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ON THE TWO-DIMENSIONAL GAS EXPANSION FOR COMPRESSIBLE EULER EQUATIONS*

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Abstract. We investigate the problem of two-dimensional, unsteady expansion of an inviscid, polytropic gas, which can be interpreted as the collapse of a wedge-shaped dam containing water initially with a uniform velocity. We model this problem by isentropic Euler equations. The flow is quasi-stationary, and using hodograph transform, we describe it by a partial differential equation of second order in the state space if it is irrotational initially. Furthermore, this equation is reduced to a linearly degenerate system of three partial differential equations with inhomogeneous source terms. These properties are used to prove that the flow is globally smooth when a wedge of gas expands into a vacuum, and to analyze that shocks may appear in the interaction of four planar rarefaction waves.

Key words. two-dimensional gas expansion, isentropic Euler equations, linearly degenerate equations, hodograph transform, planar rarefaction waves, shock waves

AMS subject classifications. Primary, 35L65, 35L67; Secondary, 65M99, 76N15

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1. Introduction.

We are concerned with the two-dimensional, unsteady expansion of an inviscid, polytropic gas whose adiabatic index γ lies between 1 and 3. The flow is described by isentropic compressible Euler equations, which can be written as

$$\begin{cases} c_t + uc_x + vc_y + \kappa c(u_x + v_y) = 0, \\ u_t + uu_x + vu_y + \dfrac{1}{\kappa} cc_x = 0, \\ v_t + uv_x + vv_y + \dfrac{1}{\kappa} cc_y = 0, \end{cases} \tag{1.1}$$

provided that it is smooth, where u and v are the components of velocity in x and y directions, respectively, and the local sound speed c acts as a dependent variable and is a simple power of the density ρ and a different power of the pressure $p = A\rho^\gamma$, i.e., $c = \sqrt{A\gamma}\rho^{(\gamma-1)/2}$, $A > 0$, is entropy, being assumed to be the unit by some suitable choice of gauge, $\kappa = (\gamma - 1)/2$. They are the equations of continuity (the first one) and momentum (the last two). For $\gamma = 2$, these equations can be interpreted properly as the shallow water equations if the flow is smooth, and ρ is understood as the height of water from the bottom.

The main objective of this paper is to solve the problem of expansion of a wedge of gas. Thus the initial datum is sectorial constant. Indeed, we take the same kind of initial data in many applications and/or computations. The well-known oblique shock reflection experiments [BG1, BG2] and the expansion of a wedge of gas into a vacuum [Le, M] are our familiar examples, which are the extension of one-dimensional Riemann problem into two-dimensional ones [K, LZY, Z]. Under such circumstances, self-similar solutions of the form $(c, u, v)(t, x, y) = (c, u, v)(1, \xi, \eta)$

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$(\xi = x/t, \eta = y/t)$ are considered so that the dimension is reduced from three to two because the governing equations (1.1) and the initial data are invariant under the dilation $(t, x, y) \to (\alpha t, \alpha x, \alpha y)$ $(\alpha > 0)$. We call such a reduced flow the two-dimensional quasi-stationary flow, which can be regarded as the snapshot of the flow at some fixed time, say, $t = 1$. Thus the theory of partial differential equations with two independent variables can be applied to dealing with problems involved. But we remind the reader that our equations under consideration are quasi-linear with inhomogeneous source terms if we write them in conservation form; see [ZZ1]. Throughout this paper, we will use the uniform notation c, u, and v to denote the sound speed and components of velocity and regard them as the functions of two independent variables ξ, η, i.e., $(c, u, v) = (c, u, v)(\xi, \eta)$, when there is no risk of confusion.

We achieve our objective by two steps. First we study some general properties of two-dimensional quasi-stationary flow from the analysis point of view. Recall that one of useful techniques to convert certain quasi-linear systems of partial differential equations into linear systems is the hodograph transform, which reverses the roles of the dependent and independent variables (see [E]). We apply this technique to the two-dimensional quasi-stationary flow mentioned above. Namely, we map the (ξ, η) plane into the (u, v) plane and regard c as the function of u and v instead of ξ and η. Then system (1.1) becomes an interesting scalar mixed-type partial differential equation of second order,

$$(\tau^2\sigma - \sigma_v^2)\sigma_{uu} + 2\sigma_u\sigma_v\sigma_{uv} + (\tau^2\sigma - \sigma_u^2)\sigma_{vv} = \tau(\sigma_u^2 + \sigma_v^2 - 2\tau^2\sigma), \tag{1.2}$$

if the flow is irrotational initially, where $\tau = \gamma - 1$ and $\sigma = c^2$, the square of the local sound speed. This equation strongly resembles that of the two-dimensional steady potential flow but with inhomogeneous source term. Unfortunately, Legendre transform does not work here [E, p. 278]. However, we can use Riemann invariants to convert this equation into a 3×3 quasi-linear system of first-order partial differential equations (see (2.25)) provided that the flow is supersonic. This is the crucial issue. We emphasize that this new system is linearly degenerate in the sense of Lax [La]. It is this linear degeneracy that makes the gradient estimates available.

Second, we apply these properties to investigate the problem of expansion of a wedge of gas into a vacuum. This problem is actually the study of the interaction of two-dimensional planar rarefaction waves, and it has a long history mainly from the application point of view. Particularly, it may be interpreted hydraulically as the collapse of a wedge-shaped dam containing water initially with a uniform velocity [Le], where the shape of interface of gas and vacuum was emphatically discussed. In [Su], a set of interesting explicit solutions were given. In contrast to the steady potential flow, in [M] Mackie proposed a scalar equation of second order for a potential function, studied the interface of gas and vacuum by the method of unsteady Prandtl–Meyer expansions, and related with the Pogodin–Suchkov–Ianenko (PSI) approach in [PSI], where a similar equation to (1.2) was used.

We prove here that the flow is supersonic and the solution is smooth globally by solving a Goursat-type boundary value problem for the reduced system (2.25). To solve this problem, we first map the boundary values extended by planar rarefaction waves in the (ξ, η) plane into those in the (u, v) plane, then verify the strict hyperbolicity of the flow, and finally obtain a priori estimates on the solution in its own right and the gradients. Then the global existence of smooth solutions is established by the extension from the local existence under the above a priori estimates. The estimate on the solution is made via the classical technique of invariant regions [Sm].

The gradient estimate is given by some transformations used in [Li, RY]. We point out that the strict hyperbolicity and the estimate on the solution are uniform in the whole domain under consideration, while the gradient estimate is first obtained in the compact subdomain under consideration after cutting an arbitrarily thin strip with width $\epsilon > 0$ between the level curves of σ and the interface of gas and vacuum, and then we extend the solution to the interface of gas and vacuum because of the arbitrariness of the strip width. Therefore, the global continuous solution exists. The reason that we cannot work out the similar results for $\gamma \geq 3$ is that we cannot find an invariant region for this case. The gradient estimate seems very difficult too. This has to be left for the future.

The linearly degenerate system (2.25) can also be applied to analyzing the presence of shock waves in the interaction of planar rarefaction waves, which first appeared in [ZZ1] and was further studied in the subsequent works [Sc, SCG, CCY, LL]. The shock waves were observed in [Sc, CCY] by numerical simulations. We explain here that the presence of shocks is due to the degenerate hyperbolicity of the governing system.

Here we need to mention some related results on the existence of two-dimensional global solutions of compressible Euler equations to our knowledge. Chen and Glimm in [CG] proved the existence of global weak solution with symmetry outside of core. The explicit structure of global solutions with axisymmetry continuous initial data was constructed in [ZZ2]; see [Z] for more details. A similar existence result of global solutions of a simplified model from compressible Euler equations can be found in [DZ]. (In this model the inertial effect of flow is ignored.) The general existence problem of global weak solutions of (1.1) still remains completely open. We would like to point out that the main difficulties may lie in the high-order singularities, for instance, at the origin for spherically symmetric flow or two-dimensional Riemann problems. The gas expansion of a wedge of gas into a vacuum may be the simplest two-dimensional Riemann problem and serves as a touchstone to understand more general problems.

For the results of the two-dimensional Riemann problem for hyperbolic conservation laws, besides the systematic statement in [LZY, Z], we particularly point out that Čanić, Keyfitz, and Lieberman [CKL] and Keyfitz [K] recently proved the existence of solutions in the subsonic domain with a transonic shock as a free boundary for the two-dimensional unsteady transonic small disturbance (UTSD) equations. This is one of central issues in the study of the two-dimensional Riemann problem. For stationary supersonic flows, we refer to Chen [C].

This paper consists of four sections. In section 2, we study the two-dimensional quasi-stationary flow and derive a linearly degenerate system. Then this conclusion is used to solve the problem of expansion of a wedge of gas into a vacuum in section 3. In section 4, we analyze the presence of shock waves in the interaction of four planar rarefaction waves. We derive the evolution of the vorticity field for smooth quasi-stationary flows in the appendix.

2. Two-dimensional quasi-stationary flow without rotation. This section examines some general properties of two-dimensional quasi-stationary flow, which results from the assumption of dynamic similarity of (1.1). We first derive a scalar second-order quasi-linear equation by the hodograph transform under the assumption that the flow is irrotational. Indeed, this assumption can be removed for the gas expansion problem in section 3. Surprisingly, this equation is then reduced to a linearly degenerate system of first-order partial differential equations in the sense of

Lax [La], but with inhomogeneous source terms, which cause many difficulties and complexities in the quasi-stationary flows, as is well known. The linearly degenerate system is what we need in the analysis of the expansion of a wedge of gas. For the derivation of the second-order equation, we repeat the steps in [PSI], just in order that this paper is self-contained. The only difference is that the square of sound speed is used as the dependent variable, which makes the resulting equations much simpler.

2.1. PSI approach to the quasi-stationary flow. We use $\sigma = c^2$, the square of local sound speed, as a dependent variable instead of the local sound speed c in (1.1) and seek self-similar solutions of the form $(\sigma, u, v)(t, x, y) = (\sigma, u, v)(1, \xi, \eta) =: (\sigma, u, v)(\xi, \eta)$, $(\xi, \eta) = (x/t, y/t)$. Then system (1.1) is written as

$$\begin{cases} (u-\xi)\sigma_\xi + (v-\eta)\sigma_\eta + \tau\sigma(u_\xi + v_\eta) = 0, \\ (u-\xi)u_\xi + (v-\eta)u_\eta + \dfrac{1}{\tau}\sigma_\xi = 0, \\ (u-\xi)v_\xi + (v-\eta)v_\eta + \dfrac{1}{\tau}\sigma_\eta = 0, \end{cases} \tag{2.1}$$

where $\tau = \gamma - 1$. Assume the flow is irrotational at the moment. Indeed, we show in the appendix that this is true for a global smooth flow if it is irrotational initially. Thus in self-similar variables (ξ, η) we have the irrotationality condition

$$u_\eta = v_\xi. \tag{2.2}$$

Using this condition and noticing that

$$\sigma_u = \frac{\sigma_\xi v_\eta - \sigma_\eta v_\xi}{u_\xi v_\eta - v_\xi u_\eta}, \qquad \sigma_v = \frac{\sigma_\eta u_\xi - \sigma_\xi u_\eta}{u_\xi v_\eta - v_\xi u_\eta}, \tag{2.3}$$

we get from the law of momentum conservation of (2.1) that

$$\begin{cases} \xi - u = \dfrac{1}{\tau}\sigma_u, \\ \eta - v = \dfrac{1}{\tau}\sigma_v, \end{cases} \tag{2.4}$$

provided that the solutions are nondegenerate in the sense that the Jacobian $J = \partial(u, v)/\partial(\xi, \eta)$ neither vanishes nor becomes infinite. These are two linear equations, and they play a crucial role in the study of two-dimensional irrotational and isentropic quasi-stationary flow in that we can use them not only to reduce (2.1) into a scalar equation below but to recover u and v as the functions of ξ and η as well, as long as σ (as the function of u and v) can be derived and the flow is nondegenerate.

We write the law of mass conservation of (2.1) by using (2.4) as

$$\sigma_u\sigma_\xi + \sigma_v\sigma_\eta = \tau^2\sigma(u_\xi + v_\eta), \tag{2.5}$$

which results in

$$\sigma_u(\sigma_u u_\xi + \sigma_v v_\xi) + \sigma_v(\sigma_u u_\eta + \sigma_v v_\eta) = \tau^2\sigma(u_\xi + v_\eta) \tag{2.6}$$

if σ is regarded as a function of u and v. Identities (2.4) also give

$$\begin{cases} \xi_u = \dfrac{1}{\tau}\sigma_{uu} + 1, \\ \xi_v = \dfrac{1}{\tau}\sigma_{uv}, \end{cases} \qquad \begin{cases} \eta_u = \dfrac{1}{\tau}\sigma_{uv}, \\ \eta_v = \dfrac{1}{\tau}\sigma_{vv} + 1. \end{cases} \tag{2.7}$$

Note the homogeneity of u_ξ, v_ξ, u_η, and v_η in (2.6); we can replace them by η_v, $-\eta_u$, $-\xi_v$, and ξ_u, respectively, provided that the Jacobian J is not degenerate. Thus, in view of (2.7), we deduce an equation from (2.6),

$$(\tau^2\sigma - \sigma_v^2)\sigma_{uu} + 2\sigma_u\sigma_v\sigma_{uv} + (\tau^2\sigma - \sigma_u^2)\sigma_{vv} = \tau(\sigma_u^2 + \sigma_v^2 - 2\tau^2\sigma), \tag{2.8}$$

in the state space, (σ, u, v) space. This is a nice second-order quasi-linear partial differential equation with two variables, which gets rid of the explicit dependence of coefficients on the independent variables (ξ, η). Observe that the principal part of this equation strongly resembles that of

$$(\phi_x^2 - c^2)\phi_{xx} + 2\phi_x\phi_y\phi_{xy} + (\phi_y^2 - c^2)\phi_{yy} = 0 \tag{2.9}$$

for the velocity potential ϕ in two-dimensional steady, irrotational, isentropic flow, where c is the local sound speed; see [M] for some remarks. Equation (2.8) also resembles the self-similar form of the two-dimensional linear wave equation

$$(c^2 - \xi^2)\phi_{\xi\xi} - 2\xi\eta\phi_{\xi\eta} + (c^2 - \eta^2)\phi_{\eta\eta} - 2(\xi\phi_\xi + \eta\phi_\eta) = 0. \tag{2.10}$$

Unfortunately, we cannot use Legendre transform to convert (2.8) into a linear counterpart due to the inhomogeneous source terms [E, p. 278]. But certain Riemann invariants can be used to transform this equation into a linearly degenerate system of first-order partial differential equations.

2.2. A linearly degenerate system of first-order equations. In order to use Riemann invariants, we write (2.8) as an equivalent system of first-order partial differential equations. Introduce

$$X = \sigma_u, \qquad Y = \sigma_v. \tag{2.11}$$

Then we arrive at a 3×3 system of first-order equations:

$$\begin{aligned} &\begin{pmatrix} \tau^2\sigma - Y^2 & XY & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ \sigma \end{pmatrix}_u + \begin{pmatrix} XY & \tau^2\sigma - X^2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ \sigma \end{pmatrix}_v \\ &\qquad = \begin{pmatrix} \tau(X^2 + Y^2 - 2\tau^2\sigma) \\ 0 \\ X \end{pmatrix}. \end{aligned} \tag{2.12}$$

This system is equivalent to (2.8) if the solution is C^1 under consideration. It has three eigenvalues,

$$\begin{aligned} &\lambda_0 = 0, \\ &\lambda_\pm = \frac{XY \pm \tau\sqrt{\sigma(X^2 + Y^2 - \tau^2\sigma)}}{\tau^2\sigma - Y^2} = \frac{\tau^2\sigma - X^2}{XY \mp \tau\sqrt{\sigma(X^2 + Y^2 - \tau^2\sigma)}}, \end{aligned} \tag{2.13}$$

from which we deduce that the hyperbolicity of (2.12) depends on the sign of $X^2 + Y^2 - \tau^2\sigma$ provided that $\sigma > 0$ and $\tau^2\sigma - Y^2 \neq 0$ (or $\tau^2\sigma - X^2 \neq 0$). That is, system (2.12) or (2.8) is hyperbolic (sonic, elliptic) if $X^2 + Y^2 - \tau^2\sigma > 0$ $(= 0, < 0)$. The eigenvalue λ_0 comes from the third equation of (2.12), and its role is to recover this system into the preceding second-order equation (2.8).

For the time being, we assume that the system is hyperbolic (i.e., $X^2+Y^2-\sigma>0$ and $\sigma\neq 0$). Then the three associated left eigenvectors with (2.13) are

$$l_0=(0,0,1),\qquad l_{\mp}=(\mp 1,\tau\sqrt{\sigma(X^2+Y^2-\tau^2\sigma)},0). \tag{2.14}$$

(2.12) or (2.8) is transonic and degenerate on the set of $\lambda_-=\lambda_+$. We multiply (2.12) by the left eigenmatrix $M=(l_+,l_-,l_0)^\top$ (here and after the superscript $\top$ means transpose) from the left-hand side to obtain

$$\begin{cases} X_u+\lambda_-Y_u+\lambda_+(X_v+\lambda_-Y_v)=\dfrac{\tau(X^2+Y^2-2\tau^2\sigma)}{\tau^2\sigma-Y^2},\\ X_u+\lambda_+Y_u+\lambda_-(X_v+\lambda_+Y_v)=\dfrac{\tau(X^2+Y^2-2\tau^2\sigma)}{\tau^2\sigma-Y^2},\\ \sigma_u=X. \end{cases} \tag{2.15}$$

Denote

$$Q=\sqrt{X^2+Y^2-\tau^2\sigma},\qquad \overline{Q}=\tau\sqrt{\sigma(X^2+Y^2-\tau^2\sigma)}, \tag{2.16}$$

and introduce the Riemann invariants

$$\begin{aligned} R&=\frac{-\tau\sqrt{\sigma}X+Y\sqrt{X^2+Y^2-\tau^2\sigma}}{\tau^2\sigma-Y^2},\\ S&=\frac{\tau\sqrt{\sigma}X+Y\sqrt{X^2+Y^2-\tau^2\sigma}}{\tau^2\sigma-Y^2}, \end{aligned} \tag{2.17}$$

which satisfy

$$X+\lambda_-Y=-\tau\sqrt{\sigma}R,\qquad X+\lambda_+Y=\tau\sqrt{\sigma}S. \tag{2.18}$$

The terminology "Riemann invariants" is borrowed from the homogeneous counterpart of (2.12), and a Riemann invariant takes on a fixed constant value in the region of certain rarefaction waves; see [Sm]. Then (2.12) is transformed into the form

$$\begin{cases} R_u+\lambda_+R_v=\dfrac{\tau\lambda_-}{\overline{Q}}\Big(-(\lambda_-\lambda_++1)+\dfrac{\tau}{2}RS\Big),\\ S_u+\lambda_-S_v=\dfrac{\tau\lambda_+}{\overline{Q}}\Big(-(\lambda_-\lambda_++1)+\dfrac{\tau}{2}RS\Big),\\ \sigma_u=X. \end{cases} \tag{2.19}$$

In addition, we notice that λ_- and λ_+ themselves are Riemann invariants satisfying

$$\begin{cases} (\lambda_-)_u+\lambda_+(\lambda_-)_v=\dfrac{\tau R}{\overline{Q}}\Big(-(\lambda_-\lambda_++1)+\dfrac{\tau}{2}RS\Big),\\ (\lambda_+)_u+\lambda_-(\lambda_+)_v=\dfrac{\tau S}{\overline{Q}}\Big(-(\lambda_-\lambda_++1)+\dfrac{\tau}{2}RS\Big),\\ \sigma_u=X. \end{cases} \tag{2.20}$$

In (2.19) and (2.20), the source terms contain both the Riemann variants (R,S) and (λ_-,λ_+) and are very complicated. However, note that the linear combinations of

Riemann invariants are still Riemann invariants. If we take the following Riemann invariants A and B as dependent variables, then the resulting system can be written in a compact form. Indeed, take

$$
\begin{aligned}
A &= R + \lambda_- = -\frac{X + \sqrt{X^2 + Y^2 - \tau^2\sigma}}{\tau\sqrt{\sigma} + Y}, \\
B &= S + \lambda_+ = \frac{X + \sqrt{X^2 + Y^2 - \tau^2\sigma}}{\tau\sqrt{\sigma} - Y}.
\end{aligned} \tag{2.21}
$$

Then we express all relevant quantities in terms of dependent variables A, B, and σ,

$$
X = \frac{\tau\sqrt{\sigma}(AB-1)}{A-B}, \qquad Y = -\frac{\tau\sqrt{\sigma}(A+B)}{A-B}, \qquad \overline{Q} = \frac{\tau\sqrt{\sigma}(AB+1)}{A-B}, \tag{2.22}
$$

and

$$
\begin{aligned}
R &= \frac{1+A^2}{2A}, \qquad & S &= \frac{1+B^2}{2B}, \\
\lambda_- &= \frac{A^2-1}{2A}, \qquad & \lambda_+ &= \frac{B^2-1}{2B}.
\end{aligned} \tag{2.23}
$$

Using these identities, we calculate

$$
\begin{aligned}
-(\lambda_-\lambda_+ + 1) + \frac{\tau}{2}RS &= -\frac{(1-\tau/2)(AB+1)^2 - (1+\tau/2)(A-B)^2}{4AB} \\
&=: G(A,B).
\end{aligned} \tag{2.24}
$$

Therefore, adding (2.19) and (2.20) gives

$$
\begin{cases}
A_u + \lambda_+ A_v = \dfrac{\tau A}{\sqrt{\sigma}} \dfrac{A-B}{AB+1} G(A,B) =: \dfrac{\tau}{\sqrt{\sigma}} G_1(A,B), \\
B_u + \lambda_- B_v = \dfrac{\tau B}{\sqrt{\sigma}} \dfrac{A-B}{AB+1} G(A,B) =: \dfrac{\tau}{\sqrt{\sigma}} G_2(A,B), \\
\sigma_u = \dfrac{\tau\sqrt{\sigma}(AB-1)}{A-B}.
\end{cases} \tag{2.25}
$$

The distinct features are that this system is linearly degenerate in the sense of Lax [La] since λ_+ depends only on B and λ_- on A, and that the source terms are expressed explicitly in terms of dependent variables A, B, and σ. We will take advantage of these features to obtain the a priori estimates on solutions and relevant gradients (cf. [Li]). Furthermore, if $G(A,B) = 0$, then the first two equations become homogeneous equations

$$
\begin{cases}
A_u + \lambda_+ A_v = 0, \\
B_u + \lambda_- B_v = 0,
\end{cases} \tag{2.26}
$$

which always have a unique global continuous solution provided that the corresponding initial and/or boundary data have a uniform bound for the C^1 norm (cf. [Li]). In fact, we can even obtain the explicit solution in the problem of expansion of a wedge of gas into a vacuum for this case; see Remark 3.5 below.

The mappings $(X,Y) \to (R,S)$, $(X,Y) \to (\lambda_-, \lambda_+)$, and $(X,Y) \to (A,B)$ are all bijective as long as the flow is supersonic, and $Y^2 \neq \tau^2\sigma$ or $X^2 \neq \tau^2\sigma$, which will be proved in Corollaries 3.8 and 3.9.

To summarize, we have the following theorem.

THEOREM 2.1. *The two-dimensional quasi-stationary, irrotational, isentropic flow* (2.1) *can be transformed into a linearly degenerate system of first-order partial differential equations* (2.25) *provided the flow is supersonic and has the property that* $Y^2 \neq \tau^2\sigma$ *or* $X^2 \neq \tau^2\sigma$.

We remark that there are factors $\sqrt{\sigma}$ and Q in the denominators of source terms of (2.19), (2.20), and (2.25). On the interface of the gas and the vacuum σ vanishes. Also, system (2.25) becomes parabolic degenerate on the set N such that $Q = 0$ (so $\lambda_- = \lambda_+$). For both cases, the source terms become infinite. This fact may lead to the blowup of solutions and the presence of shock waves on the interface of gas and vacuum or on the sonic curves where $\lambda_- = \lambda_+$; see the next two sections.

For the simplicity of presentation, we denote from now on

$$K = (A, B, \sigma)^\top, \tag{2.27}$$

and the eigenvectors associated with the eigenvalues λ_+, λ_-, and λ_0 of (2.25) are

$$l^+ = (1, 0, 0), \qquad l^- = (0, 1, 0), \qquad l^0 = (0, 0, 1). \tag{2.28}$$

Using the notation of directional derivatives along characteristics,

$$\frac{d_+}{du} = \frac{\partial}{\partial u} + \lambda_+ \frac{\partial}{\partial v}, \qquad \frac{d_-}{du} = \frac{\partial}{\partial u} + \lambda_- \frac{\partial}{\partial v}, \qquad \frac{d_0}{du} = \frac{\partial}{\partial u}, \tag{2.29}$$

we write (2.25) as

$$\begin{cases} \dfrac{d_+ A}{du} = \dfrac{\tau}{\sqrt{\sigma}} G_1(A, B), \\ \dfrac{d_- B}{du} = \dfrac{\tau}{\sqrt{\sigma}} G_2(A, B), \\ \dfrac{d_0 \sigma}{du} = \dfrac{\tau\sqrt{\sigma}(AB - 1)}{A - B}. \end{cases} \tag{2.30}$$

3. The expansion of a wedge of gas into a vacuum. In this section, we employ the properties of two-dimensional unsteady irrotational isentropic flow, which we studied in the last section, to consider the problem of the expansion of a wedge of gas into a vacuum. This problem can be interpreted hydraulically as the collapse of a wedge-shaped dam containing water initially with a uniform velocity, which corresponds to the case that $\gamma = 2$, and ρ in (1.1) is then understood as the height of water from the bottom. The conclusion we will prove is that the gas expands into the vacuum smoothly and the flow is supersonic, except possibly at the interface of the gas and vacuum. This shows that we can use the characteristically oriented schemes to solve this problem from the computational point of view.

Consider a polytropic gas with adiabatic index $1 < \gamma < 3$ at uniform velocity (u_0, v_0) and at uniform sound speed c_0. It is bounded by two infinite and rigid walls to separate it from a vacuum. Initially these two walls are removed instantaneously, and the gas expands into the vacuum. The resulting flow is clearly self-similar because, in the absence of a length dimension in the data, the whole pattern grows linearly with time and is therefore subject to the analysis of the preceding section.

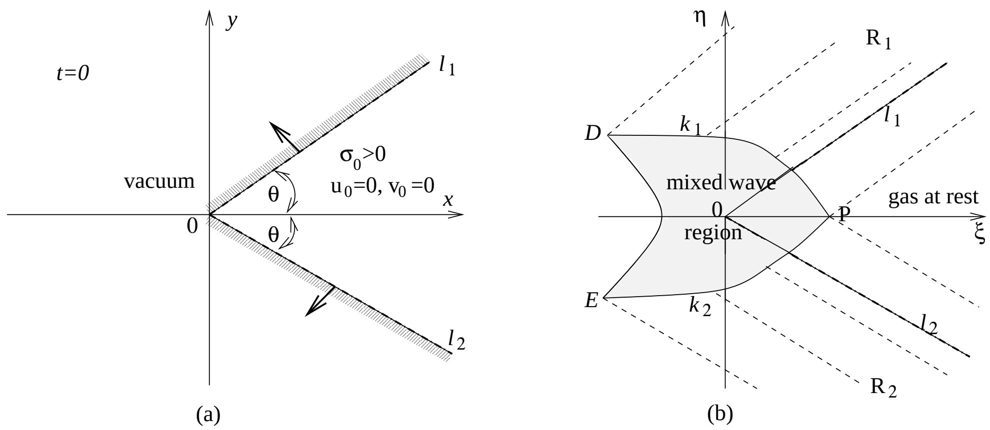

FIG. 3.1. *The expansion of a wedge of gas into a vacuum.*

For simplicity of presentation, we locate the wedge symmetrically with respect to the x-axis and the sharp corner at the origin, as in Figure 3.1(a). This problem is then formulated mathematically as seeking the solution of (1.1) with the initial data,

$$(\sigma, u, v)(t = 0, x, y) = \begin{cases} (\sigma_0, u_0, v_0), & -\theta < \alpha < \theta, \\ (0, \bar{u}, \bar{v}) & \text{otherwise}, \end{cases} \tag{3.1}$$

where $\sigma_0 > 0$, u_0 and v_0 are constant, $(\bar{u}, \bar{v})$ is the velocity of the wave front, not being specified in the state of vacuum, $\alpha = \arctan y/x$ is the polar angle, and θ is reasonable to be restricted between 0 and $\pi/2$. This is a two-dimensional Riemann problem for (1.1) with two pieces of initial data (3.1). As we will see below, this problem is actually to study the interaction of two planar rarefaction waves. Due to complicated wave patterns and difficult geometrical singularities involved for two-dimensional compressible flows, there is no analytic result with "big" initial data yet.

The gas away from the sharp corner expands into the vacuum as planar rarefaction waves R_1 and R_2 of the form $(\sigma, u, v)(t, x, y) = (\sigma, u, v)(\zeta)$ $(\zeta = (\mu x + \nu y)/t)$, where (μ, ν) is the propagation direction of waves. We assume that initially the gas is at rest, i.e., $(u_0, v_0) = (0, 0)$. Otherwise, we replace (u, v) by $(u - u_0, v - v_0)$ and (ξ, η) by $(\xi - u_0, \eta - v_0)$ in the following computations (see also (2.1)). We further assume that the initial sound speed is the unit since the transformation $(u, v, c, \xi, \eta) \to c_0(u, v, c, \xi, \eta)$ with $c_0 > 0$ can make all variables dimensionless. Then the rarefaction wave R_1 from l_1 is

$$R_1 : \begin{cases} \sqrt{\sigma} = \dfrac{1}{\tau + 2}(2 + \tau(\xi \sin\theta - \eta\cos\theta)), \\ u = \dfrac{2\sin\theta}{\tau + 2}(\xi\sin\theta - \eta\cos\theta - 1), \\ v = \dfrac{-2\cos\theta}{\tau + 2}(\xi\sin\theta - \eta\cos\theta - 1), \\ -\dfrac{2}{\tau} \le \xi\sin\theta - \eta\cos\theta \le 1, \end{cases} \tag{3.2}$$

and the rarefaction wave R_2 from l_2 is

$$R_2 : \begin{cases} \sqrt{\sigma} = \dfrac{1}{\tau+2}(2+\tau(\xi\sin\theta+\eta\cos\theta)), \\ u = \dfrac{-2\sin\theta}{\tau+2}(-\xi\sin\theta-\eta\cos\theta+1), \\ v = \dfrac{-2\cos\theta}{\tau+2}(-\xi\sin\theta-\eta\cos\theta+1), \\ -\dfrac{2}{\tau} \le \xi\sin\theta+\eta\cos\theta \le 1. \end{cases} \tag{3.3}$$

These two waves begin to interact at $P = (1/\sin\theta, 0)$ in the (ξ, η) plane due to the presence of the sharp corner, and the wave interaction region, called the mixed wave region, is formed to separate from the planar rarefaction waves by k_1, k_2,

$$\begin{cases} k_1 : (2-\tau)(\tau+2)\xi_1^2 - (\tau\eta_1+2)^2 = \alpha(\tau\eta_1+2)^{(\tau+2)/\tau}, \\ \qquad\qquad\qquad\qquad\qquad \left(\xi_1 > 0, -1 \le \eta_1 \le \dfrac{2}{\tau}\right), \\ k_2 : (2-\tau)(\tau+2)\xi_2^2 - (\tau\eta_2+2)^2 = \alpha(\tau\eta_2+2)^{(\tau+2)/\tau}, \\ \qquad\qquad\qquad\qquad\qquad \left(\xi_2 > 0, -\dfrac{2}{\tau} \le \eta_2 \le 1\right), \end{cases} \tag{3.4}$$

where k_1 and k_2 are two characteristics from P associated with the nonlinear eigenvalues of system (2.1) (see [LZY, ZZ1]),

$$\begin{aligned} \alpha = (\tau+2)\left(\frac{1}{\sqrt{\tau+1}(\tau+2)}\right)^{(\tau+2)/\tau} \\ \cdot\left((2-\tau)(\tau+1)^{-(\tau+2)/(2\tau)} + (\tau+2)(\tau+1)^{(\tau-2)/(2\tau)}\right) \end{aligned} \tag{3.5}$$

and

$$\begin{cases} \xi_1 = \xi\cos\theta+\eta\sin\theta, \\ \eta_1 = -\xi\sin\theta+\eta\cos\theta, \end{cases} \qquad \begin{cases} \xi_2 = \xi\cos\theta-\eta\sin\theta, \\ \eta_2 = \xi\sin\theta+\eta\cos\theta. \end{cases} \tag{3.6}$$

Denote the mixed wave region by Ω, which is bounded by k_1, k_2, and the interface of gas and vacuum, connecting D and E; see Figure 3.1(b). Then the solution outside Ω consists of the constant state (σ_0, u_0, v_0), the vacuum, and the planar rarefaction waves R_1 and R_2. Our next goal is to seek solutions inside Ω, subject to the boundary values on k_1 and k_2, which is the extension from the rarefaction waves R_1 and R_2. Note that k_1 and k_2 are characteristics. Hence this problem is actually the Goursat-type problem for (2.1).

Our strategy is to solve this problem in the state space, the (σ, u, v) space. Note that the initial data (3.1) are irrotational; we apply the result of the appendix to conclude that the flow is always irrotational provided that it is continuous. That is, the irrotationality condition (2.4) holds. So all conclusions in the last section can be used to deal with this problem.

For this purpose, we need to map the self-similar plane, the (ξ, η) plane, into the phase plane, the (u, v) plane. Notice that the mapping of the planar rarefaction waves R_1 and R_2 into the (u, v) plane are exactly two segments $H_1 : u\cos\theta + v\sin\theta = 0$ $(-2/\tau\sin\theta \le u \le 0)$ and $H_2 : u\sin\theta - v\cos\theta = 0$ $(-2/\tau\sin\theta \le u \le 0)$, on which we have

$$\sigma|_{H_1} = (1-\tau v'/2)^2 =: \sigma_0^1, \qquad \sigma|_{H_2} = (1+\tau v''/2)^2 =: \sigma_0^2, \tag{3.7}$$

where $v' = -u\sin\theta + v\cos\theta$ and $v'' = u\sin\theta + v\cos\theta$. Obviously,

$$0 \le \sigma_0^1,\ \sigma_0^2 \le 1. \tag{3.8}$$

Denote $m_0^2 = (1+\tau/2)/(1-\tau/2)$. As in [Le], we view (2.8) as an ordinary differential equation of some variables by suitable rotation transformations to get

$$\begin{cases} \sigma_u = \tau(1-\tau v'/2)\{[(\cot^2\theta - m_0^2)(1-\tau v'/2)^{2(1-\tau/2)/\tau} + m_0^2]^{1/2}\cos\theta + \sin\theta\}, \\ \sigma_v = \tau(1-\tau v'/2)\{[(\cot^2\theta - m_0^2)(1-\tau v'/2)^{2(1-\tau/2)/\tau} + m_0^2]^{1/2}\sin\theta - \cos\theta\} \end{cases} \tag{3.9}$$

on H_1 and

$$\begin{cases} \sigma_u = \tau(1+\tau v''/2)\{[(\cot^2\theta - m_0^2)(1+\tau v''/2)^{2(1-\tau/2)/\tau} + m_0^2]^{1/2}\cos\theta + \sin\theta\}, \\ \sigma_v = r(1+\tau v''/2)\{-[(\cot^2\theta - m_0^2)(1+\tau v''/2)^{2(1-\tau/2)/\tau} + m_0^2]^{1/2}\sin\theta + \cos\theta\} \end{cases} \tag{3.10}$$

on H_2.

We point out in passing that (3.9) and (3.10) hold for $\gamma \neq 3$. For $\gamma = 3$,

$$\begin{cases} \sigma_u = \tau(1-\tau v'/2)\{[(\cot^2\theta - 8\ln(1-v')]^{1/2}\cos\theta + \sin\theta\}, \\ \sigma_v = \tau(1-\tau v'/2)\{[(\cot^2\theta - 8\ln(1-v')]^{1/2}\sin\theta - \cos\theta\} \end{cases} \tag{3.11}$$

on H_1 and

$$\begin{cases} \sigma_u = \tau(1+\tau v''/2)\{[(\cot^2\theta - 8\ln(1+v'')]^{1/2}\cos\theta + \sin\theta\}, \\ \sigma_v = \tau(1+\tau v''/2)\{-[(\cot^2\theta - 8\ln(1+v'')]^{1/2}\sin\theta + \cos\theta\} \end{cases} \tag{3.12}$$

on H_2. In this paper, we do not work on this case.

Then for $1 < \gamma < 3$, we substitute (3.7), (3.9)–(3.10) into (2.21) to arrive at

$$\begin{aligned} A|_{H_1} &= -\frac{[(\cot^2\theta - m_0^2)(1-\tau v'/2)^{2(1-\tau/2)/\tau} + m_0^2]^{1/2}(1+\cos\theta) + \sin\theta}{[(\cot^2\theta - m_0^2)(1-\tau v'/2)^{2(1-\tau/2)/\tau} + m_0^2]^{1/2}\sin\theta + (1-\cos\theta)} \\ &=: A_0^1, \\ B|_{H_1} &= \frac{[(\cot^2\theta - m_0^2)(1-\tau v'/2)^{2(1-\tau/2)/\tau} + m_0^2]^{1/2}(1+\cos\theta) + \sin\theta}{-[(\cot^2\theta - m_0^2)(1-\tau v'/2)^{2(1-\tau/2)/\tau} + m_0^2]^{1/2}\sin\theta + (1+\cos\theta)} \\ &=: B_0^1, \\ A|_{H_2} &= -\frac{[(\cot^2\theta - m_0^2)(1+\tau v''/2)^{2(1-\tau/2)/\tau} + m_0^2]^{1/2}(1+\cos\theta) + \sin\theta}{-[(\cot^2\theta - m_0^2)(1-\tau v''/2)^{2(1+\tau/2)/\tau} + m_0^2]^{1/2}\sin\theta + (1+\cos\theta)} \\ &=: A_0^2, \\ B|_{H_2} &= \frac{[(\cot^2\theta - m_0^2)(1+\tau v''/2)^{2(1-\tau/2)/\tau} + m_0^2]^{1/2}(1+\cos\theta) + \sin\theta}{[(\cot^2\theta - m_0^2)(1-\tau v''/2)^{2(1+\tau/2)/\tau} + m_0^2]^{1/2}\sin\theta + (1-\cos\theta)} \\ &=: B_0^2. \end{aligned} \tag{3.13}$$

These boundary values have the following uniform estimates.

LEMMA 3.1. *There is a constant $C > 0$ such that*

$$-C < A_0^1, A_0^2 < -m_0, \qquad m_0 < B_0^1, B_0^2 < C, \tag{3.14}$$

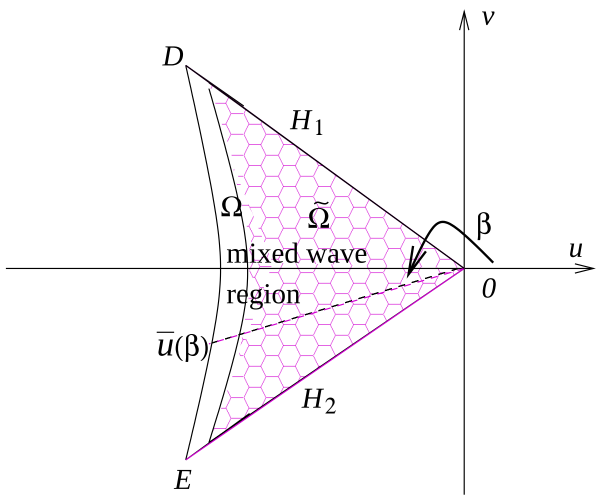

FIG. 3.2. *The mapping of solutions on the state plane, (u, v) plane.*

and the flow is uniformly supersonic in the sense that

$$(A_0^1 B_0^1 + 1)/(A_0^1 - B_0^1) > 0, \qquad (A_0^2 B_0^2 + 1)/(A_0^2 - B_0^2) > 0. \tag{3.15}$$

Proof. Obviously we have inequalities on the m_0-sides. For the other sides, we just prove the claim that $-C < A_0^1$. Since $\sin\theta > 0$ and $1 - \cos\theta > 0$, we must find sufficiently large $C > 0$ such that

$$1 + \cos\theta - C\sin\theta < 0, \qquad C(1 - \cos\theta) - \sin\theta > 0. \tag{3.16}$$

Therefore,

$$\begin{aligned} &[(\cot^2\theta - m_0^2)(1 - \tau v'/2)^{2(1-\tau/2)/\tau} + m_0^2]^{1/2}(1 + \cos\theta - C\sin\theta) \\ &\quad < C(1 - \cos\theta) - \sin\theta. \end{aligned} \tag{3.17}$$

This proves our claim. Inequalities (3.14) directly imply (3.15) by noticing that $m_0 > 1$. ∎

The boundaries H_1 and H_2 are the characteristics of (2.8) or (2.25) corresponding to λ_- and λ_+, respectively. So the problem of (2.25) subject to the boundary data (3.7)–(3.13) is a Goursat-type boundary value problem. We need to seek the solution of (2.25) and (3.7)–(3.13) in the mixed wave region Ω bounded by H_1, H_2, and the interface of gas and vacuum connecting D and E, as in Figure 3.2.

The local existence of solutions at the origin $(u, v) = (0, 0)$ follows the idea of [LY, Chapter 2] or [WW] with a routine argument. We need only to check the admissibility condition to this problem, i.e.,

$$\frac{1}{\lambda_+}\left(l^0 \cdot \frac{d_+K}{du} - \frac{\tau\sqrt{\sigma}(AB - 1)}{A - B}\right) = \frac{1}{\lambda_-}\left(l^0 \cdot \frac{d_-K}{du} - \frac{\tau\sqrt{\sigma}(AB - 1)}{A - B}\right) \tag{3.18}$$

at $(u, v) = (0, 0)$, where K and l^0 are defined in (2.27) and (2.28). Using the boundary values (3.7)–(3.13), this obviously holds. Thus we have the following lemma.

LEMMA 3.2. *There is $\delta > 0$ such that the C^1 solution of* (2.25) *and* (3.7)–(3.13) *exists uniquely in the region $\bar{\Omega} = \{(u, v) \in \Omega; -\delta < u < 0\}$, where δ depends only on the C^0 and C^1 norms of λ_-, λ_+, G_1, and G_2 on the boundaries H_1 and H_2.*

We do not give the proof. For details, see [LY, Chapter 2] or [WW].

Next, we will extend the local solution to the whole region Ω. Therefore, some a priori estimates on the solution itself and the gradients, the C^0 and C^1 norms of A, B, and σ, are needed. The norm of σ comes from the norm of A and B; see the third equation of (2.25). Therefore, we need only the estimate on A and B. Recall that the derivation of (2.25) is based on the strict hyperbolicity of flow, $\sigma > 0$, $Y^2 - \tau^2\sigma \neq 0$ (or $X^2 - \tau^2\sigma \neq 0$). These will be achieved when we estimate the C^0 norms of A and B; see subsection 3.1. The main theorem is stated as follows.

THEOREM 3.3. *Let l be the interface of the gas and the vacuum. Then there is a unique solution $(A, B, \sigma) \in C^1$ to the boundary value problem* (2.25) *and* (3.7)–(3.13) *in $\Omega - l$.*

We prove this theorem by two steps. We estimate the solution itself and then proceed with the estimates on the gradients, respectively. Due to the degeneracy of interface l, we cut off a sufficient thin strip with width ϵ between the interface l and a level curve of σ. The remaining subdomain is denoted by $\tilde{\Omega}$. We first show that there is a unique solution on $\tilde{\Omega}$. Then we extend the solution to $\Omega - l$ by the arbitrariness of $\epsilon > 0$.

3.1. The estimate on the solution (A, B, σ). Now we estimate the solution itself (A, B, σ), i.e., the C^0 norm of A, B, and σ. The technique is the method of invariant regions [Sm].

LEMMA 3.4. *Suppose that there exists a C^1 solution $(A(u, v), B(u, v), \sigma(u, v))$ to the problem* (2.25) *and* (3.7)–(3.13) *in the region $\tilde{\Omega}$. Then the C^0 norm of A and B has a uniform bound just dependent on the boundary values* (3.13) *but independent of u and v in the region in which* $0 < \sigma \leq 1$.

Proof. Since $A_0^1, A_0^2 < -m_0$ and $B_0^1, B_0^2 > m_0$, we just consider system (2.25) in the second quadrant of (A, B) plane, in which $(AB - 1)/(A - B) > 0$. Therefore, σ is strictly increasing in the domain Ω. Note that $1 \geq \sigma > 0$.

Note also that $G(A, B) = 0$, i.e., $(1 - \tau/2)(AB + 1)^2 - (1 + \tau/2)(A - B)^2 = 0$, gives a hyperbola in the second quadrant. Another hyperbola $AB + 1 = 0$ passing through $(-1, 1)$ plays an important role too. They are all strictly increasing. Denote

$$L : \frac{AB + 1}{A - B} = \sqrt{\frac{1 + \tau/2}{1 - \tau/2}} = m_0, \qquad M : AB + 1 = 0. \tag{3.19}$$

The invariant region Σ is constructed as follows. The lines $B =$ const are chosen so that the top one, L_2, is above L and the bottom one, L_3, is between L and $B = m_0$. The lines $A =$ const are taken to be on both sides of L so that the left one, L_1, is on the left-hand side of L and the right one, L_4, is between L and $A = -m_0$. See Figure 3.3. Notice that this rectangle lies totally on the left-hand side of M.

Also notice that $\frac{AB+1}{A-B} > m_0$, $A < 0$, and $B > 0$ on L_1 and L_2. Therefore,

(3.20)
$$\begin{aligned} G_1(A, B) &= -\frac{1}{4B}\frac{A - B}{AB + 1}((1 - \tau/2)(AB + 1)^2 - (1 + \tau/2)(A - B)^2) < 0 \quad \text{on } L_1, \\ G_2(A, B) &= -\frac{1}{4A}\frac{A - B}{AB + 1}((1 - \tau/2)(AB + 1)^2 - (1 + \tau/2)(A - B)^2) > 0 \quad \text{on } L_2. \end{aligned}$$

While on L_3 and L_4, $0 < (AB + 1)/(A - B) < m_0$. Therefore,

$$G_2(A, B) < 0 \quad \text{on } L_3 \tag{3.21}$$

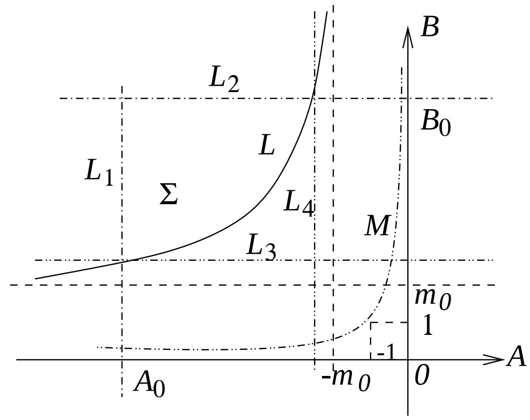

FIG. 3.3. *The invariant region of* (A, B).

and

$$G_1(A, B) > 0 \quad \text{on } L_4. \tag{3.22}$$

Thus we can construct an arbitrarily large invariant rectangle of this form. For all boundary values satisfying $-C < A_0^1, A_0^2 < -m_0$ and $m_0 < B_0^1, B_0^2 < C$, the solution (A, B, σ) lies inside the rectangle Σ. That is, the solution (A, B) has a uniform bound in $\tilde{\Omega}$. $\square$

In fact, the bounds of A and B are uniform in the whole region $\Omega - l$, in which $0 < \sigma \leq 1$.

Remark 3.5. If the angle of the wedge θ and the adiabatic index γ are related by

$$\cot^2 \theta = \frac{\gamma + 1}{3 - \gamma}, \tag{3.23}$$

then boundary value (3.13) becomes constant and is such that the source terms of (2.25) vanish; i.e., (2.25) becomes (2.26). At this moment, the solution has the explicit form

$$\sigma = \left(1 + \frac{\tau u}{2 \sin \theta}\right)^2, \tag{3.24}$$

where $-2 \sin \theta / \tau \leq u \leq 0$. This explicit solution was first observed in [Su] and is nontrivial, as said in [Le], because it is a complete solution for the expansion of a wedge of gas into a vacuum.

Remark 3.6. In the proof of Lemma 3.4, we note that $\frac{AB-1}{A-B} > \delta > 0$ for some constant δ. It follows from the third equation of (2.25) that

$$\sqrt{\sigma} < 1 + \frac{\delta}{2} u \tag{3.25}$$

for $u < 0$. Therefore, there must exist $\overline{u}(\beta) < 0$ such that $\sigma(\overline{u}(\beta), v(\overline{u})) = 0$, where $v = v(\overline{u})$ is a smooth curve in the (u, v) plane and $\pi/2 + \theta \leq \beta \leq 3\pi/2 - \theta$ is the polar angle as in Figure 3.2. This is the interface of gas and vacuum, on which $\sigma \equiv 0$. Indeed, since $(AB - 1)/(A - B)$ is uniformly bounded away from zero, we conclude from (2.4) and (2.25) that the interface in the (ξ, η) plane is the same as in the (u, v)

plane; i.e., $\xi = u$, $\eta = v$. For the explicit solution in (3.24), the interface l is a straight segment connecting D and E.

Remark 3.7. We write L as

$$L : (A + m_0)(B - m_0) = -m_0^2 - 1. \tag{3.26}$$

As $\gamma \to 3$, $m_0 \to \infty$. The branch of L in the second quadrant approaches infinity. It is therefore impossible to construct an invariant region as above for $\gamma = 3$. Namely, A and B have no uniform bound. For $\gamma > 3$, $(1-\tau/2)(A+B)^2 - (1+\tau/2)(A-B)^2 < 0$. Therefore, in view of (2.24), we cannot find an invariant region for (A, B) either.

Lemma 3.4 has the following two by-products.

COROLLARY 3.8. *The flow* (*system* (2.25)) *is uniformly supersonic* (*hyperbolic*) *in the region* $\tilde{\Omega}$.

Proof. In the proof of Lemma 3.4, we see that $\frac{AB+1}{A-B} > 0$ strictly. This shows that system (2.25) has two real eigenvalues by (2.22) and therefore is hyperbolic. Also from (2.23), $\lambda_- = \frac{A^2-1}{2A} < -\delta < 0$ and $\lambda_+ = \frac{B^2-1}{2B} > \delta > 0$ for some $\delta > 0$. So

$$\lambda_+ - \lambda_- > 2\delta > 0, \tag{3.27}$$

which gives the strict hyperbolicity. $\square$

COROLLARY 3.9. *The mappings* $(X, Y) \to (R, S)$, $(X, Y) \to (\lambda_-, \lambda_+)$, *and* $(X, Y) \to (A, B)$ *are all bijective in* $\tilde{\Omega}$.

Proof. Recall the statement before Theorem 2.1; it suffices to prove that $\tau^2\sigma - Y^2 \neq 0$. Note that

$$\tau^2\sigma - Y^2 = \tau^2\sigma(1 - (A+B)^2/(A-B)^2) = -2\tau^2\sigma AB/(A-B)^2 \neq 0 \tag{3.28}$$

due to $A < -m_0$, $B > m_0$. $\square$

Corollaries 3.8 and 3.9 show that Theorem 2.1 is true in $\tilde{\Omega}$, and, therefore, we can work on system (2.25) without hesitation.

3.2. Gradient estimates. We proceed to seek the gradient estimates for system (2.25). The result is stated in the following lemma.

LEMMA 3.10. *In* $\tilde{\Omega}$, *the* C^1 *norm of* A *and* B *has a uniform bound dependent only on the* C^0 *and* C^1 *norms of boundary values* (3.7)–(3.13).

Proof. Introduce

$$\overline{A} = \sqrt{\sigma}A, \qquad \overline{B} = \sqrt{\sigma}B. \tag{3.29}$$

Then we have

$$\begin{cases} \dfrac{\partial \overline{A}}{\partial v} = \sqrt{\sigma}\dfrac{\partial A}{\partial v} + \dfrac{\partial \sqrt{\sigma}}{\partial v}A, \\ \dfrac{\partial \overline{B}}{\partial v} = \sqrt{\sigma}\dfrac{\partial B}{\partial v} + \dfrac{\partial \sqrt{\sigma}}{\partial v}B \end{cases} \tag{3.30}$$

and

$$\begin{cases} \dfrac{\partial \overline{A}}{\partial u} + \lambda_+\dfrac{\partial \overline{A}}{\partial v} = \overline{G}_1(A, B), \\ \dfrac{\partial \overline{B}}{\partial u} + \lambda_-\dfrac{\partial \overline{B}}{\partial v} = \overline{G}_2(A, B), \end{cases} \tag{3.31}$$

where

$$
(3.32)\qquad \begin{cases} \overline{G}_1(A,B) = \tau(G_1(A,B) + AS/2) = \tau\left(G_1(A,B) + \dfrac{A}{2}\cdot\dfrac{1+B^2}{2B}\right),\\ \overline{G}_2(A,B) = \tau(G_2(A,B) - BR/2) = \tau\left(G_2(A,B) - \dfrac{B}{2}\cdot\dfrac{1+A^2}{2A}\right). \end{cases}
$$

Note that $\sigma > \delta_1 > 0$ and A, B are away from zero in $\tilde{\Omega}$. Once we have the bound of the C^1 norm of $\overline{A}$ and $\overline{B}$, the boundedness of the C^1 norm of A an B follows immediately.

Since system (2.25) is strictly hyperbolic in the sense that $\lambda_+ - \lambda_- > 2\delta > 0$ (see (3.27)), we set

$$
(3.33)\qquad \phi = (\lambda_- - \lambda_+)\frac{\partial\overline{A}}{\partial v}, \qquad \psi = (\lambda_+ - \lambda_-)\frac{\partial\overline{B}}{\partial v}.
$$

Once we obtain the boundedness of the C^0 norm of ϕ and ψ, the estimate on the gradient of $\bar{A}$ and $\bar{B}$ follows.

We calculate to obtain

$$
(3.34)\qquad \begin{aligned} \frac{\partial\phi}{\partial u} &= \frac{\partial}{\partial u}(\lambda_- - \lambda_+)\frac{\partial\overline{A}}{\partial v} + (\lambda_- - \lambda_+)\frac{\partial^2\overline{A}}{\partial u\partial v},\\ \frac{\partial\phi}{\partial v} &= \frac{\partial}{\partial v}(\lambda_- - \lambda_+)\frac{\partial\overline{A}}{\partial v} + (\lambda_- - \lambda_+)\frac{\partial^2\overline{A}}{\partial v^2}. \end{aligned}
$$

Differentiating the first equation of (3.31) with respect to v gives

$$
(3.35)\qquad \frac{\partial^2\overline{A}}{\partial u\partial v} + \lambda_+\frac{\partial^2\overline{A}}{\partial v^2} = \frac{\partial}{\partial v}\overline{G}_1(A,B) - \frac{\partial\lambda_+}{\partial v}\frac{\partial\overline{A}}{\partial v}.
$$

The combination of (3.33)–(3.35) yields

$$
(3.36)\qquad \frac{\partial\phi}{\partial u} + \lambda_+\frac{\partial\phi}{\partial v} = c_1 + d_1\phi + e_1\psi,
$$

after some routine calculation, where

$$
(3.37)\qquad \begin{cases} c_1 = c_1(A,B,\sigma) = -\dfrac{(\lambda_- - \lambda_+)\tau}{2\sqrt{\sigma}}\cdot\dfrac{A+B}{A-B}\cdot\left(A\dfrac{\partial\overline{G}_1}{\partial A} + B\dfrac{\partial\overline{G}_1}{\partial B}\right),\\ d_1 = d_1(A,B,\sigma) = \dfrac{(\lambda_- - \lambda_+)\tau}{\sqrt{\sigma}}\cdot\dfrac{A+B}{A-B}\cdot A\dfrac{\partial\overline{G}_1}{\partial A} + k_1(A,B),\\ e_1 = e_1(A,B,\sigma) = \dfrac{(\lambda_- - \lambda_+)\tau}{\sqrt{\sigma}}\cdot\dfrac{A+B}{A-B}\cdot B\dfrac{\partial\overline{G}_1}{\partial B}, \end{cases}
$$

and

$$
(3.38)\qquad k_1(A,B) = \frac{\tau(R-S)}{Q}\left(-(\lambda_-\lambda_+ + 1) + \frac{\tau}{2}RS\right).
$$

Here we make use of (2.20) and (2.22)–(2.24).

Similarly, we obtain

$$
(3.39)\qquad \frac{\partial\psi}{\partial u} + \lambda_-\frac{\partial\psi}{\partial v} = c_2 + d_2\phi + e_2\psi,
$$

where

$$\begin{cases} c_2 = c_2(A,B,\sigma) = -\dfrac{(\lambda_+ - \lambda_-)\tau}{2\sqrt{\sigma}} \cdot \dfrac{A+B}{A-B} \cdot \left(A\dfrac{\partial \overline{G}_2}{\partial A} + B\dfrac{\partial \overline{G}_2}{\partial B} \right), \\ d_2 = d_2(A,B,\sigma) = \dfrac{(\lambda_+ - \lambda_-)\tau}{\sqrt{\sigma}} \cdot \dfrac{A+B}{A-B} \cdot A\dfrac{\partial \overline{G}_2}{\partial A}, \\ e_2 = e_2(A,B,\sigma) = \dfrac{(\lambda_+ - \lambda_-)\tau}{\sqrt{\sigma}} \cdot \dfrac{A+B}{A-B} \cdot B\dfrac{\partial \overline{G}_2}{\partial B} + k_2(A,B), \end{cases} \tag{3.40}$$

and

$$k_2(A,B) = \frac{\tau(S-R)}{Q}\left(-(\lambda_-\lambda_+ + 1) + \frac{\tau}{2}RS \right). \tag{3.41}$$

The boundary values for ϕ and ψ can be obtained as follows. Since, on H_1, $u\cos\theta + v(u)\sin\theta = 0$, $A(u,v) = A_0^1(u, v(u))$, $\sqrt{\sigma} = 1 - \tau v'/2$, where $v' = -u\sin\theta + v\cos\theta$, we have $\overline{A}_0^1 = \sqrt{\sigma}A_0^1$ and

$$\frac{\partial \overline{A}}{\partial u} + \lambda_- \frac{\partial \overline{A}}{\partial v} = \frac{d\overline{A}_0^1}{du}. \tag{3.42}$$

Using the first equation of (3.31), we obtain

$$\phi = (\lambda_- - \lambda_+)\frac{\partial \overline{A}}{\partial v} = \frac{d\overline{A}_0^1}{du} - \overline{G}_1(A,B)|_{H_1} \quad \text{on } H_1. \tag{3.43}$$

Similarly, we have

$$\psi = (\lambda_+ - \lambda_-)\frac{\partial \overline{B}}{\partial v} = \frac{d\overline{B}_0^2}{du} - \overline{G}_2(A,B)|_{H_2} \quad \text{on } H_2, \tag{3.44}$$

where $\overline{B}_0^2$ is defined on H_2 similarly to $\overline{A}_0^1$ on H_1.

Thus we obtain a system for ϕ and ψ, consisting of two equations (3.36) and (3.39), with boundary values (3.43) and (3.44). By Lemma 3.4, the coefficients (c_i, d_i, e_i), $i = 1, 2$, in (3.37) and (3.40) have a uniform bound in $\tilde{\Omega}$. Then we can prove the uniform boundedness of ϕ and ψ on $\tilde{\Omega}$. Indeed, denote

$$V(\tilde{\Omega}) = \max_{(u,v)\in\tilde{\Omega}} \{\overline{A}(u,v), \overline{B}(u,v)\}. \tag{3.45}$$

Then, writing (3.36), (3.39) with the boundary value (3.43)–(3.44) into the corresponding integral equations and using Gronwall's inequality, we obtain

$$V(\tilde{\Omega}) \le c_1/\sqrt{\sigma}\exp(-c_2 u/\sqrt{\sigma}), \tag{3.46}$$

where c_1 and c_2 are positive constants depending only on the C^0 norm of A, B, σ in $\tilde{\Omega}$. Therefore, the C^1 norm of A and B is uniformly bounded due to (3.29), (3.33), and the strict hyperbolicity of system (2.25) on $\tilde{\Omega}$. Since the C^0 norm of A, B, and σ depends only on the boundary values (3.7)–(3.13) (see Lemma 3.1), so does the C^1 norm of A and B. Then Lemma 3.10 is proved. $\square$

Proof of Theorem 3.3. With the classical technique, we obtain the "global" solution in $\tilde{\Omega}$ by the extension from the local solution in $\Omega_\delta = \{(u,v) \in \tilde{\Omega}; -\delta < u < 0\}$

under the a priori estimates in Lemmas 3.4 and 3.10. Indeed, note that the extension step size δ in Lemma 3.2 depends only on the boundary values (3.7)–(3.13) and the C^0, C^1 norms of A, B, and σ. By Lemma 3.1, the boundary values A_0^1, A_0^2, B_0^1, and B_0^2 are uniformly bounded. And by Lemmas 3.4 and 3.10, σ, A, and B have uniform bounds both in the C^0 and C^1 norms in $\tilde{\Omega}$. Therefore, we can extend the solution σ, A, and B from Ω_δ to $\Omega_{2\delta}$, from $\Omega_{2\delta}$ to $\Omega_{3\delta}, \ldots$. Since $\tilde{\Omega}$ is compact, we obtain the solution $(\sigma, A, B) \in C^1$ by a finite number of steps; i.e., the C^1 solution (σ, A, B) to the problem (2.25) subject to the boundary values (3.7)–(3.13) exists uniquely in $\tilde{\Omega}$.

Since the width $\epsilon > 0$ of the strip between l and the level curves of σ is arbitrary, we can extend the C^1 solution to the interface l. That is, there is the unique C^1 solution (A, B, σ) in $\Omega - l$. □

As pointed out in section 2, the uniform boundedness of the C^1 norm of A and B cannot be obtained in the whole region Ω. This is because we cannot get the uniform bound for the C^1 norm of $\overline{G}_1(A,B)$ and $\overline{G}_2(A,B)$ with respect to $\overline{A}$ and $\overline{B}$ in (3.31). In fact, for a general linearly degenerate hyperbolic system with inhomogeneous source terms, in order to get the global solution in the domain under consideration, we need not only the strict hyperbolicity, the uniform boundedness of initial and/or boundary values, but most importantly the uniform bound for the C^1 norm of inhomogeneous source terms with respect to all variables as well. We take the following scalar equation to illustrate the strong dependence of the C^1 solution on the C^1 norm of the source term.

Consider the solution to the initial and boundary value problem in the half quadrant $x > 0$, $t > 0$,

$$\begin{cases} r_t + ar_x = 2\sqrt{r}, \quad a > 0, \\ r|_{t=0} = x, \\ r|_{x=0} = 0. \end{cases} \tag{3.47}$$

The solution can be written explicitly as

$$r(t,x) = \begin{cases} (t + \sqrt{x - at})^2, & x > at, \\ x^2/a^2, & 0 < x < at. \end{cases} \tag{3.48}$$

The first-order derivative with respect to x is

$$r_x(t,x) = \begin{cases} 1 + \dfrac{t}{\sqrt{x - at}}, & x > at, \\ \dfrac{2x}{a^2}, & 0 < x < at. \end{cases} \tag{3.49}$$

It is easy to see that the solution $r(t,x)$ is bounded in any compact domain, but r_x is discontinuous along the characteristic $x = at$ and becomes infinite in the right, which is obviously due to the nonuniform boundedness of the C^1 norm of $\sqrt{r}$ at $x = 0$.

Similar arguments may be applied for system (2.25) or (3.31) since the source terms contain the factor σ in the denominators and their C^1 norm becomes unbounded in the interface of gas and vacuum. Therefore, we cannot expect to obtain the uniform bound for the C^1 norm of A and B in the whole region.

Finally, we remark that the behavior of interface l of gas and vacuum was analyzed in [M, Le]. The convexity of the shape is closely related to the adiabatic index γ and the angle of wedge θ. The critical case is given in Remark 3.5, and the interface is a straight segment. The solution nearby can be approximately obtained by unsteady Prandtl–Meyer expansions.

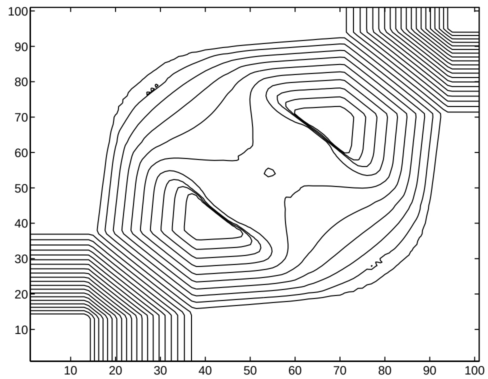

FIG. 4.1. *Contour curves of density. This solution is obtained by MmB Scheme in* [WY] *with uniform grids in which* $\Delta t/\Delta x = \Delta t/\Delta y = 0.1$ *with time steps* $n = 520$*, where* Δx *and* Δy *are the grid size and* Δt *is the time step size. The initial data are taken as* $\rho_1 = 4$, $\rho_2 = 1$, *and* $u_1 = u_2 = 0$. $\gamma = 1.4$ *for air.*

4. The application to the interaction of four rarefaction waves. In [ZZ1, Sc, SCG, CCY, LL], the four wave Riemann problem for isentropic Euler equations (1.1) was studied both theoretically and numerically. Many complicated flowfield patterns were demonstrated, including various interactions of planar elementary waves, Mach reflection, and vortex sheets, etc. An interesting example is the interaction of planar rarefaction waves. The initial data are taken as

$$(\rho, u, v)(t = 0, x, y) = (\rho_i, u_i, v_i), \quad (x, y) \text{ is in the } i\text{th quadrant}, \ i = 1, 2, 3, 4, \tag{4.1}$$

where $u_2 = u_3$, $u_1 = u_4$, $v_1 = v_2$, $v_3 = v_4$, $\rho_1 = \rho_3 > \rho_2 = \rho_4$, $u_1 - u_2 = \Theta_{12}$, $v_2 - v_3 = \Theta_{32}$, $u_3 - u_4 = \Theta_{43}$, and $v_1 - v_4 = \Theta_{14}$, where $\Theta_{ij} = \frac{2\sqrt{\gamma}}{\gamma-1}(\rho_i^{(\gamma-1)/2} - \rho_j^{(\gamma-1)/2})$. This initial data are discontinuous along each coordinate axis, from which planar rarefaction waves emit and interact instantaneously after $t = 0$. By numerical simulations, we observe that the solution is discontinuous in the interaction domain of the planar rarefaction waves, and there are two symmetric weak shocks in the solution; see Figure 4.1. This phenomenon never appears in the interaction of rarefaction waves in one-dimensional gas dynamics; see [CH]. In this section, we will attempt to explain this by making use of system (2.25). But we must admit that the theory for this phenomenon has not come out yet. By the way, since the solution is discontinuous, we take the isentropic Euler equation in conservative form and use the conservative difference scheme, the MmB scheme in [WY], to obtain the numerical result in Figure 4.1.

System (1.1) is valid until shock waves are present. Also notice that the initial data are irrotational. So from the interaction points of planar rarefaction waves, we can use (2.25) to analyze how they interact, as in section 3. We begin by considering self-similar smooth solutions of (1.1) and (4.1). Then the characteristics are defined by

$$\frac{d\eta}{d\xi} = \lambda^i, \quad i = 0, \pm, \tag{4.2}$$

where

$$\lambda^0 = \frac{v - \eta}{u - \xi}, \qquad \lambda^{\pm} = \frac{(u - \xi)(v - \eta) \pm \sqrt{\sigma((u - \xi^2) + (v - \eta)^2 - \sigma)}}{(u - \xi)^2 - \sigma}. \tag{4.3}$$

Using (2.4) in (4.3), we have

$$\lambda^{\pm} = \frac{\sigma_u\sigma_v \pm \tau\sqrt{\sigma(\sigma_u^2+\sigma_v^2-\tau^2\sigma)}}{\sigma_u^2-\tau^2\sigma} = \frac{\sigma_v^2-\tau^2\sigma}{\sigma_u\sigma_v \mp \tau\sqrt{\sigma(\sigma_u^2+\sigma_v^2-\tau^2\sigma)}}. \tag{4.4}$$

Comparing these with (2.13) yields $\lambda^- = -1/\lambda_+$ and $\lambda^+ = -1/\lambda_-$. As pointed out at the end of section 2, there are two factors that produce the singularity of (2.25), i.e., the blowup of the solution: one is the vacuum $\sigma = 0$, and the other is the parabolic degeneracy that $\lambda_- = \lambda_+$, which results from the fact that $Q = 0$. In this context, the numerical result shows that the density is away from zero. Therefore, the blowup of the gradient of solutions is due only to the latter. That is, the shock waves appear on the curves on which $\lambda^- = \lambda^+$, whose images are the degeneracy curves (such that $\lambda_- = \lambda_+$) of system (2.25) in the (u, v) plane.

With the same argument as in section 3, we can obtain the continuous solution until shock waves are present by solving (2.25). In order to achieve a global weak solution, we will have to solve a transonic problem for (1.1) with shock waves as (at least part of) the free boundary. This is a completely open problem.

Appendix. Kelvin's theorem implies the constancy of circulation. Here for our use we simply deduce the formula that the vorticity field should satisfy for smooth self-similar flows. Actually, this conclusion can be derived rigorously for weak solutions without involving slip surfaces.

From the laws of momentum conservation of (1.1), we have

$$\begin{aligned} &u_t + uu_x + vu_y + \frac{1}{\tau}\sigma_x = 0,\\ &v_t + uv_x + vv_y + \frac{1}{\tau}\sigma_y = 0. \end{aligned} \tag{A.1}$$

Differentiating the first equation of (A.1) with respect to y and the second equation with respect to x and then subtracting one from the other, we get

$$w_t + uw_x + vw_y = -(u_x + v_y)w, \tag{A.2}$$

where $w = u_y - v_x$ is the vorticity. We integrate (A.2) along the streamline $\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$ to obtain

$$w(t,x,y) = w(0,x,y)\exp\left(-\int_0^t (u_x+v_y)dt\right), \tag{A.3}$$

where $w(0, x, y)$ is the initial vorticity at $(0, x, y)$. Indeed, (A.3) is valid for the general case in $n \geq 2$ dimensions.

For quasi-stationary flows, we have

$$(u-\xi, v-\eta)\cdot(\partial_\xi, \partial_\eta) = t(1,u,v)\cdot(\partial_t, \partial_x, \partial_y). \tag{A.4}$$

Therefore, (A.2) becomes

$$(u-\xi)w_\xi + (v-\eta)w_\eta = -(u_\xi + v_\eta)w. \tag{A.5}$$

Since $\xi^2 + \eta^2 \to \infty$ as $t \to 0$, (A.5) results in

$$w(\xi,\eta) = w(\infty)\exp\left(-\int_\infty^{(\xi,\eta)} (u_\xi + v_\eta)dl\right), \tag{A.6}$$

where the integral path is the quasi streamline $d\eta/d\xi = (\eta - v)/(\xi - u)$, dl is the infinitesimal element of this path, and $w(\infty)$ is the vorticity at infinity of (ξ, η) plane.

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