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Note on the Compressible Euler Equations with Zero Temperature

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Abstract—This note presents the behavior of solution of the compressible Euler equations as the temperature drops to zero by the simple Riemann problem. © 2001 Elsevier Science Ltd. All rights reserved.

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Consider the compressible Euler equations

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + \left(\rho u^2 + p(\rho)\right)_x &= 0, \end{aligned} \tag{1}$$

where $\rho(t,x)$, $u(x,t)$ are the mean density and velocity of the flow, $p(x,t)$ is the pressure. Assume the gas under consideration is polytropic. Then the temperature, density, and pressure are related with $RT = p/\rho$ where R is a positive constant proportional to the molecular weight of gas and assumed to be unit for simplicity. So $p = T\rho$. Roughly speaking, the pressure p vanishes as T goes to zero. This note is just to investigate the behavior of solution (1) as the temperature T drops to zero by the simple Riemann problem.

The Riemann problem to (1) is well known, subject to the initial data

$$(\rho, u)(t=0, x) = (\rho_\pm, u_\pm), \qquad (\pm x > 0). \tag{2}$$

The solution can be constructed via the analysis method in phase-plane, see [1,2] for terminology. This is illustrated in Figure 1.

The rarefaction wave curves, R_1 and R_2, and shock wave curves, S_1 and S_2, divide the half plane ($\rho \geq 0$) into four parts I, II, III, and IV. We discuss our issue by two different cases.

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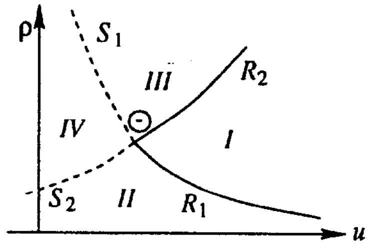

Figure 1.

CASE 1. $u_- > u_+$.

PROPOSITION 1. *If $u_- > u_+$, then there exists $T_0 > 0$ such that $(\rho_+, u_+) \in IV(\rho_-, u_-)$ when $0 < T < T_0$.*

PROOF. All states (ρ, u) connected with (ρ_-, u_-) by a backward shock wave S_1 or a forward shock wave S_2 satisfy

$$S_1 : u = u_- - \sqrt{\frac{T}{\rho\rho_-}}(\rho - \rho_-), \qquad (\rho > \rho_-)$$

or

$$S_2 : u = u_- + \sqrt{\frac{T}{\rho\rho_-}}(\rho - \rho_-), \qquad (\rho < \rho_-).$$

If $\rho_- = \rho_+$, the conclusion is obviously true. If $\rho_+ \neq \rho_-$, we have the conclusion by taking

$$T_0 = \frac{(u_- - u_+)^2 \rho_- \rho_+}{(\rho_- - \rho_+)^2}.$$

∎

This proposition shows that the curves S_1 and S_2 become steeper as T is much smaller. As $0 < t \leq T_0$, the solution consists of two shocks S_1 and S_2 and an intermediate state (ρ_*, u_*) besides the two constant state $(\rho_\pm, u_\pm)$. They have the following relations:

$$S_1 : \begin{cases} \sigma_- = u_- - \sqrt{\dfrac{T\rho_*}{\rho_-}}, \\ u_* = u_- - \sqrt{\dfrac{T}{(\rho_*\rho_-)}}(\rho_* - \rho_-), \end{cases} \qquad (\rho_* > \rho_-), \tag{3}$$

and

$$S_2 : \begin{cases} \sigma_+ = u_+ + \sqrt{\dfrac{T\rho_*}{\rho_+}}, \\ u_* = u_+ + \sqrt{\dfrac{T}{(\rho_*\rho_+)}}(\rho_* - \rho_+), \end{cases} \qquad (\rho_* > \rho_+), \tag{4}$$

where σ_- and σ_+ are the propagation speeds of S_1 and S_2, respectively. Then, we have the following propositions.

PROPOSITION 2.

$$\lim_{T\to 0} \sigma_+ = \lim_{T\to 0} \sigma_- = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}},$$
$$\lim_{T\to 0} u_* = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}.$$

PROOF. Base on (3) and (4), (ρ_*, u_*) can be expressed as

$$u_- - \sqrt{\frac{T}{(\rho_*\rho_-)}}(\rho_* - \rho_-) = u_+ + \sqrt{\frac{T}{(\rho_*\rho_+)}}(\rho_* - \rho_+),$$

i.e.,

$$\sqrt{T}(\sqrt{\rho_-}+\sqrt{\rho_+})\rho_* - (u_- - u_+)\sqrt{\rho_-\rho_+}\sqrt{\rho_*} - \sqrt{\rho_-\rho_+}(\sqrt{\rho_-}+\sqrt{\rho_+})\sqrt{T} = 0. \tag{5}$$

We solve this equation for $\sqrt{\rho_*}$ to obtain

$$\sqrt{\rho_*} = \frac{(u_- - u_+)\sqrt{\rho_-\rho_+} + \left\{(u_- - u_+)^2\rho_-\rho_+ + 4T\sqrt{\rho_-\rho_+}\left(\sqrt{\rho_-}+\sqrt{\rho_+}\right)^2\right\}^{1/2}}{2\sqrt{T}\left(\sqrt{\rho_-}+\sqrt{\rho_+}\right)}$$

because $\rho_* > \rho_+$ or ρ_-. Then

$$\begin{aligned}\sigma_- &= u_- - \frac{\sqrt{T}}{\sqrt{\rho_-}}\sqrt{\rho_*}\\ &= u_- - \frac{(u_- - u_+)\sqrt{\rho_-\rho_+} + \left\{(u_- - u_+)\rho_-\rho_+ + 4T\sqrt{\rho_-\rho_+}\left(\sqrt{\rho_-}+\sqrt{\rho_+}\right)^2\right\}^{1/2}}{2\left(\sqrt{\rho_-}+\sqrt{\rho_+}\right)}\cdot\frac{1}{\sqrt{\rho_-}}.\end{aligned}$$

We take the limit to get

$$\lim_{T\to 0}\sigma_- = u_- - \frac{(u_- - u_+)\sqrt{\rho_+}}{\sqrt{\rho_-}+\sqrt{\rho_+}} = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}}.$$

The same is true for σ_+ and u_*. ∎

This proposition shows that the two shocks coincide as the temperature T drops to zero. Next, we want to see the distribution of density on this coincidental shock. The speed is the weighted average of initial velocities.

PROPOSITION 3.

$$\lim_{T\to 0}\int_{\sigma_- t}^{\sigma_+ t}\rho_*\,dx = \sqrt{\rho_-\rho_+}(u_- - u_+)t. \tag{6}$$

PROOF. By (5), we have

$$\rho_* = \frac{\sqrt{\rho_-\rho_+}}{\sqrt{T}(\sqrt{\rho_-}+\sqrt{\rho_+})}\left\{(u_- - u_+)\sqrt{\rho_*} + (\sqrt{\rho_-}+\sqrt{\rho_-})\sqrt{T}\right\}.$$

In addition,

$$\sigma_+ - \sigma_- = u_+ - u_- + \left(\frac{1}{\sqrt{\rho_-}}+\frac{1}{\sqrt{\rho_+}}\right)\sqrt{T}\sqrt{\rho_*}.$$

Then,

$$\begin{aligned}\int_{\sigma_- t}^{\sigma_+ t}\rho_*\,dx &= \rho_*(\sigma_- - \sigma_+)t = \sqrt{\rho_-\rho_+}(\sigma_+ - \sigma_-)t + \frac{\sqrt{\rho_-\rho_+}\,(u_- - u_+)\sqrt{\rho_*}}{\sqrt{T}\left(\sqrt{\rho_-}+\sqrt{\rho_+}\right)}(\sigma_+ - \sigma_-)t\\ &= \sqrt{\rho_-\rho_+}(\sigma_+ - \sigma_-)t - \frac{\sqrt{\rho_-\rho_+}}{\sqrt{T}\left(\sqrt{\rho_-}+\sqrt{\rho_+}\right)}(u_- - u_+)^2 t + (u_- - u_+)\,\rho_* t\\ &= \sqrt{\rho_-\rho_+}\,(\sigma_+ - \sigma_-)\,t - \frac{\sqrt{\rho_-\rho_+}}{\sqrt{T}\left(\sqrt{\rho_-}+\sqrt{\rho_+}\right)}(u_- - u_+)^2 t\\ &\quad + (u_- - u_+)\frac{\sqrt{\rho_-\rho_+}\,(u_- - u_+)}{\sqrt{T}\left(\sqrt{\rho_-}+\sqrt{\rho_+}\right)}t + \sqrt{\rho_-\rho_+}\,(u_- - u_+)\,t.\end{aligned}$$

By Proposition 2, the first term $\sqrt{\rho_-\rho_+}(\sigma_+ - \sigma_-)t$ vanishes as T tends to zero, and the second and third terms are canceled. So the proof is complete. ∎

Proposition 3 shows that the measure of ρ on the coincidental shock does not vanish as the temperature T drops to zero. In other words, the density becomes a singular measure at the zero temperature, which is a linear function of t. Therefore, the solution is no longer self-similar even though equations (1) and initial data (2) are invariant under the dilation $(t,x) \to (ct, cx)(c > 0$ constant). However, the velocity, the weighted average of two initial states, still keeps bounded. This fact can be verified by directly investigating the case that the pressure vanishes in (1),

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_x + (\rho u^2)_x &= 0. \end{aligned} \tag{7}$$

This system can be used to model the process of the motion of free particles sticking under collision at zero temperature and to describe the formation of large scale structures in the universe [3,4]. It comes from Boltzman equations [5] and the flux-splitting scheme of the full compressible Euler equations [6,7]. The detailed study can be found in (e.g, [8]). We sketch some results related to the above discussion. The exact definition of measure solutions of (7) is referred to [5,8].

Consider a piecewise smooth solution of (7) of the form

$$(\rho, u)(t,x) = \begin{cases} (\rho_-, u_-)(t,x), & x < x(t), \\ (w(t)\delta(x - x(t)), u_\delta(t)), & x = x(t), \\ (\rho_+, u_+)(t,x), & x > x(t), \end{cases} \tag{8}$$

where $(\rho_\pm, u_\pm)(t,x) \in C^1$, $x(t), u_\delta(t) \in C^1$. It can be verified in [8] that (8) is a measure solution of (7) if the generalized Rankine-Hugoniot condition

$$\begin{aligned} \frac{dx(t)}{dt} &= u_\delta(t), \\ \frac{dw(t)}{dt} &= [\rho] u_\delta(t) - [\rho u], \\ \frac{d(w(t)u_\delta(t))}{dt} &= [\rho u] u_\delta(t) - [\rho u^2] \end{aligned} \tag{9}$$

is satisfied, where $[\rho] = \rho_+(x(t)+0) - \rho_-(x(t)-0)$, etc. By solving (9), we can obtain the solution of Riemann problem (7) and (2) with $u_- > u_+$, which is of the form (8) with $x(t)$, $u_\delta(t)$, and $w(t)$

$$x(t) = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}} t, \quad u_\delta(t) = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}, \quad \text{and} \quad w(t) = \sqrt{\rho_- \rho_+}(u_- - u_+)t. \tag{10}$$

This is just the limit of the solution of (1) as the temperature drops to zero.

Case 2. $u_- < u_+$. This case is much simpler. The solution will involves the vacuum.

Proposition 4. *If $u_- < u_+$, then there exists $T_0 > 0$ such that $(\rho_+, u_+) \in I(\rho_-, u_-)$ when $T < T_0$.*

Proof. It is obvious for $\rho_- = \rho_+$. For $\rho_- \neq \rho_+$, take

$$T_0 = \left(\frac{u_+ - u_-}{\ln(\rho_+/\rho_-)} \right)^2,$$

we can get our conclusion. ∎

We conclude from this proposition that the solution consists of two rarefaction waves R_1, R_2, and an intermediate state (ρ_*, u_*) besides two constant $(\rho_\pm, u_\pm)$. They satisfy

$$R_1 : \begin{cases} \lambda_1 = u - \sqrt{T}, \\ u = u_- - \sqrt{T} \ln \left(\dfrac{\rho}{\rho_-} \right), \end{cases} \qquad \rho_* \le \rho \le \rho_-, \tag{11}$$

and

$$R_2 : \begin{cases} \lambda_2 = u + \sqrt{T}, \\ u = u_+ + \sqrt{T} \ln \left(\dfrac{\rho}{\rho_+} \right), \end{cases} \qquad \rho_* \le \rho \le \rho_+. \tag{12}$$

So the intermediate state (ρ_*, u_*) can be expressed as

$$\rho_* = \sqrt{\rho_- \rho_+} \exp \left(-\frac{u_+ - u_-}{2\sqrt{T}} \right). \tag{13}$$

Equations (11)–(13) together imply the following proposition.

PROPOSITION 5. *As T drops to zero, ρ_* vanishes and two rarefaction waves, R_1 and R_2, becomes two contact discontinuities connecting the constant states $(\rho_\pm, u_\pm)$ and the vacuum $(\rho_* = 0)$.*

For (7), there is only a double and linearly degenerate eigenvalue $\lambda = u$ in the sense of Lax [2], which corresponds a contact discontinuity. With the analysis method in phase space, it is easy to get the solution of (7) and (2) with $u_- < u_+$, consisting of two contact discontinuities and the vacuum in between besides $(\rho_\pm, u_\pm)$. This is consistent with Proposition 5.

Therefore, we assert that system (7) is the limit of compressible Euler equations (1) as the temperature goes to zero. This conclusion is valid for the general initial distribution.

Some remarks are given here about the behavior of solutions of nonstrictly hyperbolic conservation laws. At zero temperature, the solution of (1) and (2) involves two extreme situations: one becomes singular measure and the other contains the vacuum, which forces us to consider the solutions of (1) beyond the traditional $L^p\,(1 \le \rho \le \infty)$ or BV spaces. This partially illustrates that it is not appropriate to take L^p or BV as the basic spaces to investigate the solutions of nonstrictly hyperbolic problems [8].

We point out that the same results are true for multidimensional cases. The interested reader is invited to check the details.

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