Journal of Differential Equations **179**, 178–194 (2002)
doi:10.1006/jdeq.2001.4025, available online at http://www.idealibrary.com on IDEAL®

Global Solution of an Initial-Value Problem for Two-Dimensional Compressible Euler Equations

Jiequan Li

Institute of Mathematics, The Chinese Academy of Sciences, Beijing, 100080, People's Republic of China; and Institute of Mathematics, The Hebrew University of Jerusalem, 91904, Givat Ram Compus, Israel
E-mail: jiequan@math.huji.ac.il

Received March 22, 2000; revised December 4, 2000

This paper is concerned with the existence of global continuous solutions of the expansion of a wedge of gas into a vacuum for compressible Euler equations. By hodograph transformation, we first prove that the flow is governed by a partial differential equation of second order, which is further reduced to a system of two nonhomogeneous linearly degenerate equations in the phase space under an irrotationality condition. Then this conclusion is applied to solving the problem that a wedge of gas expands into a vacuum, which is actually a Goursat-type problem for these two equations in the supersonic domain. © 2002 Elsevier Science

Key Words: two-dimensional gas expansion; compressible Euler equations; the existence of global continuous solutions; linearly degenerate equations; rarefaction waves.

1. INTRODUCTION

It is well known that the two-dimensional motion of inviscid ideal gas without heat conduction is governed by Euler equations,

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y = 0, \\ (\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y = 0, \\ (\rho E)_t + ((\rho E + p)\, u)_x + ((\rho E + p)\, v)_y = 0, \end{cases} \tag{1.1}$$

where ρ and p are density and pressure, u and v are the components of velocity in x and y-directions, respectively, $E = (u^2 + v^2)/2 + e$ is the total

0022-0396/02 $35.00

energy, and e is the internal energy, as usual. In this study we are concerned with the smooth flow and take the isentropic case. Then the flow is described by isentropic Euler equations,

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0, \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = 0, \end{cases} \tag{1.2}$$

where $p(\rho) = A\rho^\gamma$ for polytropic gases with index $\gamma \geqslant 1$ and $A > 0$ is entropy, being assumed to be unit by some suitable choice of gauge. The first equation is the law of mass conservation, the second and third are the law of momentum conservation. We further restrict to the case $\gamma = 1$, which can be regarded as the approximation of isentropic flow with small $\gamma > 1$. That is, $p(\rho) = \rho$.

Systems (1.2) is subject to some initial conditions. In many applications and/or computations, the initial data for (1.2) are sectorial constant. The oblique shock reflection experiments [BG] and the expansion of a wedge of gas into the vacuum [Le, M] are our familiar examples and the natural extension of one-dimensional Riemann problem into two-dimensional ones [K, LZY, Z]. Under such circumstances, self-similar solutions of the form $(\rho, u, v)(t, x, y) = (\rho, u, v)(\xi, \eta)$, $\xi = x/t, \eta = y/t$, are taken into account because the equations and the initial data are invariant under the dilation $(t, x, y) \to (ct, cx, cy)$, $c > 0$, so that the dimension is reduced from three to two by this self-similar transformation. We call such a reduced flow the two-dimensional quasi-stationary one, which can be viewed as the snapshot of the flow at some fixed time, say $t = 1$. Thus the theory of partial differential equations with two variables can be applied to dealing with problems involved.

In this paper, we consider such a quasi-stationary flow at first. By hodograph transformation $(\xi, \eta) \to (u, v)$, (1.2) is reduced into an attractive scalar partial differential equation of second order,

$$(1 - q_v^2)\, q_{uu} + 2q_u q_v q_{uv} + (1 - q_u^2)\, q_{vv} = q_u^2 + q_v^2 - 2, \tag{1.3}$$

in the phase space under irrotationality condition, similar to that in [M], where $q = \ln \rho$ for $\rho > 0$. This equation is further transformed into two nonhomogeneous linearly degenerate equations with source terms, see (2.24), by introducing some interesting interrelated Riemann invariants, provided that the flow is supersonic. Then this conclusion is used to solve the problem of the expansion of a wedge of gas into a vacuum. This problem is actually to study the interaction of two planar rarefaction waves, which is a special case and the starting point of two-dimensional

Riemann problem for gas dynamics. The existence of global solution is achieved from the extension of a local solution with *a priori* estimates on the solution of its own and the gradients. The boundedness of solution is obtained via the technique of invariant region and the gradient estimate is established through some transforms. In the meantime, we need to prove the flow is strictly supersonic (hyperbolic).

Related to the existence results of two-dimensional global solutions of compressible Euler equations, we just mention some results to our knowledge. The existence and explicit structure of global solutions with axisymmetry continuous initial data was constructed in [ZZ2], see [Z] for more details. Chen and Glimm in [CG] proved the existence of global weak solution with symmetry outside of core with the center at the origin. Hsiao, Luo and Yang studied a similar problem and obtained the global BV solutions with spherical symmetry and damping in [HLY]. But the crucial issue on the existence of solutions inside the core still keeps open because of the high geometrical singularity of initial data at the origin. One of the essential issue of two-dimensional Riemann problem is just to investigate the development of the singularity of the initial data at this point. Unfortunately, the analytic study of two-dimensional Riemann problem for gas dynamics is extremely difficult and complicated, and nothing is known up to the date. To approach this goal, a similar existence result of global solutions of a simplified model from Euler equations was obtained in [DZ]. Therefore, the present study is just the beginning of the analytic study of two-dimensional Riemann problem for physical systems.

We organize this paper into three parts. In Section 2, we investigate the two-dimensional quasi-stationary flow without rotationality and reduce the system into two nonhomogeneous linearly degenerate equations in the phase space. In Section 3, we apply the results of Section 2 to solve the existence of global continuous solutions to the problem of a wedge of gas expanding into a vacuum. Throughout this paper, we use subscripts to denote partial derivatives when no confusion is caused, C^0 and C^1 denote the spaces of continuous functions and functions with continuous first derivatives, respectively.

2. TWO-DIMENSIONAL QUASI-STATIONARY FLOW WITHOUT ROTATIONALITY

This section is devoted to the general discussion of two-dimensional quasi-stationary flow. Consider self-similar solutions of the form

$(\rho, u, v)(t, x, y) = (\rho, u, v)(\xi, \eta)$, $(\xi, \eta) = (x/t, y/t)$, of (1.2). Then (1.2) becomes

$$\begin{cases} -\xi \rho_\xi - \eta \rho_\eta + (\rho u)_\xi + (\rho v)_\eta = 0, \\ -\xi (\rho u)_\xi - \eta (\rho u)_\eta + (\rho u^2 + p)_\xi + (\rho u v)_\eta = 0, \\ -\xi (\rho v)_\xi - \eta (\rho v)_\eta + (\rho u v)_\xi + (\rho v^2 + p)_\eta = 0 \end{cases} \tag{2.1}$$

whenever solutions are smooth. We use $q = \ln \rho$ for $\rho > 0$ instead of ρ. Then (2.1) is simplified into

$$\begin{cases} (u - \xi)\, q_\xi + (v - \eta)\, q_\eta + u_\xi + v_\eta = 0, \\ (u - \xi)\, u_\xi + (v - \eta)\, u_\eta + q_\xi = 0, \\ (u - \xi)\, v_\xi + (v - \eta)\, v_\eta + q_\eta = 0. \end{cases} \tag{2.2}$$

We assume the flow is irrotational. That is, there exists a velocity potential $\psi(t, x, y)$ such that

$$\psi_x = u, \qquad \psi_y = v. \tag{2.3}$$

By introducing $\psi(t, x, y) = t\Psi(\xi, \eta)$, we obtain

$$\Psi_\xi(\xi, \eta) = u(\xi, \eta), \qquad \Psi_\eta(\xi, \eta) = v(\xi, \eta),$$

which gives the irrotationality condition,

$$u_\eta = v_\xi. \tag{2.4}$$

Using this condition and noticing that

$$q_u = \frac{q_\xi v_\eta - q_\eta v_\xi}{u_\xi v_\eta - v_\xi u_\eta}, \qquad q_v = \frac{q_\eta u_\xi - q_\xi u_\eta}{u_\xi v_\eta - v_\xi u_\eta}, \tag{2.5}$$

we get from the conservation law of momentum of (2.2) that

$$\begin{cases} \xi - u = q_u, \\ \eta - v = q_v. \end{cases} \tag{2.6}$$

These two identities play a crucial role in the study of two-dimensional irrotational and isentropic quasi-stationary flow in that we can not only use them to reduce (2.2) into a scalar equation below, but also recover u and v as the functions of ξ and η as long as q can be derived as the function of u and v and the flow is non-degenerate in the sense that the Jacobian $J = \partial(u, v)/\partial(\xi, \eta)$ neither vanishes nor becomes infinite.

We write the law of mass conservation of (2.2) by using (2.6) as

$$q_u q_\xi + q_v q_\eta = u_\xi + v_\eta,$$

which results in

$$q_u(q_u u_\xi + q_v v_\xi) + q_v(q_u u_\eta + q_v v_\eta) = u_\xi + v_\eta, \tag{2.7}$$

if q is regarded as a function of u and v. Relations (2.6) also give

$$\begin{cases} \xi_u = q_{uu} + 1, \\ \xi_v = q_{uv}, \end{cases} \qquad \begin{cases} \eta_u = q_{uv}, \\ \eta_v = q_{vv} + 1. \end{cases} \tag{2.8}$$

Note the homogeneity of u_ξ, v_ξ, u_η and v_η in (2.7), we can replace them by η_v, $-\eta_u$, $-\xi_v$ and ξ_u provided that the Jacobian J is not degenerate. Thus, in view of (2.8), we get from (2.7),

$$(1 - q_v^2)\, q_{uu} + 2q_u q_v q_{uv} + (1 - q_u^2)\, q_{vv} = q_u^2 + q_v^2 - 2. \tag{2.9}$$

This is an attractive second order quasi linear partial differential equation with two variables. The principal part of this equation strongly resembles that of

$$(\phi_x^2 - c^2)\, \phi_{xx} + 2\phi_x \phi_y \phi_{xy} + (\phi_y^2 - c^2)\, \phi_{yy} = 0,$$

for the velocity potential ϕ in two-dimensional steady flow, where c is the local sound speed, see [M] for some remarks. Equation (2.9) also resembles the self-similar form of two-dimensional linear wave equation

$$(c^2 - \xi^2)\, \phi_{\xi\xi} - 2\xi\eta\phi_{\xi\eta} + (c^2 - \eta^2)\, \phi_{\eta\eta} - 2(\xi\phi_\xi + \eta\phi_\eta) = 0.$$

To study our problem, we write (2.9) into a 2×2 system of first order by introducing

$$X = q_u, \qquad Y = q_v. \tag{2.10}$$

Then we arrive at

$$\begin{pmatrix} 1 - Y^2 & XY \\ 0 & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}_u + \begin{pmatrix} XY & 1 - X^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}_v = \begin{pmatrix} X^2 + Y^2 - 2 \\ 0 \end{pmatrix}. \tag{2.11}$$

It is well known that this system is equivalent to (2.9) as long as their solutions are smooth enough. System (2.11) has two eigenvalues

$$\lambda_- = \frac{XY - \sqrt{X^2+Y^2-1}}{1-Y^2}, \qquad \lambda_+ = \frac{XY + \sqrt{X^2+Y^2-1}}{1-Y^2} \tag{2.12}$$

with two associated left eigenvectors

$$l_- = (-1, \sqrt{X^2+Y^2-1}\,), \qquad l_+ = (1, \sqrt{X^2+Y^2-1}\,). \tag{2.13}$$

From (2.12), we observe that the flow is transonic: supersonic if $X^2+Y^2>1$, subsonic if $X^2+Y^2<1$ and parabolic degenerate on the sonic curve $X^2+Y^2-1=0$. The eigenvalues are real only in a supersonic domain. At present, we assume that the flow is supersonic. Then we multiply (2.11) by the left eigenmatrix $M=(l_-, l_+)^T$ from the left–hand side to obtain

$$\begin{cases} X_u + \lambda_- Y_u + \lambda_+ (X_v + \lambda_- Y_v) = \dfrac{X^2+Y^2-2}{1-Y^2}, \\[2ex] X_u + \lambda_+ Y_u + \lambda_- (X_v + \lambda_+ Y_v) = \dfrac{X^2+Y^2-2}{1-Y^2}, \end{cases} \tag{2.14}$$

where the superscript T represents the transpose of the matrix. Introduce Riemann invariants

$$R = \frac{-X + Y\sqrt{X^2+Y^2-1}}{1-Y^2}, \qquad S = \frac{X + Y\sqrt{X^2+Y^2-1}}{1-Y^2}, \tag{2.15}$$

which satisfy

$$\begin{cases} \dfrac{\partial R}{\partial X} = \dfrac{\lambda_-}{\sqrt{X^2+Y^2-1}}, \\[2ex] \dfrac{\partial R}{\partial Y} = \dfrac{\lambda_-^2}{\sqrt{X^2+Y^2-1}}, \end{cases} \qquad \begin{cases} \dfrac{\partial S}{\partial X} = \dfrac{\lambda_+}{\sqrt{X^2+Y^2-1}}, \\[2ex] \dfrac{\partial S}{\partial Y} = \dfrac{\lambda_+^2}{\sqrt{X^2+Y^2-1}}. \end{cases} \tag{2.16}$$

Then (2.14) takes the form

$$\begin{cases} R_u + \lambda_+ R_v = -\dfrac{1}{\sqrt{X^2+Y^2-1}} \lambda_- (\lambda_- \lambda_+ + 1), \\[2ex] S_u + \lambda_- S_v = -\dfrac{1}{\sqrt{X^2+Y^2-1}} \lambda_+ (\lambda_- \lambda_+ + 1). \end{cases} \tag{2.17}$$

Observe that λ_- and λ_+ themselves are Riemann invariants, satisfying

$$\begin{cases} \dfrac{\partial \lambda_-}{\partial X} = \dfrac{R}{\sqrt{X^2+Y^2-1}}, \\ \dfrac{\partial \lambda_-}{\partial Y} = \dfrac{R\lambda_-}{\sqrt{X^2+Y^2-1}}, \end{cases} \qquad \begin{cases} \dfrac{\partial \lambda_+}{\partial X} = \dfrac{S}{\sqrt{X^2+Y^2-1}}, \\ \dfrac{\partial \lambda_+}{\partial Y} = \dfrac{S\lambda_+}{\sqrt{X^2+Y^2-1}}. \end{cases} \tag{2.18}$$

Therefore we can write (2.14) as

$$\begin{cases} (\lambda_-)_u + \lambda_+(\lambda_-)_v = -\dfrac{1}{\sqrt{X^2+Y^2-1}} R(\lambda_-\lambda_+ + 1), \\ (\lambda_+)_u + \lambda_-(\lambda_+)_v = -\dfrac{1}{\sqrt{X^2+Y^2-1}} S(\lambda_-\lambda_+ + 1). \end{cases} \tag{2.19}$$

Notice that the linear combinations of Riemann invariants are still Riemann invariants. We take the following Riemann invariants to simplify calculations,

$$A = R + \lambda_- = -\frac{X+\sqrt{X^2+Y^2-1}}{1+Y}, \qquad B = S + \lambda_+ = \frac{X+\sqrt{X^2+Y^2-1}}{1-Y}. \tag{2.20}$$

Then we have from (2.17) and (2.19) that

$$\begin{cases} A_u + \lambda_+ A_v = -\dfrac{1}{\sqrt{X^2+Y^2-1}} A(\lambda_-\lambda_+ + 1), \\ B_u + \lambda_- B_v = -\dfrac{1}{\sqrt{X^2+Y^2-1}} B(\lambda_-\lambda_+ + 1). \end{cases} \tag{2.21}$$

The mappings $(X, Y) \to (R, S)$, $(X, Y) \to (\lambda_-, \lambda_+)$ and $(X, Y) \to (A, B)$ are all bijective as long as the flow is supersonic and $Y^2 \neq 1$; see Corollary 3.3 in the next section.

In order to make it easier to see the properties of system (2.21), we solve (2.20) to obtain

$$X = \frac{AB-1}{A-B}, \quad Y = \frac{A+B}{B-A}, \quad \sqrt{X^2+Y^2-1} = \frac{AB+1}{A-B}, \quad 1-Y^2 = \frac{-4AB}{(A-B)^2} \tag{2.22}$$

and

$$R = \frac{1+A^2}{2A}, \qquad S = \frac{1+B^2}{2B},$$
$$\lambda_- = \frac{A^2-1}{2A}, \qquad \lambda_+ = \frac{B^2-1}{2B}. \tag{2.23}$$

Explicitly in A and B, (2.21) can be written as

$$\begin{cases} A_u + \lambda_+ A_v = -AG(A, B), \\ B_u + \lambda_- B_v = -BG(A, B), \end{cases} \tag{2.24}$$

where

$$\begin{aligned} G(A, B) &= \frac{A-B}{AB+1}\left(\frac{(A^2-1)(B^2-1)}{4AB}+1\right) \\ &= \frac{(AB+1)(A-B)}{4AB}\left(1-\left(\frac{A-B}{AB+1}\right)^2\right). \end{aligned} \tag{2.25}$$

From (2.23), we see that λ_- is independent of B and λ_+ independent of A. So (2.24) is an nonhomogeneous linearly degenerate system. This kind of system has been studied with some assumptions in [Li, RY]. The problem here is to investigate the boundedness of solution and the gradient.

Thus once we obtain the solutions of (2.24), we can obtain the solutions (X, Y) of (2.11) by the recovery process (2.22) provided that $B-A \neq 0$. Moreover, using (2.6), we can get the solution by the primitive variables (ρ, u, v) because there is a one-to-one correspondence between (ξ, η)-plane and the phase plane-(u, v)-plane (cf. [LZY, Chap. 5]).

3. THE EXPANSION OF A WEDGE OF GAS INTO A VACUUM

In this section, we apply the conclusion of last section to consider the problem of the expansion of gas into a vacuum. Assume that the gas with uniform velocity is separated from a vacuum by two infinite rigid walls. Initially these two walls are moved instantaneously and the gas will expand into the vacuum immediately. For the convenience of presentation, we place the wedge symmetrically with respect to x-axis, as in Fig. 1(a). The initial data then take

$$(\rho, u, v)(t=0, x, y) = \begin{cases} (\rho_0, u_0, v_0), & -\theta < \alpha < \theta, \\ (0, \bar{u}, \bar{v}), & \text{otherwise}, \end{cases} \tag{3.1}$$

where $\rho_0 > 0$, u_0 and v_0 are constants, and $(\bar{u}, \bar{v})$ is the velocity in the wave front, not being specified, $\alpha = \text{arc tan } y/x$ is the polar angle, θ is reasonable to be restricted between 0 and $\pi/2$.

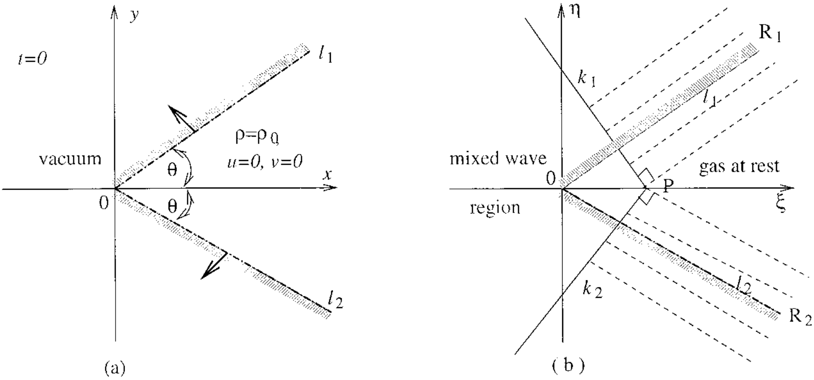

FIG. 1. The expansion of a wedge of gas into a vacuum.

From (3.1), we can see that the vorticity of initial velocity field is zero. Kelvin's theorem implies the constancy of circulation, if the solution is continuous. We conclude that the flow is irrotational at later time and so (2.4) is valid. Therefore we can use the results of last section to solve this problem.

To derive the solution of (1.2) and (3.1), we first consider planar rarefaction wave solutions of (1.2) with the initial data consisting of two pieces of constant states,

$$(\rho, u, v)(t=0, x, y) = \begin{cases} (\rho_1, u_1, v_1), & \mu x + \nu y > 0, \\ (\rho_2, u_2, v_2), & \mu x + \nu y < 0, \end{cases} \tag{3.2}$$

where (μ, ν) is the unit normal of the discontinuity line. The solution takes the form $(\rho, u, v)(t, x, y) = (\rho, u, v)((\mu x + \nu y)/t) =: (\rho, u, v)(\zeta)$. Then (1.2) takes

$$\begin{cases} -\zeta \rho_\zeta + \mu(\rho u)_\zeta + \nu(\rho v)_\zeta = 0, \\ -\zeta(\rho u)_\zeta + \mu(\rho u^2 + p)_\zeta + \nu(\rho u v)_\zeta = 0, \\ -\zeta(\rho v)_\zeta + \mu(\rho u v)_\zeta + \nu(\rho v^2 + p)_\zeta = 0, \end{cases} \tag{3.3}$$

i.e.,

$$\begin{pmatrix} -\zeta + \mu u + \nu v & \mu \rho & \nu \rho \\ \mu/\rho & -\zeta + \mu u + \nu v & 0 \\ \nu/\rho & 0 & -\zeta + \mu u + \nu v \end{pmatrix} \begin{pmatrix} \rho \\ u \\ v \end{pmatrix}_\zeta = 0,$$

which gives either a constant solution $(\rho, u, v) \equiv (\rho_0, u_0, v_0)$ or a planar rarefaction wave solutions associated with $\zeta_\pm = \mu u + \nu v \pm 1$,

$$\begin{cases} \zeta_+ = V + 1, \\ dq = dV, \\ dU = 0, \end{cases} \qquad \begin{cases} \zeta_- = V - 1, \\ dq = -dV, \\ dU = 0, \end{cases} \tag{3.4}$$

where $U = -\nu u + \mu v$ and $V = \mu u + \nu v$ are the tangential and normal components along the propagation of the wave.

As a special case that the wave propagates in y-direction, the solution is

$$\begin{cases} \eta = v + 1, \\ q - q_0 = v - v_0, \\ u \equiv u_0, \end{cases} \qquad \begin{cases} \eta = v - 1, \\ q - q_0 = -(v - v_0), \\ u \equiv u_0. \end{cases} \tag{3.5}$$

Now we turn back to the problem of gas expansion into the vacuum. Assume that $(u_0, v_0) = (0, 0)$, i.e., the gas is at rest initially. Otherwise, we simply take a translation transform. After the wedge-shaped walls are removed, the gas away from the sharp corner expands uniformly to infinity as planar rarefaction waves explicitly expressed as

$$\begin{aligned} &R_1 : \begin{cases} q = q_0 + \xi \sin\theta - \eta\cos\theta - 1, \\ u = \sin\theta(\xi\sin\theta + \eta\cos\theta - 1), \\ v = -\cos\theta(\xi\sin\theta - \eta\cos\theta - 1), \\ -\infty < \xi\sin\theta - \eta\cos\theta \leqslant 1, \end{cases} \\ &R_2 : \begin{cases} q = q_0 + \xi\sin\theta + \eta\cos\theta - 1, \\ u = \sin\theta(\xi\sin\theta + \eta\cos\theta - 1), \\ v = \cos\theta(\xi\sin\theta + \eta\cos\theta - 1), \\ -\infty < \xi\sin\theta + \eta\cos\theta \leqslant 1, \end{cases} \end{aligned} \tag{3.6}$$

where R_1 emits from l_1 and R_2 from l_2, $(\xi, \eta) = (x/t, y/t)$ as before. These two rarefaction waves interact at $P = (1/\sin\theta, 0)$ and they are separated from the mixed wave region, the interaction region of the rarefaction waves, by

$$\begin{cases} k_1 : -\xi\sin\theta + \eta\cos\theta = -1, \\ k_2 : \xi\sin\theta + \eta\cos\theta = 1, \end{cases} \tag{3.7}$$

which are the characteristics of (2.2) from P, see Fig. 1(b).

To use the results of last section, we consider the projection of solution in the mixed wave region into (u, v)-plane. Then the planar rarefaction wave (3.5) is a semi-infinite line on $u = u_0$. For the former of (3.5), the line is $u = u_0$, $v \leqslant 0$, on which $q_v = 1$ and $q_{vv} = 0$. Therefore we can view (2.9) as an ODE of q_u. Consequently,

$$q_u = \{e^v(c^2 + e^{-v} - 1)\}^{1/2}, \qquad q_v = 1, \tag{3.8}$$

where c is the value of q_u at $(u, v) = (0, 0)$. While for the latter of (3.5), the line is $v \geqslant 0$, and

$$q_u = \{e^{-v}(\bar{c}^2 + e^v - 1)\}^{1/2}, \qquad q_v = -1, \tag{3.9}$$

where $\bar{c}$ is also the value of q_u at $(u, v) = (0, 0)$.

Take the transformations $(x', y') = (x \cos\beta + y \sin\beta, -x \sin\beta + y \cos\beta)$ and $(u', v') = (u \cos\beta + v \sin\beta, -u \sin\beta + v \cos\beta)$. Then we have

$$\begin{pmatrix} \dfrac{\partial}{\partial u} \\ \dfrac{\partial}{\partial v} \end{pmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial u'} \\ \dfrac{\partial}{\partial v'} \end{pmatrix}. \tag{3.10}$$

When the planar rarefaction wave R_1 emanating from l_1 is considered, we take $\beta = \theta$ and the wave satisfies the latter case of (3.5) and the image of the semi-infinite line in (u, v)-plane is $u \cos\theta + v \sin\theta = 0$ $(u \leqslant 0)$. So

$$q_{u'} = \{e^{-v'}(c^2 - 1) + 1\}^{1/2}, \qquad q_{v'} = -1,$$

which gives from (3.10) that

$$\begin{cases} q_u = \{\exp(u \sin\theta - v \cos\theta)(c^2 - 1) + 1\}^{1/2} \cos\theta + \sin\theta, \\ q_v = \{\exp(u \sin\theta - v \cos\theta)(c^2 - 1) + 1\}^{1/2} \sin\theta - \cos\theta. \end{cases} \tag{3.11}$$

Since the solution is symmetric with respect to u-axis, we have

$$q_u(u, v) = q_u(u, -v), \qquad q_v(u, v) = -q_v(u, -v). \tag{3.12}$$

Thus $q_v(u, 0) \equiv 0$, which implies from (3.11) that $c = \cot\theta$.

With the same method, we obtain that the rarefaction wave R_2 from l_2 satisfies the former of (3.5) and corresponds to $u\cos\theta - v\sin\theta = 0$ $(u \leqslant 0)$ in (u, v)-plane on which

$$\begin{cases} q_u = \{\exp(u\sin\theta + v\cos\theta)(\cot^2\theta - 1) + 1\}^{1/2}\cos\theta + \sin\theta, \\ q_v = -\{\exp(u\sin\theta + v\cos\theta)(\cot^2\theta - 1) + 1\}^{1/2}\sin\theta + \cos\theta. \end{cases} \tag{3.13}$$

Therefore problem (1.2) and (3.1) is reduced into a boundary value problem for (2.24) with the boundary data on $H_1: -u\sin\theta + v\cos\theta = 0$ $(u \leqslant 0)$ and H_2: $u\sin\theta + v\cos\theta = 0$ $(u \leqslant 0)$,

$$A|_{H_1} = A_0^1, \qquad A|_{H_2} = A_0^2, \qquad B_{H_1} = B_0^1, \qquad B|_{H_2} = B_0^2, \tag{3.14}$$

where

$$\begin{aligned} A_0^1 &= -\frac{\{\exp(u\sin\theta - v\cos\theta)(\cot^2\theta - 1) + 1\}^{1/2}(1 + \cos\theta) + \sin\theta}{\{\exp(u\sin\theta - v\cos\theta)(\cot^2 - 1) + 1\}^{1/2}\sin\theta + (1 - \cos\theta)}, \\ A_0^2 &= -\frac{\{\exp(u\sin\theta + v\cos\theta)(\cot^2\theta - 1) + 1\}^{1/2}(1 + \cos\theta) + \sin\theta}{-\{\exp(u\sin\theta + v\cos\theta)(\cot^2 - 1) + 1\}^{1/2}\sin\theta + (1 + \cos\theta)}, \\ B_0^1 &= \frac{\{\exp(u\sin\theta - v\cos\theta)(\cot^2\theta - 1) + 1\}^{1/2}(1 + \cos\theta) + \sin\theta}{-\{\exp(u\sin\theta - v\cos\theta)(\cot^2\theta - 1) + 1\}^{1/2}\sin\theta + (1 + \cos\theta)}, \\ B_0^2 &= \frac{\{\exp(u\sin\theta + v\cos\theta)(\cot^2\theta - 1) + 1\}^{1/2}(1 + \cos\theta) + \sin\theta}{\{\exp(u\sin\theta + v\cos\theta)(\cot^2\theta - 1) + 1\}^{1/2}\sin\theta + (1 - \cos\theta)}. \end{aligned} \tag{3.15}$$

This is a Goursat-type problem because H_1 and H_2 are characteristics of (2.24). It is easy to check that $-\epsilon_1 < A_0^1, A_0^2 < -1$ and $1 < B_0^1, B_0^2 < \epsilon_2$ for some bounded positive constants ϵ_1 and ϵ_2. Furthermore, on H_1 and H_2, the flow is uniformly supersonic, i.e., $X^2 + Y^2 - 1 > 0$, if u and v is finite. Therefore we draw an isopleth L of $\sqrt{X^2 + Y^2 - 1}$ through two symmetric points P and Q on H_1 and H_2. Now our problem is to seek the solution of (2.24) and (3.14) in the angular domain $\Xi = \{(u, v); \pi/2 + \theta \leqslant \arctan v/u \leqslant 3\pi/2 - \theta, 0 > u > -C\}$ for any fixed $C > 0$. This domain is bounded by H_1, H_2 and the isopleth L; see Fig. 2.

The proof of local existence of solutions to (2.24) and (3.14) is quite easy. Consult [LY]. That is, there is a $\delta > 0$ such that the solution of (2.24) and (3.14) exists uniquely in $\Xi_\delta := \{(u, v) \in \Xi; -\delta < u < 0\}$.

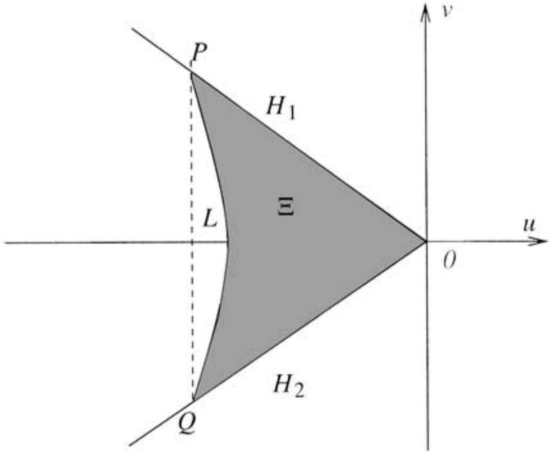

FIG. 2. The projection of solutions on phase plane-(u, v)-plane.

The existence of the global smooth solution in Ξ will be achieved by the extension from the local solution. For this purpose, we need some *a priori* estimates. First we prove the boundedness of C^0-norm of solution. The technique of invariant region is used here; see Smoller [Sm].

LEMMA 3.1. *Suppose that there exists a C^1 solution $(A(u, v), B(u, v))$ on the domain under consideration. Then C^0 norm has a uniform bound independent of u for all $-\infty < u \leqslant 0$.*

Proof. First we notice that the curve L: $\frac{AB+1}{A-B} = 1$ gives a hyperbola L. We construct an invariant region as follows. The lines $B = const.$ are chosen so that the top one L_2 is above L and the bottom one L_3 is between L and $B = 1$. The lines $A = const.$ are taken so that the left one L_1 is on the left–hand side of L and the right one L_4 is between L and $A = 1$. This rectangle has two opposite vertexes on L, as shown in Fig. 3.

Note that inside the rectangle $(AB+1)/(A-B) > 1$. On L_1 and L_2, we have

$$-AG(A, B) = -\frac{(AB+1)(A-B)}{4B}\left(1-\left(\frac{A-B}{AB+1}\right)^2\right) < 0,$$

and

$$-BG(A, B) = -\frac{(AB+1)(A-B)}{4A}\left(1-\left(\frac{A-B}{AB+1}\right)^2\right) > 0.$$

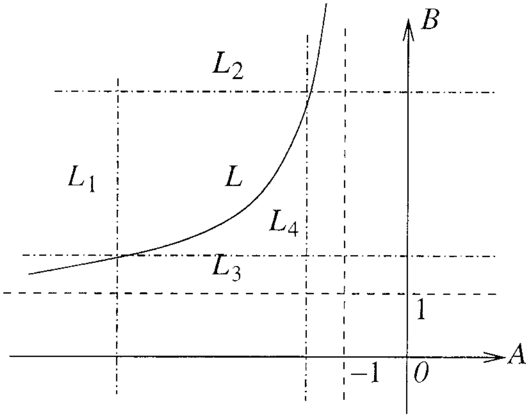

FIG. 3. The invariant region for (A, B)

While on L_3 and L_4, we have

$$-AG(A, B) > 0, \qquad -BG(A, B) < 0.$$

Therefore, according to [Sm], this rectangle is the invariant region for (A, B) because the boundary values of A and B lies in the second quadrant of (A, B)-plane and the rectangle can be made sufficiently large if necessary. ▮

Remark 3.2. As $\theta = \pi/4$, we can get the explicit solution because $A = -(1+\sqrt{2})$ on H_1 and $B = 1+\sqrt{2}$ on H_2. So $X = \sqrt{2}$ and $Y = 0$ from (2.22). Thus the exact solution is

$$q = q_0 + \sqrt{2}\,u,$$

and u and v can be recovered from (2.6). The Jacobian $J = \partial(u, v)/\partial(\xi, \eta)$ is obviously non–degenerate. ▮

COROLLARY 3.3. *The flow is uniformly supersonic (hyperbolic) in Ξ and non–degenerate in the sense that* $1 - Y^2 \neq 0$.

Proof. The uniform hyperbolicity is obvious due to the construction of Ξ in Fig. 2. Note

$$1 - Y^2 = \frac{-4AB}{(A-B)^2},$$

and we conclude that the flow is non-degenerate from the construction of invariant region in Lemma 3.1. ▮

This corollary shows that the transformations $(X, Y) \to (R, S)$, $(X, Y) \to (\lambda_-, \lambda_+)$ and $(X, Y) \to (A, B)$ are all bijective (cf. Section 2).

Next, we obtain the gradient estimate. We use the idea in [Li].

LEMMA 3.4. *In Ξ, the C^1-norm of (A, B) is uniformly bounded.*

Proof. Note the uniform hyperbolicity of the flow in Ξ. That is, $\lambda_- - \lambda_+ \neq 0$. We introduce

$$\phi = (\lambda_- - \lambda_+)\frac{\partial A}{\partial v}, \qquad \psi = (\lambda_+ - \lambda_-)\frac{\partial B}{\partial v}. \tag{3.16}$$

Then

$$\begin{aligned} \frac{\partial \phi}{\partial u} &= (\lambda_- - \lambda_+)\frac{\partial^2 A}{\partial u \partial v} + \frac{\partial(\lambda_- - \lambda_+)}{\partial u}\frac{\partial A}{\partial v}, \\ \frac{\partial \phi}{\partial v} &= (\lambda_- - \lambda_+)\frac{\partial^2 A}{\partial v^2} + \frac{\partial(\lambda_- - \lambda_+)}{\partial v}\frac{\partial A}{\partial v}. \end{aligned} \tag{3.17}$$

Differentiating the first equation of (2.24) with respect to v gives

$$\frac{\partial^2 A}{\partial u \partial v} + \lambda_+ \frac{\partial^2 A}{\partial v^2} = -\frac{\partial}{\partial v}(AG(A, B)) - \frac{\partial \lambda_+}{\partial v}\frac{\partial A}{\partial v}. \tag{3.18}$$

Therefore, from (3.17) and (3.18), we have

$$\frac{\partial \phi}{\partial u} + \lambda_+ \frac{\partial \phi}{\partial v} = a_1 \phi + b_1 \psi, \tag{3.19}$$

where

$$\begin{aligned} a_1 &= -G(A, B) - A\frac{\partial G}{\partial A} + \frac{1}{\lambda_- - \lambda_+} \cdot \frac{R - S}{\sqrt{X^2 + Y^2 - 1}}(\lambda_- \lambda_+ + 1), \\ b_1 &= A\frac{\partial G}{\partial B}. \end{aligned} \tag{3.20}$$

Similarly we get from the second equation of (2.24),

$$\frac{\partial \psi}{\partial u} + \lambda_+ \frac{\partial \psi}{\partial v} = a_2 \phi + b_2 \psi, \tag{3.21}$$

where

$$\begin{aligned} a_2 &= B\frac{\partial G}{\partial A}, \\ b_2 &= -G(A, B) - B\frac{\partial G}{\partial B} + \frac{1}{\lambda_+ - \lambda_-} \cdot \frac{S - R}{\sqrt{X^2 + Y^2 - 1}}(\lambda_- \lambda_+ + 1). \end{aligned} \tag{3.22}$$

The boundary values for ϕ and ψ can be obtained as follows. Note on H_1: $u\cos\theta+v(u)\sin\theta=0$, $A(u,v)=A_0^1(u,v(u))$, and

$$\frac{\partial A}{\partial u}+\lambda_-\frac{\partial A}{\partial v}=\frac{dA_0^1}{du}. \tag{3.23}$$

Using the first equation of (2.24) and (3.16), we obtain

$$\phi=(\lambda_--\lambda_+)\frac{\partial A}{\partial v}=\frac{dA_0^1}{du}-AG(A,B)|_{H_1}, \qquad \text{on } H_1. \tag{3.24}$$

Similarly, we have

$$\psi=(\lambda_+-\lambda_-)\frac{\partial B}{\partial v}=\frac{dB_0^2}{du}-BG(A,B)|_{H_2}, \qquad \text{on } H_2. \tag{3.25}$$

Note that the coefficients a_i, b_i, $i=1,2$, in (3.20) and (3.22) are uniformly bounded, and so are boundary values (3.24) and (3.25). We write (3.19) and (3.21) into the integral form along characteristics and apply Gronwall's inequality, the desired result is obtained. ∎

Then we have the final conclusion.

THEOREM 3.5. *There exists a unique global smooth solution* $(A(u,v), B(u,v))$ *of* (2.24) *and* (3.14) *in the domain* Ξ *for* $U<u<0$, *where* U *is any fixed constant.*

Proof. The local existence is classical because the boundary values are continuous at the origin. Then we use *a priori* estimates in Lemmas 3.1 and 3.4 to conclude that the C^1 solution exists in Ξ by extension step by step. ∎

ACKNOWLEDGMENTS

The author thanks the DFG–Graduiertenkolleg "Modellierung, Berechnung und Identifikation mechanischer Systeme" in Magdeburg, Germany, NSF of China and GIF Grant I0318–195.06/93 for the support of this research. Special thanks are given to G. Warnecke, T. Zhang, and S. X. Chen for their helpful discussions and comments.

REFERENCES

[BG] G. Ben-dor and I. I. Glass, Domains and boundaries of non-stationary oblique shock wave reflection, *J. Fluid Mech.* **92** (1979), 459–496; **96** (1980), 735–756.

[CCY] T. Chang, G.-Q. Chen, and S.-L. Yang, On the 2-D Riemann problem for the compressible Euler equations. I. Interaction of shock waves and rarefaction waves, *Discrete Cont. Dynam. Systems* **1** (1995), 555–584.

[CG] G.-Q. Chen and J. Glimm, Global solutions to the compressible Euler equations with geometrical structure, *Comm. Math. Phys.* **180** (1996), 124–147.

[DZ] Z.-H. Dai and T. Zhang, Global smooth solution of degenerate Goursat problem for a quasilinear mixed type equation, *Arch. Ration. Mech. Anal.* **155** (2000), 277–298.

[HLY] L. Hsiao, T. Luo, and T. Yang, Global BV solutions of compressible Euler equations with spherical symmetry and damping, *J. Differantial Equations* **146** (1998), 203–225.

[Le] L. E. Levine, The expansion of a wedge of gas into a vacuum, *Math. Proc. Cambridge Philos. Soc.* **64** (1968), 1151–1163.

[Li] T.-T. Li, "Global Classical Solutions for Quasilinear Hyperbolic Systems," Wiley, New York, 1994.

[LL] P. D. Lax and X.-D. Liu, Solutions of two-dimensional Riemann problem of gas dynamics by positive schemes, *SIAM J. Sci. Comput.* **19** (1998), 319–340.

[LY] T.-T. Li and W.-C. Yu, "Boundary Value Problems for Quasilinear Hyperbolic Systems," Duke University Mathematics Series, Vol. V, Duke University, 1985.

[LZY] J. Q. Li, T. Zhang, and S.-L. Yang, "The Two-Dimensional Riemann Problem in Gas Dynamics," Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 98, Longman, Harlow, 1998.

[M] A. G. Mackie, Two-dimensional quasi-stationary flows in gas dynamics, **68** (1968), 1099–1108.

[RY] B. L. Rozdestvenskii and N. N. Yanenko, "Systems of Quasi Linear Equations," Nauka, Moscow, 1968.

[Sc] C. W. Schulz-Rinne, Classification of the Riemann problem for two-dimensional Riemann problem, *SIAM J. Math. Anal.* **24** (1993), 76–88.

[Sm] J. Smoller, "Shock Waves and Reaction-Diffusion Equations," 2nd ed., Springer-Verlag, New York/Berlin, 1994.

[SCG] C. W. Schulz-Rinne, J. P. Collins, and H. M. Glaz, Numerical solution of the Riemann problem for two-dimensional gas dynamics, *SIAM J. Sci. Comput.* **4** (1993), 1394–1414.

[Z] Y. X. Zheng, Two-dimensional Riemann problems for systems of conservation laws, *in* "Progress in Nonlinear Differential Equations," Birkäuser, Basel, 2001.

[ZZ1] T. Zhang and Y. X. Zheng, Conjecture on the structure of solutions of the Riemann problem for two-dimensional gas dynamics systems, *SIAM J. Math. Anal.* **21** (1990), 593–630.

[ZZ2] T. Zhang and Y. X. Zheng, Axisymmetric solutions of the Euler equations for polytropic gases, *Arch Rational Mech. Anal.* **142** (1998), 253–279.