(F)$: the threshold for F-containment property of $G(n,p)$.

Meta Theorem: Robustness.

Suppose that every n-vertex graph G with $\delta(G) \geq \alpha n$ contains a subgraph F. Let $p \gg th(F)$. Let G be an n-vertex graph G with $\delta(G) \geq (\alpha + \varepsilon)n$. Then aas G_p contains a copy of F.

Known for hypergraph perfect matchings, (certain) hypergraph Hamilton cycles under minimum degree conditions.

Results: Robustness

Robust Hajnal–Szemerédi Theorem. [Allen et al., Pham et al. 2024+]

Let G be an n-vertex graph with $\delta(G) \geq (1-1/r)n$. Let $p \geq Cn^{-2/r}(\log n)^{1/\binom{r}{2}}$. Then aas G_p has a K_r-factor.

Generalized to F-factors for all strictly balanced graphs F by Kelly–Müyesser–Pokrovskiy.

$th(F)$: the threshold for F-containment property of $G(n,p)$.

Meta Theorem: Robustness.

Suppose that every n-vertex graph G with $\delta(G) \geq \alpha n$ contains a subgraph F. Let $p \gg th(F)$. Let G be an n-vertex graph G with $\delta(G) \geq (\alpha+\varepsilon)n$. Then aas G_p contains a copy of F.

Known for hypergraph perfect matchings, (certain) hypergraph Hamilton cycles under minimum degree conditions.

Results: Robustness

Robust Hajnal–Szemerédi Theorem. [Allen et al., Pham et al. 2024+]

Let G be an n-vertex graph with $\delta(G) \geq (1 - 1/r)n$. Let $p \geq Cn^{-2/r}(\log n)^{1/\binom{r}{2}}$. Then aas G_p has a K_r-factor.

Generalized to F-factors for all strictly balanced graphs F by Kelly–Müyesser–Pokrovskiy.

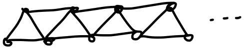
$th(F)$: the threshold for F-containment property of $G(n,p)$.

Meta Theorem: Robustness.

Suppose that every n-vertex graph G with $\delta(G) \geq \alpha n$ contains a subgraph F. Let $p \gg th(F)$. Let G be an n-vertex graph G with $\delta(G) \geq (\alpha + \varepsilon)n$. Then aas G_p contains a copy of F.

Known for hypergraph perfect matchings, (certain) hypergraph Hamilton cycles under minimum degree conditions.

Our result: Robust Thresholds



Robust Pósa–Seymour. [Chen–H.–Luo, 2024++, independently by Joos et al.]

Let G be an n-vertex graph with $\delta(G) \geq (1 - \frac{1}{r+1} + o(1))n$. Let $p \geq Cn^{-1/r}$. Then aas G_p contains an r-th power of Hamilton cycle.

We derive this from a general result which implies new threshold results.

Our result: Robust Thresholds

北京理工大學
BEIJING INSTITUTE OF TECHNOLOGY

Robust Pósa–Seymour. [Chen–H.–Luo, 2024++, independently by Joos et al.]

Let G be an n-vertex graph with $\delta(G) \geq (1 - \frac{1}{r+1} + o(1))n$. Let $p \geq Cn^{-1/r}$. Then aas G_p contains an r-th power of Hamilton cycle.

We derive this from a general result which implies new threshold results.

Our result: Thresholds of degenerate graphs

$$m_1(F) = \max\left\{\frac{e(F')}{v(F')-1} : F' \subseteq F \text{ and } v(F') > 1\right\}.$$

Definition

An n-vtx graph F is called a (d,α)-*degenerate* graph if $m_1(F) \le d$ and for any subset $U \subseteq V(F)$ with size $|U| = o(n)$, we have $e(G[F]) \le d(|U|-1) - \alpha$.

Theorem. [Chen–H.–Luo, 24++]

Let F be a (d,α)-degenerate graph with $\alpha > 0$. Then $th(F) \le n^{-1/d}$.

- (Kelly et al. JCTB, '24) $th(F) \le n^{-1/m_1(F)} \log n$ for ALL F.
- (Riordan, CPC, '20) This is true for (d,d)-degenerate graphs.

Our result: Thresholds of degenerate graphs

$$m_1(F) = \max\left\{\frac{e(F')}{v(F')-1} : F' \subseteq F \text{ and } v(F') > 1\right\}.$$

Definition

An n-vtx graph F is called a (d,α)-*degenerate* graph if $m_1(F) \le d$ and for any subset $U \subseteq V(F)$ with size $|U| = o(n)$, we have $e(G[F]) \le d\,(|U|-1) - \alpha$.

Theorem. [Chen–H.–Luo, 24++]

Let F be a (d,α)-degenerate graph with $\alpha > 0$. Then $th(F) \le n^{-1/d}$.

- (Kelly et al. JCTB, '24) $th(F) \le n^{-1/m_1(F)} \log n$ for ALL F.
- (Riordan, CPC, '20) This is true for (d,d)-degenerate graphs.

Our result: Thresholds of degenerate graphs

$$m_1(F) = \max\left\{\frac{e(F')}{v(F')-1} : F' \subseteq F \text{ and } v(F') > 1\right\}.$$

Definition

An n-vtx graph F is called a (d,α)-*degenerate* graph if $m_1(F) \le d$ and for any subset $U \subseteq V(F)$ with size $|U| = o(n)$, we have $e(G[F]) \le d\,(|U|-1) - \alpha$.

Theorem. [Chen–H.–Luo, 24++]

Let F be a (d,α)-degenerate graph with $\alpha > 0$. Then $th(F) \le n^{-1/d}$.

- (Kelly et al. JCTB, '24) $th(F) \le n^{-1/m_1(F)} \log n$ for ALL F.
- (Riordan, CPC, '20) This is true for (d,d)-degenerate graphs.

Our result: Thresholds of degenerate graphs

$$m_1(F) = \max\left\{\frac{e(F')}{v(F')-1} : F' \subseteq F \text{ and } v(F') > 1\right\}.$$

Definition

An n-vtx graph F is called a (d,α)-*degenerate* graph if $m_1(F) \le d$ and for any subset $U \subseteq V(F)$ with size $|U| = o(n)$, we have $e(G[F]) \le d\,(|U|-1) - \alpha$.

Theorem. [Chen–H.–Luo, 24++]

Let F be a (d,α)-degenerate graph with $\alpha > 0$. Then $th(F) \le n^{-1/d}$.

- (Kelly et al. JCTB, '24) $th(F) \le n^{-1/m_1(F)} \log n$ for ALL F.
- (Riordan, CPC, '20) This is true for (d,d)-degenerate graphs.

Our result: Robust Thresholds

北京理工大學 BEIJING INSTITUTE OF TECHNOLOGY

Definition

An n-vtx graph F is called a (d,α)-*degenerate* graph if $m_1(F) \le d$ and for any subset $U \subseteq V(F)$ with size $|U| = o(n)$, we have $e(G[F]) \le d\,(|U|-1) - \alpha$.

Examples of (d,α)-degenerate graphs

- Every d-degenerate graph is (d,α)-degenerate ($e(U) \le \binom{d}{2} + d(|U|-d)$ for $|U| \ge d$).
- Every planar graph is $(3,3)$-degenerate ($e(U) \le 3|U| - 6$).
- Every K_4-minor-free graph is $(2,1)$-degenerate ($e(U) \le 2|U| - 3$).
- The r-th power of a cycle is (r,α)-degenerate (r-degenerate for proper subset U).

Our result: Robust Thresholds

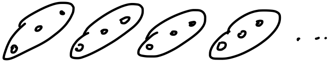
Definition

An n-vtx graph F is called a (d,α)-*degenerate* graph if $m_1(F) \le d$ and for any subset $U \subseteq V(F)$ with size $|U| = o(n)$, we have $e(G[F]) \le d\,(|U|-1) - \alpha$.

Examples of (d,α)-degenerate graphs

- Every d-degenerate graph is (d,α)-degenerate ($e(U) \le \binom{d}{2} + d(|U|-d)$ for $|U| \ge d$).
- Every planar graph is $(3,3)$-degenerate ($e(U) \le 3|U|-6$).
- Every K_4-minor-free graph is $(2,1)$-degenerate ($e(U) \le 2|U|-3$).
- The r-th power of a cycle is (r,α)-degenerate (r-degenerate for proper subset U).

Perfect matchings in k-graphs



Minimum degree conditions forcing perfect matchings (PM) in hypergraphs has been an intriguing problem.

Theorem, Rödl–Ruciński–Szemerédi, '09

For $k \geq 3$, and n large, let H be a k-graph with $\delta_{k-1}(H) \geq n/2 - k + 3$. Then H contains a PM.

Determining the sharp $\delta_d(H)$ condition for $d \geq 1$ is a major open problem.

Robust PM, Kang et al., Combinatorica, '24

Let $p \geq Cn^{1-r} \log n$. Then H_p aas contains a PM.

They indeed solved the "robustness" part for all d.

Perfect matchings in k-graphs

BEIJING INSTITUTE OF TECHNOLOGY

Minimum degree conditions forcing perfect matchings (PM) in hypergraphs has been an intriguing problem.

Theorem, Rödl–Ruciński–Szemerédi, '09

For $k \geq 3$, and n large, let H be a k-graph with $\delta_{k-1}(H) \geq n/2 - k + 3$. Then H contains a PM.

Determining the sharp $\delta_d(H)$ condition for $d \geq 1$ is a major open problem.

Robust PM, Kang et al., Combinatorica, '24

Let $p \geq Cn^{1-r} \log n$. Then H_p aas contains a PM.

They indeed solved the "robustness" part for all d.

Perfect matchings in k-graphs

Minimum degree conditions forcing perfect matchings (PM) in hypergraphs has been an intriguing problem.

Theorem, Rödl–Ruciński–Szemerédi, '09

For $k \geq 3$, and n large, let H be a k-graph with $\delta_{k-1}(H) \geq n/2 - k + 3$. Then H contains a PM.

Determining the sharp $\delta_d(H)$ condition for $d \geq 1$ is a major open problem.

Robust PM, Kang et al., Combinatorica, '24

Let $p \geq Cn^{1-r} \log n$. Then H_p aas contains a PM.

They indeed solved the "robustness" part for all d.

Perfect matchings in k-graphs

Minimum degree conditions forcing perfect matchings (PM) in hypergraphs has been an intriguing problem.

Theorem, Rödl–Ruciński–Szemerédi, '09

For $k \geq 3$, and n large, let H be a k-graph with $\delta_{k-1}(H) \geq n/2 - k + 3$. Then H contains a PM.

Determining the sharp $\delta_d(H)$ condition for $d \geq 1$ is a major open problem.

Robust PM, Kang et al., Combinatorica, '24

Let $p \geq Cn^{1-r} \log n$. Then H_p aas contains a PM.

They indeed solved the "robustness" part for all d.

Our result: Robust Perfect Matchings

Theorem. [Keevash et al. '15, H., '17]

Let H be a k-graph with $\delta_{k-1}(H) \geq (1/k + o(1))n$. Then H has a PM iff H satisfies certain divisibility conditions defined on certain vertex partition.

We prove a robust version and a counting version.

Theorem. [H.–Zhao, '24++]

Let H be a k-graph with $\delta_{k-1}(H) \geq (1/k + o(1))n$. Let M be a matching of size $\leq k$ in H. Suppose $H - M$ satisfies certain divisibility conditions defined on certain vertex partition. Let $p \geq Cn^{1-r} \log n$. Then $(H - M)_p$ aas has a PM.

Theorem. [H.–Zhao, '24++]

Such H either has no PM, or $\geq (\varepsilon n)^{n/k - O(1)}$ PMs.

Our result: Robust Perfect Matchings

Theorem. [Keevash et al. '15, H., '17]

Let H be a k-graph with $\delta_{k-1}(H) \geq (1/k + o(1))n$. Then H has a PM iff H satisfies certain divisibility conditions defined on certain vertex partition.

We prove a robust version and a counting version.

Theorem. [H.–Zhao, '24++]

Let H be a k-graph with $\delta_{k-1}(H) \geq (1/k + o(1))n$. Let M be a matching of size $\leq k$ in H. Suppose $H - M$ satisfies certain divisibility conditions defined on certain vertex partition. Let $p \geq Cn^{1-r} \log n$. Then $(H - M)_p$ aas has a PM.

Theorem. [H.–Zhao, '24++]

Such H either has no PM, or $\geq (\varepsilon n)^{n/k - O(1)}$ PMs.

Our result: Robust Perfect Matchings

Theorem. [Keevash et al. '15, H., '17]

Let H be a k-graph with $\delta_{k-1}(H) \geq (1/k + o(1))n$. Then H has a PM iff H satisfies certain divisibility conditions defined on certain vertex partition.

We prove a robust version and a counting version.

Theorem. [H.–Zhao, '24++]

Let H be a k-graph with $\delta_{k-1}(H) \geq (1/k + o(1))n$. Let M be a matching of size $\leq k$ in H. Suppose $H - M$ satisfies certain divisibility conditions defined on certain vertex partition. Let $p \geq Cn^{1-r} \log n$. Then $(H - M)_p$ aas has a PM.

Theorem. [H.–Zhao, '24++]

Such H either has no PM, or $\geq (\varepsilon n)^{n/k - O(1)}$ PMs.

Transversal versions

Transversal versions: H-containment

Let H be an n-vertex m-edge graph. Find the smallest $f(H)$ satisfying the following. Let $G_1, \ldots, G_m$ be graphs on the same vertex set V with $|V| = n$ such that $\delta(G_i) \geq (f(H) + o(1))n$. Then there is a copy of H that consists of exactly one edge from each G_i, $i \in [m]$.

Example: Transversal Dirac by Joos–Kim, '20

Let $G_1, \ldots, G_n$ be graphs on the same vertex set V with $|V| = n$ such that $\delta(G_i) \geq n/2$. Then there is a Hamiltonian cycle that consists of exactly one edge from each G_i, $i \in [n]$.

Transversal versions

Transversal versions: H-containment

Let H be an n-vertex m-edge graph. Find the smallest $f(H)$ satisfying the following. Let $G_1, \ldots, G_m$ be graphs on the same vertex set V with $|V| = n$ such that $\delta(G_i) \geq (f(H) + o(1))n$. Then there is a copy of H that consists of exactly one edge from each G_i, $i \in [m]$.

Example: Transversal Dirac by Joos–Kim, '20

Let $G_1, \ldots, G_n$ be graphs on the same vertex set V with $|V| = n$ such that $\delta(G_i) \geq n/2$. Then there is a Hamiltonian cycle that consists of exactly one edge from each G_i, $i \in [n]$.

Our result: Transversal versions of Robust Theorems

BEIJING INSTITUTE OF TECHNOLOGY

Transversal Hajnal–Szemerédi [Cheng–H.–Wang–Wang, Forum Math. Sigma, '23, Montgomery–Müyesser–Pehova, Adv. Comb. '22]

Let $m = \frac{n}{r}\binom{r}{2}$. Let $G_1, \ldots, G_m$ be graphs on the same vertex set V with $|V| = n$ such that $\delta(G_i) \geq (1 - 1/r + o(1))n$. Then there is a K_r-factor that consists of exactly one edge from each G_i, $i \in [m]$.

We establish a robust version of the result above.

Robust Transversal Hajnal–Szemerédi [H.–Hu–Yang, '24++]

(From above) let $p \geq Cn^{-1-2/r} \log n^{1/\binom{r}{2}}$. For $i \in [m]$, let $G_i(n,p)$ be independent copies of $G(n,p)$ on the same vertex set. Then aas there is a K_r-factor that consists of exactly one edge from each $G_i \cap G_i(n,p)$, $i \in [m]$.

Our result: Transversal versions of Robust Theorems

北京理工大學 BEIJING INSTITUTE OF TECHNOLOGY

Transversal Hajnal–Szemerédi [Cheng–H.–Wang–Wang, Forum Math. Sigma, '23, Montgomery–Müyesser–Pehova, Adv. Comb. '22]

Let $m = \frac{n}{r}\binom{r}{2}$. Let $G_1, \ldots, G_m$ be graphs on the same vertex set V with $|V| = n$ such that $\delta(G_i) \geq (1 - 1/r + o(1))n$. Then there is a K_r-factor that consists of exactly one edge from each G_i, $i \in [m]$.

We establish a robust version of the result above.

Robust Transversal Hajnal–Szemerédi [H.–Hu–Yang, '24++]

(From above) let $p \geq Cn^{-1-2/r} \log n^{1/\binom{r}{2}}$. For $i \in [m]$, let $G_i(n,p)$ be independent copies of $G(n,p)$ on the same vertex set. Then aas there is a K_r-factor that consists of exactly one edge from each $G_i \cap G_i(n,p)$, $i \in [m]$.

Proof ideas – vertex spread

- Consider $H \to G$.
- Let $\mathcal{H}$ be the family of all embeddings (functions) of H to G.
- Let μ be a probability measure on $\mathcal{H}$.
- If for every s, $u_1, \dots, u_s \in V(H)$, $v_1, \dots, v_s \in V(G)$, we have

$$\mu(\{\phi : u_i \to v_i, i \in [s]\}) \le q^s,$$

then we say μ is a q-vertex-spread measure.

Example

If $G = K_n$, $v(H) = n$ and μ is the uniform measure, then μ is $\frac{e}{n}$-vertex-spread.
This is because

$$\mu(\{\phi : u_i \to v_i, i \in [s]\}) = \frac{(n-s)!}{n!} \le \left(\frac{e}{n}\right)^s.$$

Proof ideas – vertex spread

北京理工大学 BEIJING INSTITUTE OF TECHNOLOGY

- Consider $H \to G$.
- Let $\mathcal{H}$ be the family of all embeddings (functions) of H to G.
- Let μ be a probability measure on $\mathcal{H}$.
- If for every s, $u_1, \ldots, u_s \in V(H)$, $v_1, \ldots, v_s \in V(G)$, we have

$$\mu(\{\phi : u_i \to v_i, i \in [s]\}) \leq q^s,$$

then we say μ is a q-vertex-spread measure.

Example

If $G = K_n$, $v(H) = n$ and μ is the uniform measure, then μ is $\frac{e}{n}$-vertex-spread.
This is because

$$\mu(\{\phi : u_i \to v_i, i \in [s]\}) = \frac{(n-s)!}{n!} \leq \left(\frac{e}{n}\right)^s.$$

Proof ideas – vertex spread

北京理工大学 BEIJING INSTITUTE OF TECHNOLOGY

Kelly–Müyesser–Pokrovskiy proved that
having an $\frac{O(1)}{n}$-vertex-spread measure $\Longrightarrow$ can determine the almost optimal threshold.
To find the (optimal) vertex-spread, they proved the following lemma.

Partition Lemma. [KMP, 2023+]

There exists $C \in \mathbb{N}$ s.t. TFH. Given G with $\delta(G) \geq (\alpha + \varepsilon)n$. Then there exists a distribution of partitions of $V(G) = V_0 \cup V_1 \cup V_2 \cup \cdots \cup V_m$ s.t.

- $|V_0| \leq 2C^2$ and $|V_i| = C$ for all $i \geq 1$,
- $\delta(G[V_i]) \geq \alpha |V_i|$ for all $i \geq 0$,
- aas, given a vertex v the probability that $v \in V_i$ is at most $O(C)/n$.

Proof ideas – vertex spread

北京理工大學 BEIJING INSTITUTE OF TECHNOLOGY

Kelly–Müyesser–Pokrovskiy proved that
having an $\frac{O(1)}{n}$-vertex-spread measure $\Longrightarrow$ can determine the almost optimal threshold.
To find the (optimal) vertex-spread, they proved the following lemma.

Partition Lemma. [KMP, 2023+]

There exists $C \in \mathbb{N}$ s.t. TFH. Given G with $\delta(G) \geq (\alpha + \varepsilon)n$. Then there exists a distribution of partitions of $V(G) = V_0 \cup V_1 \cup V_2 \cup \cdots \cup V_m$ s.t.

- $|V_0| \leq 2C^2$ and $|V_i| = C$ for all $i \geq 1$,
- $\delta(G[V_i]) \geq \alpha |V_i|$ for all $i \geq 0$,
- aas, given a vertex v the probability that $v \in V_i$ is at most $O(C)/n$.

Open Problems

Meta Theorem: Robustness.

Suppose that every n-vertex graph G with $\delta(G) \geq \alpha n$ contains a subgraph F. Let $p \gg th(F)$. Let G be an n-vertex graph G with $\delta(G) \geq (\alpha + \varepsilon)n$. Then aas G_p contains a copy of F.

- Other graph parameters (that are reserved after sparsification): average degree? maximum degree?
- More theorems?
- Digraphs? Hypergraphs? Set systems?

Thank you for your attention!

Questions?

han.jie@bit.edu.cn