ON THE UNIQUENESS AND EXISTENCE PROBLEM FOR A MULTIDIMENSIONAL REACTING AND CONVECTION SYSTEM

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Abstract

The paper is concerned with the uniqueness and existence problem for a multidimensional reacting and convection system with the vanishing viscosity method. The uniqueness theorem is obtained from the stability with respect to the initial data. To solve the existence problem, the uniqueness and existence of solutions to a viscous system are first proved, and then the L^1-modulus of continuity of solutions independent of the small viscosity is obtained.

1. *Introduction*

Consider the Cauchy problem for a multidimensional reacting and convection system

$$\begin{cases} \partial_t(u+qz)+\sum_i \partial_i f_i(u) = 0, \\ \partial_t z + K\phi(u)\, z = 0 \end{cases} \tag{1.1}$$

with the initial data

$$(u,z)(t=0,x) = (u_0,z_0)(x) \in L^\infty(R^n). \tag{1.2}$$

Here and throughout this paper $x = (x_1,\dots,x_n) \in R^n$, $t \geqslant 0$, $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$, and $\sum$ designates the summation taken from 1 to n unless exceptional cases are pointed out. This model is the natural generalization of a one-dimensional model to display dynamic combustion in several dimensions (cf. [**10**, **11**] and so on). The first equation of (1.1) resembles the energy conservation laws in gas dynamics, leading to shock waves and rarefaction waves, and the second equation is a reacting equation with contact discontinuities, along which the chemical reaction occurs. We interpret the dependent variable u as a lumped variable to represent density, velocity and temperature, and z as the percentage of unburnt gas. The constant q is the chemical binding energy, K denotes the reaction rate and $\phi(u)$ $(0 \leqslant \phi \leqslant 1)$ is an ignition type function taken as

$$\phi(u) = \begin{cases} 1, & u > \bar{u}+\eta, \\ 0, & u < \bar{u}, \end{cases} \tag{1.3}$$

where $\bar{u}$ is the ignition temperature and can be considered as zero, $\eta > 0$. We assume that $f_i \in C^2(R)$, $\phi \in C^1(R)$, and L_f and L_ϕ are their respective Lipschitz constants.

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Our purpose is to prove the uniqueness and existence of weak entropy solutions to (1.1) and (1.2). The uniqueness theorem is based on the framework of Kruzkov [**4**] and obtained from the stability relative to the changes in the initial data. The existence problem is solved with the vanishing viscosity method; see [**3–5**]. In this connection, we first prove the uniqueness and existence of solutions to the Cauchy problem for a reacting–diffusion–convection system

$$\begin{cases} \partial_t(u+qz)+\sum_i \partial_i f_i(u) = \epsilon\Delta u, \quad \epsilon > 0, \\ \partial_t z + K\phi(u)\, z = 0, \end{cases} \tag{1.4}$$

where ϵ is the viscosity, and Δ is the Laplacian. This result lays a solid foundation for the theory of multidimensional gas dynamic combustion [**7**]. Then we derive the modulus of continuity in L^1 of the solutions independent of ϵ.

Relating to this work, the existence and uniqueness problem for the one-dimensional case of (1.1) and (1.2) was shown in [**5**, **11**] with the difference method and the vanishing viscosity method respectively. The corresponding Riemann problem was studied in [**9**, **11**]. The one-dimensional case of (1.4) was taken by Majda [**10**] to model the dynamic combustion and the limit of viscous profile related to the strong detonation was proved to be a travelling wave solution as $t \to \infty$. The proof of the stability of viscous strong detonation as $t \to \infty$ was due to [**6**, **8**].

This paper is organized as follows. We give the definition of the weak entropy solution to (1.1) and (1.2) in Section 2. On the basis of this definition, we follow Kruzkov and Levy [**4**, **5**] to prove the uniqueness of solutions. The result is obtained from the stability of solutions with respect to the initial data, for which a more exact estimate than that in [**5**] is given. This is the content of Section 3. Section 4 contributes to the uniqueness and existence problem for the viscous system (1.4) and (1.2). In Section 5, we prove the existence of a weak entropy solution to (1.1) and (1.2) by using the estimates on the L^1-modulus of continuity.

2. *Definition of a weak entropy solution and entropy condition on simple discontinuities*

As is well known, the solutions of (1.1) and (1.2) may not exist after a finite time $T^* > 0$ in the classical sense even though the initial data (1.2) is smooth enough, and some additional condition should be posed in order to characterize an admissible solution among all their solutions in the generalized sense. In this section we will derive the definition of weak entropy solutions of (1.1), which contains the entropy condition. The method is frequently used for conservation laws; see for example [**3**]. For convenience of statement, we often use the notation

$$\Pi_T = [0, T] \times R^n. \tag{2.1}$$

For simplicity, assume that (u, z) is a smooth solution to (1.4). Then (1.4) is equivalent to the system

$$\begin{cases} \partial_t u + \sum_i \partial_i f_i(u) - Kq\phi(u)\, z = \epsilon\Delta u, \\ \partial_t z + K\phi(u)\, z = 0. \end{cases} \tag{2.2}$$

We multiply the first equation of (2.2) by a convex function $\Phi'(u)$ ($\Phi''(u) > 0$) to obtain

$$\begin{aligned}\partial_t \Phi(u) + \sum_i \partial_i F_i(u) - Kq\phi(u) z\Phi'(u) &= \epsilon \Delta u \Phi'(u) \\ &= \epsilon\left(\Delta\Phi(u) - \Phi''(u)\sum_i (\partial_i u)^2\right) \leqslant \epsilon\Delta\Phi(u), \end{aligned} \tag{2.3}$$

where

$$F_i(u) = \int f_i'(u)\,\Phi'(u)\,du. \tag{2.4}$$

Since the solutions u_ϵ of (2.2) are locally bounded and have a limit as $\epsilon \to 0$, which will be seen in later sections, we obtain

$$\int_{\Pi_T} \left\{\Phi(u)\,\partial_t g + \sum_i F_i(u)\,\partial_i g + Kq\phi(u)\,z\Phi'(u)\,g\right\} dx\,dt \geqslant 0, \tag{2.5}$$

for all $0 \leqslant g(t,x) \in C_0^\infty(\Pi_T)$.

Taking

$$\Phi(u) = ((u-v)^2 + \mu^2)^{1/2} \tag{2.6}$$

with $v \in R$ and $\mu > 0$, we conclude that

$$\Phi(u) = |u - v| \quad \text{and} \quad F_i(u) = \operatorname{sign}(u-v)\,(f_i(u) - f_i(v))$$

as μ tends to zero, where $\operatorname{sign}(x) = \pm 1$ if $\pm x > 0$. Therefore, the definition of weak entropy solutions to (1.1) and (1.2) can be given as follows.

DEFINITION 2.1. A pair of bounded measurable functions $(u,z)(t,x)$ is a *weak entropy solution* to the Cauchy problem (1.1) and (1.2) in the strip Π_T for a bounded $T > 0$ if $(u,z)(t,x) \in C([0,T], L^1_{\mathrm{loc}}(R^n))$,

$$(u,z)(0,x) = (u_0, z_0)(x), \quad \text{almost everywhere in } \Pi_T, \tag{2.7}$$

and

$$\begin{cases} \displaystyle\int_{\Pi_T} \operatorname{sign}(u-v)\left\{(u-v)\,\partial_t g + \sum_i (f_i(u) - f_i(v))\,\partial_i g + Kq\phi(u)\,zg\right\} dx\,dt \geqslant 0, \\ \displaystyle\int_{\Pi_T} \{z\partial_t g - K\phi(u)\,zg\}\,dx\,dt = 0, \end{cases} \tag{2.8}$$

for any $0 \leqslant g \in C_0^\infty(\Pi_T)$ and $v \in R$.

This definition implies the integral indentity by setting $v = \pm\sup_{(t,x)\in\Pi_T} |u(t,x)|$ so that

$$\int_{\Pi_T} \left\{u\partial_t g + \sum_i f_i(u)\,\partial_i g + Kq\phi(u)\,zg\right\} dx\,dt = 0, \tag{2.9}$$

for any $g \in C_0^\infty(\Pi_T)$ and $v \in R$. The second equation of (2.8) is equivalent to the equation

$$\int_{\Pi_T} \operatorname{sign}(z(t,x) - w)\,\{(z(t,x) - w)\,\partial_t g - K\phi(u)\,zg\}\,dx\,dt = 0 \tag{2.10}$$

for all $w \in R$.

For a discontinuity, this definition also gives an entropy condition. Let $\nu = (\nu_t, \nu_1, \ldots, \nu_n)$ be a normal to a surface of discontinuity S with $(u^{\pm}, z^{\pm})$ as the limit value of the weak entropy solution (u, z) on the side to which $\pm\nu$ is pointing. Then, from (2.8), there holds either

$$\nu_t = 0 \tag{2.11}$$

or

$$(u^+ - u^-)\,\nu_t + \sum_i (f_i(u^+) - f_i(u^-))\,\nu_i = 0. \tag{2.12}$$

For the former, u is continuous and z may undergo a jump. This is a contact discontinuity. For the latter, z is continuous (that is, no reaction occurs) and u undergoes a jump for which the following entropy condition should be satisfied:

$$(u^- - v)\,\nu_t + \sum_i (f_i(u^-) - f_i(v))\,\nu_i \leqslant 0, \tag{2.13}$$

for all $v \in (u^-, u^+)$. This inequality is consistent with the entropy condition in [**11**].

3. *Uniqueness of the weak entropy solution to* (1.1) *and* (1.2) *and stability with respect to the initial condition*

Denote

$$M = \|u\|_{L^\infty(\Pi_T)}, \quad N = \max_{|u| \leqslant M} \left\{ \sum_i (f_i'(u))^2 \right\}^{1/2}, \tag{3.1}$$

and

$$\Omega = \{(t, x); |x| < R, 0 \leqslant t \leqslant T_0 = \min(T, RN^{-1})\}, \tag{3.2}$$

$$S_\tau = \Omega \cap \{t = \tau; 0 \leqslant \tau \leqslant T_0\}. \tag{3.3}$$

Then the uniqueness of the weak entropy solution of the Cauchy problem (1.1) and (1.2) follows from the following theorem concerning the stability with respect to the initial data in the norm of the L^1_{loc}-space.

THEOREM 3.1. *Let (u_1, z_1) and (u_2, z_2) be two weak entropy solutions of the problem* (1.1) *and* (1.2) *with the initial data (u_0^1, z_0^1) and (u_0^2, z_0^2) respectively. Then, for almost all $t \in [0, T_0]$,*

$$\int_{S_t} \{|u_1(t,x) - u_2(t,x)| + |z_1(t,x) - z_2(t,x)|\}\,dx \leqslant C \int_{S_0} \{|u_0^1 - u_0^2| + |z_0^1 - z_0^2|\}\,dx, \tag{3.4}$$

where $C = \exp(K(q+1)\max(1, L_\phi)\,t)$.

Proof. Let $0 \leqslant g(t, x, \tau, y) \in C_0^\infty(\Pi_T \times \Pi_T)$. We set $v = u_2(\tau, y)$, $g(t, x) = g(t, x, \tau, y)$ for a fixed point (τ, y) in (2.8) and then integrate over Π_T with respect to (τ, y) to obtain

$$\begin{aligned}\iint_{\Pi_T \times \Pi_T} &\mathrm{sign}(u_1(t,x) - u_2(\tau,y)) \Big\{ (u_1(t,x) - u_2(\tau,y))\,\partial_t g(t,x,\tau,y) \\ &+ \sum_i (f_i(u_1(t,x)) - f_i(u_2(\tau,y)))\,\partial_i g(t,x,\tau,y) \\ &+ Kq\phi(u_1(t,x))\,z_1(t,x)\,g(t,x,\tau,y) \Big\}\,dx\,dt\,d\tau\,dy \geqslant 0.\end{aligned} \tag{3.5}$$

Adding the analogue of the above inequality where u_1, z_1, t, x are interchanged with u_2, z_2, τ, y, we get

$$\iint_{\Pi_T \times \Pi_T} \operatorname{sign}(u_1(t,x)-u_2(\tau,y)) \Big\{ (u_1(t,x)-u_2(\tau,y))\,(\partial_t g + \partial_\tau g) + \sum_i (f_i(u_1(t,x))-f_i(u_2(\tau,y)))\,(\partial_{x_i} g + \partial_{y_i} g) + Kq(\phi(u_1(t,x))\,z_1(t,x) - \phi(u_2(\tau,y))\,z_2(\tau,y))\,g \Big\}\,dx\,dt\,d\tau\,dy \geqslant 0. \quad (3.6)$$

Here and in the remainder of this section $\partial_{x_i} = \partial/\partial x_i$ and $\partial_{y_i} = \partial/\partial y_i$.

Take

$$g(t,x,\tau,y) = p\left(\frac{t+\tau}{2}, \frac{x+y}{2}\right)\chi_h\left(\frac{t-\tau}{2}\right)\prod_{i=1}^{n}\chi_h\left(\frac{x_i-y_i}{2}\right), \quad (3.7)$$

where $p(t,x) \in C_0^\infty([\rho, T-2\rho] \times K_{r-2\rho})$, $K_r = \{x \in R^n; |x| < r\}$, $0 \leqslant \chi_h \in C_0^\infty(|x| < h < \rho, x \in R)$ with $\int_R \chi_h(y)\,dy = 1$. Then, with the same method as in [**4**], we get from (3.6)

$$\int_{\Pi_T} \operatorname{sign}(u_1(t,x)-u_2(t,x)) \Big\{ (u_1(t,x)-u_2(t,x))\,\partial_t g + \sum_i (f_i(u_1(t,x))-f_i(u_2(t,x)))\,\partial_i g + Kq(\phi(u_1(t,x))\,z_1(t,x) - \phi(u_1(t,x))\,z_2(t,x))\,g \Big\}\,dx\,dt \geqslant 0 \quad (3.8)$$

by letting h approach zero.

Denote by Υ_u the set which does not contain Lebesgue points of the function u on $[0, T]$. Then it has Lebesgue measure zero; see [**4**]. Set

$$\alpha_h(\sigma) = \int_{-\infty}^{\sigma} \chi_h(y)\,dy, \quad 0 \leqslant \delta_h^\gamma(t,x) = 1 - \alpha_h(|x| + Nt - R + \gamma), \quad \gamma > 0,$$
$$0 \leqslant f_h^\gamma(t,x) = [\alpha_h(t-\rho) - \alpha_h(t-\tau)]\,\delta_h^\gamma(t,x), \quad (3.9)$$

for some $0 < \rho < \tau \leqslant \min(\rho, T-\tau, (\tau-\rho)/2, \rho, \tau \in [0, T_0] \backslash (\Upsilon_{u_1} \cup \Upsilon_{u_2} \cup \Upsilon_\mu)$, where the notation

$$\mu(t) = \int_{S_t} |u_1(t,x) - u_2(t,x)|\,dx \quad (3.10)$$

is used. Then we have $\operatorname{supp} \delta_h^\gamma(t,x) \subset \Omega$, and

$$0 \equiv \partial_t \delta_h^\gamma + N\left|\sum_i \partial_i \delta_h^\gamma\right| \geqslant \partial_t \delta_h^\gamma + \sum_i \frac{f_i(u_1) - f_i(u_2)}{u_1 - u_2}\,\partial_i \delta_h^\gamma. \quad (3.11)$$

Taking $g(t,x) = f_h^\gamma(t,x)$, one can obtain

$$\int_{\Pi_T} \{|u_1(t,x) - u_2(t,x)|\,[\chi_h(t-\rho) - \chi(t-\tau)]\,\delta_h^\gamma + Kq|\phi(u_1)\,z_1 - \phi(u_2)\,z_2|\,f_h^\gamma\}\,dx\,dt \geqslant 0. \quad (3.12)$$

Since ρ and τ are the Lebesgue points of $\mu(t)$, it follows that, as $\gamma \to 0$,

$$\int_0^{T_0} \left\{ [\chi_h(t-\rho) - \chi_h(t-\tau)] \int_{S_t} |u_1(t,x) - u_2(t,x)|\, dx \right\} dt$$
$$+ Kq \int_0^T \left\{ [\alpha_h(t-\rho) - \alpha_h(t-\tau)] \int_{S_t} |\phi(u_1)\, z_1 - \phi(u_2)\, z_2|\, dx \right\} dt \geqslant 0. \tag{3.13}$$

Letting $h \to 0$, and noting the assumption that $u_1, u_2 \in C([0,T], L^1_{\mathrm{loc}}(R^n))$, we get

$$\mu(\rho) - \mu(\tau) + Kq \int_\rho^\tau \int_{S_t} |\phi(u_1)\, z_1 - \phi(u_2)\, z_2|\, dx\, dt \geqslant 0. \tag{3.14}$$

Also, with the same arguments as above, we obtain from (2.10) that

$$\nu(\rho) - \nu(\tau) + K \int_\rho^\tau \int_{S_t} |\phi(u_1)\, z_1 - \phi(u_2)\, z_2|\, dx\, dt \geqslant 0, \tag{3.15}$$

where

$$\nu(t) = \int_{S_t} |z_1(t,x) - z_2(t,x)|\, dx. \tag{3.16}$$

Since

$$|\phi(u_1)\, z_1 - \phi(u_2)\, z_2| \leqslant |z_1 - z_2| + L_\phi |u_1 - u_2|,$$

we add (3.14) and (3.15) and then let ρ approach zero to arrive at

$$\mu(\tau) + \nu(\tau) \leqslant \mu(0) + \nu(0) + K(q+1) \max(1, L_\phi) \int_0^\tau (\mu(t) + \nu(t))\, dt. \tag{3.17}$$

Hence, it follows from Gronwall's inequality that

$$\mu(\tau) + \nu(\tau) \leqslant (\mu(0) + \nu(0)) \cdot \exp(K(q+1) \max(1, L_\phi)\, \tau). \tag{3.18}$$

The proof is complete. $\square$

We point out that this theorem shows the unique weak entropy solution of (1.1) and (1.2) has the finite propagation speed property.

4. *Existence and uniqueness problem for the viscous system*

In this section, we solve the uniqueness and existence problem for (1.4) and (1.2). First, we assume that $(u_0, z_0)(x) \in C_0^3(R^n)$. Then the following maximum principle is satisfied.

THEOREM 4.1. *If* $(u(t,x), z(t,x))$ *is a classical bounded solution of* (1.4) *and* (1.2) *in* Π_T *for a bounded* $T > 0$, *then*

$$\inf_{x \in R^n} \{u_0(x)\} \leqslant u(t,x) \leqslant \sup_{x \in R^n} \{u_0(x)\} + Kqt,$$
$$\inf_{x \in R^n} \{z_0(x)\} \leqslant z(t,x) \leqslant \sup_{x \in R^n} \{z_0(x)\}. \tag{4.1}$$

Proof. The method is classical. We only prove the upper bound claim of $u(t,x)$. Assume that $v(t,x) = u(t,x) - Kqt - \delta(2\epsilon t + |\delta x|^2/2n)$ $(\delta > 0)$ attains its maximum value at $(\bar{t}, \bar{x})$ in Π_T. Then, at $(\bar{t}, \bar{x})$, $\partial_t v = \partial_t u - Kq - 2\delta\epsilon \geqslant 0$, $\Delta v = \Delta u - \delta^3 \leqslant 0$ and $\partial_j v = \partial_j u - \delta^3 \bar{x}_j/n = 0$, so

$$\begin{aligned} 0 &\leqslant \partial_t v - \epsilon\Delta v = \partial_t u - Kq - 2\delta\epsilon - \epsilon\Delta u + \epsilon\delta^3 \\ &= -\sum_i f_i'(u(\bar{t},\bar{x}))\,\partial_i u + Kq\phi(u(\bar{t},\bar{x}))\,z(\bar{t},\bar{x}) - Kq - 2\delta\epsilon + \epsilon\delta^3 \\ &= -\sum_i f_i'(u(\bar{t},\bar{x}))\frac{\delta^3\bar{x}_j}{n} - Kq(1-\phi(u(\bar{t},\bar{x}))\,z(\bar{t},\bar{x})) - 2\delta\epsilon + \epsilon\delta^3 < 0 \end{aligned}$$

if δ is small enough. This contradiction proves our claim. □

Next we consider the existence problem. We make use of the iterative method in [**3**, **5**] to solve it. It is evident that the viscous system (1.4) is equivalent to

$$\begin{cases} \partial_t u - \epsilon\Delta u = -\sum_i \partial_i f_i(u) + Kq\phi(u)\,z, \\ \partial_t z + K\phi(u)\,z = 0. \end{cases} \tag{4.2}$$

Define inductively $(u^{(m)}(t,x), z^{(m)}(t,x))$ $(m = 0, 1, \ldots)$ by the solutions of the Cauchy problems

$$\begin{cases} \partial_t u^{(m)} - \epsilon\Delta u^{(m)} = -\sum_i \partial_i f_i(u^{(m-1)}) + Kq\phi(u^{(m-1)})\,z^{(m-1)}, \\ \partial_t z^{(m-1)} + K\phi(u^{(m-1)})\,z^{(m-1)} = 0 \end{cases} \tag{4.3}$$

subject to the initial data

$$(u^{(m)}, z^{(m)})\,(t=0,x) = (u_0, z_0)\,(x). \tag{4.4}$$

Here $u^{(-1)}$ should be read as zero and $z^{(-1)}$ as $z_0(x)$. Then the solutions of (4.3) and (4.4) can be given, by Duhamel's principle, as

$$\begin{cases} u^{(m)} = -\int_0^t \int_{R^n} \sum_i \partial_i f_i(u^{(m-1)}(s,y))\,E_\epsilon(t-s, x-y)\,ds\,dy \\ \qquad + Kq\int_0^t \int_{R^n} \phi(u^{(m-1)}(s,y))\,z^{(m-1)}(s,y)\,E_\epsilon(t-s,x-y)\,ds\,dy \\ \qquad + \int_{R^n} u_0(y)\,E_\epsilon(t, x-y)\,dy, \\ z^{(m-1)} = z_0(x)\exp\left(-K\int_0^t \phi(u^{(m-1)}(s,x))\,ds\right), \end{cases} \tag{4.5}$$

where $E_\epsilon(t,x)$ is the heat kernel

$$E_\epsilon(t,x) = (4\pi\epsilon t)^{-n/2}\exp\left(-\frac{|x|^2}{4\epsilon t}\right) \tag{4.6}$$

satisfying

$$\int_{R^n} |\partial_x^\alpha E_\epsilon(t,x)|\,dx = C_\alpha(\epsilon t)^{-|\alpha|/2}, \quad t > 0, \tag{4.7}$$

$\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and C_α is a constant depending only on α and $C_0 = 1$.

Set

$$v^{(m)} = u^{(m+1)} - u^{(m)}, \quad w^{(m)} = z^{(m+1)} - z^{(m)}. \tag{4.8}$$

Then we have, from (4.5),

$$\begin{cases} v^{(m)} = -\displaystyle\int_0^t \int_{R^n} \sum_i \partial_i [f_i(u^{(m)}(s,y)) - f_i(u^{(m-1)}(s,y))]\, E_\epsilon(t-s, x-y)\, ds\, dy \\ \qquad + Kq \displaystyle\int_0^t \int_{R^n} [\phi(u^{(m)}(s,y))\, z^{(m)}(s,y) - \phi(u^{(m-1)}(s,y))\, z^{(m-1)}(s,y)] \\ \qquad \cdot E_\epsilon(t-s, x-y)\, ds\, dy \\ w^{(m-1)} = z_0(x) \left[\exp\left(-K \displaystyle\int_0^t \phi(u^{(m)}(s,x))\, ds \right) - \exp\left(-K \displaystyle\int_0^t \phi(u^{(m-1)}(s,x))\, ds \right) \right]. \end{cases} \tag{4.9}$$

Noting that if $x_1, x_2 < 0$,

$$|e^{x_1} - e^{x_2}| < |x_1 - x_2|,$$

we get

$$\begin{aligned} |w^{(m-1)}| &\leqslant K \int_0^t |\phi(u^{(m)}(s,x)) - \phi(u^{(m-1)}(s,x))|\, ds \\ &\leqslant K L_\phi \int_0^t |v^{(m-1)}(s,x)|\, ds. \end{aligned}$$

Therefore

$$\|w^{(m-1)}\|_{L^\infty(\Pi_t)} \leqslant K L_\phi \int_0^t \|v^{(m-1)}\|_{L^\infty(\Pi_s)}\, ds. \tag{4.10}$$

Integrating by parts in (4.9), we arrive at

$$\begin{cases} v^{(m)}(t,x) = \displaystyle\int_0^t \int_{R^n} \sum_i [f_i(u^{(m)}(s,y)) - f_i(u^{(m-1)}(s,y))]\, \partial_i E_\epsilon(t-s, x-y)\, ds\, dy \\ \qquad + Kq \displaystyle\int_0^t \int_{R^n} [\phi(u^{(m)}(s,y))\, z^{(m)}(s,y) - \phi(u^{(m-1)}(s,y))\, z^{(m-1)}(s,y)] \\ \qquad \cdot E_\epsilon(t-s, x-y)\, ds\, dy. \end{cases}$$

This gives, through the use of (4.7) and (4.10),

$$\begin{cases} |v^{(m)}(t,x)| \leqslant \displaystyle\int_0^t \int_{R^n} \sum_i |f_i(u^{(m)}(s,y)) - f_i(u^{(m-1)}(s,y))|\, |\partial_i E_\epsilon(t-s, x-y)|\, ds\, dy \\ \qquad + Kq \displaystyle\int_0^t \int_{R^n} \{ |\phi(u^{(m)}(s,y)) - \phi(u^{(m-1)}(s,y))|\, z^{(m)}(s,y) \\ \qquad + \phi(u^{(m-1)}(s,y))\, |z^{(m)}(s,y) - z^{(m-1)}(s,y)| \} \cdot E_\epsilon(t-s, x-y)\, ds\, dy \\ \qquad \leqslant C_1 \displaystyle\int_0^t \|v^{(m-1)}\|_{L^\infty(\Pi_s)} ((t-s)^{-1/2} + 1)\, ds, \end{cases}$$

if $0 \leqslant t \leqslant T$ for any bounded $T > 0$, that is,

$$\|v^{(m)}\|_{L^\infty(\Pi_t)} \leqslant C_1 \int_0^t \|v^{(m-1)}\|_{L^\infty(\Pi_s)}((t-s)^{-1/2}+1)\, ds, \tag{4.11}$$

where $C_1 = C_1(K, q, L_f, L_\phi, T)$. Hence, for $0 \leqslant t \leqslant T$,

$$\|v^{(m)}\|_{L^\infty(\Pi_t)} \leqslant \|u_0\|_{L^\infty(R^n)}\, C_T^m\, t^{m/2}\Gamma\left(\frac{m+2}{2}\right)^{-1}, \tag{4.12}$$

if $C_T > C_1(\sqrt{T}+1)\,\Gamma(1/2)$, where $\Gamma(s)$ is the gamma function.

In fact, (4.12) is true when $m = 0$, and since

$$\int_0^t (t-s)^{-1/2} s^{m/2}\, ds = t^{(m+1)/2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{m+2}{2}\right)^{-1}\Gamma\left(\frac{m+3}{2}\right),$$

the estimate (4.12) follows inductively; see [**3**, **5**]. The consequence is that

$$\sum_{m=0}^{\infty} \|v^{(m)}\|_{L^\infty(\Pi_t)} < \infty, \tag{4.13}$$

if $0 \leqslant t \leqslant T$. This implies that the sequence $\{u^{(m)}\}_{m=0}^{\infty}$ has a uniform limit in any strip $0 \leqslant t \leqslant T$, and so does $\{z^{(m)}\}_{m=0}^{\infty}$ from (4.10). That is,

$$\lim_{m\to\infty} u^{(m)}(t,x) = u(t,x), \quad \lim_{m\to\infty} z^{(m)}(t,x) = z(t,x), \tag{4.14}$$

in $0 \leqslant t \leqslant T$ for any bounded $T > 0$.

Now our task is to prove that $\{u(t,x), z(t,x)\}$ defined in (4.14) is the classical solution of (1.4) and (1.2). Obviously, owing to (4.5), $u(t,x)$ and $z(t,x)$ satisfy

$$\begin{aligned} u(t,x) = &\int_0^t \int_{R^n} \sum_i f_i(u(s,y))\, \partial_i E_\epsilon(t-s, x-y)\, ds\, dy \\ &+ Kq \int_0^t \int_{R^n} \phi(u(s,y))\, z(s,y)\, E_\epsilon(t-s, x-y)\, ds\, dy + \int_{R^n} u_0(y)\, E_\epsilon(t, x-y)\, dy, \end{aligned} \tag{4.15}$$

$$z(t,x) = z_0(x) \exp\left(-K \int_0^t \phi(u(s,x))\, ds\right). \tag{4.16}$$

Differentiating (4.5) with respect to x_j ($j = 1, \ldots, n$), we obtain

$$\begin{aligned} \partial_j u^{(m)}(t,x) = &-\int_0^t \int_{R^n} \sum_i \partial_i f_i(u^{(m-1)}(s,y))\, \partial_j E_\epsilon(t-s, x-y)\, ds\, dy \\ &+ Kq \int_0^t \int_{R^n} \phi(u^{(m-1)}(s,y))\, z^{(m-1)}(s,y)\, \partial_j E_\epsilon(t-s, x-y)\, ds\, dy \\ &+ \int_{R^n} \partial_j u_0(y)\, E_\epsilon(t, x-y)\, dy. \end{aligned} \tag{4.17}$$

It follows by using (4.7) that

$$\|\partial_j u^{(m)}\|_{L^\infty(\Pi_t)} \leqslant C_2 \int_0^t \|\partial_j u^{(m-1)}\|_{L^\infty(\Pi_s)} (t-s)^{-1/2}\, ds + C_3, \tag{4.18}$$

where $C_2 = C_2(L_f, \epsilon)$ and $C_3 = C_3(\epsilon, K, q, \|\partial_j u_0\|_{L^\infty(R^n)})$. Therefore, we obtain inductively from (4.18) that for all m

$$\|\partial_j u^{(m)}\|_{L^\infty(\Pi_t)} \leqslant 2C_3 e^{C_4 T}, \tag{4.19}$$

if C_4 is so large that

$$C_2 \int_0^T s^{-1/2} e^{-C_4 s}\, ds < \tfrac{1}{2}. \tag{4.20}$$

Noting that $\partial_t E_\epsilon = \epsilon \Delta E_\epsilon$, we get

$$\int_{R^n} |\partial_j (E_\epsilon(t, x+\Delta x) - E_\epsilon(t,x))|\, dx \leqslant C(\epsilon)\, t^{-(1+\theta)/2} |\Delta x|^\theta,$$

$$\int_{R^n} |\partial_j (E_\epsilon(t+\Delta t, x) - E_\epsilon(t,x))|\, dx \leqslant \tilde{C}(\epsilon)\, t^{-(1+2\tau)/2} |\Delta t|^\tau, \tag{4.21}$$

for $0 \leqslant \theta < 1$, $0 \leqslant \tau < \frac{1}{2}$, where $\Delta x \in R^n$ and $\Delta t \in R$, $C(\epsilon)$ and $\tilde{C}(\epsilon)$ are constants depending only on ϵ. Therefore we conclude that $\partial_j u^{(m)}$ is uniformly locally Hölder continuous in Π_T with respect to x for any exponent $0 \leqslant \theta < 1$, and with respect to t for any exponent $0 \leqslant \tau < \frac{1}{2}$, respectively, since the right-hand sides of (4.21) are integrable in t. Using (4.14) and the Arzela–Ascoli theorem, one can obtain

$$\partial_j u^{(m)}(t,x) \to \partial_j u(t,x), \quad j = 1, \ldots, n, \tag{4.22}$$

uniformly in every compact subset of Π_T because $u^{(m)}$ converges to $u(t,x)$ uniformly. Hence it follows from (4.5) that

$$\partial_j z^{(m)}(t,x) \to \partial_j z(t,x), \quad j = 1, \ldots, n. \tag{4.23}$$

With almost the same method, we get uniform bounds for $\Delta u^{(m)}$ by differentiating (4.17) with respect to x_j and using the uniform boundedness of $\partial_j u^{(m)}$, $\partial_j z^{(m)}$ and $f_i''(u^{(m)})$. We can further obtain the uniform local Hölder continuity of $\Delta u^{(m)}$. Therefore, as in (4.22), it holds that

$$\Delta u^{(m)} \to \Delta u \tag{4.24}$$

uniformly in every compact subset of Π_T. Hence $(u(t,x), z(t,x))$ is a classical solution of (1.4) and (1.2).

Such a classical solution is unique. As a matter of fact, assume (u_1, z_1) and (u_2, z_2) to be two solutions of (1.4) with the same initial data (u_0, z_0), and denote $\tilde{u} = u_1 - u_2$, $\tilde{z} = z_1 - z_2$; then, with the same arguments leading to (4.10) and (4.11), we obtain

$$\|\tilde{u}\|_{L^\infty(\Pi_t)} \leqslant C \int_0^t \|\tilde{u}\|_{L^\infty(\Pi_s)} ((t-s)^{-1/2} + 1)\, ds, \quad \|\tilde{z}\|_{L^\infty(\Pi_t)} \leqslant K L_\phi \int_0^t \|\tilde{u}\|_{L^\infty(\Pi_s)}\, ds. \tag{4.25}$$

This results in $\tilde{u} = \tilde{z} \equiv 0$.

Thus we prove the following theorem.

THEOREM 4.2. *There exists a unique classical solution* $u(t,x)$ *and* $z(t,x)$ *to* (1.4) *and* (1.2) *in any strip* Π_T.

5. *The existence of the entropy solution to* (1.1) *and* (1.2)

We will follow [**4**, **5**] to prove that there exists a solution of (1.1) and (1.2) in the sense of Definition 2.1 which is the limit of solutions to (1.4) and (1.2) as the viscosity ϵ vanishes. To this end, we need to derive an *a priori* estimate of the modulus of continuity in L^1-space of the viscous solutions independent of ϵ. The following lemma is necessary.

LEMMA 5.1. *For* $\tau > 0$, $\gamma > 0$, *the equation*

$$L(g) = \partial_t g + \sum_i A_i(t,x)\,\partial_i g + \gamma \Delta g = 0 \tag{5.1}$$

admits a solution $g(t,x)$ *in* Π_τ *such that* $g(\tau,x) = g_0(x) \in C_0^\infty(\{x; |x| \leqslant r\})$ *and*

$$|g(t,x)| \leqslant \|g_0\|_{L^\infty(R^n)} \exp\{\gamma^{-1}[(C\gamma + H)(\tau - t) + r - |x|]\}, \tag{5.2}$$

where $A_i(t,x)$ *are bounded differentiable with bounded derivatives, C is some constant depending only on n, and*

$$H = 1 + \sup_{(t,x)\in\Pi_\tau} \left(\sum_i |A_i|^2\right)^{1/2}. \tag{5.3}$$

The one-dimensional version of this lemma can be found in [**5**].

Proof of Lemma 5.1. The existence follows exactly from Theorem 4.2 by the setting of $\tilde{\tau} = \tau - t$. Now we prove the remaining parts.

Take $\Phi = \gamma \log \|g_0\|_{L^\infty(R^n)} + r - |x|$, $\Psi = 1 + \Phi * \chi$, where $*$ represents convolution, $0 \leqslant \chi \leqslant 1$ is a mollifier satisfying $\chi \in C_0^\infty(R^n)$, $\int_{R^n} \chi\, dx \equiv 1$ and $\chi = 0$ outside the unit ball. Then $\Phi < \Psi < \Phi + 2$ and $|\Delta\Psi| < C$ for some C depending only on n. Denote

$$h(t,x) = \exp\{\gamma^{-1}[(C_\gamma + H)(\tau - t) + \Psi(x)]\}; \tag{5.4}$$

then

$$\begin{aligned} L(h \pm g) &= \partial_t h + \sum_i A_i(t,x)\,\partial_i h + \gamma \Delta h \\ &= \gamma^{-1}\left(-H - C\gamma + \sum_i A_i(t,x)\,\partial_i \Psi + \sum_i (\partial_i \Psi)^2 + \gamma \Delta \Psi\right) h \leqslant 0. \end{aligned} \tag{5.5}$$

Therefore, by the maximum principle,

$$\begin{aligned} \pm g &\leqslant \exp\{\gamma^{-1}[(C\gamma + H)(\tau - t) + \Psi(x)]\} \\ &< \exp\{\gamma^{-1}[(C\gamma + H)(\tau - t) + \Phi + 2]\} \\ &= \|g_0\|_{L^\infty(R^n)} \exp\{\gamma^{-1}[(C\gamma + H)(\tau - t) + r - |x|]\}. \end{aligned}$$

This proves (5.2). □

In what follows, we derive the uniform estimate of solutions to (1.4) and (1.2) for the L^1-modulus of continuity, which is independent of ϵ.

LEMMA 5.2. *Denote*

$$I(g,y) = \int_{R^n} |g(x+y) - g(x)|\, dx, \quad y \in R^n, \tag{5.6}$$

and assume that

$$\|\nabla u_0\|_{L^\infty(R^n)}, \|\nabla z_0\|_{L^\infty(R^n)} < \infty. \tag{5.7}$$

Then, for any bounded $T > 0$, it holds that

$$I(u(t,\cdot),y)+I(z(t,\cdot),y) \leqslant C(I(u(0,\cdot),y)+I(z(0,\cdot),y)), \tag{5.8}$$

where C depends only on q, K, L_ϕ and T.

Proof. First, we claim that

$$\|\nabla u(t,\cdot)\|_{L^1(R^n)}, \|\nabla z(t,\cdot)\|_{L^1(R^n)} < C, \tag{5.9}$$

where $C = C(\epsilon, K, q, L_f, L_\phi, \|u_0\|_{L^\infty(R^n)}, \|\nabla u_0\|_{L^1(R^n)}, \|\nabla z_0\|_{L^1(R^n)}, T)$.

In fact, from (4.5), we have

$$\int_{R^n} |\nabla z^{(m-1)}(t,x)|\,dx \leqslant \int_{R^n} |\nabla z_0(x)|\,dx + KL_\phi \int_0^t \int_{R^n} |\nabla u^{(m-1)}(s,x)|\,dx\,ds. \tag{5.10}$$

While we obtain from (4.17)

$$\begin{aligned}\int_{R^n} |\nabla u^{(m)}(t,x)|\,dx \leqslant{}& \int_{\Pi_T} \left| \int_{R^n} \sum_i f_i'(u^{(m-1)}(s,y))\,\partial_i u^{(m-1)}(s,y)\,\nabla E_\epsilon(t-s,x-y)\,ds\,dy \right| dx \\ &+ Kq \int_{\Pi_T} \int_{R^n} \{|\phi'(u^{(m-1)}(s,y))\,\nabla u^{(m-1)}(s,y)\,z^{(m-1)}(s,y)| \\ &+ |\phi(u^{(m-1)}(s,y))\,\nabla z^{(m-1)}(s,y)|\}\,E_\epsilon(t-s,x-y)\,ds\,dy\,dx \\ &+ \int_{R^n} \int_{R^n} |E_\epsilon(t,x-y)|\,|\nabla u_0(y)|\,dy\,dx.\end{aligned}$$

Using the uniform boundedness of $u^{(m-1)}$, $z^{(m-1)}$, $f_i'(u^{(m-1)})$ and $\phi(u^{(m-1)})$, we obtain

$$\begin{aligned}\|\nabla u^{(m)}(t,\cdot)\|_{L^1(R^n)} \leqslant{}& C \int_0^t [\|\nabla u^{(m-1)}(s,\cdot)\|_{L^1(R^n)} \\ &+ \|\nabla z^{(m-1)}(s,\cdot)\|_{L^1(R^n)}]\,[(t-s)^{-1/2}+1]\,ds + \|\nabla u_0\|_{L^1(R^n)},\end{aligned} \tag{5.11}$$

where $C = C(\epsilon, K, q, L_f, L_\phi)$. It follows through the use of (5.10) that

$$\begin{aligned}\|\nabla u^{(m)}(t,\cdot)\|_{L^1(R^n)} \leqslant{}& \tilde{C}(C,T) \int_0^t \|\nabla u^{(m-1)}(s,\cdot)\|_{L^1(R^n)}[(t-s)^{-1/2}+1]\,ds \\ &+ \bar{C}(\|\nabla u_0\|_{L^1(R^n)}, \|\nabla z_0\|_{L^1(R^n)}, T).\end{aligned}$$

Since $\{\nabla u^{(m)}\}_{m=0}^\infty$ converges to ∇u, we conclude by Fatou's lemma that $\|\nabla u(t,\cdot)\|_{L^1(R^n)} \leqslant C(\tilde{C}, \bar{C}, \|\nabla u_0\|_{L^1(R^n)}, \|\nabla z_0\|_{L^1(R^n)})$, for all $0 \leqslant t \leqslant T$. Furthermore, this together with (5.10) shows that $\|\nabla z(t,\cdot)\|_{L^1(R^n)}$ has similar bounds for $0 \leqslant t \leqslant T$. These prove our claim (5.9).

Set $(u_1, z_1) = (u(t,x+y), z(t,x+y))$. Then the differences $v = u_1 - u$ and $w = z_1 - z$ satisfy

$$\begin{cases} \partial_t v + \sum_i \partial_i (a_i(t,x)\,v) = \epsilon \Delta v + Kq(b(t,x)\,v z_1 + \phi(u)\,w), \\ w = z_0(x+y)\exp\left(-K\int_0^t \phi(u_1(s,x))\,ds\right) - z_0(x)\exp\left(-K\int_0^t \phi(u(s,x))\,ds\right), \end{cases} \tag{5.12}$$

where

$$a_i(t,x)=\frac{f_i(u_1)-f_i(u)}{u_1-u},\quad b(t,x)=\frac{\phi(u_1)-\phi(u)}{u-u_1}. \tag{5.13}$$

It follows by the assumption on $f_i(u)$ and $\phi(u)$ that $a_i(t,x)$ $(i=1,\dots,n)$ have bounded continuous first order derivatives and $b(t,x)$ is bounded.

Multiplying the first equation of (5.12) by $g(t,x)$, which is the solution

$$\partial_t g+\sum_i a_i(t,x)\,\partial_i g+\epsilon\Delta g=0 \tag{5.14}$$

with $g(\tau,x)\in C_0^\infty(x;|x|<R)$, integrating by parts and using Lemma 5.1 and the decay property of g, one can get

$$\int_{R^n} v(\tau,x)\,g(\tau,x)\,dx=\int_{R^n} v(0,x)\,g(0,x)\,dx + Kq\int_0^\tau\int_{R^n}[b(t,x)\,vz_1+\phi(u)\,w]\,g(t,x)\,dx\,dt, \tag{5.15}$$

from which it follows that

$$\int_{R^n} v(\tau,x)\,g(\tau,x)\,dx\leqslant \|g(\tau,\cdot)\|_{L^\infty(R^n)}\Big\{I(u(0,\cdot),y) + Kq\max(1,L_\phi)\int_0^\tau[I(u(t,\cdot),y)+I(z(t,\cdot),y)]\,dt\Big\}. \tag{5.16}$$

Here we use the fact that $|g(0,x)|\leqslant\|g(\tau,x)\|_{L^\infty(R^n)}$ by the maximum principle for (5.1).

Since $L^\infty(R^n)$ is the dual of $L^1(R^n)$ and $C_0^\infty(R^n)$ is dense in L^∞ endowed with $\omega*$-topology, we obtain from (5.16)

$$I(u(\tau,\cdot),y)\leqslant I(u(0,\cdot),y)+Kq\max(1,L_\phi)\int_0^\tau[I(u(t,\cdot),y)+I(z(t,\cdot),y)]\,dt. \tag{5.17}$$

Similar arguments for the second equation of (5.12) give

$$I(z(\tau,\cdot),y)\leqslant I(z(0,\cdot),y)+K\max(1,L_\phi)\int_0^\tau[I(u(t,\cdot),y)+I(z(t,\cdot),y)]\,dt. \tag{5.18}$$

The summation of (5.17) and (5.18) is

$$I(u(\tau,\cdot)\,y)+I(z(\tau,\cdot),y)\leqslant I(u(0,\cdot),y)+I(z(0,\cdot),y) + K(q+1)\max(1,L_\phi)\int_0^\tau[I(u(t,\cdot),y)+I(z(t,\cdot),y)]\,dt. \tag{5.19}$$

This gives the proof of Lemma 5.2 by Gronwall's inequality, the boundedness of $\|\nabla u(t,\cdot)\|_{L^1(R^n)}$ and $\|\nabla z(t,\cdot)\|_{L^1(R^n)}$, and the inequality

$$\int_{R^n}|g(x+y)-g(x)|\,dx\leqslant\|\nabla g\|_{L^1(R^n)}|y|. \tag{5.20}$$

□

To prove the existence of a weak entropy solution of (1.1) and (1.2), we need to obtain the L^1-continuity of the solutions of (1.4) and (1.2) with respect to t independent of ϵ.

LEMMA 5.3. *Assume the conditions of Lemma* 5.2. *Then, for all* $0 \leqslant t < t+\delta t \leqslant T$,

$$\int_{|x|<R} |u(t+\delta t, x) - u(t,x)|\, dx \leqslant w_R(\delta t), \quad \int_{|x|<R} |z(t+\delta t, x) - z(t,x)|\, dx \leqslant w_R(\delta t), \tag{5.21}$$

where $w_R(\delta t)$ *is a continuous nondecreasing function, being independent of* ϵ *and satisfying* $w_R(0) = 0$.

Proof. Set

$$p(x) = u_\epsilon(t+\delta t, x) - u_\epsilon(t,x), \quad p_h(x) = \begin{cases} \operatorname{sign} v(x), & |x| < R+h, \\ 0, & \text{otherwise}, \end{cases} \tag{5.22}$$

for $\delta t > 0$ and the regularization of $p_h(x)$

$$g^h(x) = \int_{R^n} p_h(x-hy)\, \chi(y), \quad 0 < h \in R, \tag{5.23}$$

where $0 \leqslant \chi \in C_0^\infty(\{x \in R^n; |x| < 1\})$ satisfying $\int_{R^n} \chi(y)\, dy = 1$. Then $|g^h| < 1$, $|\nabla g^h| \leqslant C/h$ and $|\Delta g^h| \leqslant C/h^2$; here C only depends on the dimension n. Since

$$\int_{R^n} |p(x)|\, dx - \int_{R^n} p(x)\, g^h(x)\, dx = \iint_{R^n \times R^n} (p(x-hy) - p(x))\, p^h(x-hy)\, \chi(y)\, dx\, dy, \tag{5.24}$$

it follows by Lemma 5.2 that

$$\int_{R^n} |p(x)|\, dx \leqslant \left| \int_{R^n} p(x)\, g^h(x)\, dx \right| + C_5(I(u(0,\cdot), h) + I(z(0,\cdot), h)). \tag{5.25}$$

Using Fubini's theorem, the smoothness of $\partial_t u$ and the first equation of (1.4), and integrating by parts, we calculate

$$\begin{aligned}
\left| \int_{R^n} g^h(x)\, p(x)\, dx \right| &\leqslant \left| \int_{|x|<R+2h} g^h(x)\, p(x)\, dx \right| + 2 \int_{R<|x|<r+2h} p(x)\, dx \\
&\leqslant \left| \int_{|x|<R+2h} g^h(x) \int_t^{t+\delta t} \partial_t u(\tau, x)\, d\tau\, dx \right| + C_4 h \\
&= \left| \int_t^{t+\delta t} \int_{|x|<R+2h} g^h(x) \left[-\sum_i \partial_i f_i(u) + Kq\phi(u(\tau,x))\, z(\tau,x) + \epsilon \Delta u(\tau,x) \right] dx\, d\tau \right| + C_4 h \\
&= \left| \int_t^{t+\delta t} \int_{|x|<R+2h} \left[\sum_i f_i(u(\tau,x))\, \partial_i g^h(x) + Kq\phi(u(\tau,x))\, z(\tau,x)\, g^h(x) \right.\right. \\
&\qquad \left.\left. + \epsilon u(\tau,x)\, \Delta g^h(x) \right] dx\, d\tau \right| + C_4 h \\
&\leqslant \delta t \cdot C_3 \sup_{|x|<R+2h} \{|g| + |\nabla g| + |\Delta g|\} + C_4 h \\
&\leqslant \delta t \cdot C_3 (1 + C(n)/h + C(n)/h^2) + C_4 h.
\end{aligned}$$

It follows by Lemma 5.2 and (5.25) that

$$\int_{R^n} |v|\, dx \leqslant \delta t \cdot C_3(1 + C(n)/h + C(n)/h^2) + C_4 h + C_5(I(u(0,\cdot),h) + I(z(0,\cdot),h)). \tag{5.26}$$

The L^1-modulus of continuity of $z(t,x)$ with respect to t is rather simple. By Fubini's theorem, from the second equation of (1.4), we have

$$\begin{aligned}\int_{|x|<R} |z(t+\delta t, x) - z(t,x)|\, dx &= \int_{|x|<R} \left| \int_t^{t+\delta t} \partial_t z(s,x)\, dt \right| dx \\ &= \int_{|x|<R} \left| \int_t^{t+\delta t} -K\phi(u(s,x))\, z(s,x)\, ds \right| dx \\ &\leqslant K \cdot C_R \cdot \delta t,\end{aligned} \tag{5.27}$$

where C_R is the measure of $\{x; |x| < R\}$. The proof of this lemma is complete. □

On the basis of these lemmas, we can show the following theorem.

THEOREM 5.1. *Let $u_0, z_0 \in C_0^\infty(R^n)$ such that*

$$\nabla u_0, \nabla z_0 \in L^1(R^n) \quad \textit{and} \quad \|\nabla^i u_0\|_{L^\infty(R^n)}, \|\nabla^i z_0\|_{L^\infty(R^n)} < \infty, \quad i = 1,2,3. \tag{5.28}$$

Then there exists a weak entropy solution to the Cauchy problem (1.1) *and* (1.2).

Proof. The uniqueness has been proved in Theorem 3.1. Here we only need solve the existence problem. Let $u^\epsilon(t,x)$ and $z^\epsilon(t,x)$ be the global solution of (1.4) and (1.2) in any strip Π_T for a bounded $T > 0$. Lemma 5.2 guarantees that $\{u^\epsilon\}_{\epsilon>0}$ and $\{z^\epsilon\}_{\epsilon>0}$ are precompact in $L^1_{\text{loc}}(R^n)$ for $0 \leqslant t \leqslant T$. Lemma 5.3 shows that $\{u^\epsilon\}_{\epsilon>0}$ and $\{z^\epsilon\}_{\epsilon>0}$ are equi-continuous from $[0,T]$ to $L^1_{\text{loc}}(R^n)$. By the Arzela–Ascoli theorem, we conclude that there exists a subsequence $\{\epsilon_k\}_{k=1}^\infty$ of $\{\epsilon; \epsilon > 0\}$ such that $\{u^{\epsilon_k}\}_{k=1}^\infty$ and $\{z^{\epsilon_k}\}_{k=1}^\infty$ converge uniformly to $u(t,\cdot)$ and $z(t,\cdot)$ in L^1_{loc}-norm, respectively, as k tends to infinity, where $u(t,\cdot), z(t,\cdot) \in C([0,T]. L^1_{\text{loc}}(R^n))$. Therefore the pair $(u(t,\cdot), z(t,\cdot))$ is a weak solution and satisfies the initial data almost everywhere.

With the same procedure as in Section 2, we can prove this solution is entropy. □

THEOREM 5.2. *Let $u_0(x), z_0(x) \in L^\infty(R^n)$, then there exists a weak solution to* (1.1) *and* (1.2).

Proof. The finite propagation speed property of solutions in Theorem 3.1 shows that it suffices to prove the existence problem with $u_0, z_0 \in L^\infty(R^n)$ having compact support.

If $u_0, z_0 \in L^\infty(R^n)$, $|u_0| \leqslant M$, $|z_0| \leqslant 1$, the regularization and a cutoff far away gives sequences $\{u_0^h(x)\}$ and $\{z_0^h(x)\}$, with $|u_0^h(x)| \leqslant M$ and $|z_0^h(x)| \leqslant 0$, which converge to $u_0(x)$ and $z_0(x)$ in $L^1_{\text{loc}}(R^n)$ respectively. Also we have $I(u_0^h, y) \leqslant I(u_0, y)$ and $I(z_0^h, y) \leqslant I(z_0, y)$.

From the proof of the lemmas and theorems above, we can conclude that the solutions of (1.1) and (1.2) with the initial data $(u_0^h, z_0^h)(x)$ is equi-continuous from $[0,T]$ to $L^1_{\text{loc}}(R^n)$. The same proof as in Theorem 5.1 gives the conclusion of this theorem. □

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