DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **23**, Number **1&2**, January & February **2009**

doi:10.3934/dcds.2009.23.1

pp. **1–27**

THE CONVERGENCE OF THE GRP SCHEME

Matania Ben–Artzi and Joseph Falcovitz

Institute of Mathematics, the Hebrew University of Jerusalem, 91904, Israel

Jiequan Li

School of Mathematical Sciences, Capital Normal University, 100037, Beijing, China

Dedicated to Professor Daqian Li on his 70th birthday

Abstract. This paper deals with the convergence of the second-order GRP (Generalized Riemann Problem) numerical scheme to the entropy solution for scalar conservation laws with strictly convex fluxes. The approximate profiles at each time step are linear in each cell, with possible jump discontinuities (of functional values and slopes) across cell boundaries. The basic observation is that the discrete values produced by the scheme are **exact averages** of an **approximate conservation law**, which enables the use of properties of such solutions in the proof. In particular, the "total-variation" of the scheme can be controlled, using analytic properties. In practice, the GRP code allows "sawteeth" profiles (i.e., the piecewise linear approximation is not monotone even if the sequences of averages is such). The "reconstruction" procedure considered here also allows the formation of "sawteeth" profiles, with an hypothesis of "Godunov Compatibility", which limits the slopes in cases of non-monotone profiles. The scheme is proved to converge to a weak solution of the conservation law. In the case of a monotone initial profile it is shown (under a further hypothesis on the slopes) that the limit solution is indeed the entropy solution. The constructed solution satisfies the "finite propagation speed", so that no rarefaction shocks can appear in intervals such that the initial function is monotone in their domain of dependence. However, the characterization of the limit solution as the unique entropy solution, for general initial data, is still an open problem.

1. **Introduction.** In this paper we study the convergence of high resolution second order numerical schemes to entropy solutions of the initial value problem for scalar conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \qquad x \in \mathbb{R}, \qquad t \geq 0, \tag{1}$$

subject to initial data

$$u(x, t = 0) = u_0(x) \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R}), \tag{2}$$

2000 *Mathematics Subject Classification.* Primary: 65M06, 65M12; Secondary: 35L65, 35L67.

Key words and phrases. Hyperbolic conservation laws, the GRP scheme, Convergence, TVD, entropy.

Matania Ben–Artzi is partially supported by GIF grant I0318–195.06/93. Jiequan Li is partially supported by the Key Program from Beijing Educational Commission with no. KZ200510028018, Program for New Century Excellent Talents in University (NCET) and Funding Project for Academic Human Resources Development in Institutions of Higher Learning Under the Jurisdiction of Beijing Municipality (PHR-IHLB) as well as 973 Key Project with no. 2006CB805902.

where BV is the space of functions of bounded variation. We assume that the flux function $f:\mathbb{R}\to\mathbb{R}$ is C^2 and strictly convex, $f''(u)\geq\mu>0$.

As is well–known [25, 9] the nonlinearity of the flux leads to the formation of singularities of solutions $u(x,t)$ in a finite time even for very smooth initial data (2). Thus *a global solution* $u(x,t)\in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ *must be understood in the weak (distribution) sense, namely for every test function* $\phi(x,t)\in C_0^\infty(\mathbb{R}\times\overline{\mathbb{R}}_+)$

$$\int_0^\infty\int_{-\infty}^\infty(u\frac{\partial\phi}{\partial t}+f(u)\frac{\partial\phi}{\partial x})dxdt+\int_{-\infty}^\infty u_0(x)\phi(x,0)dx=0. \tag{3}$$

Furthermore such weak solutions are not unique and a suitable condition is needed to select the "correct" solution. The latter is usually meant to be the one obtained by Kruzkov's vanishing viscosity approach [16]. We refer the reader to [11, 9] for general background on various entropy conditions.

The purpose of this paper is to give a proof of the convergence of the fully discrete high resolution second order GRP scheme to solutions of (1)–(2). The scheme studied in this paper belongs to the class of *"high-resolution second-order Godunov-type"* schemes , where a main ingredient, one way or the other, is the treatment of Riemann problems. The general strategy can be outlined as follows. At each time level t_n, it is assumed that the solution $u(x,t_n)$ is approximated by a piecewise linear function. More specifically, we take an equally spaced grid in $\mathbb{R}$, $x_{j+\frac12}=(j+\frac12)h$, $j\in\mathbb{Z}$, and assume that

$$v^n(x)=v_j^n+(x-x_j)s_j^n,\qquad x\in(x_{j-\frac12},x_{j+\frac12}). \tag{4}$$

In particular, the value v_j^n is the average of $v^n(x)$ in "cell j" $(=(x_{j-\frac12},x_{j+\frac12}))$ and is associated with its center $x_j=\frac12(x_{j-\frac12}+x_{j+\frac12})$, and s_j^n is the slope of $v^n(x)$ in cell j.

In general terms, the idea (initiated by van–Leer [26]) is to obtain the approximation $v^{n+1}(x)$ at time $t_{n+1}=t_n+k$ by solving (1) with initial data $U(x,t_n)=v^n(x)$. Let $U(x,t)$ be the solution. The function $v^{n+1}(x)$ is then evaluated as a suitable approximation to the exact solution $U(x,t_{n+1}-)$. In the piecewise constant case ($s_j^n=0$ for all j), this leads to the Godunov scheme [12]. The values v_j^{n+1} are the exact averages of the solution $U(x,t_{n+1})$. This exact evaluation is of course due to the fact that the solution $U(x,t)$ consists in this case of a family of Riemann problems and the resulting fluxes $f(U(x_{j+\frac12},t))$, $(t_n\leq t\leq t_{n+1})$, are constant in time.

In the second-order case ($s_j^n\neq0$ in general) a similar study for analytic evaluation of the fluxes $f(U(x_{j+\frac12},t))$ leads to the generalized Riemann problem (**GRP**). For the given initial distribution (4), one cannot expect to have a full knowledge of the fluxes $f(U(x_{j+\frac12},t))$, and they are replaced by some linear approximations in t. One particular way of doing it is the GRP method [1, 2, 5, 20], which is the focus of our discussion here, and which we recall (for the scalar case) in the next section. It is based on a local analysis of the solution $f(U(x_{j+\frac12},t))$ at jump discontinuities. The GRP scheme was originally devised for compressible non-isentropic) fluid flows [1] and has been extensively used in the study of various physical models [2, 5, 4]. We mention also the application of the GRP scheme to two dimensional scalar conservation laws as discussed in [3]. It leads to very interesting genuinely two-dimensional wave patterns, that could serve as test-cases for numerical schemes.

An alternative approach to the GRP (and its numerical implementation), using series expansions and asymptotic analysis, was developed in [19, 7, 17].

The essential idea in our treatment of (1), (2) is the following.

The linearized approximations of the fluxes $f(U(x_{j+\frac{1}{2}},t))$ are actually exact fluxes of a conservation equation

$$\begin{aligned} &\frac{\partial \tilde{u}}{\partial t} + \frac{\partial \tilde{f}(\tilde{u})}{\partial x} = 0, \qquad t_n \le t < t_{n+1}, \\ &\tilde{u}(x,t_n) = v^n(x), \qquad x \in \mathbb{R}, \end{aligned} \tag{5}$$

where $\tilde{f}$ is a smooth flux function, which serves for the solution in the time interval $[t_n, t_{n+1})$.

In particular, the updated averages $\{v_j^{n+1}\}_{j=-\infty}^{\infty}$ are **exact averages** *of $\tilde{u}(x,t_{n+1})$ over the cells $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$.*

The proof can therefore rely on properties of (exact) entropy solutions to (5), such as the maximum-minimum principle and TVD (total variation diminishing) property. The solution $\tilde{u}(x,t)$ is reconstructed at the end of the time step ($t = t_{n+1}$), so as to obtain the piecewise linear profile $v^{n+1}(x)$. In the course of this reconstruction we have a choice of restricting the new slopes s_j^{n+1} so that the total variation of $v^{n+1}(x)$ does not exceed that of $\tilde{u}(x,t_{n+1}-)$ or to relax this restriction and allow the formation of "sawteeth" of the profile. The former is the "canonical" choice of reconstruction (see [11, Ch. 4]), and is the one used in all references discussed below. However, the more relaxed version fits more closely the exact solution (of (5)), reduces the amount of numerical dissipation in the reconstruction step and has proved to yield good numerical results in a wide variety of cases. Thus we do not exclude this possibility, but impose certain hypotheses (see details at the end of Section 2 below) that (a) ensure that the total-variation remains **bounded**, namely, that the scheme is TVB and therefore converges to a weak solution of (1) by compactness and (b) limit the number of "large slopes" and ensure that (in the case of monotone initial profiles) the limit solution satisfies all the entropy conditions. We emphasize, however, that for general initial data the results here do not show that the limit solution indeed satisfies the entropy conditions, and it remains an open problem.

The literature concerning the (entropy) convergence of high resolution numerical schemes for (1) is very extensive. We comment here briefly on those works that are closely related to the present paper. The convergence for the semi-discrete MUSCL scheme was discussed in [23], and for a high resolution scheme with modified numerical flux in [24]. In [22] a second order central difference scheme was proposed and the entropy convergence was investigated with the modification of numerical fluxes.

In [21] a backward MUSCL-type scheme was applied for the Hamilton-Jacobi equation and then for the equation of type (1) with the analysis of convergence. In [18] and [28] the convergence of fully discrete MUSCL-type schemes was treated by tracking local extremal values. In the last three papers, it is shown that the limit solution satisfies the entropy conditions. However, the schemes proposed have never (to the best of our knowledge) been implemented in numerical codes, so one cannot assess their actual performance. In [15] the convergence for discontinuous Galerkin methods was proved by finding a cell entropy inequality via a special reconstruction projection. In [27] the convergence of a second order scheme is proved assuming the full knowledge of fluxes $f(U(x_{j+1/2},t))$. In [6] a MUSCL-type method, with a special reconstruction of slopes, was proposed to satisfy all numerical entropy inequalities. Another approach, which has some resemblance to our treatment, is given in [13]. However, the flux function in [13] is replaced by a piecewise linear one, so that even in the case of vanishing slope their scheme does not reduce to the Godunov scheme. Our construction of the approximate flux $\tilde{f}$ (in (5)) is more delicate and reduces to the Godunov scheme (for the original flux f) in the case of vanishing slopes. In addition to the assumption of strict convexity, all the references (of the "Godunov-type") cited here are based on various assumptions, and involve schemes that (to the best of our knowledge) have not been implemented in actual codes. In contrast, our treatment here applies precisely to the fully discrete GRP-scheme as implemented in practice.

The structure of the paper is as follows. In section 2 we give all details of the scheme and the reconstruction of slopes ("Limiter Algorithm"). In Section 3 we deal with the basic ingredient of the paper–the construction of the "approximate flux". In Section 4 we introduce the concept of "monotone chains" and establish the relation between the *exact solution* of the *approximate equation* and the discrete GRP scheme. It leads to the necessary bounds on the total variation, which enable us to prove in Section 5 the convergence of the discrete scheme to a weak solution (of (1)). In Section 6 we prove, in the case of *monotone* initial data, that the weak solution satisfies the entropy inequalities. As a local corollary we obtain that "local monotone profiles" cannot lead to "rarefaction shocks" within their domain of influence.

We set

$$\|u_0\|_\infty := \sup_{x\in\mathbb{R}} |u_0(x)| = M, \qquad TV(u_0) = K, \tag{6}$$

where TV is the total variation of u_0 over $\mathbb{R}$. Recall that for a function $\phi(x)$ on $\mathbb{R}$, and any interval $[a,b] \subseteq \mathbb{R}$, the total variation of ϕ over $[a,b]$ is defined by

$$TV(\phi;[a,b]) = \sup_{a=y_0<y_1<\cdots<y_m=b} \sum_{k=1}^{m} |\phi(y_k) - \phi(y_{k-1})|. \tag{7}$$

(if $a = -\infty$ we take $\phi(a) = \lim_{y\to-\infty}\phi(y)$, similarly if $b = +\infty$).

2. GRP scheme–basic properties.

We let $U(x,t)$ be the exact solution to (1), (4), $t \geq t_n$, with $U(x,t_n) = v^n(x)$.

With notations as introduced above, the approximate averages v_j^{n+1} are determined by

$$v_j^{n+1} = v_j^n - \lambda(f_{j+\frac{1}{2}}^{n+\frac{1}{2}} - f_{j-\frac{1}{2}}^{n+\frac{1}{2}}), \tag{8}$$

where $\lambda = \frac{k}{h}$ and the numerical flux $f_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ should approximate the time average of $f(U(x_{j+\frac{1}{2}},t))$, $t_n \leq t \leq t_{n+1} = t_n + k$. Recall that the conservation law (1.1) is now solved subject to the initial condition $v^n(x)$ as in (4), where the cells are of uniform size $h = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$.

In the first step, we evaluate $v_{j+\frac{1}{2}}^n$ by

$$v_{j+\frac{1}{2}}^n = R(0; v_{j+\frac{1}{2},\mp}^n), \tag{9}$$

where $R(\frac{x}{t}; w_\mp)$ is the (self–similar) solution to the Riemann problem for (1), having initial data $w_\mp$ as $\mp x > 0$. The limiting values $v_{j+\frac{1}{2},\mp}^n$ appearing in (9) are defined by

$$v^n_{j+\frac{1}{2},\mp} = \lim_{x \to x_{j+\frac{1}{2}}\mp} v^n(x). \tag{10}$$

Thus, the value $v^n_{j+\frac{1}{2}}$ is the "instantaneous" value obtained by solving the Riemann problem with the limiting values of $v^n(x)$ at the cell boundary $x_{j+\frac{1}{2}}$. In our case (f strictly convex), the Riemann solution is determined simply as follows.

(i) $v^n_{j+\frac{1}{2}} = v_{\min}$, the minimum point of $f(v)$, if $v^n_{j+\frac{1}{2},-} \le v_{\min} \le v^n_{j+\frac{1}{2},+}$. In this case we say that $x_{j+\frac{1}{2}}$ is a sonic point. Otherwise,
(ii) $v^n_{j+\frac{1}{2}} = w$ such that $f(w) = \min\{f(v); v \in [v^n_{j+\frac{1}{2},-}, v^n_{j+\frac{1}{2},+}]\}$ if $v^n_{j+\frac{1}{2},-} \le v^n_{j+\frac{1}{2},+}$ (rarefaction), or,
(iii) $v^n_{j+\frac{1}{2}} = w$ such that $f(w) = \max\{f(v); v \in [v^n_{j+\frac{1}{2},+}, v^n_{j+\frac{1}{2},-}]\}$ if $v^n_{j+\frac{1}{2},+} \le v^n_{j+\frac{1}{2},-}$ shock).

Note that if $s^n_j = s^n_{j+1} = 0$, the value $v^{n+\frac{1}{2}}_{j+\frac{1}{2}} = v^n_{j+\frac{1}{2}}$ would be the one used in the Godunov scheme.

We also note that the wave moves to the right (resp. the left) if $f'(v^n_{j+\frac{1}{2}}) > 0$ (resp. $f'(v^n_{j+\frac{1}{2}}) < 0$).

The crucial ingredient in the GRP method is the assumption that both $U(x_{j+\frac{1}{2}}, t)$ and $f(U(x_{j+\frac{1}{2}}, t))$ are approximated linearly (in $t \in [t_n, t_{n+1}]$).

We obtain therefore the linear expressions,

$$\begin{aligned}
&(i) \qquad \tilde{u}(x_{j+\frac{1}{2}}, t) = v^n_{j+\frac{1}{2}} + \left(\frac{\partial U}{\partial t}\right)^n_{j+\frac{1}{2}} (t - t_n),\\
&(ii) \qquad \tilde{f}(\tilde{u}(x_{j+\frac{1}{2}}, t)) = f(v^n_{j+\frac{1}{2}}) + f'(v^n_{j+\frac{1}{2}}) \left(\frac{\partial U}{\partial t}\right)^n_{j+\frac{1}{2}} (t - t_n).
\end{aligned} \tag{11}$$

The exact instantaneous value of $\left(\frac{\partial U}{\partial t}\right)^n_{j+\frac{1}{2}}$ at the cell boundary is obtained from (1),

$$\left(\frac{\partial U}{\partial t}\right)^n_{j+\frac{1}{2}} = \begin{cases} -f'(v^n_{j+\frac{1}{2}}) s^n_j, & \text{if the wave moves to the right,}\\ -f'(v^n_{j+\frac{1}{2}}) s^n_{j+1}, & \text{if the wave moves to the left,}\\ 0, & \text{if } x_{j+\frac{1}{2}} \text{ is a sonic point.} \end{cases} \tag{12}$$

Remark 2.1. (Stationary shocks). If the Riemann solution $R(\frac{x - x_{j+\frac{1}{2}}}{t - t_n}; v^n_{j+\frac{1}{2},\pm})$ yields a stationary shock along $x = x_{j+\frac{1}{2}}$ we note that $f(v^n_{j+\frac{1}{2},-}) = f(v^n_{j+\frac{1}{2},+})$, $v^n_{j+\frac{1}{2},-} > v^n_{j+\frac{1}{2},+}$, so that the shock speed

$$s(t) = \frac{f(U^+(x_{j+\frac{1}{2}}, t)) - f(U^-(x_{j+\frac{1}{2}}, t))}{U^+(x_{j+\frac{1}{2}}, t)) - U^-(x_{j+\frac{1}{2}}, t)} \tag{13}$$

($U^{\pm}(x_{j+\frac{1}{2}},t)$ are the values of U on the two sides of the shock, $U^{\pm}(x_{j+\frac{1}{2}},t_n) = v^n_{j+\frac{1}{2},\pm}$) can be differentiated to yield,

$$s'(t)|_{t=t_n} = \frac{-f'(v^n_{j+\frac{1}{2},+})^2 s^n_{j+1} + f'(v^n_{j+\frac{1}{2},-})^2 s^n_j}{v^n_{j+\frac{1}{2},+} - v^n_{j+\frac{1}{2},-}}. \tag{14}$$

The value of $\left(\dfrac{\partial U}{\partial t}\right)^n_{j+\frac{1}{2}}$ in (12) is determined according to whether $\pm s'(t_n) > 0$ and $v^n_{j+\frac{1}{2}}$ is replaced by one of the values of $v^n_{j+\frac{1}{2},\pm}$, taken from the "smooth side".

Using the above formulae in (11), we DEFINE the GRP fluxes:

$$f^{n+\frac{1}{2}}_{j+\frac{1}{2}} = \tilde{f}(\tilde{u}(x_{j+\frac{1}{2}}, t_n + \frac{\Delta t}{2})), \qquad -\infty < j < \infty. \tag{15}$$

and then v^{n+1}_j is determined by (8). The new slopes s^{n+1}_j are calculated in three steps.

Step 1. Determine $v^{n+1}_{j+\frac{1}{2}} = \tilde{u}(x_{j+\frac{1}{2}}, t_{n+1})$ by (11) (i) and then set

$$\tilde{s}^{n+1}_j = \frac{1}{\Delta x}(v^{n+1}_{j+\frac{1}{2}} - v^{n+1}_{j-\frac{1}{2}}). \tag{16}$$

Step 2 (" Limiter Algorithm"). Set the final value for some $\theta \in (0,2]$,

$$s^{n+1}_j = \frac{1}{\Delta x}\mathrm{minmod}((2-\theta)(v^{n+1}_{j+1} - v^{n+1}_j), \Delta x \tilde{s}^{n+1}_j, (2-\theta)(v^{n+1}_j - v^{n+1}_{j-1})), \tag{17}$$

where the minmod function is defined by,

$$\mathrm{minmod}(a,b,c) = \begin{cases} s\min(|a|,|b|,|c|), & \text{if } s = \mathrm{sign}(a) = \mathrm{sign}(b) = \mathrm{sign}(c), \\ 0, & \text{otherwise}, \end{cases}$$

Step 3 ("Sonic Limiter"). If $(v^{n+1}_{j-\frac{1}{2},+} - v_{\min})(v^{n+1}_{j+\frac{1}{2},-} - v_{\min}) < 0$, where $v_{\min}$ is such that $f'(v_{\min}) = 0$, then set

$$s^{n+1}_j = 0. \tag{18}$$

Remark 2.2. Step 3 is technically needed in some extreme cases, in order to maintain the CFL condition always as given by (19) below. We refer to Case III in Claim 3.5 in the following section for details. Observe that the condition means that "cell j" is "sonic", namely, the approximate values there are around the sonic point $v_{\min}$.

In addition , we impose in Section 4 a "Godunov Compatibility Hypothesis I", which ensures that the reconstruction step does not increase "too much" the total variation (this hypothesis is not needed if we take $\theta \geq 1$). In Section 6 we impose the "Godunov Compatibility Hypothesis II", which enables us to prove the entropy inequalities. This hypothesis is certainly valid when the slopes are sufficiently small and, in any case, can be monitored in the *actual code.*

Remark 2.3. Geometrically speaking, our limiter satisfies the "minimal" change needed in implementing the following "5–point rule":

If $\{v_{j-1}^{n+1}, v_j^{n+1}, v_{j+1}^{n+1}\}$ form a monotone increasing sequence, then so are the five values $\{v_{j-1}^{n+1}, v_j^{n+1} - \dfrac{\Delta x}{2} s_j^{n+1}, v_j^{n+1}, v_j^{n+1} + \dfrac{\Delta x}{2} s_j^{n+1}, v_{j+1}^{n+1}\}$, see Figure 2.1. However $\theta < 1$ leads to "sawtooth".

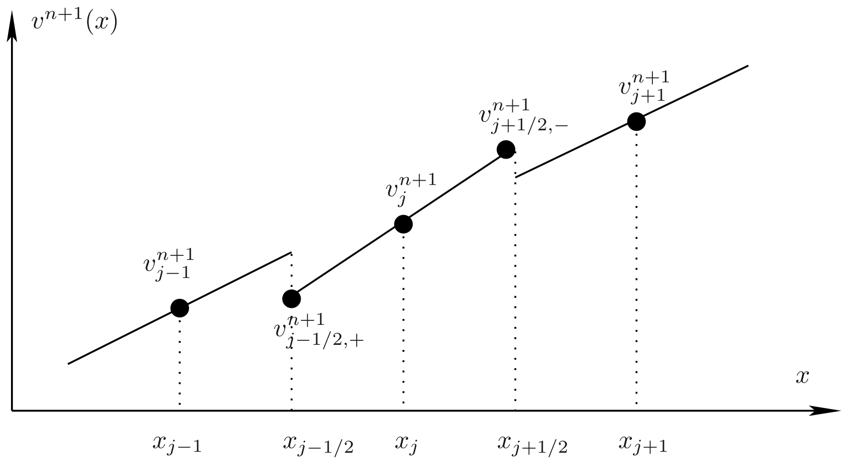

FIGURE 1. Distribution of new data $v^{n+1}(x)$. The notations $v_{j\mp\frac{1}{2},\pm}^{n+1}$ are for $v_j^{n+1} \mp \frac{\Delta x}{2} s_j^{n+1}$ respectively.

As we shall see in Section 3 (Notation 3), there exists a constant $r_0 \in (0, 1/2)$, related to the convexity of f over the interval $[-M, M]$, such that the CFL condition for the scheme can be written as,

$$\lambda \max_{|v|\le M} |f'(v)| < \frac{1}{2} r_0, \qquad \lambda = \frac{\Delta t}{\Delta x} = \frac{k}{h}, \tag{19}$$

It ensures that the solution $\tilde{u}(x,t)$, $t_n \le t \le t_{n+1}$, is determined only by the values of $v^n(x)$ for $|x - x_{j+\frac{1}{2}}| \le \dfrac{1}{2} r_0 \Delta x$. We note that for the Burgers equation one can take any $r_0 < 1/2$.

3. **Construction of local approximate fluxes and the Maximum–minimum principle for GRP.** In the previous section the GRP method for the evaluation of $\{v_j^{n+1}\}_{j=-\infty}^{\infty}$, $\{s_j^{n+1}\}_{j=-\infty}^{\infty}$ (given the piecewise linear distribution $v^n(x)$ (4)) was presented in detail. It was shown that the fluxes $f_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ are obtained via a linear approximation (in t) of $f(U(x_{j+\frac{1}{2}}, t))$, where $u(x_{j+\frac{1}{2}}, t)$ is the exact solution (to (1)) for the same initial data. As noted above (see (9)), the

value $v^n_{j+\frac{1}{2}}$ is actually given by

$$v^n_{j+\frac{1}{2}} = \begin{cases} v^n_{j+\frac{1}{2},-}, & \text{if the wave moves to the right,} \\ v^n_{j+\frac{1}{2},+}, & \text{if the wave moves to the left,} \\ v_{\min}, & \text{at a sonic point} \end{cases} \tag{20}$$

where $v_{\min}$ is defined such that $f(v_{\min}) < f(v)$ for all $v \neq v_{\min}$.

Fixing j, we shall now show that v^{n+1}_j, s^{n+1}_j are obtained from <u>exact</u> solutions of an <u>approximate</u> conservation law (i.e., a conservation law for a flux approximating the original flux f),

$$\tilde{u}_t + \tilde{f}(\tilde{u})_x = 0, \qquad \tilde{u}(x, t_n) = v^n(x). \tag{21}$$

We emphasize that $\tilde{f}$ is of a <u>local</u> character, depending on j, but we suppress j for simplicity. Basically, therefore, we shall be concerned with constructing $\tilde{f}(v)$ in a neighborhood of the values $v^n_{j\pm\frac{1}{2},\pm}$, v^n_j.

The convexity of f plays an important role in our treatment as we seek to construct $\tilde{f}$ without increasing the maximal propagation speed, i.e., so that $\|\tilde{f}'\|_\infty \le \|f'\|_\infty$. In view of the maximum principle to be proved later (see Corollary 3.10), we shall actually need only that

$$\max_{|v|\le M} |\tilde{f}'(v)| \le \max_{|v|\le M} |f'(v)|, \qquad M = \|u_0\|_\infty.$$

Let $-M \le a < b \le M$ and let $y_r = f(a) + rf'(a)(b-a)$, $z_r = f(b) - rf'(b)(b-a)$, where $0 \le r < \frac{1}{2}$ (see Fig. 3.1).

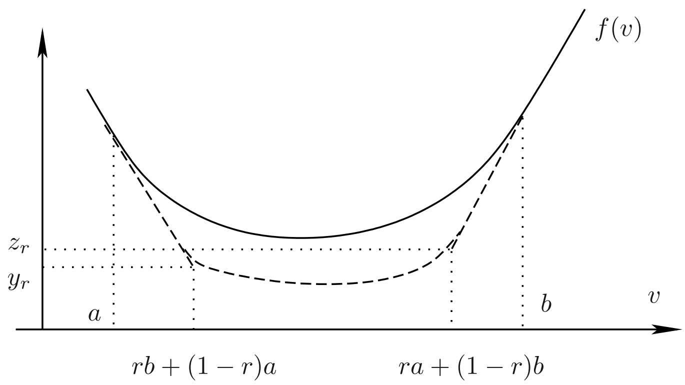

FIGURE 2. Construction of L_f.

We want to connect the two tangent segments (i.e., $(a, f(a))$ to $(a + r(b-a), y_r)$ and $(b, f(b))$ to $(b - r(b-a), z_r)$) by a smooth curve whose derivative lies between $f'(a)$, $f'(b)$. In other words, we are looking for a function $L_f(v; a, b, r)$ such that,

$$\begin{aligned} &(i) && L_f(v;a,b,r) \in C^1[a,b], && \\ &(ii) && L_f(v;a,b,r) = f(a) + f'(a)(v-a), && a \le v \le a + r(b-a), \\ &(iii) && L_f(v;a,b,r) = f(b) + f'(b)(v-b), && b - r(b-a) \le v \le b, \\ &(iv) && f'(a) \le L_f'(v;a,b,r) \le f'(b), && a \le v \le b. \end{aligned} \tag{22}$$

The following claim is obvious.

Claim 3.1. *A necessary and sufficient condition for the existence of L_f satisfying (22) is given by*

$$f'(a) \leq \frac{z_r - y_r}{(b-a)(1-2r)} \leq f'(b). \tag{23}$$

Next we show that it is possible to find $r_0 \in (0, \frac{1}{2})$ so that the condition (23) is satisfied, with $r = r_0$, for all $-M \leq a < b \leq M$.

Claim 3.2. *There exists $r_0 \in (0, \frac{1}{2})$ such that the inequality (23) is satisfied, with $r = r_0$, for all $-M \leq a < b \leq M$.*

Proof. Consider the function

$$g(a, b; r) = \frac{z_r - y_r}{b-a} - (1-2r)f'(b) = \frac{f(b) - f(a)}{b-a} - r(f'(b) + f'(a)) - (1-2r)f'(b),$$

which is defined continuously (for fixed $r \in (0, \frac{1}{2})$) in $-M \leq a < b \leq M$. Clearly, it can be extended continuously to the closed triangle by setting $g(a, a; r) = 0$, $-M \leq a \leq M$. By the strict convexity of f we have, if $a < b$,

$$\lim_{r \to 0^+} g(a, b; r) < 0,$$

and $g(a, b; r)$ is linearly increasing in r. It vanishes when

$$r = r_{a,b} = \frac{\frac{f(b)-f(a)}{b-a} - f'(b)}{f'(a) - f'(b)} > 0.$$

But the strict convexity of f implies that

$$\lim_{a \to b-0} r_{a,b} = \frac{1}{2},$$

so that $\underline{r} = \inf\limits_{-M \leq a < b \leq M} r_{a,b} > 0$. Thus, taking any $0 < r_0 < \underline{r}$, we obtain the right–hand side of inequality (3.4) for all $-M \leq a < b \leq M$. We proceed similarly with the left–hand side. $\square$

Notation. *Fixing $r = r_0$ as in Claim 3.2, we suppress r and denote by $L_f(v; a, b)$, for any $-M \leq a < b \leq M$, the function satisfying (22).*

Remark 3.3. It is easy to see that for the Burgers equation $f(v) = v^2/2$ we can take any $r_0 < \frac{1}{2}$ and (3.4) is satisfied for any $-\infty < a < b < \infty$.

Returning to the problem of constructing $\tilde{f}(v)$, we assume first that the triplet v_{j-1}^n, v_j^n, v_{j+1}^n is monotone, and without loss of generality, we assume

$$v_{j-1}^n \leq v_j^n \leq v_{j+1}^n. \tag{24}$$

we have, by Remark 2.3,

$$v_{j-1}^n \leq v_{j-\frac{1}{2}}^n \leq v_j^n \leq v_{j+\frac{1}{2}}^n \leq v_{j+1}^n.$$

Assume first $v^n_{j-\frac12} < v^n_{j+\frac12}$, then we take

$$\tilde f(v) = L_f(v; v^n_{j-\frac12}, v^n_{j+\frac12}), \qquad \text{(see Notation 3.3)}. \tag{25}$$

We then further extend the linear segments of $\tilde f(v)$ on $(v^n_{j-\frac12} - r_0(v^n_{j-\frac12} - v^n_{j-1}), v^n_{j-\frac12}]$ and $[v^n_{j+\frac12}, v^n_{j+\frac12} + r_0(v^n_{j+1} - v^n_{j+\frac12}))$ (there is no extension if the interval is null, e.g., $v^n_{j+1} = v^n_{j+\frac12}$). If $v^n_{j-\frac12} = v^n_{j+\frac12} (= v^n_j)$, we simply take

$$\tilde f(v) = f(v^n_j) + f'(v^n_j)(v - v^n_j), \quad v \in (v^n_j - r_0(v^n_j - v^n_{j-1}), v^n_j + r_0(v^n_{j+1} - v^n_j)). \tag{26}$$

Thus, in both cases $\tilde f(v)$ is defined in $J_j = \{v^n_{j-\frac12} - r_0(v^n_{j-\frac12} - v^n_{j-1}) \le v \le v^n_{j+\frac12} + r_0(v^n_{j+1} - v^n_{j+\frac12})\}$.

We shall prove now the following basic claim.

Claim 3.4. *Assume (24). Consider Eq. (21) and take $k = \Delta t$ such that the following CFL condition is satisfied,*

$$\lambda \max_{|v|\le M} |f'(v)| \le \frac12 r_0. \tag{27}$$

Then the GRP fluxes $f^{n+\frac12}_{j\pm\frac12}$, as given by (11), (15), satisfy

$$f^{n+\frac12}_{j\pm\frac12} = \frac1k \int_{t_n}^{t_{n+1}} \tilde f(\tilde u(x_{j\pm\frac12}, t))dt. \tag{28}$$

Proof. We prove for $j + \frac12$. Note that the condition (27) restricts the domain of dependence of the solution $\tilde u(x_{j+\frac12}, t)$, $t_n \le t \le t_n + k$, to the interval $(x_{j+\frac12} - \dfrac12 r_0\Delta x, x_{j+\frac12} + \dfrac12 r_0\Delta x)$. If $x_{j+\frac12}$ is non–sonic, then clearly the solution for the Riemann problem associated with $\tilde f$ coincides with that of f, namely, $v^n_{j+\frac12}$. Furthermore, the characteristics for (21) emanating from $(x_{j+\frac12}, t)$, $t_n \le t \le t_n + k$, intersect $t = t_n$ at $x_{j+\frac12} - \dfrac12 r_0\Delta x \le x \le x_{j+\frac12} + \dfrac12 r_0\Delta x$. The values of $v^n(x)$ there are contained in an interval where $\tilde f(v)$ is linear (with slope $f'(v^n_{j+\frac12})$). We obtain therefore,

$$\tilde u(x_{j+\frac12}, t) = v^n_{j+\frac12} - s^n_{j+\frac12} f'(v^n_{j+\frac12}) \cdot (t - t_n), \quad 0 \le t - t_n \le k, \tag{29}$$

where $s^n_{j+\frac12} = s^n_j$ (resp. $s^n_{j+\frac12} = s^n_{j+1}$) if $f'(v^n_{j+\frac12}) > 0$, (resp. $f'(v^n_{j+\frac12}) < 0$), and by the linearity of $\tilde f(v)$ in this interval,

$$\tilde f(\tilde u(x_{j+\frac12}, t)) = f(v^n_{j+\frac12}) - s^n_{j+\frac12} f'(v^n_{j+\frac12})^2 \cdot (t - t_n), \quad 0 \le t - t_n \le k. \tag{30}$$

Finally, if $x_{j+\frac12}$ is sonic, we have $v^n_{j+\frac12,-} \le v_{\min} \le v^n_{j+\frac12,+}$ and $v^n_{j+\frac12} = v_{\min}$. By the construction of $\tilde f$ in J_j, we have $\tilde f(v) \equiv f(v_{\min})$ in a neighborhood of $v_{\min}$, which contains the interval $(v^n_{j+\frac12,-} - r_0(v^n_{j+\frac12,-} - v^n_j), v^n_{j+\frac12,+} + r_0(v^n_{j+1} - v^n_{j+\frac12,+}))$. Thus, once again $\tilde f(v) = f(v_{\min})$ in the domain of dependence of $(x_{j+\frac12}, t)$. While the solution $\tilde u(x_{j+\frac12}, t)$ remains fixed as a "rarefaction shock" ($\tilde u(x_{j+\frac12} \pm 0, t) = v^n_{j+\frac12,\pm}$), the flux is $f(v_{\min})$, thus satisfying again (30) (since $f'(v^n_{j+\frac12}) = f'(v_{\min}) = 0$). The equality (28) now follows from (30) and (11)–(12), (15). □

We are left with the more delicate case of constructing $\tilde{f}(v)$ and proving (28) when (24) is not satisfied, and without loss of generality we take,

$$v_{j-1}^n < v_j^n > v_{j+1}^n. \tag{31}$$

Note that necessarily, by (17), we have $s_j^n = 0$. Furthermore, the wave at $x_{j+\frac{1}{2}}$ is a shock while the one at $x_{j-\frac{1}{2}}$ is a rarefaction.

Claim 3.5. *Assume (31) and the CFL condition (27). Then we can construct $\tilde{f}(v)$ so that the solution to (21) satisfies (28).*

Proof. We study the situation by case.

(I) The rarefaction at $x_{j-\frac{1}{2}}$ moves to the left, hence $v_{j-\frac{1}{2}}^n = v_j^n \geq v_{j+\frac{1}{2}}^n$. We take $\tilde{f}(v) = L_f(v; v_{j+\frac{1}{2}}^n, v_{j-\frac{1}{2}}^n)$ and if $v_{j+\frac{1}{2}}^n = v_{j-\frac{1}{2}}^n$ we take $\tilde{f}(v) = f(v_j^n) + f'(v_j^n)(v - v_j^n)$ (compare (25)–(26)). With the CFL condition (27) it is clear that now

$$\tilde{u}(x_{j-\frac{1}{2}}, t) = v_j^n, \qquad t_n \leq t \leq t_n + k,$$

since the characteristics for (21) emanating from $(x_{j-\frac{1}{2}}, t)$ intersect $t = t_n$ at $x_{j-\frac{1}{2}} \leq x \leq x_{j-\frac{1}{2}} + \frac{1}{2} r_0 \Delta x$. Thus, $\tilde{u}(x_{j-\frac{1}{2}}, t)$ is equal to $U(x_{j-\frac{1}{2}}, t)$ (see (11)) since $\left(\frac{\partial U}{\partial t}\right)_{j-\frac{1}{2}}^n = 0$ and we get (28) at the boundary $x_{j-\frac{1}{2}}$. The validity of (28) at $x_{j+\frac{1}{2}}$ is obtained exactly as in the proof of Claim 3.4. Note that since $f'(v_{j+\frac{1}{2}}^n) \leq f'(v_j^n)) \leq 0$, the characteristics for (21) emanating from $(x_{j+\frac{1}{2}}, t)$ intersect $t = t_n$ at $x_{j+\frac{1}{2}} \leq x \leq x_{j+\frac{1}{2}} + \frac{1}{2} r_0 \Delta x$, where the values of $v^n(x)$ are in the "linear" domain of $\tilde{f}$.

(II) The point $x_{j-\frac{1}{2}}$ is sonic. Now we have

$$v_{j-\frac{1}{2},-}^n \leq v_{\min} \leq v_j^n.$$

If the shock at $x_{j+\frac{1}{2}}$ moves to the right we take $\tilde{f}(v) = L_f(v; v_{\min}, v_j^n)$ (or $\tilde{f}(v) = f(v_{\min})$ if $v_{\min} = v_j^n$). We extend $\tilde{f}(v)$ as $f(v_{\min})$ in $(v_{j-\frac{1}{2},-}^n, v_{\min})$. Then clearly the solution to (21) at $x_{j-\frac{1}{2}}$ is a rarefaction shock, with $\tilde{f}(\tilde{u}(x_{j-\frac{1}{2}}, t)) \equiv f(v_{\min})$ and (since $\tilde{f}'(v_j^n) \geq 0$), $\tilde{u}(x_{j+\frac{1}{2}}, t) \equiv v_j^n$ and $\tilde{f}(\tilde{u}(x_{j+\frac{1}{2}}, t)) \equiv f(v_j^n)$, thus obtaining again (28).

The other case here is (still with $x_{j-\frac{1}{2}}$ sonic) when the shock at $x_{j+\frac{1}{2}}$ moves to the left, and in particular $v_{j+\frac{1}{2},+}^n < v_{\min}$. We take then $\tilde{f}(v) = L_f(v; v_{j+\frac{1}{2},+}^n, v_{\min})$ (extended as usual by linearity at the two ends). Note that now $\tilde{f}'(v) \leq 0$ throughout its domain, and for $v > v_{\min}$ we have $\tilde{f}'(v) = 0$ (in particular for $v = v_j^n$). We therefore obtain $\tilde{f}(\tilde{u}(x_{j-\frac{1}{2}}, t)) \equiv f(v_{\min})$ while $\tilde{u}(x_{j+\frac{1}{2}}, t) = v_{j+\frac{1}{2},+}^n - s_{j+1}^n f'(v_{j+\frac{1}{2},+}^n)(t - t_n)$, in agreement with (29) (since now $v_{j+\frac{1}{2},+}^n = v_{j+\frac{1}{2}}^n$). Once again the validity of (28) is established.

(III) The last case is when the rarefaction at $x_{j-\frac{1}{2}}$ moves to the right, hence

$$v_{\min} < v_{j-\frac{1}{2},-}^n \leq v_j^n.$$

Now clearly the values of $\tilde{u}(x_{j-\frac{1}{2}}, t)$ are determined by $v^n(x)$, $x \leq x_{j-\frac{1}{2}}$. If these values decrease too rapidly (i.e., $v_{j-1}^n < v_{\min}$), then we might have a conflict trying to define $\tilde{f}(v)$ also in a neighborhood of $v^n_{j+\frac{1}{2}} \le v^n_j$. Thus, IT IS HERE that we use our further limiter restriction (18), which forces $v^n_{j-1} > v_{\min}$. We now look at $v^n_{j+\frac{1}{2}}$. If $v^n_{j+\frac{1}{2}} = v^n_j$, meaning (since $v^n_j > v_{\min}$) that the shock at $x_{j+\frac{1}{2}}$ moves to the right, we take $\tilde{f}(v) = L_f(v; v^n_{j-\frac{1}{2},-}, v^n_j)$ (or $\tilde{f}(v) = f(v^n_j) + f'(v^n_j)\cdot(v - v^n_j)$ if $v^n_{j-\frac{1}{2},-} = v^n_j$). Since here $\tilde{f}'(v) \ge 0$, we obtain the solutions

$$\begin{aligned}
&\tilde{u}(x_{j-\frac{1}{2}}, t) = v^n_{j-\frac{1}{2},-} - s^n_{j-1} f'(v^n_{j-\frac{1}{2},-})(t - t_n),\\
&\tilde{u}(x_{j+\frac{1}{2}}, t) \equiv v^n_j,
\end{aligned}$$

yielding once again (28).

Finally, if $v^n_{j+\frac{1}{2},+} = v^n_{j+\frac{1}{2}} < v^n_j$, meaning that the shock at $x_{j+\frac{1}{2}}$ moves to the left, we set

$$\tilde{f}(v) = L_f(v; v^n_{j+\frac{1}{2}}, v^n_{j-\frac{1}{2},-}),$$

and extend $\tilde{f}(v)$ by linearity at $v^n_{j+\frac{1}{2}}$. Note that now $v^n_{j+\frac{1}{2}} < v_{\min} < v^n_{j-1} < v^n_{j-\frac{1}{2},-}$, so the solution $\tilde{u}(x_{j-\frac{1}{2}}, t)$ is determined linearly as before by values $v^n(x)$, $x_{j-\frac{1}{2}} - \frac{1}{2}r_0\Delta x \le x \le x_{j-\frac{1}{2}}$ while the solution $\tilde{u}(x_{j+\frac{1}{2}}, t)$ is determined by the values $v^n(x) \le v^n_{j+\frac{1}{2}}$ for $x_{j+\frac{1}{2}} \le x \le x_{j+\frac{1}{2}} + \frac{1}{2}r_0\Delta x$. □

Remark 3.6. Because we allow $\theta > 0$ to be small, we cannot avoid the "sawtooth" phenomenon, namely, that $0 < v^n_j - v^n_{j-\frac{1}{2}}$ can be very small (see Figure 2.1). Thus, we must restrict ((27)) $\lambda \max_{|v|\le M} |f'|$ to $\frac{1}{2}r_0$. It ensures that the characteristic lines (issuing from $(x_{j+\frac{1}{2}}, t)$) intersect (at $t = t_n$) cell j in its right–hand half, where the values of $v^n(x)$ are contained in the "linear" domain of $\tilde{f}(v)$. A more "rigid" choice of θ (say, $\theta = 1$, eliminating the sawtooth effect) would enable us to take $\lambda \max_{|v|\le M} |f'| \le r_0$.

Remark 3.7. We emphasize that the extra limiter condition (18) was used in the proof of Claim 3.5 only in Case III, that of a rarefaction moving to the right (at $x_{j-\frac{1}{2}}$) while a shock is moving to the left (at $x_{j+\frac{1}{2}}$), under the situation (31), or the analogous situation with a minimum $\tilde{u}$. It enables us to define $\tilde{f}(v)$ in a way that (28) is satisfied without further restricting the CFL condition (27).

We call the reader's attention to the fact that while rarefaction waves for the solution $\tilde{u}(x,t)$ (to (21)) become (because of the linearity of $\tilde{f}$) "rarefaction shocks", the solution $\tilde{u}(x_{j\pm\frac{1}{2}}, t)$ is identical to the linear (in t) expansion of $U(x_{j\pm\frac{1}{2}}, t)$, the exact solution to (1), (4) in all cases.

We can summarize all the above as follows.

Proposition 3.8. *(Exact character of GRP solution). For every j, construct $\tilde{f}$ as in Claims 3.4 and 3.5. Then v^{n+1}_j, the updated value by the GRP method, and s^{n+1}_j, the updated slope, are exactly those obtained from the solution $\tilde{u}(x,t)$ of the* *approximate equation (21). More precisely, we have, under CFL condition (27),*

$$\begin{aligned} v_j^{n+1} &= \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \tilde{u}(x,t_{n+1})dx \\ &= v_j^n - \lambda\left[\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (\tilde{f}(\tilde{u}(x_{j+\frac{1}{2}},t)) - \tilde{f}(\tilde{u}(x_{j-\frac{1}{2}},t)))dt\right]. \end{aligned} \tag{32}$$

While s_j^{n+1} is the slope obtained by modifying $(\tilde{u}(x_{j+\frac{1}{2}},t_{n+1}) - \tilde{u}(x_{j-\frac{1}{2}},t_{n+1}))/\Delta x$ according to the algorithm of the preceding section (16)–(18).

Corollary 3.9. *(The Maximum–Minimum Principle for the GRP–scheme). The values $\{v_j^n\}_{j=-\infty}^{\infty}$ (and indeed, the functions $v^n(x)$) satisfy, for all $n \geq 0$,*

$$\sup_j v_j^n \leq \|u_0\|_\infty, \qquad \inf_j v_j^n \geq -\|u_0\|_\infty. \tag{33}$$

Proof. As expressed in Proposition 3.8, the values $v_j^{n+1}$are derived as exact averages of a conservation law ((21)), so that for all

$$l \in \mathbb{Z}, \quad \inf_j v_j^n = \inf_x v^n(x) \leq v_l^{n+1} \leq \sup_x v^n(x) = \sup_j v_j^n.$$

Note that the equalities $\inf_x v^n(x) = \inf_j v_j^n$ and $\sup_x v^n(x) = \sup_j v_j^n$ follow from the construction of $\{s_j^n\}_{j=-\infty}^{\infty}$ in the preceding section. □

As another interesting feature of the GRP scheme, we derive from Proposition 3.8 the property that at a local maximum (resp. minimum) the approximate solution $v^n(x)$ cannot increase (resp. decrease). Obviously, this is a property of the exact solution to the conservation law.

Corollary 3.10. *(Control of Local Extrema). Suppose that $v_j^n \geq \max(v_{j-1}^n, v_{j+1}^n)$ (resp. $v_j^n \leq \min(v_{j-1}^n, v_{j+1}^n)$). Then $v_j^{n+1} \leq v_j^n$ (resp. $v_j^{n+1} \geq v_j^n$).*

Proof. Note that in this case $s_j^n = 0$, so that $v^n(x) \leq v_j^n$ (resp. $v^n(x) \geq v_j^n$) for $x \in [x_{j-\frac{3}{2}}, x_{j+\frac{3}{2}}]$. Thus $\tilde{u}(x,t_{n+1}) \leq v_j^n$ (resp. $\tilde{u}(x,t_{n+1}) \geq v_j^n$) for $x \in [x_{j-1}, x_{j+1}]$ and the claim follows from (32). □

Remark 3.11. We note that Corollary 3.10 can be proved directly from the GRP definitions, using the fact that $s_j^n = 0$ and inspecting the various wave possibilities at $x_{j\pm\frac{1}{2}}$ to obtain that always $f_{j+\frac{1}{2}}^{n+\frac{1}{2}} \geq f_{j-\frac{1}{2}}^{n+\frac{1}{2}}$ in this case for local maximum and $f_{j+\frac{1}{2}}^{n+\frac{1}{2}} \leq f_{j-\frac{1}{2}}^{n+\frac{1}{2}}$ for local minimum.

Note that if v_j^n is a local maximum the construction in the preceding section yields (in addition to $s_j^n = 0$) also

$$s_{j-1}^n \geq 0, \quad s_{j+1}^n \leq 0, \quad \max(v_{j-\frac{1}{2},-}^n, v_{j+\frac{1}{2},+}^n) \leq v_j^n.$$

Similar statements apply in the case of a local minimum.

For the treatment of the convergence properties of the GRP scheme, we shall require an estimate on the difference between of the GRP fluxes $f_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ (given always by (28)) and the corresponding time integrals for the original flux function $f(\tilde{u})$. We have

Proposition 3.12. *Let k satisfy the CFL condition (27). The GRP fluxes $f_{j+\frac{1}{2}}^{n+\frac{1}{2}}$, $-\infty < j < \infty$, are of second order accuracy in the following sense,*

$$\left| f_{j+\frac{1}{2}}^{n+\frac{1}{2}} - \frac{1}{k}\int_{t_n}^{t_{n+1}} f(\tilde{u}(x_{j+\frac{1}{2}}, t))dt \right| \le C[|(\Delta v)_j^n| + |(\Delta v)_{j+1}^n|]^2, \tag{34}$$

where $(\Delta v)_j^n = \Delta x \cdot s_j^n$, and $C > 0$ is independent of j, n.

Proof. In view of (29) we have, for $t_n \le t \le t_{n+1}$,

$$|\tilde{u}(x_{j+\frac{1}{2}}, t) - v_{j+\frac{1}{2}}^n| \le |(\Delta v)_j^n| + |(\Delta v)_{j+1}^n|, \tag{35}$$

since the CFL condition implies $|f'(v_{j+\frac{1}{2}}^n)(t - t_n)| \le \Delta x$. Denoting the right–hand side of (35) by A_j^n, we get, by Taylor's theorem and (29)–(30),

$$\begin{aligned} f(\tilde{u}(x_{j+\frac{1}{2}}, t)) &= f(v_{j+\frac{1}{2}}^n) + f'(v_{j+\frac{1}{2}}^n)(\tilde{u}(x_{j+\frac{1}{2}}, t) - v_{j+\frac{1}{2}}^n) + O((A_j^n)^2) \\ &= f(v_{j+\frac{1}{2}}^n) - f'(v_{j+\frac{1}{2}}^n)^2 s_{j+\frac{1}{2}}^n (t - t_n) + O((A_j^n)^2) \\ &= \tilde{f}(\tilde{u}(x_{j+\frac{1}{2}}, t)) + O((A_j^n)^2). \end{aligned} \tag{36}$$

Integrating this equality and using (28) we obtain (34). □

4. **Monotone chains and the TVD properties.** The construction of the approximate flux $\tilde{f}$ was local, pertaining only to the evaluation of v_j^{n+1}, s_j^{n+1}, for a fixed j. However, we now show that an approximate flux can be constructed so that it serves to compute a sequence of values $\{v_j^{n+1}\}_{j=j_1}^{j_2}$, $\{s_j^{n+1}\}_{j=j_1}^{j_2}$ provided that a certain monotonicity requirement on v^n is fulfilled.

Definition 4.1. Let $j_1 < j_2$ be two integers. We say that the sequence $\{v_j^n\}_{j=j_1}^{j_2}$ is a (maximal) monotone chain if

(i) $v_{j_1}^n \le v_{j_1+1}^n \le \cdots \le v_{j_2}^n$ (or $v_{j_1}^n \ge v_{j_1+1}^n \ge \cdots \ge v_{j_2}^n$),

(ii) $v_{j_1-1}^n > v_{j_1}^n$, $v_{j_2+1}^n < v_{j_2}^n$ (or $v_{j_1-1}^n < v_{j_1}^n$, $v_{j_2+1}^n > v_{j_2}^n$).

(We allow the case $j_1 = -\infty$ or $j_2 = \infty$, and then the corresponding condition (ii) is void.)

An inspection of the construction of $\tilde{f}$ in Section 3 shows that it can be constructed simultaneously for $j_1 \le j \le j_2$, if $\{v_j^n\}_{j=j_1}^{j_2}$ is a monotone chain. The approximate flux $\tilde{f}(v)$ is constructed separately near $v_{j\pm\frac{1}{2}}^n$ (see (25)) and then continued to values of v near $v_{j\pm\frac{3}{2}}^n$ (namely, the evaluation of $v_{j\pm1}^{n+1}$) etc. This construction of $\tilde{f}(v)$ can be summarized in the following proposition.

Proposition 4.2. *Let $\{v_j^n\}_{j=j_1}^{j_2}$ be a maximal monotone chain. Then a C^1 flux function $\tilde{f}$ defined over a neighborhood of $[v_{j_1-\frac{1}{2},-}^n, v_{j_2+\frac{1}{2},+}^n]$ can be constructed so that if $\tilde{u}(x, t_{n+1})$ is the solution of (21) for $x \in [x_{j_1-\frac{1}{2}}, x_{j_2+\frac{1}{2}}]$, and if the CFL condition (27) is satisfied, the following properties of $\tilde{u}$ hold true.*

(a) For every $j_1 - 1 \le j \le j_2$, the GRP fluxes $f^{n+\frac{1}{2}}_{j+\frac{1}{2}}$ are given by (28).

(b) For every $j_1 \le j \le j_2$, the GRP updated values v_j^{n+1} are given by (32).

(c) For every $j_1 - 1 \le j \le j_2$, the GRP new "edge values" $v^{n+1}_{j+\frac{1}{2}}$ satisfy $v^{n+1}_{j+\frac{1}{2}} = \tilde{u}(x_{j+\frac{1}{2}}, t_{n+1})$, so that the new slopes s_j^{n+1} are obtained by processing $\tilde{u}(x_{j+\frac{1}{2}}, t_{n+1}) - \tilde{u}(x_{j-\frac{1}{2}}, t_{n+1})$ as described in Section 2, see (16)–(18).

Note that the solutions at the "edges" of the maximal monotone chain, $\tilde{u}(x_{j_1-\frac{1}{2}}, t_{n+1})$ and $\tilde{u}(x_{j_2+\frac{1}{2}}, t_{n+1})$ are obtained, in view of (21) and the construction of $\tilde{f}(v)$, as values of $v^n(\tilde{x}_{j_1})$, $v^n(\tilde{x}_{j_2})$, respectively, where $x_{j_1-1} \le \tilde{x}_{j_1} \le x_{j_1}$, $x_{j_2} \le \tilde{x}_{j_2} \le x_{j_2+1}$. (If one of these points, say $x_{j_1-\frac{1}{2}}$, is sonic, the corresponding value is $v_{\min}$ and $\tilde{x}_{j_1} = x_{j_1-\frac{1}{2}}$. It will be clear in the sequel how to incorporate this case, and we shall not mention it again).

By the definition of total variation of functions in (7), for the piecewise linear function $v^n(x)$ we get

$$\begin{aligned} TV(v^n; [a,b]) = |v^n_{k-\frac{1}{2},-} - v^n(a)| \quad &+ \textstyle\sum_{m=k}^{l+1} |v^n_{m-\frac{1}{2},+} - v^n_{m-\frac{1}{2},-}| \\ &+ \textstyle\sum_{m=k}^{l} |s_m^n| \Delta x + |v^n(b) - v^n_{l+\frac{1}{2},+}|, \end{aligned} \tag{37}$$

where $a \in [x_{k-1}, x_k]$, $b \in [x_l, x_{l+1}]$. (if $a = x_{k-\frac{1}{2}}$, we take $v^n(a) = v^n_{k-\frac{1}{2},-}$, similarly in b).

The fact that $\tilde{u}$ is a solution to (21) and the classical theory of scalar conservation laws now implies the following proposition, where we are using the notation $\tilde{x}_{j_1}$, $\tilde{x}_{j_2}$ introduced above.

Proposition 4.3. *Let $\{v_j^n\}_{j=j_1}^{j_2}$ be a maximal monotone chain. Then the solution $\tilde{u}(x, t_{n+1})$ (where the CFL condition (27) is satisfied), satisfies the following "TV property",*

$$TV(\tilde{u}(x, t_{n+1}); [x_{j_1-\frac{1}{2}}, x_{j_2+\frac{1}{2}}]) \le TV(v^n(x); [\tilde{x}_{j_1}, \tilde{x}_{j_2}]). \tag{38}$$

If $j_1 = -\infty$ (resp. $j_2 = \infty$), we take in (38) $\tilde{x}_{j_1} = -\infty$ (resp. $\tilde{x}_{j_2} = \infty$).

Since the reconstruction of steps (16)–(18) lead to "sawtooth" in $v^{n+1}(x)$, we cannot simply deduce that the total variation of $v^{n+1}(x)$ is less than that of $\tilde{u}(x, t_{n+1})$. However, these steps do not violate the maximum–minimum principle, which allows us to control the total variation of $v^{n+1}(x)$ by tracking minimum and maximum values.

Definition 4.4. The value v_j^n is said to be maximum (or minimum) if

$$v_j^n \ge \max\{v_{j-1}^n, v_{j+1}^n\}, \qquad (\text{ or } v_j^n \le \min\{v_{j-1}^n, v_{j+1}^n\}. \tag{39}$$

Then x_j is a maximum (or minimum) point. Maximum and minimum values (points) are all called extremum values (points).

As explained in Section 2, the variation hs_j^{n+1} in "cell j" (before modification by the "Limiter Algorithm") is $\tilde{u}(x_{j+\frac{1}{2}}, t_{n+1}) - \tilde{u}(x_{j-\frac{1}{2}}, t_{n+1})$. In view of Proposition 4.3 we have

$$h \sum_{j=j_1}^{j_2} |s_j^{n+1}| \le TV(v^n(x); [\tilde{x}_{j_1}, \tilde{x}_{j_2}]).$$

However, adjusting the linear profile to the average v_j^{n+1} involves an additional jump of $2(v_j^{n+1} - \tilde{u}(x_j, t_{n+1}))$, where the factor 2 comes from counting the jumps at both $x_{j\pm\frac{1}{2}}$. As remarked already, if the limiter parameter $\theta \geq 1$ (see (2.10)), then the reconstructed profile is monotone and we get (see [11, Ch. 4, Lemma 3.1])

$$TV(v^{n+1}(x); [x_{j_1-\frac{1}{2}}, x_{j_2+\frac{1}{2}}]) \leq TV(v^n(x); [\tilde{x}_{j_1}, \tilde{x}_{j_2}]). \tag{40}$$

In view of the average property (32), the additional jumps in "cell j" do not exceed $TV(\tilde{u}(x, t_{n+1}); [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}])$. If, however, $\theta < 1$, we cannot ensure that the contribution of the "sawteeth" jumps does not exceed $TV(v^n(x); [\tilde{x}_{j_1}, \tilde{x}_{j_2}])$. We therefore impose the following further restriction on the "limiter parameter" θ.

GODUNOV COMPATIBILITY HYPOTHESIS I: There exists a constant $C \geq 0$, independent of k, such that we have, instead of (40)

$$TV(v^{n+1}(x); [x_{j_1-\frac{1}{2}}, x_{j_2+\frac{1}{2}}]) \leq (1 + Ck) TV(v^n(x); [\tilde{x}_{j_1}, \tilde{x}_{j_2}]). \tag{41}$$

Remark 4.5. As already observed, this condition is satisfied, with $C = 0$, if $\theta \geq 1$. In particular, it holds for the Godunov scheme (where all slopes are zero, namely, $\theta = 2$. This explains our terminology of "Godunov compatibility". Furthermore, this condition can be "built into" the "Limiter algorithm" (in other words, into the actual numerical code) by properly adjusting θ at each time level. Since always $\theta \geq 1$, the resulting scheme remains second-order, and the adjustment amounts to the addition of "numerical dissipation".

In the sequel we assume that this hypothesis is satisfied.

Suppose now that $j_2 < \infty$ and that the next maximal monotone chain is $\{v_j^n\}_{j=j_2}^{j_3}$, $j_3 > j_2$. Let $\tilde{u}_1(x, t)$ be the function constructed as above (for the chain $\{v_j^n\}_{j=j_2}^{j_3}$), defined for $x \in [x_{j_2+\frac{1}{2}}, x_{j_3+\frac{1}{2}}]$. The functions $\tilde{u}(x, t)$, $\tilde{u}_1(x, t)$ coincide along $(x_{j_2+\frac{1}{2}}, t)$, $t_n \leq t \leq t_{n+1}$. Indeed, they are both equal to the linear part of $U(x_{j_2+\frac{1}{2}}, t)$ (see the paragraph preceding Proposition 3.9). We retain the notation $\tilde{u}(x, t)$ for the joint function

$$\tilde{u}(x, t) = \begin{cases} \tilde{u}(x, t), & x \in [x_{j_1-\frac{1}{2}}, x_{j_2+\frac{1}{2}}], \\ \tilde{u}_1(x, t), & x \in [x_{j_2+\frac{1}{2}}, x_{j_3+\frac{1}{2}}]. \end{cases}$$

Going over all monotone chains, we finally get the function $\tilde{u}(x, t)$, $x \in \mathbb{R}$. The point $\tilde{x}_{j_2}$ (in Proposition 4.3) was obtained as a trace of a characteristic line, corresponding to the flux function $\tilde{f}(v)$ in the neighborhood of $v^n_{j_2+\frac{1}{2}}$, where $\tilde{f}(v)$ is linear, so that $\tilde{x}_{j_2} = x_{j_2+\frac{1}{2}} - \tilde{f}'(v^n_{j+\frac{1}{2}}) \cdot k$. By construction, the same function is used at $v = v^n_{j_2+\frac{1}{2}}$ in the computation of $v^{n+1}_{j_2+\frac{1}{2}}$ (as a "left" flux now). Hence the same $\tilde{x}_{j_2}$ is obtained when evaluating the dependence domain for $\tilde{u}_1$ over the chain $\{v_j^n\}_{j=j_2}^{j_3}$. Thus the inequality (38) remains valid with j_1, j_2 (resp. $\tilde{x}_{j_1}$, $\tilde{x}_{j_2}$) replaced by j_2, j_3 (resp. $\tilde{x}_{j_2}$, $\tilde{x}_{j_3}$). We therefore get Proposition 4.3 for this adjacent chain. By summing over all chains we finally obtain the following theorem.

Theorem 4.1. *Let the CFL condition (27) be satisfied, let $v^{n+1}(x)$ be constructed by the GRP method and let $\tilde{u}(x,t)$ be the solution constructed above. Then, for all $t \in [t_n, t_{n+1}]$,*

$$TV(v^{n+1}(x); \mathbb{R}) \le (1 + Ck) TV(\tilde{u}(x,t); \mathbb{R}) \le (1 + Ck) TV(v^n(x); \mathbb{R}). \tag{42}$$

Recall that $\tilde{f}$ is constructed as a C^1–function such that

$$\max_{|v| \le \|u_0\|_\infty} |\tilde{f}'(v)| \le \tilde{M} = \max_{|v| \le \|u_0\|_\infty} |f'(v)|. \tag{43}$$

Thus, the classical theory of conservation laws (see [11, Th. 3.1, P. 69]) yields, in addition to Proposition 4.3, also the following claim.

Proposition 4.6. *Under the condition of Proposition 4.3, we have for any $t_n \le \tau < \tau + \Delta\tau \le t_{n+1}$,*

$$\int_{x_{j_1 - \frac{1}{2}}}^{x_{j_2 + \frac{1}{2}}} |\tilde{u}(x, \tau + \Delta\tau) - \tilde{u}(x,\tau)| dx \le \tilde{M} \cdot TV(v^n(x); [\tilde{x}_{j_1}, \tilde{x}_{j_2}]) \cdot \Delta\tau. \tag{44}$$

In particular, by adding over all monotone chains, we get, in addition to (42),

$$\int_{\mathbb{R}} |\tilde{u}(x, \tau + \Delta\tau) - \tilde{u}(x,\tau)| dx \le \tilde{M} \cdot TV(u_0) \cdot \Delta\tau. \tag{45}$$

Fix $T > 0$, and let $0 < k \ll T$. We approximate the initial function $u_0(x)$ by $v^0(x)$ as in (4), with $h = k/\lambda$,

$$v_j^0(x) = \frac{1}{h} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_0(x) dx, \qquad s_j^0 = \frac{1}{h}(u_0(x_{j+\frac{1}{2}}) - u_0(x_{j-\frac{1}{2}})), \tag{46}$$

(s_j^0 is further modified as in (17)–(18). Clearly, we have

$$TV(v^0(x); \mathbb{R}) \le TV(u_0(x); \mathbb{R}) = K. \tag{47}$$

Let $\Gamma_k^n = \mathbb{R} \times [t_n, t_n + k)$, $t_n = nk$. In view of the foregoing discussion we can construct the function $\tilde{u}(x,t)$ in Γ_k^0, so that $\tilde{u}(x, 0) = v^0(x)$. Proceeding from one strip Γ_k^n to the next one Γ_k^{n+1}, we can continue this construction. As expressed in Proposition 3.8, the piecewise linear function $v^{n+1}(x)$ is obtained from $\tilde{u}(x, t_{n+1}-)$ by an averaging–and– linearization procedure (namely, the "postprocessing" mechanism of the GRP method). We then construct $\tilde{u}(x,t)$ in Γ_k^{n+1}, subject to the initial condition $\tilde{u}(x, t_{n+1}) = v^{n+1}(x)$. We continue this construction up to the strip Γ_k^N, where $Nk \le T \le (N+1)k$.

Notation. *We designate by $\tilde{u}^k(x,t)$, $x \in \mathbb{R}$, $t \in [0,T]$, the function obtained by the above construction. The function $\tilde{u}(x, t_{n+1})$ (as in (39)) obtained by solving (21) in Γ_k^n will now be denoted by $\tilde{u}(x, t_{n+1}-)$ (it is reconstructed as $v^{n+1}(x) = \tilde{u}^k(x, t_{n+1})$).*

The function $\tilde{u}^k(x,t)$ is the "product" of the GRP scheme (for the time step $\Delta t = k$). In view of Corollary 3.9 and Theorem 4.1, we have

Corollary 4.7. *The function $\tilde{u}^k(x,t)$ satisfies ,for all $k > 0$, the Maximum–Minimum principle,*

$$\sup_{k>0} \tilde{u}^k(x,t) \le \|u_0\|_\infty, \qquad \inf_{k>0} \tilde{u}^k(x,t) \ge -\|u_0\|_\infty. \tag{48}$$

Furthermore, the function $t \to TV(\tilde{u}^k(x,t);\mathbb{R})$ is monotone non-increasing in each time interval $[t_n, t_{n+1})$ and is uniformly bounded in $[0,T]$ by $e^{CT}TV(u_0(x))$.

The reconstruction of the piecewise linear functions $v^n(x)$ from $\tilde{u}(x, t_n-)$ breaks possibly the continuity properties of $\tilde{u}^k$ at $t = t_n$. However it is remarkable that the most crucial continuity property, namely, the (Lipschitz in time) continuity with values in $L^1(\mathbb{R})$, is approximately preserved. To show this, we shall use the finite propagation speed property.

Let $(x,t) \in \mathbb{R} \times [0,T]$ and assume that $nk \le t < (n+1)k$. With $\tilde{M}$ as in (43), we denote, for any $m \le n$,

$$\begin{aligned} &(i) \qquad D_k^m(x,t) = [x - \tilde{M}(t-mk), x + \tilde{M}(t-mk)], \\ &(ii) \qquad D_{k,h}^m(x,t) = [([\frac{x - \tilde{M}(t-mk)}{h}] - \frac{1}{2})h, ([\frac{x + \tilde{M}(t-mk)}{h}] + \frac{3}{2})h], \end{aligned} \tag{49}$$

In (49)(ii) we use $h = \dfrac{k}{\lambda}$ and $[\alpha]$ = the largest integer $\le \alpha$. Note that λ is given by (19) where $\max\{|\tilde{f}'(v)|\} \le \max\{|f'(v)|\}$. Thus $D_{k,h}^m(x,t)$ is the smallest "grid interval" (of the type $(x_{j-\frac{1}{2}}, x_{l+\frac{1}{2}})$) containing $D_k^m(x,t)$.

In the strip Γ_k^n the function $\tilde{u}^k$ solves Eq. (21) and, in view of (43), the domain of dependence of (x,t) is contained in $D_k^n(x,t)$. The standard theory of conservation laws now yields

$$\inf_{y \in D_k^n(x,t)} v^n(y) \le \tilde{u}^k(x,t) \le \sup_{y \in D_k^n(x,t)} v^n(y), \qquad t_n \le t < t_{n+1}. \tag{50}$$

However, it is clear from the reconstruction algorithm (16)–(18) that for every $x \in \mathbb{R}$,

$$\inf_{y \in D_{k,h}^n(x,t_{n+1})} \tilde{u}^k(y, t_{n+1}-) \le v^{n+1}(x) \le \sup_{y \in D_{k,h}^n(x,t_{n+1})} \tilde{u}^k(y, t_{n+1}-). \tag{51}$$

We can now prove the following continuity property.

Theorem 4.2. *The function $t \to \tilde{u}^k(\cdot,t)$ is "almost" Lipschitz continuous with values in $L^1(\mathbb{R})$. More precisely, if $0 \le t < t' \le T$,*

$$\int_{\mathbb{R}} |\tilde{u}^k(x,t') - \tilde{u}^k(x,t)| dx \le 4\tilde{M} \exp(CT) \cdot TV(u_0;\mathbb{R}) \cdot (|t'-t| + k). \tag{52}$$

Proof. Suppose that $t' \in [t_n, t_n + k)$ and $t \in [t_m, t_m + k)$, $m < n$ (for $m = n$ this is (45)). Extending the estimates (50)–(51) through the strips Γ_k^n, Γ_k^{n-1}, $\cdots$, Γ_k^m, we obtain

$$\inf_{E_{k,h}^m(x,t)} \tilde{u}^k(x,t_m) \le \tilde{u}^k(x,t') \le \sup_{E_{k,h}^m(x,t)} \tilde{u}^k(x,t_m), \tag{53}$$

where $E_{k,h}^m = D_{k,h}^m + \{y; |y| \le (n-m)h\}$ (since each step "down" from Γ_k^j to Γ_k^{j-1} forces us, by (51), to add a full cell on each side of the domain of dependence. This is essentially the " price" paid for the reconstruction procedure).

Observe that $(n-m)h = (n-m)\dfrac{k}{\lambda} \le \tilde{M}(t'-t+k)$, so we infer from (53), for any $x \in \mathbb{R}$,

$$|\tilde{u}^k(x,t') - \tilde{u}^k(x,t)| \le TV(\tilde{u}^k(\cdot,t); [x - 2\tilde{M}(t'-t+k), x + 2\tilde{M}(t'-t+k)]). \tag{54}$$

Integrating (54) over $\mathbb{R}$ and using Fubini's theorem, we conclude,

$$\begin{aligned}\int_{\mathbb{R}} |\tilde{u}^k(x,t') - \tilde{u}^k(x,t)|dx &\le \int_{\mathbb{R}} \int_{x-2\tilde{M}(t'-t+k)}^{x+2\tilde{M}(t'-t+k)} |d\tilde{u}^k(\cdot,t)|dx \\ &\le 4\tilde{M} \cdot TV(\tilde{u}^k(x,t);\mathbb{R}) \cdot (t'-t+k) \\ &\le 4\tilde{M} \exp(CT) \cdot TV(u_0;\mathbb{R}) \cdot (t'-t+k).\end{aligned} \tag{55}$$

□

Remark 4.8. In (52) the Lipschitz constant depends on T. This is in contrast with the theory of conservation laws (see [11, Ch.2, Th. 3.1]), where this constant is independent of t . The reason, of course, is the fact that the total variation is increased by the appearance of "sawteeth". However, the *boundedness* of the total variation permits us to follow the compactness arguments used in the theory, as is explained in the next section.

5. **Convergence to a weak solution.** The functions $\tilde{u}^k(x,t)$ constructed in Section 4 (see Notation 4) can be defined for all $t \in [0,\infty)$. In this section, we show that every sequence $\tilde{u}^{k_i}(x,t)$, $k_i \to 0$, contains a subsequence converging boundedly almost everywhere (in $\mathbb{R}\times[0,\infty)$) to a weak solution $u(x,t)$ of the conservation law (1)–(2).

Our first lemma is a standard application of Theorem 4.2. We recall its proof for completeness.

Lemma 5.1. *Let $k_i \to 0$. There exist a function $u(x,t)$ and a subsequence $\{l_i\}_{i=1}^{\infty} \subseteq \{k_i\}_{i=1}^{\infty}$ such that, for every $X, T > 0$,*

$$\int_{-X}^{X} |\tilde{u}^{l_i}(x,t) - u(x,t)|dx \to 0, \qquad \text{for every } t \in [0,\infty), \tag{56}$$

and

$$\int_0^T \int_{-X}^{X} |\tilde{u}^{l_i}(x,t) - u(x,t)|dx\, dt \to 0, \tag{57}$$

as $i \to \infty$ ($l_i \to 0$). In addition, $|u(x,t)| \le \|u_0\|_\infty$ for $(x,t) \in \mathbb{R}\times[0,\infty)$ and

$$\tilde{u}^{l_i}(x,t) \to u(x,t), \tag{58}$$

almost everywhere in $\mathbb{R}\times[0,\infty)$.

Proof. By Corollary 4.7 and the compact imbedding of $BV(\mathbb{R})$ in $L^1_{loc}(\mathbb{R})$ (see [11, Ch. 2, Th. 1.2, P. 53]) and by a standard diagonal process we can find a dense sequence $\{t_j\}_{j=1}^{\infty} \subseteq [0,\infty)$ such that (56) is satisfied for all $t = t_j$, $X > 0$ (with a limit function $u(x,t_j)$). From (52) we infer that $\{\tilde{u}^{l_i}(\cdot,t)\}_{i=1}^{\infty}$ is a Cauchy sequence in $L^1_{loc}(\mathbb{R})$ for every $t \in [0,\infty)$, hence converges to a limit $u(x,t)$. By the maximum–minimum principle (Corollary 4.7), $|u(x,t)| \le \|u_0\|_\infty$ and (57) is derived from (56) by integration.

Finally, the pointwise convergence (58) is obtained from (58) by a further passage to a subsequence (and a diagonal process). □

We now show that the limit function $u(x,t)$ is a weak solution of (1)–(2), as in (3).

Theorem 5.2. *The limit function $u(x,t)$, obtained in Lemma 5.1, is a weak solution of (1) –(2) in the sense of (3).*

Proof. Let $\phi(x,t) \in C_0^3([-X,X]\times[0,T])$ for some $X,T>0$. Set $\phi_j^n = \phi(j\Delta x, n\Delta t)$ and multiply (2.1) by ϕ_j^n to obtain,

$$h\sum_{j,n}(v_j^{n+1}-v_j^n)\phi_j^n = -k\sum_{j,n}(f_{j+\frac12}^{n+\frac12}-f_{j-\frac12}^{n+\frac12})\phi_j^n. \tag{59}$$

(We use $h=\Delta x$, $k=\Delta t$, $\lambda = k/h$ as in (19). We rewrite (59) as

$$-h\sum_{j,n}v_j^{n+1}(\phi_j^{n+1}-\phi_j^n) - h\sum_{j,n}v_j^0\phi_j^0 = k\sum_{j,n}f_{j+\frac12}^{n+\frac12}(\phi_{j+1}^n-\phi_j^n). \tag{60}$$

In view of Proposition 3.12, the right–hand side of (60) can be estimated as follows,

$$k\sum_{j,n}f_{j+\frac12}^{n+\frac12}(\phi_{j+1}^n-\phi_j^n) = kh\sum_{j,n}\frac{1}{k}\int_{t_n}^{t_{n+1}} f(\tilde u^k(x_{j+\frac12},t))dt\cdot(\phi_x)_j^n+\delta_1, \tag{61}$$

where, by (34) and the uniform boundedness of $\{f_{j+\frac12}^{n+\frac12}\}_{j,n}$,

$$|\delta_1| \le Ck\cdot\sum_{j,n}\{[(\Delta v)_j^n]^2\cdot h|(\phi_x)_j^n| + h^2|\phi_{xx}(\xi_j^n,n\Delta t)|\} \tag{62}$$

with $j\Delta x \le \xi_j^n \le (j+1)\Delta x$. Since

$$\sum_{j,n}h^2|\phi_{xx}(\xi_j^n,n\Delta t)| \le C\|\phi_{xx}\|_\infty, \tag{63}$$

(where $C=C(\phi)$ is essentially $1/\lambda$ times the area of supp$\phi(x,t)$), we get, by (33),

$$\begin{aligned}|\delta_1| &\le Ck\{h\sum_{j,n}|(\Delta v)_j^n|\|\phi_x\|_\infty + \|\phi_{xx}\|_\infty\}\\ &\le Ck\{\sum_n h\cdot TV(v^n(x);\mathbb{R})\|\phi_x\|_\infty + \|\phi_{xx}\|_\infty)\}\\ &\le CTh\cdot\|\phi_x\|_\infty\cdot TV(u_0(x),\mathbb{R}) + Ck\|\phi_{xx}\|_\infty,\end{aligned} \tag{64}$$

(we have used Theorem 4.1 in the last step).

Recall (see (32)) that v_j^{n+1} is the exact average of $\tilde u^k(x,t_{n+1})$ over $[x_{j-\frac12},x_{j+\frac12}]$. Thus, the first term in the left–hand side of (60) can be written as,

$$\begin{aligned}-h\sum_{j,n}v_j^{n+1}(\phi_j^{n+1}-\phi_j^n) &= -\sum_{j,n}(\phi_j^{n+1}-\phi_j^n)\cdot\int_{x_{j-\frac12}}^{x_{j+\frac12}}\tilde u^k(x,t_{n+1}-)dx\\ &= -k\sum_{j,n}\int_{x_{j-\frac12}}^{x_{j+\frac12}}\tilde u^k(x,t_{n+1}-)dx\cdot(\phi_t)_j^{n+1}+\delta_2,\end{aligned} \tag{65}$$

where, by L^1 contraction property applied to $\tilde u^k(x,x)$,

$$|\delta_2| \le Ck^2\sum_n\int_{-X}^{X}|\tilde u^k(x,t_n)|dx \le CkTX. \tag{66}$$

Clearly, by (47), the second term in the left–hand side of (60) satisfies,

$$-h\sum_j v_j^0\phi_j^0 = -\int_{\mathbb{R}} u_0(x)\phi(x,0)dx + \delta_3, \tag{67}$$

where $\delta_3 \to 0$ as $h \to 0$.

Using (61), (65) and (67) in (60), we get

$$\begin{aligned} &-k\sum_{j,n}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \tilde{u}^k(x,t_{n+1}-)dx\cdot(\phi_t)_j^{n+1} - \int_{\mathbb{R}} u_0(x)\phi(x,0)dx \\ = \;& h\sum_{j,n}\int_{t_n}^{t_{n+1}} f(\tilde{u}^k(x_{j+\frac{1}{2}},t))dt\cdot(\phi_x)_j^n + \delta, \end{aligned} \tag{68}$$

where $\delta \to 0$ as $k \to 0$.

Taking a subsequence $l_i \to 0$ as in Lemma 5.1, we have

$$-l_i\sum_{j,n}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \tilde{u}^{l_i}(x,t_{n+1}-)dx\cdot(\phi_t)_j^{n+1} \to -\int_0^\infty\int_{\mathbb{R}} u(x,t)\phi_t dxdt. \tag{69}$$

On the other hand, fixing k, j, n,

$$\begin{aligned} &\left| h\int_{t_n}^{t_{n+1}} f(\tilde{u}^k(x_{j+\frac{1}{2}},t))dt - \int_{t_n}^{t_{n+1}}\int_{x_j}^{x_{j+1}} f(\tilde{u}^k(x,t))dxdt\right| \\ &\le C\int_{t_n}^{t_{n+1}}\int_{x_j}^{x_{j+1}} |\tilde{u}^k(x_{j+\frac{1}{2}},t) - \tilde{u}^k(x,t)|dxdt \\ &\le Ckh\cdot\sup_{t_n\le t\le t_{n+1}} TV(\tilde{u}^k(x,t);[x_j,x_{j+1}]), \end{aligned} \tag{70}$$

so that, by Theorem 4.1 and Lemma 5.1,

$$h\cdot\sum_{j,n}\int_{t_n}^{t_{n+1}} f(\tilde{u}^{l_i}(x_{j+\frac{1}{2}},t))dt\cdot(\phi_x)_j^n \to \int_0^\infty\int_{\mathbb{R}} f(u(x,t))\phi_x dxdt. \tag{71}$$

Thus, $u(x,t)$ satisfies (3) in view of (68), (69) and (71). □

The fact that the approximating solution $\tilde{u}^k$ is an exact solution (hence satisfies the "finite propagation property") and the fact that the reconstruction step does not increase the domain of dependence beyond an additional cell size h (see the discussion following Corollary 4.7) yield the following result.

Corollary 5.1. *The solution $u(x,t)$ satisfies the "finite propagation speed" in the following sense. Given $0 \le \tau < t \le T$, the value $u(x_0,t)$ depends only on the values $u(y,\tau)$, $\quad y \in [x_0 - \tilde{M}(t-\tau), x_0 + \tilde{M}(t-\tau)]$. In other words, changing the values of $u(y,\tau)$ for $|y-x_0| > \tilde{M}(t-\tau)$ does not change the value $u(x_0,t)$ in the process of the GRP approximation.*

6. **The entropy condition.** In this section we show that , in the case of a *monotone* initial data $\{v_j^0\}_j$, the weak solution obtained in Section 5 is indeed the unique entropy solution to (1)–(2). Note that the fact that the solution $\tilde{u}^k$ is an exact solution to (5) (in each time-step) implies that the sequence of averages $\{v_j^n\}_j$ is monotone for all n.

Recall that for a given $k = \Delta t > 0$ the solution constructed by the GRP method was denoted by $\tilde{u}^k(x,t)$ (see Notation 4), and its (piecewise linear) values at discretized time level $t = t_n = nk$ are $v^n(x) = \tilde{u}^k(x,t)$. The updated values $v_j^{n+1} = v^{n+1}(x_j) = \dfrac{1}{\Delta x}\displaystyle\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \tilde{u}^k(x, t_{n+1}-)dx$ are obtained by the GRP scheme (8), (36). As in Proposition 3.5, we set $(\Delta v)_j^n = \Delta x \cdot s_j^n$, where s_j^n is the (constant) slope of $v^n(x)$ in "cell j"$=(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$. We solve up to time $T = Nk$. The slopes s_j^n were obtained by the "Limiter Algorithm" (17). When all slopes s_j^n are zero, the scheme coincides with the Godunov scheme. In this case, the fluxes $\tilde{f}(\tilde{u}(x_{j+\frac{1}{2}}, t))$ (see (36)) are constant for $t_n \le t \le t_{n+1}$ and the ONLY RELEVANT requirement on $\tilde{f}$ is the zero–order part in (23), namely, that $\tilde{f}(v^n_{j+\frac{1}{2}}) = f(v^n_{j+\frac{1}{2}})$. We summarize this as follows.

Observation 6.1. *If all slopes $s_j^n = 0$ (in (4)), the scheme (2.1) coincides with the Godunov scheme for any choice of the approximate flux functions $\tilde{f}$, provided $\tilde{f}(v^n_{j+\frac{1}{1}}) = f(v^n_{j+\frac{1}{2}})$ for all $-\infty < j < \infty$. Here $v^n_{j+\frac{1}{2}} = R(0; v^n_{j+\frac{1}{2},\mp})$ (see (9)), the Riemann solution to the Riemann problem for (1), using the exact flux function f.*

Let $S(u)$ be a convex entropy function and let $F(u)$ be the corresponding entropy flux, namely, $F'(u) = U'(u)f'(u)$. The entropy condition for (1)-(2) is that, for every test function $0 \le \phi(x,t) \in C_0^\infty(\mathbb{R} \times (0,T))$,

$$-\int_{\mathbb{R}}\int_0^T [U(u(x,t))\phi_t(x,t) + F(u(x,t))\phi_x(x,t)]dxdt \le 0. \tag{72}$$

We proceed to show that the solution u obtained in Section 5 satisfies this condition, subject to the monotonicity assumption of the initial data. Let $\tilde{u}^k(x,t)$, be the approximate solution as defined in Notation 4. We consider its restriction to the strip (see the paragraph following Eq. (47)) $\Gamma_k^n = \mathbb{R} \times [t_n, t_n + k)$. As observed above, the sequence of averages $\{v_j^n\}_{-\infty<j<\infty}$ is monotone, so that there is only one chain (namely, in Definition 4.1, $j_1 = -\infty$ and $j_2 = \infty$). Then in Γ_k^n we have only one flux function $\tilde{f}$ and we can define the corresponding entropy flux by

$$\tilde{F}'(u) = U'(u)\tilde{f}'(u). \tag{73}$$

Since the solution $\tilde{u}^k(x,t)$, was constructed as an entropy solution, it satisfies the entropy inequality

$$\begin{aligned} &-\int_{\mathbb{R}}\int_{t_n}^{t_{n+1}} [U(\tilde{u}^k(x,t))\phi_t(x,t) + \tilde{F}(\tilde{u}^k(x,t))\phi_x(x,t)]dxdt \\ &+\int_{\mathbb{R}} U(\tilde{u}^k(x,t_{n+1}-))\phi(x,t_{n+1})dx - \int_{\mathbb{R}} U(v^n(x))\phi(x,t_n)dx \le 0. \end{aligned} \tag{74}$$

Note that the boundary terms result from the fact that ϕ does not (necessarily) vanish on the lines $\{t = t_n\}$ and $\{t = t_{n+1}\}$. Suppose next that the same monotonicity assumption is imposed in the strip Γ_k^{n+1}, so that we have the inequality (74) with n replaced by $n+1$, and with the appropriate functions $\tilde{f}$ and $\tilde{F}$. In the sequel we refer to these functions as $\tilde{f}$ and $\tilde{F}$, in the union $\Gamma_k^n \cup \Gamma_k^{n+1}$, by retaining their definition in each strip. Taking the sum of the two inequalities, we obtain

$$\begin{aligned}
&-\int_{\mathbb{R}}\int_{t_n}^{t_{n+2}}[U(\tilde{u}^k(x,t))\phi_t(x,t)+\tilde{F}(\tilde{u}^k(x,t))\phi_x(x,t)]dxdt\\
&+\int_{\mathbb{R}}U(\tilde{u}^k(x,t_{n+2}-))\phi(x,t_{n+2}dx\\
+\int_{\mathbb{R}}&[U(\tilde{u}^k(x,t_{n+1}-))-U(v^{n+1}(x))]\phi(x,t_{n+1})dx-\int_{\mathbb{R}}U(v^n(x))\phi(x,t_n)dx\le 0.
\end{aligned}\tag{75}$$

We now estimate the difference

$$\begin{aligned}
I=&\int_{\mathbb{R}}[U(\tilde{u}^k(x,t_{n+1}-))-U(v^{n+1}(x))]\phi(x,t_{n+1})dx\\
=&\sum_j\int_{x_{j-\frac12}}^{x_{j+\frac12}}[U(\tilde{u}^k(x,t_{n+1}-))-U(v_j^{n+1})]\phi_j^{n+1})dx\\
&-\sum_j\int_{x_{j-\frac12}}^{x_{j+\frac12}}[U(v^{n+1}(x))-U(v_j^{n+1})]\phi_j^{n+1}dx\\
&+\sum_j\int_{x_{j-\frac12}}^{x_{j+\frac12}}[U(\tilde{u}^k(x,t_{n+1}-))-U(v^{n+1}(x))](\phi(x,t_{n+1})-\phi_j^{n+1})dx,
\end{aligned}$$

where we have used the notation $\phi_j^{n+1}=\phi(x_j,t_{n+1})$. Recall that $hv_j^{n+1}=\int_{x_{j-\frac12}}^{x_{j+\frac12}}\tilde{u}^k(x,t_{n+1}-)dx$ so in view of the convexity of U, the Jensen inequality yields

$$\int_{x_{j-\frac12}}^{x_{j+\frac12}}[U(\tilde{u}^k(x,t_{n+1}-))-U(v_j^{n+1})]\phi_j^{n+1}dx\ge 0,$$

and we get

$$\begin{aligned}
I\ge&-\sum_j\int_{x_{j-\frac12}}^{x_{j+\frac12}}[U(v^{n+1}(x))-U(v_j^{n+1})]\phi_j^{n+1}dx\\
&+\sum_j\int_{x_{j-\frac12}}^{x_{j+\frac12}}[U(\tilde{u}^k(x,t_{n+1}-))-U(v^{n+1}(x))](\phi(x,t_{n+1})-\phi_j^{n+1})dx\\
=&\sum_j J_j^n+\sum_j K_j^n.
\end{aligned}\tag{76}$$

In what follows we use $C>0$ to represent any constant depending only on U, u_0, f, ϕ... but not on h, k.

To estimate K_j^n we use that $|\phi(x,t_{n+1})-\phi_j^{n+1}|\le Ch$, so that

$$|K_j^n|\le Ch|\int_{x_{j-\frac12}}^{x_{j+\frac12}}|\tilde{u}^k(x,t_{n+1}-)-v^{n+1}(x)|dx,$$

so that

$$\sum_j|K_j^n|\le Ch\int_{\mathbb{R}}|\tilde{u}^k(x,t_{n+1}-)-v^{n+1}(x)|dx\le Chk,\tag{77}$$

where in the second inequality we have used the result of Theorem 4.2.

We now proceed to estimate the sum of the J^n_j. Note that by construction (see Eq. (4))

$$|J^n_j| \le C s^{n+1}_j h^2.$$

Recall (Claim 3.4) that we keep fixed the ratio $\lambda = \frac{k}{h}$, so that

$$\sum_j |J^n_j| \le Chk \sum_j |s^{n+1}_j| \le CkTV(v^{n+1}(x); \mathbb{R}). \tag{78}$$

Assume next that the monotonicity assumption, imposed already in the union $\Gamma^n_k \cup \Gamma^{n+1}_k$, is valid in all the strips $\Gamma^n_k, \quad 0 < n \le N$. We can then sum over n to obtain

$$\sum_{j\in\mathbb{R}, 0\le n\le N} (|J^n_j| + |K^n_j|) \le Ch(1 + k \sum_{n=0}^{N} \sum_{j\in\mathbb{Z}} |s^n_j|). \tag{79}$$

Adding up the inequalities in (75), noting that ϕ vanishes on $\{t = 0\}$ and $\{t = T\}$ and using the estimate (79) we have

$$\begin{aligned} &- \int_{\mathbb{R}} \int_0^T [U(\tilde{u}^k(x,t))\phi_t(x,t) + \tilde{F}(\tilde{u}^k(x,t))\phi_x(x,t)] dxdt \\ &\le Ch(1 + k \sum_{n=0}^{N} \sum_{j\in\mathbb{Z}} |s^n_j|). \end{aligned} \tag{80}$$

Note that $\tilde{f}$, hence also $\tilde{F}$, are defined separately in each strip Γ^n_k.

We note that the right-hand side in (80) is bounded by $C(h + TV(u_0(x); \mathbb{R}))$ but it does not necessarily vanish as $h \to 0$.

We therefore need to restrict further the reconstructed slopes . This is done as follows.

Let $\eta > 0$ and denote

$$\Lambda_\eta = \{n, \quad 0 \le n \le N = k^{-1}T \quad \text{such that} \quad h \sum_j |s^n_j| > \eta\}.$$

We now impose the following hypothesis concerning the reconstructed slopes.

GODUNOV COMPATIBILITY HYPOTHESIS II: For any $\eta > 0$ the size of Λ_η is $o(h^{-1})$ as $h \to 0$.

Remark 6.1. Observe that in the Godunov scheme all slopes vanish. Thus, this hypothesis can be viewed as a further "limiter" on slopes (that can be checked in calculations). It is actually a "dissipative" mechanism, which ensures that shock profiles are sufficiently smeared so as to avoid "too many large slopes".

With this hypothesis, we can rewrite the estimate (80) as

$$\begin{aligned} &- \int_{\mathbb{R}} \int_0^T [U(\tilde{u}^k(x,t))\phi_t(x,t) + \tilde{F}(\tilde{u}^k(x,t))\phi_x(x,t)] dxdt \\ &\le C(h + \eta) + o(1), \quad \text{as} \quad h \to 0, \end{aligned} \tag{81}$$

for any $\eta > 0$, where $C > 0$ is independent of η.

Our next step is to replace the approximate entropy-flux $\tilde{F}$ by the exact entropy-flux F (we still retain the assumption on the monotonicity of the profiles). From (81) we have

$$
\begin{aligned}
&-\int_{\mathbb{R}}\int_0^T [U(\tilde{u}^k(x,t))\phi_t(x,t) + F(\tilde{u}^k(x,t))\phi_x(x,t)]dxdt \\
&\le \int_{\mathbb{R}}\int_0^T |F(\tilde{u}^k(x,t)) - \tilde{F}(\tilde{u}^k(x,t))|\phi_x(x,t)dxdt + C(h+\eta) + o(1), \quad \text{as} \quad h \to 0.
\end{aligned} \tag{82}
$$

Claim 6.2. *The approximate entropy-flux function $\tilde{F}$ can be constructed in such a way that*

$$\sup_{0\le n\le N}\ \sup_{(x,t)\in\Gamma_k^n} |F(\tilde{u}^k(x,t)) - \tilde{F}(\tilde{u}^k(x,t))| \to 0, \quad \textit{as} \quad h \to 0.$$

Proof. Recall that (see (73)) the approximate entropy flux is constructed separately in each strip Γ_k^n . Furthermore, at every value v_j^n, $j \in \mathbb{Z}$, we have $\tilde{F}'(v_j^n) = U'(v_j^n)\tilde{f}'(v_j^n) = F'(v_j^n)$. Since the sequence $\{v_j^n\}_{j\in\mathbb{Z}}$ is monotone, it suffices to consider the construction in an interval of the form $[v_j^n, v_{j+1}^n]$. Since $\{v_j^n\}_{j\in\mathbb{Z}} \subseteq [-M, M]$ we have , in view of (22)(iv), $|\tilde{F}'(v) - F'(v)| \le C$, hence

$$|\tilde{F}(v) - F(v)| \le C|v - v_j^n|, \quad v \in [v_j^n, v_{j+1}^n]. \tag{83}$$

Let $\varepsilon > 0$ be given. If $|v_j^n - v_{j+1}^n| \le \varepsilon$ then $|\tilde{F}(v) - F(v)| \le C\varepsilon$ for $v \in [v_j^n, v_{j+1}^n]$. If $|v_j^n - v_{j+1}^n| > \varepsilon$ we consider the construction of the approximate flux (see (22) and Figure 3.1) in the interval. Let $\delta > 0$ be such that $|f'(\alpha) - f'(\beta)| < \varepsilon$ when $|\alpha-\beta| < \delta$, $\alpha, \beta \in [-M, M]$. We then restrict r_0 (see Claim 3.2) so that $2Mr_0 < \delta$. Let $v_1 = v_j^n + r_0(v_{j+1}^n - v_j^n)$, $v_2 = v_{j+1}^n - r_0(v_{j+1}^n - v_j^n)$. Then $|f'(v) - f'(v_j^n)| = |f'(v) - \tilde{f}'(v)| < \varepsilon$, $v \in [v_j^n, v_1]$, and $|f'(v) - f'(v_{j+1}^n)| = |f'(v) - \tilde{f}'(v)| < \varepsilon$, $v \in [v_2, v_{j+1}^n]$. For every $w \in [v_1, v_2]$ we now set

$$\tilde{f}'(w) = f'(v_w), \quad v_w = v_j^n + \frac{1}{1-2r_0}(w - v_j^n).$$

It is clear that $|f'(v) - \tilde{f}'(v)| < \varepsilon$ for $v \in [v_j^n, v_{j+1}^n]$. This implies also that $|F'(v) - \tilde{F}'(v)| < C\varepsilon$ hence $|F(v) - \tilde{F}(v)| < C\varepsilon|v - v_j^n|$.

To conclude the proof, we use a diagonal process. We choose $\varepsilon(h) \to 0$ as $h \to 0$ and use the above construction (for mesh size h) with $\varepsilon = \varepsilon(h)$. □

Given this claim, and noting that (82) is valid with any $\eta > 0$, we obtain from Lemma 5.1 the following result.

Theorem 6.2. *Assume that the initial sequence of average values $\{v_j^0\}_{j\in\mathbb{Z}}$ is monotone. Assume further that the Godunov Compatibility Hypothesis II is satisfied by the reconstructed slopes $\{s_j^n\}_{j\in\mathbb{Z}}$. Then the weak solution u obtained in Theorem 5.2 is the unique entropy solution to (1)-(2), in the sense that it satisfies the inequalities*

$$-\int_{\mathbb{R}}\int_0^T [U(u(x,t))\phi_t(x,t) + F(u(x,t))\phi_x(x,t)]dxdt \le 0, \tag{84}$$

for every entropy entropy-flux pair U, F, and every nonnegative test function ϕ.

Corollary 6.3. *Let $u_0(x)$ be a general initial function (in the sense of Section 5). Fix an interval $I = [x_0 - \eta, x_0 + \eta]$, and let $0 < t \le T$. Then , if u_0 is monotone in the interval $J = [x_0 - \eta - \tilde{M}t, x_0 + \eta + \tilde{M}t]$, the solution $u(x,t)$ (obtained by the GRP approximation) has no "rarefaction shocks" in I.*

Proof. According to Corollary 5.1 we can modify the values of u_0 in J without affecting the values of the solution $u(x,t)$ in I. In particular, the modification can be such that the initial profile is globally monotone. We can then use Theorem 6.2. □

Acknowledgements. We would like to thank the referees very much for their valuable comments and suggestions.

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Received December 2007; revised May 2008.

E-mail address: mbartzi@math.huji.ac.il
E-mail address: ccjf@math.huji.ac.il
E-mail address: jiequan@mail.cnu.edu.cn