Explicit construction of measure solutions of Cauchy problem for transportation equations*

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Abstract The transportation equations are a mathematical model of zero-pressure flow in gas dynamics and the adhesion particle dynamics system to explain the formation of large scale structures in the universe. With the help of convex hull of a potential function, the solution is explicitly constructed here. It is straightforward to prove that the solution is a global measure one. And Dirac delta-shocks explained as the concentration of particles may develop in the solution.

Keywords: **transportation equations, Cauchy problem, measure solution, convex hull, Dirac delta-shocks.**

1 Motivation and main theorem

Let us consider the Cauchy problem for the Burgers equation[1]

$$\begin{cases} u_t + (u^2/2)_x = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $u_0(x) \in C^1(\mathbb{R}^1)$. Its smooth solution is woven by characteristic lines

$$\Gamma_t(x_0):\ x = x_0 + tu_0(x_0) = h(x_0;t), \tag{1.2}$$

where $h(\cdot\,;t)$ is a map from $\mathbb{R}^1$ to $\mathbb{R}^1$ for any fixed $t>0$.

In the simplest case where $u_0'(x_0) \geqslant 0$ $(x_0 \in \mathbb{R}^1)$, $\dfrac{\partial h(x_0;t)}{\partial x_0} = 1 + tu_0'(x_0) > 0$. Therefore the inverse of $h(x_0;t)$, $h^{-1}(x;t)$ exists, and is strictly increasing; in other words, $\Gamma_t(x_0)$ never intersect each other for all $t \geqslant 0$. Thus we obtain an explicit formula of solution of (1.1):

$$u(x,t) = u_0(h^{-1}(x;t)). \tag{1.3}$$

In general, $u_0'(x_0)$ may be less than zero at some point, say, $x_0 = \bar{x}_0$. This means that $\Gamma_t(x_0)(x_0 \in (\bar{x}_0 - \varepsilon, \bar{x}_0 + \varepsilon), \varepsilon > 0)$ must overlap in some region where the solution is multivalued; that is, $h(\cdot\,;t)$ is not one-to-one in this region. Hence we have to make the solution single-valued to obtain the global weak solution.

For a fixed $t \geqslant 0$, introduce a potential function of $h(x_0;t)$[2]:

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$$G(x_0;t) = \int^{x_0}(\eta + tu_0(\eta))\mathrm{d}\eta. \tag{1.4}$$

Obviously, $G(x_0;t)$ is convex for the simplest case mentioned above and

$$h(x_0;t) = \frac{\partial G(x_0;t)}{\partial x_0} \tag{1.5}$$

is increasing. However, the convexity of $G(x_0;t)$ fails once $u_0(x_0)$ is nonincreasing. Therefore, we consider the convex hull $\tilde{G}(x_0;t)$ of $G(x_0;t)$ at this moment. Then

$$x = \frac{\partial \tilde{G}(x_0;t)}{\partial x_0} = \tilde{h}(x_0;t) \tag{1.6}$$

gives a nondecreasing map, although $x_0 = \tilde{h}^{-1}(x, t)$ may have some discontinuous points. Instead of (1.3), the formula

$$u(x,t) = u_0(\tilde{h}^{-1}(x;t)) \tag{1.7}$$

just presents a weak solution of (1.1). Shocks may develop in solution (1.7), and the characteristic lines coming from the initial time $t = 0$ are cut off by shocks.

Now we turn to the transportation equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = 0, \end{cases} \tag{1.8}$$

where $\rho(x,t)$ and $u(x,t)$ represent the density and velocity of particles in the universe, respectively. This is a system of free particles sticking under collision, which can be used to describe the formation of large-scale structures in the universe[3]. It is noted that the characteristic lines of (1.8) are the same as those of (1.1). Instead of shocks, weighted Dirac delta-shocks develop in the solution of (1.8). These delta-shock waves are exactly interpreted as the galaxies in the universe. Since the density may develop a Dirac measure in a finite time, it is natural for us to seek for the solution of (1.8) in the sense of measures rather than in the class of bounded variation or boundedly measurable functions.

In two earlier papers[1),2)], the Riemann problem for (1.8) has been solved completely and the classical Rankine-Hugoniot condition of shocks has been generalized to those of Dirac delta-shocks to describe the relationship among the propagation speed, the location and the weight of delta-shocks. Ref. [4] studied the existence of solutions to the Cauchy problem by use of the so-called generalized variational principle and the method of discrete approximations when particles at the initial time move at continuous velocity.

The Cauchy problem for (1.8) is solved completely in the present paper. After introducing the convex hull of a potential function similar to (1.4), we succeed in constructing explicitly a global solution, which is then verified to be a measure solution directly. Let it be noted that we give up the restrict on the initial velocity and need not use the so-called generalized variational principle in ref. [4], and that the method adopted here can be extended to the multidimensional case[1)].

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1) Sheng, W., Zhang, T., The Riemann problem for transportation equations in gas dynamics, to appear in *Memoirs of American Mathematical Society*.

2) Li, J., Zhang, T., Generalized Rankine-Hugoniot conditions of weighted Dirac delta-shocks of transportation equations, Preprint, 1996.

Now let us state the main result in this paper. Let the initial data be

$$(u, M)(x, 0) = (u_0(x), \ M_0(x)), \tag{1.9}$$

where $M_0(x) \geqslant 0$ is the mass distribution at the initial time $t = 0$. We require that $u_0(x)$ be boundedly measurable with respect to $M_0(x)$ and the total mass $\int_{R^1} M_0(\mathrm{d}x)$ be finite. If Supp (M_0) is unbounded, the condition

$$\int_0^x \eta M_0(\mathrm{d}\eta) \to +\infty \quad \text{as } |x| \to +\infty \tag{1.10}$$

is assumed.

Next we present the definition of the measure solution of (1.8) and (1.9).

Definition 1. $(u(x, t), M(\Delta; t))$ is the measure solution of (1.8) with the initial data (1.9) if, for any ϕ, $\psi \in C_0^\infty(\mathbb{R}_+^2)$ $(\mathbb{R}_+^2 = \mathbb{R} \times [0, \infty))$, the equalities

$$\begin{cases} \displaystyle\iint_{\mathbb{R}_+^2} (\phi_t + u\phi_x) M(\mathrm{d}x; t)\mathrm{d}t + \int_{t=0} \phi(x, 0) M_0(\mathrm{d}x) = 0, \\ \displaystyle\iint_{\mathbb{R}_+^2} u(\psi_t + u\psi_x) M(\mathrm{d}x; t)\mathrm{d}t + \int_{t=0} u_0(x) \psi(x, 0) M_0(\mathrm{d}x) = 0 \end{cases} \tag{1.11}$$

hold where $M(\Delta; t) \geqslant 0$ is the mass distribution on the Borel measurable set Δ at moment t, and $u(x, t)$ is boundedly measurable with respect to $M(\Delta, t)$.

Theorem 1.1. *If the initial data* (1.9) *satisfies the conditions imposed, then there exists a measure solution* $(u, (x, t), \ M(\Delta, t))$ *of* (1.8) *and* (1.9) *satisfying* (1.11).

In the study of the two-dimensional Riemann problem for a simplified model of Euler system

$$\begin{cases} u_t + (u^2)_x + (uv)_y = 0, \\ v_t(uv)_x + (v^2)_y = 0, \end{cases} \tag{1.12}$$

ref. [5] found the delta-shock waves independently. Tan *et al*. proved that such a solution containing Delta-shocks satisfies one-dimensional case of (1.12) in the sense of measures and that this kind of waves is stable for viscous perturbations[6]. Furthermore, they proved the existence of solutions to the one-dimensional case of (1.12), in the class of piecewise smooth functions with piecewise convex or concave smooth initial data by introducing a potential function. For the first time, as far as we know, Ding and Wang completely solved the existence and uniqueness of solution to Cauchy problem in the one-dimensional case of (1.12) by use of Lebesgue-Stieltjes integral in reference [7].

2 Construction of solution

We will construct the solution of (1.8) and (1.9) in this section and study the properties of the solution for convenience in proving Theorem 1.1.

First of all, introduce two maps F: $\mathbb{R}^1 \to \mathbb{R}^1$ and B: $\mathbb{R}^1 \to \mathbb{R}^1$ defined as

$$F(x_0; t) = \int_A (\eta + t u_0(\eta)) M_0(\mathrm{d}\eta), \qquad z = B(x_0) = \int_{-\infty}^{x_0} M_0(\mathrm{d}\eta), \tag{2.1}$$

for any fixed t, where $A = [0, x_0)$ if $x_0 > 0$ or $A = [x_0, 0)$ if $x_0 < 0$. When there is no confusion, $F(x_0; t)$ is abbreviated to $F(x_0)$ later. Obviously, $B(x_0)$ is nondecreasing and then its inverse is denoted by $x_0 = x_0(z) = B^{-1}(z)$. Set

$$G(z) = F \circ x_0(z) = F \circ B^{-1}(z). \tag{2.2}$$

Here we supplement the definition of $G(z)$ at the interval $[z_1, z_2]$ on which $x = B^{-1}(z)$ takes a constant $\bar{x}$. Let z_1 and z_2 be the left and right limits of $B(x_0)$ at $\bar{x}$. Then

$$G(z) = F(x_0(z_1)) + \frac{F(x_0(z_2)) - F(x_0(z_1))}{(z_2 - z_1)} \cdot (z - z_1), \quad z \in (z_1, z_2). \tag{2.2'}$$

$G(z)$ is convex only when $u_0(x)$ is a nondecreasing function. At this moment, we observe that the slope of $G(z)$ is

$$x = \frac{\mathrm{d}G}{\mathrm{d}z} = x_0(z) + tu_0(x_0(z)), \qquad \text{a.e. with respect to } M_0, \tag{2.3}$$

where x is just the location of particle x_0 at time t. However, once $u_0(x)$ is not increasing, $G(z)$ is no longer convex, which means that particles must collide with and stick to each other after a finite time; in other words, delta-shocks develop in the solution. Therefore, we need to construct the convex hull of $G(z)$ denoted by $\tilde{G}(z)$

$$\tilde{G}(z) = \inf\left\{ y; (x, y) \in \Pi, \Pi = \bigcap_{\Omega \supset H(G, I)} \Omega, \Omega \text{ is a closed set} \right\}, \tag{2.4}$$

where $H(G, I) = \{(x, y); y > G(x), x \in I\}$, $I = \left[0, \int_{-\infty}^{+\infty} M_0 \mathrm{d}x)\right]$. Then $\tilde{G}(x)$ has the following properties.

(i)
$$\left.\frac{\mathrm{d}\tilde{G}(z)}{\mathrm{d}z}\right|_{z = B(-\infty)} < \left.\frac{\mathrm{d}\tilde{G}(z)}{\mathrm{d}z}\right|_{z = B(+\infty)}. \tag{2.5}$$

(ii) The straight line $\mathbb{R}^1$ consists of two kinds of sets $\mathbb{R}^1 = L_1 \cup L_2$, where $L_1 = \{x_0 \in \mathbb{R}^1; G(z(x_0)) = \tilde{G}(z(x_0))\}$. At almost every point x_0 of this set, there holds

$$\left.\frac{\mathrm{d}G(z)}{\mathrm{d}z}\right|_{z = B(x_0 - 0)} \leqslant \left.\frac{\mathrm{d}\tilde{G}(z)}{\mathrm{d}z}\right|_{z = B(x_0 - 0)} \leqslant \left.\frac{\mathrm{d}\tilde{G}(z)}{\mathrm{d}z}\right|_{z = B(x_0 + 0)} \leqslant \left.\frac{\mathrm{d}G(z)}{\mathrm{d}z}\right|_{z = B(x_0 + 0)}. \tag{2.6}$$

It follows that

$$\frac{\mathrm{d}G(z)}{\mathrm{d}z} = \frac{\mathrm{d}\tilde{G}(z)}{\mathrm{d}z} = x_0 + tu_0(x_0) \tag{2.7}$$

holds almost everywhere with respect to M_0 when $\left.\frac{\mathrm{d}G}{\mathrm{d}z}\right|_{z = B(x_0 - 0)} = \left.\frac{\mathrm{d}G}{\mathrm{d}z}\right|_{z = B(x_0 + 0)}$; while $L_2 = \{x_0 \in \mathbb{R}^1;\ G(z(x_0)) \neq \tilde{G}(z(x_0))\}$, which consists of disjoint intervals $L_2 = \bigcup_j \Omega_j$, $\Omega_i \cap \Omega_j = \emptyset, j \in \Lambda$, Λ being a countable set. The slope of $\tilde{G}(z)$ is a constant on each interval Ω_j and differs at different intervals. The slope of $\tilde{G}(z)$ at every interval is

$$\frac{\mathrm{d}\tilde{G}(z)}{\mathrm{d}z} = \frac{G(z_1) - G(z_2)}{z_1 - z_2} = \frac{\int_{(x_1, x_2)} (\eta + tu_0(\eta)) M_0(\mathrm{d}\eta)}{\int_{(x_1, x_2)} M_0(\mathrm{d}\eta)}, \tag{2.8}$$

where $z_1 = B(x_1)$, $z_2 = B(x_2)$, and x_1 and x_2 are the endpoints of the interval.

Since $\tilde{G}(z)$ has its left and right derivatives everywhere, it is reasonable to define a map $h(\cdot; t)$: $\mathbb{R}^1 \to \mathbb{R}^1$ by

$$x = h(x_0; t) = \left.\frac{\mathrm{d}\tilde{G}(z)}{\mathrm{d}z}\right|_{x = B(x_0)} \tag{2.9}$$

From the above discussion, we deduce that $h(\cdot;t)$ maps either a single point or an interval into a point. At a rate, $h(\cdot;t)$ is nondecreasing for a fixed $t\geqslant 0$; that is, if $x_0^1\leqslant x_0^2$, then $h(x_0^1,t)\leqslant h(x_0^2,t)$. Denote the inverse of $h(\cdot;t)$ by $h^{-1}(\cdot;t)$. By (2.9), if $\frac{\mathrm{d}\tilde{G}(z)}{\mathrm{d}z}$ is discontinuous at some point, say, at x_0, i.e. $x_1=\left.\frac{\mathrm{d}\tilde{G}(z)}{\mathrm{d}z}\right|_{z=B(x_0-0)}<x_2=\left.\frac{\mathrm{d}\tilde{G}(z)}{\mathrm{d}z}\right|_{z=B(x_0+0)}$, then we define

$$h^{-1}(x;t)=h^{-1}(x_2;t)\text{ for } x\in(x_1,x_2). \tag{2.9'}$$

Thus $h^{-1}(\cdot;t)$ maps a point into a single or an interval. Furthermore, we call $h^{-1}(x;t)$ a cell of $h(\cdot;t)$.

By a method completely analogous to that in ref. [2], we can prove the following.

Lemma 2.1. *Let x_1, $x_2\in\mathbb{R}^1$, for a fixed $t\geqslant 0$. If $h(x_1;t)\leqslant h(x_2;t)$, then $x_1\leqslant x_2$.*

Lemma 2.2. *For $0<t^*\leqslant t$, each cell of $h(\cdot;t^*)$ is contained in a cell of $h(\cdot;t)$.*

(2.7) and (2.9) show that if $h^{-1}(\cdot;t)$ maps a single point x_0 into a point x, then

$$x=x_0+tu_0(x_0),\ \text{a.e. with respect to } M_0; \tag{2.10}$$

if $h(\cdot;t)$ maps an interval (x_1,x_2) into a point x, then x is just the location of the center of mass of the particles on (x_1,x_2) at the moment t, i.e.

$$x=h(x_0;t)=\frac{\int_{(x_1,x_2)}(\eta+tu_0(\eta))M_0(\mathrm{d}\eta)}{\int_{(x_1,x_2)}M_0(\mathrm{d}\eta)},\quad x_0\in(x_1,x_2). \tag{2.11}$$

Lemma 2.3. *For a fixed $t\geqslant 0$ and x, if $x_0\in h^{-1}(x;t)$, then*

$$|x_0-x|\leqslant t\max_{x\in\mathbb{R}^1}|u_0(x)|. \tag{2.12}$$

Proof. If $h^{-1}(x;t)=\{x_0\}$, by (2.10), (2.12) obviously holds; if $h^{-1}(\cdot;t)$ maps x into an interval, let x_1 and x_2 be the left and right endpoints of the interval. Then

$$|x-x_0|\leqslant\max\{|x-x_1|,\ |x-x_2|\}. \tag{2.13}$$

It is easy to see that (2.12) holds.

Having made all the above analyses, we construct the measure solution of (1.8) and (1.9), which will be proved in the next section, to satisfy (1.11), as follows:

$$u(x,t)=\begin{cases}u_0(x_0),\ \text{a.e. with respect to } M_0, & \text{if } h^{-1}(x;t)=\{x_0\},\\[2mm] \dfrac{\int_{(x_1,x_2)}u_0(\eta)M_0(\mathrm{d}\eta)}{\int_{(x_1,x_2)}M_0(\mathrm{d}\eta)}, & \text{if } h^{-1}(x;t)=(x_1,x_2)\end{cases} \tag{2.14}$$

and

$$M(\Delta;t)=M_0(h^{-1}(\Delta;t)), \tag{2.15}$$

where $M(\Delta;t)$ is the mass distribution function on the Borel measurable set Δ at moment t.

3 Proof of Theorem 1.1

For the convenience of later statement, denote by $h(\cdot;t,t+\Delta_t)=h(h^{-1}(\cdot;t);t+\Delta_t)$

the maps from the location of particles at moment t into the location of the particle at time $t+\Delta_t$ $(\Delta_t \geqslant 0)$.

Lemma 3.1. *For all $0 \leqslant t^* \leqslant t$, if $h^{-1}(x;t)$ is an interval (x_1, x_2), then*

$$x = \frac{\int_{h((x_1,x_2);t^*)} (\eta + (t - t^*)u(\eta, t^*))M(\mathrm{d}\eta, t^*)}{\int_{h((x_1,x_2);t^*)} M(\mathrm{d}\eta, t^*)}. \tag{3.1}$$

Proof. From (2.11), we have

$$x = \frac{\int_{(x_1,x_2)} (\eta + tu_0(\eta))M_0(\mathrm{d}\eta)}{\int_{(x_1,x_2)} M_0(\mathrm{d}\eta)}, \tag{3.2}$$

which is rewritten as

$$x \cdot \int_{(x_1,x_2)} M_0(\mathrm{d}\eta) = \int_{(x_1,x_2)} (\eta + tu_0(\eta))M_0(\mathrm{d}\eta). \tag{3.2$'$}$$

From (2.14) and (2.15), we arrive at

$$\int_{(x_1,x_2)} M_0(\mathrm{d}\eta) = \int_{h((x_1,x_2);t^*)} M(\mathrm{d}\eta; t^*), \tag{3.3}$$

and

$$\int_{(x_1,x_2)} tu_0(\eta)M_0(\mathrm{d}\eta) = t\int_{h((x_1,x_2);t^*)} u(\eta;t^*)M(\mathrm{d}\eta;t^*). \tag{3.4}$$

The interval (x_1, x_2) can be divided into two kinds of sets D_1 and D_2. On the set D_1, $h(\cdot;t^*)$ is one-to-one, and

$$\eta = h(x_0;t^*) = x_0 + t^* u_0(x_0), \quad x_0 \in D_1 \tag{3.5}$$

and

$$u(\eta, t^*) = u_0(x_0), \quad x_0 \in D_1.$$

Therefore, we have

$$x_0 = \eta - t^* u(\eta, t^*), \quad x_0 \in D_1. \tag{3.6}$$

It follows that

$$\int_{D_1} x_0 M_0(\mathrm{d}x_0) = \int_{h(D_1;t^*)} (\eta - t^* u(\eta, t^*))M(\mathrm{d}\eta;t^*); \tag{3.7}$$

while D_2 can be expressed as $D_2 = \bigcup_j D_j^*$, $D_i^* \cap D_j^* = \emptyset (i \neq j)$, $h(\cdot;t^*)$ maps every D_j^* into a single point, which is denoted by $(D_j^*;t^*)$. Then

$$x_j^* = \frac{\int_{D_j^*} (x_0 + t^* u_0(x_0))M_0(\mathrm{d}x_0)}{\int_{D_j^*} M_0(\mathrm{d}x_0)} = \frac{\int_{D_j^*} x_0 M_0(\mathrm{d}x_0) + t^* \int_{D_j^*} u_0(x_0)M_0(\mathrm{d}x_0)}{\int_{D_j^*} M_0(\mathrm{d}x_0)}, \tag{3.8}$$

from which we obtain

$$\begin{aligned}\int_{D_j^*} x_0 M_0(\mathrm{d}x_0) &= x_j^* \int_{D_j^*} M_0(\mathrm{d}x_0) - t^* \int_{D_j^*} u_0(x_0)M_0(\mathrm{d}x_0) \\ &= x_j^* M(x_j^*;t^*) - t^* \int_{h(D_j^*)} u(\eta;t^*)M(\mathrm{d}\eta;t^*)\end{aligned}$$

$$= (x_j^* - t^* u(x_j^*;t^*))M(x_j^*;t^*). \tag{3.9}$$

So

$$\int_{D_2} x_0 M_0(\mathrm{d}x_0) = \sum_j \int_{D_j^*} x_0 M_0(\mathrm{d}x_0) = \sum_j (x_j^* - t^* u(x_j^*;t^*))M(x_j^*;t^*). \tag{3.10}$$

Thus we calculate

$$\int_{(x_1,x_2)} x_0 M_0(\mathrm{d}x_0) = \int_{D_1} x_0 M_0(\mathrm{d}x_0) + \int_{D_2} x_0 M_0(\mathrm{d}x_0)$$

$$= \int_{h(D_1;t^*)} (\eta - t^* u(\eta,t^*))M(\mathrm{d}\eta;t^*) + \sum_j (x_j^* - t^* u(x_j^*))M(x_j^*;t^*)$$

$$= \int_{h((x_1,x_2);t^*)} (\eta - t^* u(\eta;t^*))M(\mathrm{d}\eta;t^*). \tag{3.11}$$

By combining it with (3.2′), the proof of this lemma is completed.

Lemma 3.2. *Let $x^* \in h(h^{-1}(x;t);t^*)$ $(0 \leqslant t^* \leqslant t)$. Then*

$$|x - x^*| \leqslant \max_{x \in \mathbb{R}^1} |u(x,t^*)|(t - t^*). \tag{3.12}$$

The proof of this lemma can be obtained without any substantial difficulty, and therefore it is omitted.

Lemma 3.3. *Let $\tilde{u}(t) = \max_{x \in \mathbb{R}^1} |u(x,t)|$. Then $\tilde{u}(t)$ is nonincreasing.*

Proof. For $x \in \mathbb{R}^1$, if $h(h^{-1}(x;t);t)$ is a single x^*, then $u(x,t) = u(x^*,t^*)$; otherwise, since

$$u(x,t)M(x,t) = \int_{h(h^{-1}(x;t);t^*)} u(\eta,t^*)M(\mathrm{d}\eta;t^*), \tag{3.13}$$

we have

$$|u(x,t)|M(x,t) \leqslant \max_{x \in h(h^{-1}(x;t);t^*)} |u(x,t^*)| \int_{h(h^{-1}(x;t);t^*)} M(\mathrm{d}\eta;t^*). \tag{3.14}$$

It follows from (2.15) that

$$|u(x,t)| \leqslant \max_{x \in h(h^{-1}(x;t);t^*)} |u(x,t^*)|. \tag{3.15}$$

Now let us prove Theorem 1.1. At the outset we will prove that the solution constructed in (2.14) and (2.15) satisfies the first equality of (1.11).

Let $\phi(x,t) \in C_0^\infty(\mathbb{R}_+^2)$, because of the compactness of $\phi(x,t)$, for a sufficiently large time T and an interval (a,b). Then there holds

$$\int_{(a,b)} (\phi(x,T)M(\mathrm{d}x;T) - \phi(x,0)M_0(\mathrm{d}x)) + \int_{(a,b)} \phi(x,0)M_0(\mathrm{d}x) = 0. \tag{3.16}$$

Therefore, it suffices to show that the following equality holds:

$$\int_{(a,b)} (\phi(x,T)M(\mathrm{d}x;T) - \phi(x,0)M_0(\mathrm{d}x)) = \int_0^T \int_{(a,b)} (\phi_t + u\phi_x)M(\mathrm{d}x;t)\mathrm{d}t. \tag{3.17}$$

We mesh $[0,T]$ by n parts $[t_i, t_{i+1}]$ $(i = 0,1,\cdots,n-1)$, where $t_0 = 0$, $t_n = T$ and denote $\Delta t_i = t_{i+1} - t_i$. Then (3.17) becomes

$$\sum_{i=0}^{n-1}\int_{(a,b)}(\phi(x,t_{i+1})M(\mathrm{d}x;t_{i+1})-\phi(x,t_i)M(\mathrm{d}x;t_i))$$
$$=\sum_{i=0}^{n-1}\Big[\int_{(a,b)}(\phi(x,t_{i+1})-\phi(x,t_i))M(\mathrm{d}x;t_{i+1})$$
$$+\int_{(a,b)}(\phi(x,t_i)M(\mathrm{d}x;t_{i+1})-\phi(x,t_i)M(\mathrm{d}x;t_i))\Big]$$
$$=\sum_{i=0}^{n-1}\Big[\int_{(a,b)}\phi_t(x,\bar{t}_i)M(\mathrm{d}x;t_{i+1})\cdot\Delta t_i$$
$$+\int_{(a,b)}(\phi(h(x;t_i,t_{i+1}),t_i)-\phi(x,t_i))M(\mathrm{d}x;t_i)\Big]$$
$$=J_1+J_2, \tag{3.18}$$
where $\bar{t}_i\in(t_i,t_{i+1})$. Since
$$\int_{(a,b)}\phi_t(x,t+\Delta t)M(\mathrm{d}x;t+\Delta t)-\phi_t(x,t)M(\mathrm{d}x;t)$$
$$=\int_{(a,b)}(\phi_t(h(x;t,t+\Delta t),t)-\phi_t(x,t))M(\mathrm{d}x;t)$$
$$=\int_{(a,b)}\phi_{tt}(x+\theta(h(x;t,t+\Delta t)-x),t)(h(x;t,t+\Delta t)-x)M(\mathrm{d}x;t), \tag{3.19}$$
where $\theta\in(0,1)$, and
$$h(x;t,t+\Delta t)-x\to 0 \quad\text{as}\quad \Delta t\to 0, \tag{3.20}$$
we assert that $\int_{(a,b)}\phi_t(x,t)M(\mathrm{d}x;t)$ is continuous with respect to t, and
$$J_1\to\int_0^T\int_{(a,b)}\phi_t(x,t)M(\mathrm{d}x;t),\quad\text{as}\quad n\to\infty\quad\text{and}\quad \max_i(\Delta t_i)\to 0. \tag{3.21}$$
For J_2, we have
$$J_2=\sum_{i=0}^{n-1}\int_{(a,b)}\phi_x(\bar{x},t_i)(h(x;t_i,t_{i+1})-x)M(\mathrm{d}x,t_i), \tag{3.22}$$
where $\bar{x}=x+\theta(h(x;t_i,t_{i+1})-x)(0\leqslant\theta\leqslant 1)$. The interval (a,b) can be written as the union of two kinds of sets. On the first kind of set Σ_1, $h(\cdot;t_i,t_{i+1})$ is one-to-one; while the second of set Σ_2 is the union of disjoint intervals Δ_j; that is, $\Sigma_2=\bigcup_j\Delta_j$, $\Delta_i\cap\Delta_j=\emptyset(i\neq j)$, and $h(\cdot;t_i,t_{i+1})$ maps each of them into a single point. Therefore,
$$J_2=\sum_{i=0}^{n-1}\Big[\int_{\Sigma_1}\phi_x(\bar{x},t_i)(h(x;t_i,t_{i+1})-x)M(\mathrm{d}x;t_i)$$
$$+\int_{\Sigma_2}\phi_x(\bar{x},t_i)(h(x;t_i,t_{i+1})-x)M(\mathrm{d}x;t_i)\Big].$$
$$=J_{21}+J_{22}. \tag{3.23}$$
For J_{21}, we have obviously
$$J_{21}=\sum_{i=0}^{n-1}\int_{\Sigma_1}\phi_x(\bar{x},t_i)u(x,t_i)M(\mathrm{d}x;t_i)\Delta t_i. \tag{3.24}$$
And we have for J_{22}, with (2.11),
$$J_{22}=\sum_{i=0}^{n-1}\sum_j\int_{\Delta_j}\phi_x(\bar{x},t_i)\frac{\int_{\Delta_j}(\eta-x)M(\mathrm{d}\eta;t_i)}{\int_{\Delta_j}M(\mathrm{d}\eta;t_i)}\cdot M(\mathrm{d}x;t_i)$$

$$+\sum_{i=0}^{n-1}\sum_{j}\int_{\Delta_j}\phi_x(\bar{x},t_i)\left(\frac{\int_{\Delta_j}u(\eta,t_i)M(\mathrm{d}\eta;t_i)}{\int_{\Delta_j}M(\mathrm{d}\eta;t_i)}-u(x,t_i)\right)\cdot M(\mathrm{d}x;t_i)\Delta t_i$$

$$+\sum_{i=0}^{n-1}\sum_{j}\int_{\Delta_j}\phi_x(\bar{x},t_i)u(x,t_i)\cdot M(\mathrm{d}x;t_i)\Delta t_i$$

$$=J_{22}^{(1)}+J_{22}^{(2)}+J_{22}^{(3)},\tag{3.25}$$

where Δ_j is a cell containing x. Now we need to estimate $J_{22}^{(1)}$ and $J_{22}^{(2)}$. First, we prove that $J_{22}^{(1)}$ tends to zero as $\max_i(\Delta t_i)$ goes to zero. Because for all $\xi,\ \eta\in\Delta_j$, there holds

$$|n-\xi|\leqslant|\Delta_j|\leqslant 2\max_{x\in\Delta_j}|u(x,t_i)|\cdot\Delta t_i\leqslant 2\max_{x\in\mathbb{R}^1}|u_0(x)|\cdot\Delta t_i=C_1\Delta t_i,\tag{3.26}$$

where $|\Delta_j|$ is the measure of Δ_j, $C_1=\max_{x\in\mathbb{R}^1}|u_0(x)|$, we obtain

$$J_{22}^{(1)}=\sum_{i=0}^{n-1}\sum_{j}\frac{1}{2\int_{\Delta_j}M(\mathrm{d}\eta;t_i)}\int_{\Delta_j}\int_{\Delta_j}(\phi_x(\bar{x},t_i)(\eta-x)+\phi_x(\bar{\eta},t_i)(x-\eta))$$
$$\cdot M(\mathrm{d}\eta;t_i)M(\mathrm{d}x;t_i)$$
$$=\sum_{i=0}^{n-1}\sum_{j}\frac{1}{2\int_{\Delta_j}M(\mathrm{d}\eta;t_i)}\int_{\Delta_j}\int_{\Delta_j}(\phi_x(\bar{x},t_i)-\phi_x(\bar{\eta},t_i))(\eta-x)$$
$$\cdot M(\mathrm{d}\eta;t_i)\cdot M(\mathrm{d}x;t_i)$$
$$=\sum_{i=0}^{n-1}\sum_{j}\frac{1}{2\int_{\Delta_j}M(\mathrm{d}\eta;t_i)}\int_{\Delta_j}\int_{\Delta_j}\phi_{xx}(\tilde{x},t_i)(\bar{\eta}-\bar{x})(\eta-x)$$
$$\cdot M(\mathrm{d}\eta;t_i)\cdot M(\mathrm{d}x;t_i),\tag{3.27}$$

where $\bar{\eta}=\eta+\theta(h(\eta;t_i)-\eta))$ $(\theta\in(0,1)$, $\tilde{x}\in(\bar{x},\bar{\eta})$. So

$$|J_{22}^{(1)}|=\sum_{i=0}^{n-1}\sum_{j}\frac{1}{2\int_{\Delta_j}M(\mathrm{d}\eta;t_i)}\int_{\Delta_j}\int_{\Delta_j}|(\phi_{xx}(\tilde{x},t_i)||\bar{\eta}-\bar{x}||\eta-x|$$
$$\cdot M(\mathrm{d}\eta;t_i)\cdot M(\mathrm{d}x;t_i)$$
$$\leqslant\sum_{i=0}^{n-1}\sum_{j}\frac{C_1^2C_2(\Delta t_i)^2}{\int_{\Delta_j}M(\mathrm{d}\eta,t_i)}\int_{\Delta_j}\int_{\Delta_j}\cdot M(\mathrm{d}\eta;t_i)M(\mathrm{d}x;t_i)$$
$$=\sum_{i=0}^{n-1}\frac{C_1^2C_2}{2}\int_{\Sigma_2}M(\mathrm{d}\eta;t_i)(\Delta t_i)^2$$
$$\leqslant\frac{C_1^2C_2TC_3}{2}\cdot\max_i(\Delta t_i),\tag{3.28}$$

where $C_2=\max_{x\in(a,b)}|\phi_{xx}(x,t)|$, $C_3=\int_{(a,b)}M_0(\mathrm{d}\eta)$. Obviously, $J_{22}^{(1)}$ tend to zero as $\max_i(\Delta t_i)$ goes to zero.

Next we prove that $J_{22}^{(2)}$ also tends to zero as $\max_i(\Delta t_i)$ goes to zero. Similar to the estimate of $J_{22}^{(1)}$, we have

$$|J_{22}^{(2)}| = \left| \sum_{i=0}^{n-1} \sum_j \frac{1}{2\int_{\Delta_j} M(\mathrm{d}\eta; t_i)} \int_{\Delta_j}\int_{\Delta_j} (\phi_x(\bar{\eta}, t_i) - \phi_x(\bar{x}, t_i))(u(\eta, t_i) - u(x, t_i)) \right.$$

$$\left. \cdot M(\mathrm{d}\eta, t_i) \cdot M(\mathrm{d}x; t_i)\Delta t_i \right|$$

$$\leqslant \sum_{i=0}^{n-1} \sum_j \frac{1}{2\int_{\Delta_j} M(\mathrm{d}\eta, t_i)} \int_{\Delta_j}\int_{\Delta_j} |\phi_{xx}(\tilde{x}, t_i)| \, |u(\eta, t_i) - u(x, t_i)| \, |\bar{\eta} - \bar{x}|$$

$$\cdot M(\mathrm{d}\eta, t_i) \cdot M(\mathrm{d}x; t_i)\Delta t_i$$

$$\leqslant \sum_{i=0}^{n-1} \sum_j \frac{C_1^2 C_2}{\int \Delta_j M(\mathrm{d}\eta, t_i)} \int_{\Delta_j}\int_{\Delta_j} M(\mathrm{d}\eta; t_i) \cdot M(\mathrm{d}x; t_i)(\Delta t_i)^2$$

$$\leqslant C_1^2 C_2 T C_3 \max_i(\Delta t_i). \tag{3.29}$$

Combining (3.23—3.25) and (3.28—3.29), we arrive at

$$J_2 \to \int_0^T \int_{(a,b)} \phi_x(x, t) u(x, t) M(\mathrm{d}x; t)\mathrm{d}t, \quad \text{as} \quad n \to \infty \text{ and } \max_i(\Delta t_i) \to 0. \tag{3.30}$$

Before verifying that (2.14) and (2.15) satisfy the second equality of (1.11), we introduce the momentum $I(\Delta; t)$

$$I(\Delta; t) = \int_\Delta u(\eta, t) M(\mathrm{d}\eta; t), \tag{3.31}$$

where $I(\Delta; t)$ can be understood as a Radon measure. Then we have

$$I(\Delta; t_{i+1}) = \int_{h^{-1}(\Delta; t_i, t_{i+1})} u(\eta, t_i) M(\mathrm{d}\eta; t_i)$$

$$= I(h^{-1}(\Delta; t_i, t_{i+1}); t_i). \tag{3.32}$$

Therefore, for a test function $\psi \in C_0^\infty(\mathbb{R}_+^2)$, there is

$$0 = \int_{(a,b)} (\psi, (x, T) u(x, T) M(\mathrm{d}x; T) - \psi(x, 0) u_0(x) M_0(\mathrm{d}x))$$

$$+ \int_{(a,b)} u_0(x) \psi(x, 0) M_0(\mathrm{d}x)$$

$$= \int_{(a,b)} \psi, (x, T) I(\mathrm{d}x; T) - \psi(x, 0) I(\mathrm{d}x; 0) + \int_{(a,b)} \psi(x, 0) I(\mathrm{d}x; 0)$$

$$= H + \int_{(a,b)} \psi(x, 0) I(\mathrm{d}x; 0). \tag{3.33}$$

Hence, it is sufficient to verify that

$$H = \int_0^T \int_{(a,b)} u(\psi_t + u\psi_x) M(\mathrm{d}x; t)\mathrm{d}t. \tag{3.34}$$

Similar to the method adopted above, we divided H into two parts:

$$H = H_1 + H_2, \tag{3.35}$$

where

$$H_1 = \sum_{i=0}^{n-1} \int_{(a,b)} \psi_t(x, \bar{t}_i) I(\mathrm{d}x; t_{i+1})\Delta t_i$$

$$\to \int_0^T \int_{(a,b)} \psi_t(x, t) u(x, t) M(\mathrm{d}x; t), \quad \text{as} \quad n \to +\infty \text{ and } \max_i(\Delta t_i) \to 0, \tag{3.36}$$

and

$$H_2 = \sum_{i=0}^{n-1}\left[\int_{\Sigma_1}\varphi_x(\bar{x}, t_i)u(x, t_i)\Delta t_i I(\mathrm{d}x; t_i) + \int_{\Sigma_2}\varphi_x(\bar{x}, t_i)(h(x; t_i, t_{i+1}) - x)I(\mathrm{d}x; t_i)\right] = H_{21} + H_{22}, \tag{3.37}$$

where $\bar{x}_i \in (x, h(x; t_i, t_{i+1}))$. Since

$$H_{22} = \sum_{i=0}^{n-1}\left[\int_{\Delta_j}\varphi_x(\bar{x}, t_i)\frac{\int_{\Delta_j}(\xi - x)M(\mathrm{d}\xi; t_i)}{\int_{\Delta_j}M(\mathrm{d}\xi; t_i)}I(\mathrm{d}x; t_i) + \int_{\Delta_j}\varphi_x(\bar{x}, t_i)\frac{\int_{\Delta_j}(u(\xi, t_i) - u(x, t_i))M(\mathrm{d}\xi; t_i)}{\int_{\Delta_j}M(\mathrm{d}\xi; t_i)}I(\mathrm{d}x; t_i)\cdot\Delta t_i + \int_{\Delta_j}\varphi_x(\bar{x}, t_i)u(x, t_i)I(\mathrm{d}x; t_i)\Delta t_i\right]$$
$$= H_{22}^{(1)} + H_{22}^{(2)} + H_{22}^{(3)}, \tag{3.38}$$

it follows that as $n \to +\infty$ and $\max_n(\Delta t_i) \to 0$,

$$H_{21} + H_{22}^{(3)} \to \int_0^T\int_{(a,b)}\varphi_x(x, t)u^2(x, t)M(\mathrm{d}x; t)\mathrm{d}t. \tag{3.39}$$

In what follows, it is necessary to prove that as $n \to +\infty$ and $\max_n(\Delta t_i) \to 0$, $H_{22}^{(1)}$ and $H_{22}^{(2)}$ all tend to zero.

$$H_{22}^{(1)} = \sum_{i=0}^{n-1}\sum_j \frac{1}{\int_{\Delta_j}M(\mathrm{d}\xi; t_i)}\int_{\Delta_j}\varphi_x(\bar{x}, t_i)(\xi - x)\cdot u(x, t_i)M(\mathrm{d}\xi; t_i)M(\mathrm{d}x; t_i)$$
$$= \sum_{i=0}^{n-1}\sum_j \frac{1}{2\int_{\Delta_j}M(\mathrm{d}\xi; t_i)}\int_{\Delta_j}(\varphi_x(\bar{x}, t_i) - \varphi_x(\bar{\xi}_i, t_i))(\xi - x)\cdot u(x, t_i)\cdot M(\mathrm{d}\xi; t_i)M(\mathrm{d}x; t_i)$$
$$+ \sum_{i=0}^{n-1}\sum_j \frac{1}{2\int_{\Delta_j}M(\mathrm{d}\xi; t_i)}\int_{\Delta_j}\varphi_x(\bar{x}, t_i)(\xi - x)\cdot(u(x, t_i) - u(\xi, t_i))\cdot M(\mathrm{d}\xi; t_i)M(\mathrm{d}x; t_i)$$
$$= H_{22}^{(11)} + H_{22}^{(12)}. \tag{3.40}$$

Then we can prove that $H_{22}^{(11)}$ tends to zero as $n \to +\infty$ and $\max_i(\Delta t_i) \to 0$ with a method similar to the estimate of $J_{22}^{(2)}$. The difficulty lies in the estimate of $H_{22}^{(12)}$. In order to overcome this difficulty, we give a well-known fact.

Lemma 3.4. *Let $\mu > 0$ be a measure on the domain Ω, and let $f(x)$ be a boundedly measurable function, defined on Ω, with respect to μ. Then for arbitrary $\varepsilon > 0$, there exists a* *continuous function* $g(x)$ *such that*

$$\int_{\Omega} | f(x) - g(x) | \mathrm{d}\mu < \varepsilon \tag{3.41}$$

holds.

Now we continue the proof of Theorem 1. Note that for $t > 0$, $u(x, t)$ is boundedly measurable with respect to $M(x, t)$. Then we deduce, by Lemma 3.4, that for all $\varepsilon > 0$, there exists a continuous function $g_t(x)$ dependent on t such that

$$\int_{\Omega} | u(x, t) - g_t(x) | M(\mathrm{d}x; t) < \varepsilon. \tag{3.42}$$

Define

$$s(t) = \int_{\Omega} | u(x, t) - g_t(x) | M(\mathrm{d}x; t). \tag{3.43}$$

Then $s(t)$ is continuous. It follows that for all $t > 0$, there is a neighborhood of $(t - \Delta_t, t + \Delta_t)$ $(\Delta_t > 0)$ such that (3.42) holds on it. Denote $\tau_t = (t - \Delta_t, t + \Delta_t)$, and the corresponding $g_t(x)$ by $g_{\tau_t}(x)$. Then $[0, T] = \bigcup_{t \in [0, T]} \tau_t$. Therefore, by Haire-Borel theorem, there exist τ_{t_1}, $\tau_{t_2}, \cdots, \tau_{t_m}$ such that

$$[0, T] \in \bigcup_{k=1}^{m} \tau_{t_k}. \tag{3.44}$$

Hence, when n is sufficiently large and $\max_i(\Delta t_i)$ is sufficiently small, we define $p(x, t)$ as

$$p(x, t) = g_{\tau_{t_k}}(x, y), \quad t \in \tau_{t_k} \quad (k = 1, 2, \cdots, m). \tag{3.45}$$

Then $p(x, t)$ has the following properties.

Lemma 3.5. *For all* $\varepsilon > 0$, *there exists a* $\Delta > 0$ *such that when* $|P - Q| < \Delta$ *for two points* P *and* Q *on* (a, b),

$$| p(P, t) - p(Q, t) | < \varepsilon \tag{3.46}$$

holds for all $t \in [0, T]$; *in other words*, $p(x, t)$ *is uniformly continuous for all* $t \in [0, T]$.

Based on Lemma 3.5, we have, for $H_{22}^{(12)}$,

$$\begin{aligned} H_{22}^{(12)} = & \sum_{i=0}^{n-1} \sum_{j} \frac{1}{2\int_{\Delta_j} M(\mathrm{d}\xi; t_i)} \int_{\Delta_j} \psi_x(\bar{x}, t_i)(\xi - x) \cdot (u(x, t_i) - p(x, t_i)) \\ & \cdot M(\mathrm{d}\xi; t_i) M(\mathrm{d}x; t_i) \\ & + \sum_{i=0}^{n-1} \sum_{j} \frac{1}{2\int_{\Delta_j} M(\mathrm{d}\xi; t_i)} \int_{\Delta_j} \psi_x(\bar{x}, t_i)(\xi - x) \cdot (p(x, t_i) - p(\xi, t_i)) \\ & \cdot M(\mathrm{d}\xi; t_i) M(\mathrm{d}x; t_i) \\ & + \sum_{i=0}^{n-1} \sum_{j} \frac{1}{2\int_{\Delta_j} M(\mathrm{d}\xi; t_i)} \int_{\Delta_j} \psi_x(\bar{x}, t_i)(\xi - x) \cdot (p(\xi, t_i) - u(\xi, t_i)) \\ & \cdot M(\mathrm{d}\xi; t_i) M(\mathrm{d}x; t_i) \\ = & H_{22}^{(121)} + H_{22}^{(122)} + H_{22}^{(123)}. \end{aligned} \tag{3.47}$$

According to the construction of $p(x, t)$, we obtain

$$| H_{22}^{(12i)} | < C\varepsilon, i = 1, 2, 3, \tag{3.48}$$