8.2

(a) t_α (t-distribution with degree of freedom α) has pdf $\frac{C}{(1+x^2/\alpha)^{\frac{\alpha+1}{2}}} \sim \frac{C'}{x^{\alpha+1}}$ when $x \to \infty$ and hence we have $1 - F(x) \sim \frac{C''}{x^\alpha}$, where C, C', C'' are some constants. So it is of polynomial type, i.e., $x^\alpha(1 - F(x)) \to C''$ as $x \to \infty$. By a theorem, F in the domain of attraction of

$$G_{1,k} = \begin{cases} e^{-x^{-k}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(b) For $N(0,1)$, it can be shown that $\frac{F^{(j)}(x)}{F^{(j-1)}(x)} \sim -x = -\frac{f(x)}{1-F(x)}$ as $x \to \infty$. That is, it is of exponential type. For $C(0,1)$ which is t_1, we have shown it is of polynomial type. As $x \to \infty$, $C(0,1)$ dominates over $N(0,1)$. Thus we suspect $F = (1-\epsilon)N(0,1) + \epsilon C(0,1)$ is of polynomial type and thus in the domain of attraction of $G_{1,1}$. To show this rigorously, we first find its pdf by taking the convolution of the pdfs of $N(0,(1-\epsilon)^2)$ and $C(0,\epsilon)$

$$f(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1-\epsilon)^2}} e^{-\frac{z^2}{2(1-\epsilon)^2}} \frac{1}{\epsilon\pi[1 + \frac{(x-z)^2}{\epsilon^2}]} dz$$

We want to $F(x)$ is of regular variation....

(c) χ_k^2 has pdf $f(x) = Cx^{\frac{k}{2}-1}e^{-\frac{x}{2}}$. Note that

$$\begin{aligned}
\lim_{t\to\infty} \frac{1-F(t+2x)}{1-F(t)} &= \lim_{t\to\infty} \frac{\int_{t+2x}^{\infty} f(x)dx}{\int_t^\infty f(x)dx} \\
&= \lim_{t\to\infty} \frac{f(t+xx)}{f(t)} \\
&= \lim_{t\to\infty} \frac{(t+2x)^{\frac{k}{2}-1}e^{-\frac{t+2x}{2}}}{t^{\frac{k}{2}-1}e^{-\frac{t}{2}}} \\
&= \lim_{t\to\infty} (1+\frac{2x}{t})^{\frac{k}{2}-1}e^{-x} \\
&= e^{-x}
\end{aligned}$$

Then by THM8.6 F in the domain of contraction of G_3.

9.2 Let $Y_{i+1} = X_{i+2} - \mu$. Adding

$$\begin{aligned}
Y_{i+1} - \rho Y_i &= \sigma Z_i \\
\rho Y_i - \rho^2 Y_{i-1} &= \rho\sigma Z_{i-1} \\
\rho^2 Y_{i-1} - \rho^3 Y_{i-2} &= \rho^2\sigma Z_{i-2} \\
&\cdots \\
\rho^{i-1} Y_2 - \rho^i Y_1 &= \rho^{i-1}\sigma Z_1
\end{aligned}$$

gives $Y_{i+1} = \rho^i Y_1 + \sigma(Z_i + \rho Z_{i-1} + \cdots + \rho^{i-1} Z_1)$. It follow that

$$\begin{aligned}\sqrt{n}\bar{Y}_n &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} Y_i \\ &= \frac{1}{\sqrt{n}} Y_1 \sum_{i=1}^{n} \rho^i + \frac{\sigma}{\sqrt{n}}(Z_1 \sum_{i=1}^{n} \rho^{i-1} + Z_2 \sum_{i=1}^{n-1} \rho^{i-1} + \cdots + Z_{n-1}) \\ &\sim N(0, \tau^2)\end{aligned}$$

where $\tau^2 = \frac{\sigma^2}{1-\rho}$. That is, $\sqrt{n}(\bar{X} - \mu) \sim N(0, \tau^2)$

12.1 By invariance principle and reflection principle,

$$\max_{0 \le k \le n} \frac{S_k - k\mu}{\sigma\sqrt{n}} \sim \sup_{0 \le t \le 1} W(t) \sim |N(0,1)|$$

$U[-1,1]$ has mean $\mu = 0$ and variance $\sigma^2 = 1/3$. It follow that $\max_{0 \le k \le n} \frac{S_k}{\sqrt{n}} \sim \frac{1}{\sqrt{3}} |N(0,1)|$. Thus $P(\max_{0 \le k \le n} \frac{S_k}{\sqrt{n}} > 2) \approx P(|N(0,1| > 2\sqrt{3}) = 2P(N(0,1) < -2\sqrt{3})$.

$U[-1,1]$ does not matter but its mean and variance matter.

12.3 according to the result given in class(second arcsin law), $1 - \frac{2}{\pi} \arcsin \sqrt{0.6}$.

note: in the previous two problems, n=25,50 does not matter for the result either. It's provided for simulation purpose.

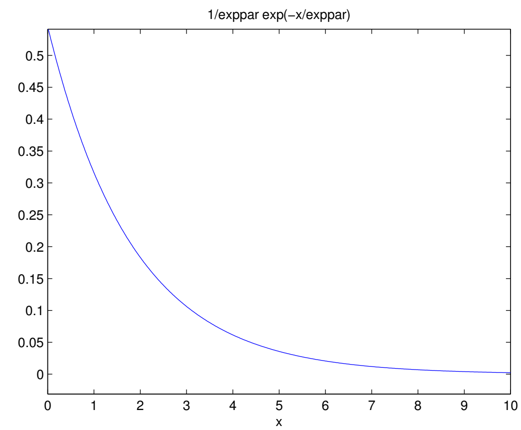
12.4

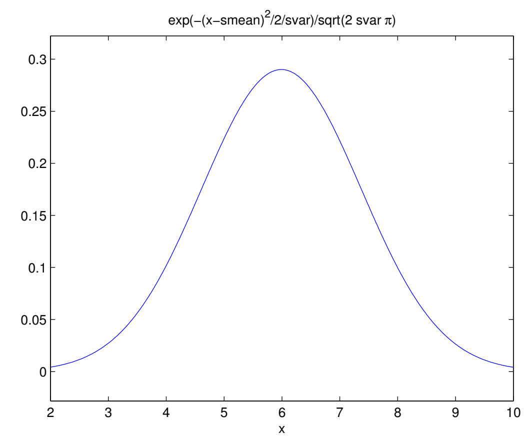
$f(x) = \begin{cases} 0 & x < 0.5 \\ 1 & x \ge 0.5 \end{cases}$, $f(x) = \begin{cases} x & x < 0.5 \\ x+1 & x \ge 0.5 \end{cases}$, $f(x) = \begin{cases} x & x < 0.5 \\ 2x & x \ge 0.5 \end{cases}$

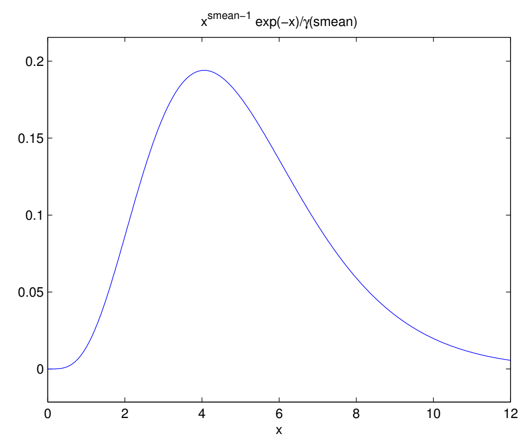
2.

(a)

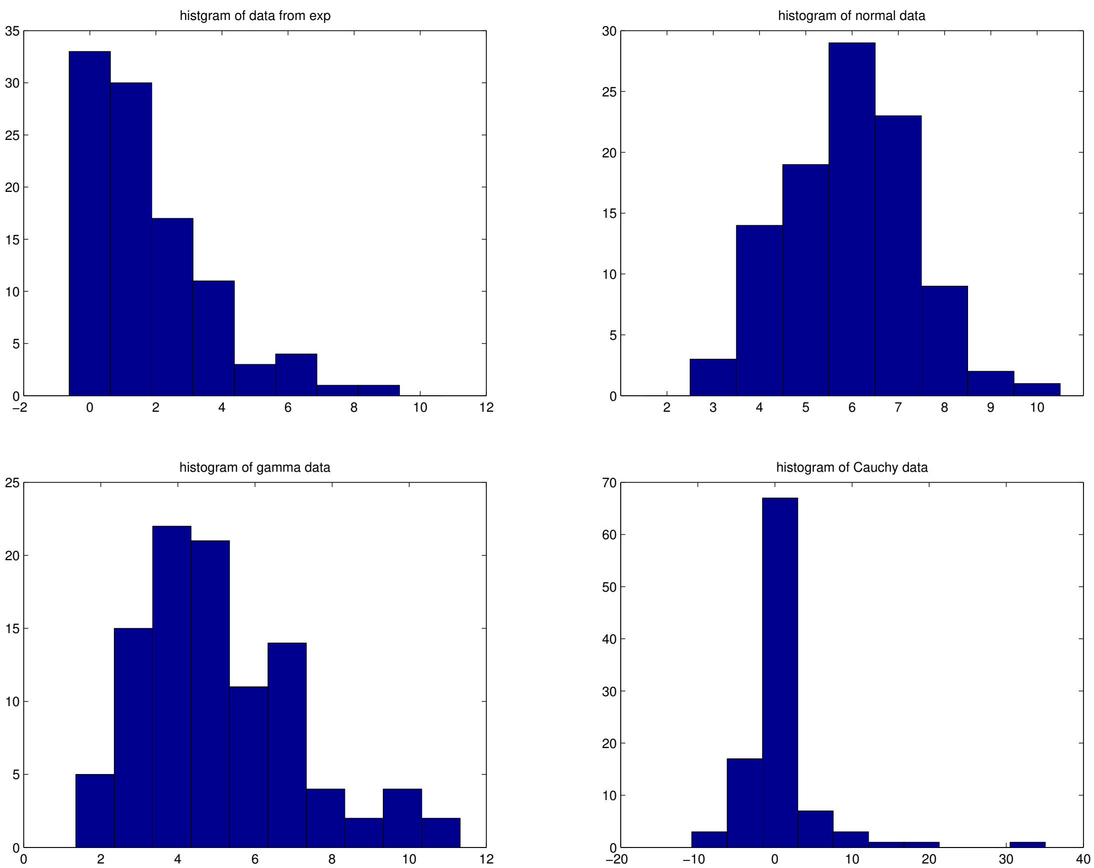
distribution	estimated parameter
$\exp(\theta)$	$\hat{\theta} = \frac{\sum_{i=1}^{n} X_i}{n} = 1.8341$
$N(\mu, \sigma^2)$	$\hat{\mu} = \frac{\sum_{i=1}^{n} X_i}{n} = 5.9902$, $\hat{\theta}^2 = \frac{\sum_{i=1}^{n}(X_i - \hat{\mu})^2}{n-1} = 1.8914$
$\text{Gamma}(\theta, 1)$	$\hat{\theta} = \frac{\sum_{i=1}^{n} X_i}{n} = 5.0566$


1/exppar exp(−x/exppar)
x

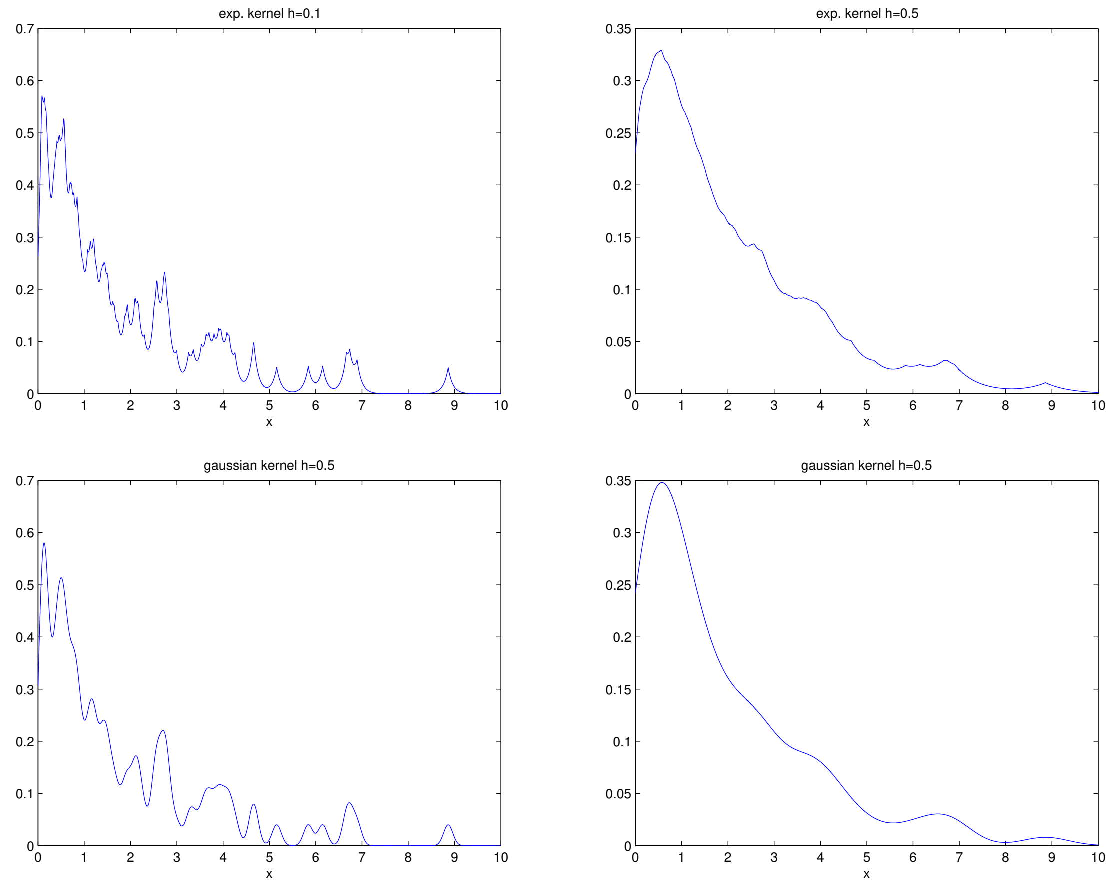

exp(−(x−smean)^2/2/svar)/sqrt(2 svar π)
x


x^smean−1 exp(−x)/γ(smean)
x

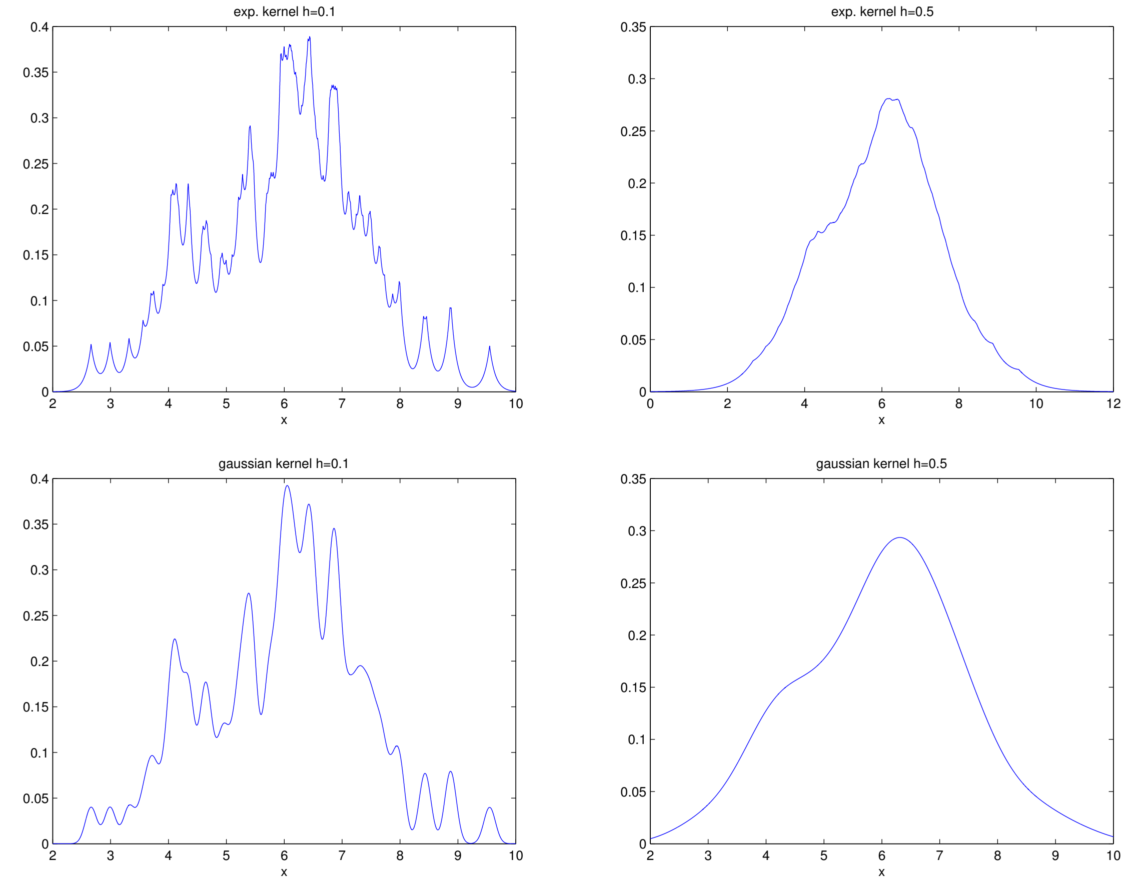
(b)


histgram of data from exp
35
30
25
20
15
10
5
0
−2
0
2
4
6
8
10
12
histogram of normal data
30
25
20
15
10
5
0
2
3
4
5
6
7
8
9
10
histogram of gamma data
25
20
15
10
5
0
0
2
4
6
8
10
12
histogram of Cauchy data
70
60
50
40
30
20
10
0
−20
−10
0
10
20
30
40

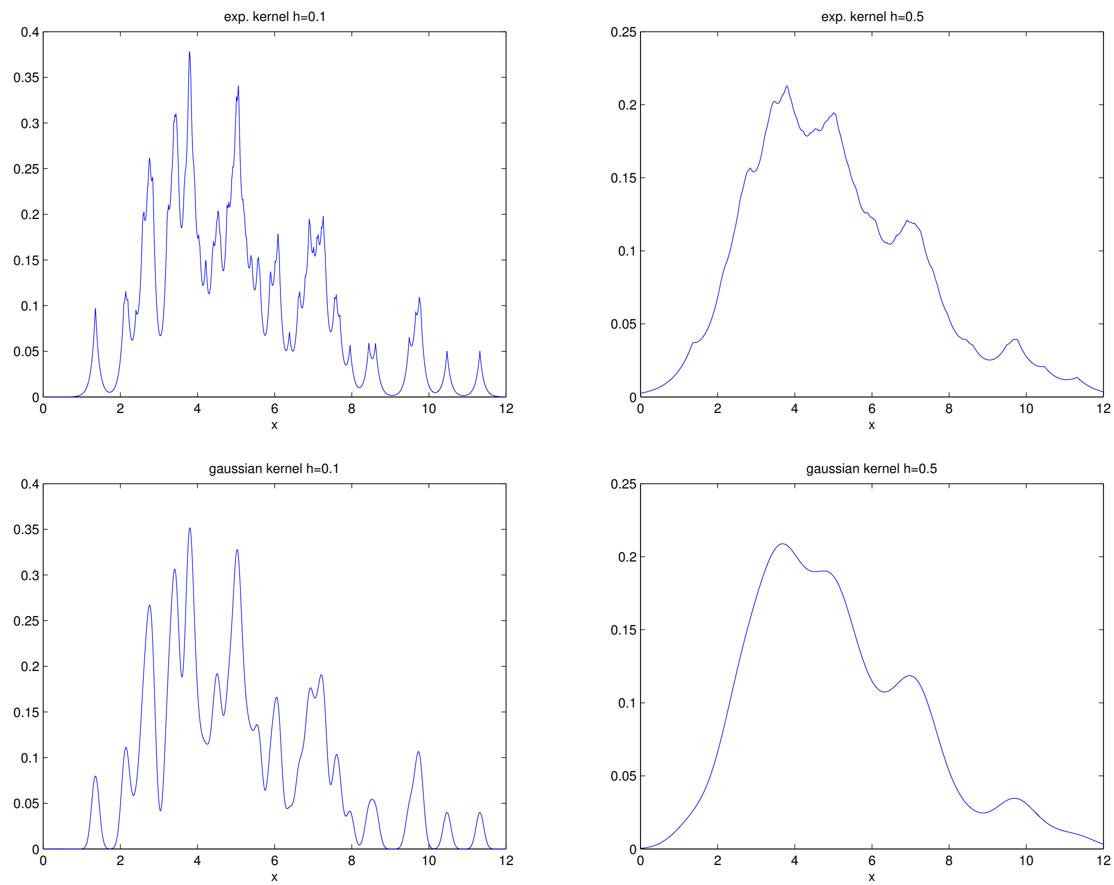
(c) kernel estimator for exponential data set:



kernel estimator for normal data set:



kernel estimator for gamma data set:



kernel estimator for Cauch data set: