ResearchGate

See discussions, stats, and author profiles for this publication at: https://www.researchgate.net/publication/257319063

Degenerate Goursat-type boundary value problems arising from the study of two-dimensional isothermal Euler equations

Article *in* Zeitschrift für angewandte Mathematik und Physik ZAMP · December 2012

DOI: 10.1007/s00033-012-0203-2

CITATIONS
15

READS
32

3 authors:

Yanbo Hu
Hangzhou Normal University
15 PUBLICATIONS **68** CITATIONS

SEE PROFILE

Jiequan Li
Institute of Applied Physics and Computational Mathematics
60 PUBLICATIONS **1,051** CITATIONS

SEE PROFILE

Wancheng Sheng
Shanghai University
70 PUBLICATIONS **600** CITATIONS

SEE PROFILE

Some of the authors of this publication are also working on these related projects:

Project The Riemann problem for the hyperbolic conservation laws View project

All content following this page was uploaded by Jiequan Li on 16 September 2016.

The user has requested enhancement of the downloaded file.

Z. Angew. Math. Phys. 63 (2012), 1021–1046
© 2012 Springer Basel AG
0044-2275/12/061021-26
published online March 9, 2012
DOI 10.1007/s00033-012-0203-2

Zeitschrift für angewandte Mathematik und Physik ZAMP

Degenerate Goursat-type boundary value problems arising from the study of two-dimensional isothermal Euler equations

Yanbo Hu, Jiequan Li and Wancheng Sheng

Abstract. As a ladder step to study transonic problems, we investigate two families of degenerate Goursat-type boundary value problems arising from the two-dimensional pseudo-steady isothermal Euler equations. The first family is about the genuinely two-dimensional full expansion of gas into a vacuum with a wedge; the other is a semi-hyperbolic patch that starts on sonic curves and ends at transonic shocks. Both the vacuum and the sonic sets cause parabolic degeneracy that results in substantial difficulties such as singularities of solutions and uniform a priori estimates. Main ingredients in this study are various characteristic decompositions for the pseudo-steady Euler equations in order to obtain necessary a priori estimates. Furthermore, we are able to verify the uniform Hölder continuity of solutions with exponent 1/2 for the gas expansion problem and up to 2/7 for the semi-hyperbolic problem.

Mathematics Subject Classification. Primary 35L65, 35J70, 35R35; Secondary 35J65.

Keywords. Isothermal Euler equations · Degenerate Goursat-type problems · Interaction of rarefaction waves · Semi-hyperbolic patch · characteristic decomposition.

1. Introduction

As a ladder step toward transonic (mixed-type) flow problems, we study two families of degenerate Goursat-type problems arising from the two-dimensional isentropic compressible Euler equations

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y = 0, \\ (\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y = 0, \end{cases} \tag{1.1}$$

where ρ is the density, (u, v) is the velocity and p is the pressure given by $p(\rho) = \rho$ for the isothermal case. The first family is the full gas expansion into a vacuum with a wedge, often interpreted as the dam collapse problem in hydraulics [12,13,19,24]. The other is the semi-hyperbolic patch whose formulation starts from [23] for a system of pressure-gradient equations and [21] for the polytropic case $\gamma > 1$ of the Euler equations. In comparison with general polytropic gases, the isothermal gas (1.1) has its own unique features, for example, the sonic speed is uniformly unit, and the gas expands into the vacuum with infinite speed.

These problems boil down to the interaction of simple waves in two spatial dimensions. The wave interaction is fundamental in the field of aerodynamics, such as the Mach experiment that disclosed the phenomenon of an oblique shock hitting a ramp. There are series of significant progress recently. In [1], a weak shock reflection problem was solved for the UTSD model. The regular reflection of shock by large-angle wedges was solved in [2] for the potential flow equation and in [27] for the system of

Yanbo Hu and Wancheng Sheng were supported by NSFC (10971130), Shanghai Leading Academic Discipline Project (J50101) and Shanghai Municipal Education Commission of Scientific Research Innovation Project: 11ZZ84. Jiequan Li was supported by the Key Program from Beijing Educational Commission (KZ200910028002), 973 project (2006CB805902), PHR(IHLB) and NSFC (10971142).

Birkhäuser

pressure-gradient equations. In [3], the Mach configuration of shocks was confirmed to be stable in a certain sense. In [8], the weak shock stability was established for the problem of a supersonic flow against a solid wedge. The present problems, compared to the oblique shock reflection problem, are shock-free and relatively simple, yet they catch a substantial class of wave interaction with degenerate free boundaries, sonic curves and the vacuum front (although this front runs to infinity immediately for the isothermal case). The degeneracy exists extensively in transonic problems.

In order to solve these problems, we develop various characteristic decompositions for the inclination angles of characteristics, Mach lines and the sound speed directly in the pseudo-steady plane, $(x/t, y/t)$-plane, expanding the work of [4], which allow us to establish a priori C^0, C^1 and $C^{1,1}$-norms estimates in the interaction domains. Due to the degeneracy, the C^1 and $C^{1,1}$-norms estimates are not uniform. In addition, the classical approach to extend local solutions to "global" solutions does not work here. Hence, we have to study the non-characteristic and regularity of the isolines of Mach number, so that it can be taken as the "Cauchy" support at each extension step, specifically in the semi-hyperbolic patch problem since the sonic boundary is not fixed. In particular, the uniform Hölder continuity property of solutions is obtained, with exponent $1/2$ for the gas expansion problem and up to $2/7$ for the semi-hyperbolic patch problem. The regularity of the sonic boundary is of order $2/7$, but locally. In our forthcoming paper, we can verify the regularity of the sonic boundary up to order $1/3$. We remark that the regularity seems difficult to be improved better, mainly due to some technical reason in our viewpoint.

This work is the continuation of our earlier contributions. For the gas expansion problem, the first result was given in [14] for the isothermal case in the hodograph plane, but it was not converted into the physical $(x/t, y/t)$-plane. General polytropic cases were treated first in [13] in the hodograph plane and then completed subsequently in [4,16,19]. This problem was studied for the pressure-gradient equations [7,11]. The semi-hyperbolic patch is quite ubiquitous in transonic problems. See [5,6,9] and some recent numerical simulations [10]. The formulation of such a problem was first extracted for the pressure-gradient equations [23] and then extended to the isentropic Euler equations in [21]. For more related results see the survey paper [15]. The present paper, in addition to filling with a special isothermal case, provides more uniform regularity estimates.

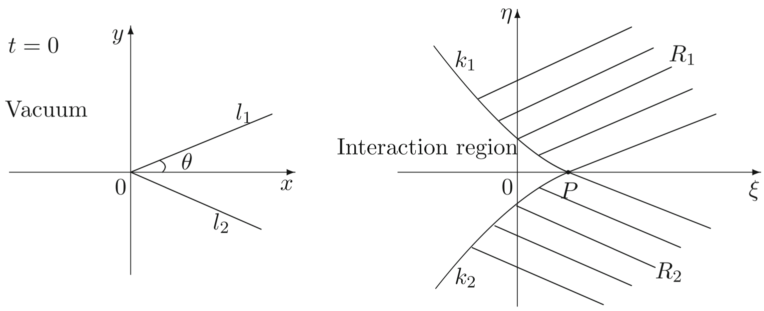
This paper consists of six sections. Besides the introduction section here, we set up the problems in Sect. 2. In Sect. 3, we organize primary self-similar formulation of (1.1), characteristic forms, Riemann invariants and diagonalization, etc. In Sect. 4, we provide various characteristic decompositions in order to establish a priori estimates. In Sect. 5, we solve the gas expansion problem, and in Sect. 6, we investigate the semi-hyperbolic patch problem.

2. Setup of problems

In this section, we set up two families of degenerate Goursat-type (hyperbolic) problems that are investigated in this paper. One is the gas expansion problem, which has the vacuum interface as the degenerate boundary, and the other is the semi-hyperbolic patch problem, which has a sonic curve as the degenerate boundary. The gas expansion problem was studied only in the hodograph plane in [14] for the isothermal case. Here, we avoid using the hodograph approach. Instead, we adopt a direct approach in the self-similar plane to solve the problems. This direct approach is significant for the semi-hyperbolic patch problem because the hodograph transformation becomes extremely involved near the sonic boundary.

2.1. The gas expansion problem

For the convenience of presentation, we place the wedge symmetrically with respect to $x-$axis and the sharp corner at the origin, as in Fig. 1a. This problem is then formulated mathematically as seeking the solution of (1.1) with the initial data at $t = 0$,



(a) Initial Data **(b)** Interaction of rarefaction waves

Fig. 1. The expansion of a wedge of gas

$$(\rho, u, v)(0, x, y) = \begin{cases} (\rho_0, u_0, v_0), & -\theta < \omega < \theta, \\ (0, \bar{u}, \bar{v}), & \text{otherwise}, \end{cases} \tag{2.1}$$

where $\rho_0 > 0, u_0$ and v_0 are constants. $(\bar{u}, \bar{v})$ is the velocity of the wave front, not being specified in the state of vacuum, $\omega = \arctan(y/x)$ is the polar angle, and θ is the half-angle of the wedge restricted between 0 and $\pi/2$. This can be considered as a two-dimensional Riemann problem for (1.1) with two pieces of initial data (2.1). We note that the solution we construct is valid for any *portion* of (2.1) since the solutions are hyperbolic.

Similar to [4,13,14,16,19], this problem boils down to the study of interaction of planar rarefaction waves. Hence, we first write down a planar rarefaction wave solution. Assume that the initial data for (1.1) are

$$(\rho, u, v)(x, y, 0) = \begin{cases} (\rho_1, 0, 0), & \text{for } n_1 x + n_2 y > 0, \\ \text{vaccum}, & \text{for } n_1 x + n_2 y < 0, \end{cases} \tag{2.2}$$

where $n_1^2 + n_2^2 = 1$, ρ_1 is a constant. The solution of (1.1) and (2.2) takes the form, see [17,26],

$$(\rho, u, v)(x, y, t) = \begin{cases} (\rho_1, 0, 0), & \text{for } \zeta > 1, \\ (\rho, u, v)(\zeta), & \text{for } -\infty < \zeta \leq 1, \end{cases} \tag{2.3}$$

where $\zeta = n_1 x/t + n_2 y/t$. The rarefaction wave solution $(q, u, v)(\zeta)$ satisfies

$$\begin{cases} \zeta = n_1 u + n_2 v + 1, \\ \mathrm{d}q = \mathrm{d}(n_1 u + n_2 v), \\ \mathrm{d}(-n_2 u + n_1 v) = 0, \end{cases} \tag{2.4}$$

where $q = \ln \rho$ for $\rho > 0$, $-n_2 u + n_1 v$ and $n_1 u + n_2 v$ are the tangential and normal components along the propagation of the wave, respectively. As a special case that the wave propagates in the x-direction, that is, $(n_1, n_2) = (1, 0)$, the solution is

$$x/t = u + 1,\ q - q_1 = u,\ v \equiv 0,\ -\infty < u \leq 0 \tag{2.5}$$

The gas away from the sharp corner expands uniformly to infinity as planar rarefaction waves R_1 and R_2 of the form $(q, u, v)(t, x, y) = (q, u, v)(\zeta)(\zeta = n_1 x/t + n_2 y/t)$, where (n_1, n_2) is the propagation direction of waves, see Fig. 1b. We assume that initially the gas is at rest, that is, $(u_0, v_0) = (0, 0)$. Otherwise, we replace (u, v) by $(u - u_0, v - v_0)$ and $(\xi, \eta)(\xi = x/t, \eta = y/t)$ by $(\xi - u_0, \eta - v_0)$ in the following computations (see also (3.1)). Then the rarefaction waves R_1, R_2 emitting from the initial discontinuities l_1, l_2 are expressed in (2.4) with $(n_1, n_2) = (\sin\theta, -\cos\theta)$ and $(n_1, n_2) = (\sin\theta, \cos\theta)$, respectively. These two waves begin to interact at $P = (1/\sin\theta, 0)$ in the (ξ, η) plane due to the presence of the sharp corner, and a wave interaction region, called the *wave interaction region* D, is formed to separate from the planar rarefaction waves by k_1, k_2,

$$k_1: \quad \sin^2\theta(\xi_1^2-1)=\cos(2\theta)\exp(-1-\eta_1), \quad (\xi_1>0,\ -1\le\eta_1<+\infty), \tag{2.6}$$
$$k_2: \quad \sin^2\theta(\xi_2^2-1)=\cos(2\theta)\exp(-1+\eta_2), \quad (\xi_2>0,\ -\infty<\eta_2\le 1), \tag{2.7}$$

where k_1 and k_2 are two characteristics from P, associated with the non-linear eigenvalues of system (3.1), and

$$\begin{cases} \xi_1=\xi\cos\theta+\eta\sin\theta, \\ \eta_1=-\xi\sin\theta+\eta\cos\theta, \end{cases} \qquad \begin{cases} \xi_2=\xi\cos\theta-\eta\sin\theta, \\ \eta_2=\xi\sin\theta+\eta\cos\theta, \end{cases} \tag{2.8}$$

So the wave interaction region D is bounded by k_1 and k_2, see Fig. 1b. The solution outside D consists of the constant state (q_0,u_0,v_0) and the planar rarefaction waves R_1 and R_2.

Problem 2.1. *Find a solution inside the wave interaction region D, subject to the boundary values on k_1 and k_2, which are determined continuously from the rarefaction waves R_1 and R_2.*

This problem is a Goursat-type boundary value problem for (3.1), which will be given in Sect. 3, since k_1 and k_2 are characteristics. We remark that since the planar rarefaction waves expand into the vacuum with infinite speed, the interaction region D is unbounded, different from the general polytropic cases $\gamma>1$ in earlier studies [4,13,16,19,20].

2.2. A semi-hyperbolic patch problem

For the semi-hyperbolic patch problem, we refer to [23] for the pressure-gradient equation case and to [21] for the case $\gamma>1$. More backgrounds are referred to the introduction in [23].

Suppose that (u_1,v_1,ρ_1) and $(u_4,v_4,\rho_4)(\rho_1>\rho_4)$ are two constant states, denoted by (1) and (4), respectively. For convenience, we assume that the state (4) of the gas is at rest, that is, $(u_4,v_4)=0$. Otherwise, we replace (u,v) by $(u-u_4,v-v_4)$ and (ξ,η) by $(\xi-u_4,\eta-v_4)$ in the following computations. Let $R_{14}(\eta)$ be a planar rarefaction wave, connecting the two constant states (1) and (4), defined by

$$\begin{cases} \eta=v+1, & (\eta_4\le\eta\le\eta_1) \\ v=v_4+q-q_4, & (0<\rho_4\le\rho\le\rho_1) \\ u=u_1=u_4=0 \end{cases} \tag{2.9}$$

in the domain $\xi>0$, where $\eta_i=v_i+1,\ i=1,4$. See Fig. 2. Note that v_1 is not free to choose—it is determined by the solution. Now denote the point $(\xi,\eta)=(0,v_1+1)$ by A. Let a positive characteristic curve passing through A in the rarefaction wave region intersect its bottom boundary at B, see Fig. 2. Under the above assumptions, we consider the following problem.

Problem 2.2. *For a given convex negative characteristic curve $\widehat{BC}$ where the end C is a sonic point, build a solution with a maximal hyperbolic domain which ends at either the sonic curve, or a single characteristic curve which both starts and ends at sonic points or an envelope $\widehat{CD}$ of the positive family of characteristic. See Fig. 2.*

This problem is actually a Goursat-type problem for system (3.1) since both the curves $\widehat{AB}$ and $\widehat{BC}$ are characteristics. Note that C is a sonic point, and thus, this problem is degenerate in the sense that the possible sonic curve $\widehat{AC}$ is a part of boundary of interaction domain bounded by $\widehat{AC}$, $\widehat{BC}$ and $\widehat{AB}$. Besides, this problem is different from the gas expansion problem in Sect. 2.1 since the density on $\widehat{AB}$ and $\widehat{BC}$ becomes larger and larger from B to A and from B to C, respectively, as we specify below. It is exciting to mention that this problem will give us some hint about sonic curve, which makes it possible to extend the solution into the subsonic domain in the future work.

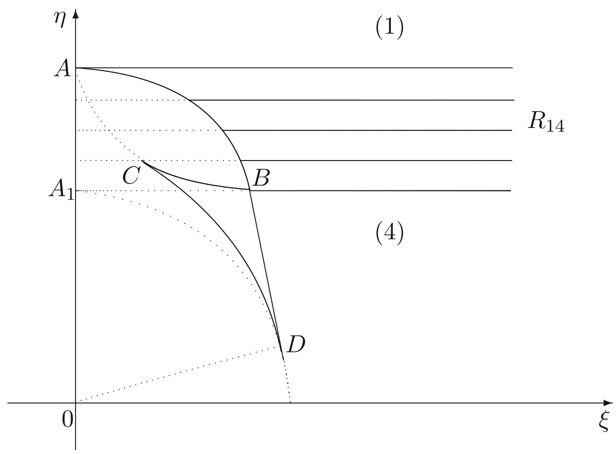


FIG. 2. Setup for a semi-hyperbolic patch

3. Self-similar Euler equations and characteristic forms

In order to solve the problems proposed previously, we will use the self-similar Euler equations, as conventionally done in the context of two-dimensional Riemann problem [26]. In terms of self-similar variables $(\xi, \eta) = (x/t, y/t)$, the system is

$$\begin{cases} Uq_\xi + Vq_\eta + u_\xi + v_\eta = 0, \\ Uu_\xi + Vu_\eta + q_\xi = 0, \\ Uv_\xi + Vv_\eta + q_\eta = 0, \end{cases} \tag{3.1}$$

where $(U, V) = (u - \xi, v - \eta)$ is the pseudo-velocity and $q = \ln\,\rho$ for $\rho > 0$. As the flow is ir-rotational $u_\eta = v_\xi$, or

$$U_\eta = V_\xi, \tag{3.2}$$

we can decouple a subsystem in terms of (u, v) from (3.1)

$$\begin{cases} (1 - U^2)u_\xi - UV(u_\eta + v_\xi) + (1 - V^2)v_\eta = 0, \\ u_\eta = v_\xi. \end{cases} \tag{3.3}$$

Then, we restore the density (the variable q) from the momentum equations of (3.1) or from the Bernoulli law

$$\frac{U^2 + V^2}{2} + q = -\phi, \quad \phi_\xi = U, \quad \phi_\eta = V. \tag{3.4}$$

Note that the sound speed is unit for the present isothermal case.

We introduce eigenvalues of (3.3)

$$\Lambda_\pm = \frac{UV \pm \sqrt{U^2 + V^2 - 1}}{U^2 - 1}, \tag{3.5}$$

which define pseudo-characteristics as the integral curves of

$$\Gamma^\pm : \ \frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \Lambda_\pm. \tag{3.6}$$

Then, we can write the system (3.3) in the characteristic form

$$\partial^\pm u + \Lambda_\mp \partial^\pm v = 0, \quad \partial^\pm = \partial_\xi + \Lambda_\pm \partial_\eta. \tag{3.7}$$

3.1. Characteristic form in terms of inclination angles of characteristics

We use inclination angle variables of characteristics as in [19]. Let

$$\tan\alpha = \Lambda_+, \qquad \tan\beta = \Lambda_-. \tag{3.8}$$

We express (U, V) in terms of α, β, we have

$$u = \xi - \frac{\cos\sigma}{\sin\delta}, \qquad v = \eta - \frac{\sin\sigma}{\sin\delta}, \tag{3.9}$$

where $\sigma = (\alpha+\beta)/2$ is the inclination angle of flow characteristics, and $\delta = (\alpha-\beta)/2$ is the Mach angle satisfying

$$\frac{1}{\sin^2\delta} = U^2 + V^2. \tag{3.10}$$

Introduce again scaled characteristic fields $\bar\partial^+ = \cos\alpha\partial_\xi + \sin\alpha\partial_\eta$ and $\bar\partial^- = \cos\beta\partial_\xi + \sin\beta\partial_\eta$. Then using the equations (3.9), we obtain

$$\bar\partial^- u = \cos\beta + \frac{1}{2}\frac{\cos\beta\bar\partial^-\alpha - \cos\alpha\bar\partial^-\beta}{\sin^2\delta}, \tag{3.11}$$

$$\bar\partial^- v = \sin\beta + \frac{1}{2}\frac{\sin\beta\bar\partial^-\alpha - \sin\alpha\bar\partial^-\beta}{\sin^2\delta}, \tag{3.12}$$

Putting (3.11) and (3.12) into (3.7) and noticing that $\Lambda_+ = \tan\alpha$, we find

$$\bar\partial^-\beta = \cos(2\delta)(2\sin^2\delta + \bar\partial^-\alpha). \tag{3.13}$$

In a similar way, we also have

$$\bar\partial^+\alpha = \cos(2\delta)(-2\sin^2\delta + \bar\partial^+\beta). \tag{3.14}$$

Thus, we obtain the characteristic form in terms of α and β,

$$\begin{cases} \bar\partial^+\alpha = \cos(2\delta)(-2\sin^2\delta + \bar\partial^+\beta), \\ \bar\partial^-\beta = \cos(2\delta)(2\sin^2\delta + \bar\partial^-\alpha). \end{cases} \tag{3.15}$$

By direct calculation, we have the following formulas:

$$\begin{cases} \bar\partial^+\delta = -\cos(2\delta)\sin^2\delta - \sin^2\delta\bar\partial^+\beta, \\ \bar\partial^-\delta = -\cos(2\delta)\sin^2\delta + \sin^2\delta\bar\partial^-\alpha, \end{cases} \tag{3.16}$$

and

$$\begin{cases} \bar\partial^+\sigma = -\cos(2\delta)\sin^2\delta + \cos^2\delta\bar\partial^+\beta, \\ \bar\partial^-\sigma = \cos(2\delta)\sin^2\delta + \cos^2\delta\bar\partial^-\alpha. \end{cases} \tag{3.17}$$

3.2. Riemann variables and diagonalization

It is useful to notice that the system (3.15) can be reduced to a diagonal form

$$\begin{cases} \bar\partial^+(-\beta + \cot\delta) = \cos(2\delta), \\ \bar\partial^-(\alpha + \cot\delta) = \cos(2\delta). \end{cases} \tag{3.18}$$

The Riemann variables $\cot\delta - \beta$ and $\cot\delta + \alpha$ correspond to the classical Riemann invariants for homogeneous systems, and there is a one-to-one correspondence from (α, β) to $(\cot\delta - \beta, \cot\delta + \alpha)$. In this paper, we use this diagonal system to show the boundedness of $C^{1,1}$ norms of solutions. This diagonal form is useful in the context of estimates of higher-order derivatives and uniform continuity of solutions.

4. Characteristic decompositions

We derive various characteristic decompositions for α, β and σ in this section. These formulas will be extensively used to construct invariant regions of solutions and establish a prior gradient estimates or higher-order derivative estimates. We first quote the commutator relations of $\partial^{\pm}$ and $\bar{\partial}^{\pm}$ from [19].

Proposition 4.1. (Commutator relation of $\partial^{\pm}$). *For any quantity $I(\xi, \eta)$, there holds*

$$\partial^{-}\partial^{+}I - \partial^{+}\partial^{-}I = \frac{\partial^{-}\Lambda_{+} - \partial^{+}\Lambda_{-}}{\Lambda_{-}\Lambda_{+}}(\partial^{-}I - \partial^{+}I). \tag{4.1}$$

Proposition 4.2. (Commutator relation of $\bar{\partial}^{\pm}$). *For any quantity $I(\xi, \eta)$, there holds*

$$\bar{\partial}^{-}\bar{\partial}^{+}I - \bar{\partial}^{+}\bar{\partial}^{-}I = \frac{1}{\sin(2\delta)}\Big\{[\cos(2\delta)\bar{\partial}^{+}\beta - \bar{\partial}^{-}\alpha]\bar{\partial}^{-}I - [\bar{\partial}^{+}\beta - \cos(2\delta)\bar{\partial}^{-}\alpha]\bar{\partial}^{+}I\Big\}. \tag{4.2}$$

Using Proposition 4.2, we can obtain the characteristic decompositions for the inclination angles of α and β.

Proposition 4.3. *For the inclination angles of α and β, we have*

$$\begin{cases} \bar{\partial}^{+}\bar{\partial}^{-}\alpha + M_1\bar{\partial}^{-}\alpha = \frac{\sin(2\delta)}{2}(3\tan^2\delta - 1)\bar{\partial}^{+}\alpha, \\ \bar{\partial}^{-}\bar{\partial}^{+}\beta + M_2\bar{\partial}^{+}\beta = \frac{\sin(2\delta)}{2}(3\tan^2\delta - 1)\bar{\partial}^{-}\beta, \end{cases} \tag{4.3}$$

where

$$\begin{aligned} M_1 &= \tfrac{1}{\sin(2\delta)}\Big\{-8\sin^6\delta - \bar{\partial}^{-}\alpha + [1 - \tfrac{1}{2}(1-\tan^2\delta)\sin^2(2\delta)]\bar{\partial}^{+}\beta\Big\}, \\ M_2 &= \tfrac{1}{\sin(2\delta)}\Big\{-8\sin^6\delta + \bar{\partial}^{+}\beta - [1 - \tfrac{1}{2}(1-\tan^2\delta)\sin^2(2\delta)]\bar{\partial}^{-}\alpha\Big\}. \end{aligned} \tag{4.4}$$

Proof. We only derive the first one of (4.3), and the other can be obtained in a similar way. We apply the commutator relation (4.2) for α to obtain

$$\begin{aligned} &\bar{\partial}^{+}\bar{\partial}^{-}\alpha - \bar{\partial}^{-}\bar{\partial}^{+}\alpha + \frac{1}{\sin(2\delta)}\Big\{[\cos(2\delta)\bar{\partial}^{+}\beta - \bar{\partial}^{-}\alpha]\bar{\partial}^{-}\alpha - [\bar{\partial}^{+}\beta - \cos(2\delta)\bar{\partial}^{-}\alpha]\bar{\partial}^{+}\alpha\Big\} \\ &= 0. \end{aligned} \tag{4.5}$$

Differentiating the first equation of (3.15), we have

$$\bar{\partial}^{-}\bar{\partial}^{+}\alpha = \cos(2\delta)\bar{\partial}^{-}\bar{\partial}^{+}\beta - 4\sin\delta\cos(3\delta)\bar{\partial}^{-}\delta - 2\sin(2\delta)\bar{\partial}^{-}\delta\bar{\partial}^{+}\beta. \tag{4.6}$$

Then we substitute (4.6) into (4.5) to yield

$$\begin{aligned} &\bar{\partial}^{+}\bar{\partial}^{-}\alpha - \cos(2\delta)\bar{\partial}^{-}\bar{\partial}^{+}\beta + 4\sin\delta\cos(3\delta)\bar{\partial}^{-}\delta + 2\sin(2\delta)\bar{\partial}^{-}\delta\bar{\partial}^{+}\beta \\ &\quad + \frac{1}{\sin(2\delta)}\Big\{[\cos(2\delta)\bar{\partial}^{+}\beta - \bar{\partial}^{-}\alpha]\bar{\partial}^{-}\alpha - [\bar{\partial}^{+}\beta - \cos(2\delta)\bar{\partial}^{-}\alpha]\bar{\partial}^{+}\alpha\Big\} = 0. \end{aligned} \tag{4.7}$$

We next apply the commutator relation (4.2) for β to obtain

$$\begin{aligned} &\bar{\partial}^{+}\bar{\partial}^{-}\beta - \bar{\partial}^{-}\bar{\partial}^{+}\beta \\ &\quad + \frac{1}{\sin(2\delta)}\Big\{[\cos(2\delta)\bar{\partial}^{+}\beta - \bar{\partial}^{-}\alpha]\bar{\partial}^{-}\beta - [\bar{\partial}^{+}\beta - \cos(2\delta)\bar{\partial}^{-}\alpha]\bar{\partial}^{+}\beta\Big\} = 0. \end{aligned} \tag{4.8}$$

Similarly, we differentiate the second equation of (3.15) and substitute it into (4.8) to eliminate the term $\bar{\partial}^{+}\bar{\partial}^{-}\beta$ so as to yield

$$\begin{aligned} &-\bar{\partial}^{-}\bar{\partial}^{+}\beta + \cos(2\delta)\bar{\partial}^{+}\bar{\partial}^{-}\alpha + 4\sin\delta\cos(3\delta)\bar{\partial}^{+}\delta - 2\sin(2\delta)\bar{\partial}^{+}\delta\bar{\partial}^{-}\alpha \\ &\quad + \frac{1}{\sin(2\delta)}\Big\{[\cos(2\delta)\bar{\partial}^{+}\beta - \bar{\partial}^{-}\alpha]\bar{\partial}^{-}\beta - [\bar{\partial}^{+}\beta - \cos(2\delta)\bar{\partial}^{-}\alpha]\bar{\partial}^{+}\beta\Big\} = 0. \end{aligned} \tag{4.9}$$

We use (4.9) to eliminate the term $\bar\partial^-\bar\partial^+\beta$ in (4.7) and obtain

$$\begin{aligned}
&\sin^2(2\delta)\bar\partial^+\bar\partial^-\alpha - 2\cos(2\delta)[2\sin\delta\cos(2\delta) - \sin(2\delta)\bar\partial^-\alpha]\bar\partial^+\delta\\
&+2[2\sin\delta\cos(2\delta) + \sin(2\delta)\bar\partial^+\beta]\bar\partial^-\delta\\
&+\frac{1}{\sin(2\delta)}\Big\{(\cos(2\delta)\bar\partial^+\beta - \bar\partial^-\alpha)(\bar\partial^-\alpha - \cos(2\delta)\bar\partial^-\beta)\\
&-(\bar\partial^+\beta - \cos(2\delta)\bar\partial^-\alpha)(\bar\partial^+\alpha - \cos(2\delta)\bar\partial^+\beta)\Big\} = 0.
\end{aligned} \tag{4.10}$$

Recall the equations (3.15) and (3.16). Then, we obtain the first equation of (4.3) by a direct calculation and simplification. □

Using Proposition 4.3, we can derive the characteristic decompositions for σ.

Proposition 4.4. *For the inclination angle σ, we have*

$$\begin{cases} \bar\partial^-\bar\partial^+\sigma + N_1\bar\partial^+\sigma = \tan\delta\cdot a(\delta)\bar\partial^-\sigma,\\ \bar\partial^+\bar\partial^-\sigma + N_2\bar\partial^-\sigma = \tan\delta\cdot a(\delta)\bar\partial^+\sigma, \end{cases} \tag{4.11}$$

where

$$\begin{aligned}
&N_1 = \tan\delta(1-4\sin^2\delta) + \tfrac{1}{\cos^2\delta}\Big[\tfrac12\tan\delta\cos(2\delta) + \tfrac{1}{\sin(2\delta)}(\bar\partial^+\sigma - \cos(2\delta)\bar\partial^-\sigma)\Big],\\
&N_2 = \tan\delta(1-4\sin^2\delta) + \tfrac{1}{\cos^2\delta}\Big[\tfrac12\tan\delta\cos(2\delta) - \tfrac{1}{\sin(2\delta)}(\bar\partial^-\sigma - \cos(2\delta)\bar\partial^+\sigma)\Big],\\
&a(\delta) = \tfrac12[2\tan^2\delta - (1-\tan^2\delta)\cos(2\delta)].
\end{aligned} \tag{4.12}$$

Proof. Differentiating the first equation of (3.17), we obtain

$$\begin{aligned}
\bar\partial^-\bar\partial^+\sigma &= \bar\partial^-(-\cos(2\delta)\sin^2\delta + \cos^2\delta\bar\partial^+\beta)\\
&= \frac12\sin(2\delta)[\sin^2\delta - \cos(2\delta) - \bar\partial^+\beta](\bar\partial^-\alpha - \bar\partial^-\beta) + \cos^2\delta\bar\partial^-\bar\partial^+\beta.
\end{aligned} \tag{4.13}$$

We substitute the second equation of (4.3) into (4.13) to eliminate the term $\bar\partial^-\bar\partial^+\beta$. Then, we recall the relation of $\bar\partial^\pm\alpha$, $\bar\partial^\pm\beta$ and $\bar\partial^\pm\sigma$ in (3.15) and (3.17) to obtain

$$\bar\partial^-\bar\partial^+\sigma = A\bar\partial^+\sigma + B\bar\partial^-\sigma + C, \tag{4.14}$$

where

$$A = \frac{\cos(2\delta)\sin(2\delta)\sin^2\delta}{\cos^4\delta} - \frac{\cos(2\delta)\sin^2\delta}{\sin(2\delta)\cos^2\delta} - \frac{\sin(2\delta)\sin^2\delta}{\cos^4\delta}\bar\partial^-\sigma - M_2, \tag{4.15}$$

$$\begin{aligned}
B = {}&\frac{\sin(2\delta)\sin^2\delta}{\cos^2\delta}\Big[2\sin^2\delta - \cos(2\delta) - \frac{\cos(2\delta)\sin^2\delta}{\cos^2\delta}\Big]\\
&+\frac{\sin(2\delta)}{2}\cos(2\delta)(3\tan^2\delta - 1) + \frac{\cos(2\delta)\sin^2\delta}{\cos^2\delta\sin(2\delta)}[1 - 2\sin^2\delta\cos(2\delta)],
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
C = {}&-\frac{\sin(2\delta)\cos(2\delta)\sin^2\delta}{\cos^2\delta}\Big[2\sin^2\delta - \cos(2\delta) - \frac{\cos(2\delta)\sin^2\delta}{\cos^2\delta}\Big]\\
&+\frac{\sin(2\delta)}{2}\cos(2\delta)\sin^2\delta(3\tan^2\delta - 1)\\
&-\frac{\cos(2\delta)\sin^2\delta}{\sin(2\delta)}\Big[-8\sin^6\delta + \frac{2\cos(2\delta)\sin^2\delta(1-\sin^2\delta\cos(2\delta))}{\cos^2\delta}\Big].
\end{aligned} \tag{4.17}$$

We continue to compute directly and simplify to obtain the expressions in (4.12). □

Remark 4.5. *Note that $a(\delta)$ has the following factorization:*

$$\begin{aligned} a(\delta) &:= \frac{1}{2}[2\tan^2\delta - (1-\tan^2\delta\cos(2\delta))] \\ &= \frac{1}{2}\cos^2\delta\left(\tan^2\delta - \frac{1}{2+\sqrt{5}}\right)(\tan^2\delta + 2 + \sqrt{5}). \end{aligned} \tag{4.18}$$

Denote $\bar{\theta}$ to be given by $\tan^2\bar{\theta} = 1/(2+\sqrt{5})$. Then if $\delta > \bar{\theta}$, we have $a(\delta) > 0$.

Using Proposition 4.3 again, we have the following characteristic decompositions:

Proposition 4.6. *For the variables α and β satisfying $\alpha - \beta \neq \pi/2, \pi$, we have the following characteristic decompositions:*

$$\begin{cases} \bar{\partial}^-\bar{\partial}^+\alpha = \left\{\frac{\sin(2\delta)}{2}(3\tan^2\delta - 1) + \frac{\cos(2\delta)+2\sin^2\delta\cos(2\delta)-\sin^2(2\delta)}{\sin(2\delta)\cos(2\delta)}\bar{\partial}^-\alpha - \frac{\bar{\partial}^+\beta}{\sin(2\delta)}\right\}\bar{\partial}^+\alpha, \\ \bar{\partial}^+\bar{\partial}^-\beta = \left\{\frac{\sin(2\delta)}{2}(3\tan^2\delta - 1) - \frac{\cos(2\delta)+2\sin^2\delta\cos(2\delta)-\sin^2(2\delta)}{\sin(2\delta)\cos(2\delta)}\bar{\partial}^+\beta + \frac{\bar{\partial}^-\alpha}{\sin(2\delta)}\right\}\bar{\partial}^-\beta. \end{cases} \tag{4.19}$$

Proposition 4.7. *For the variables α and β satisfying $\alpha - \beta \neq \pi/2, \pi$, we have the following characteristic decompositions:*

$$\begin{cases} \bar{\partial}^-\left(\frac{-\bar{\partial}^+\alpha}{\cos(2\delta)}\right) = \frac{-\bar{\partial}^+\alpha}{\cos(2\delta)}\left\{(-2-\cos(2\delta))\tan\delta + f_1\frac{-\bar{\partial}^+\alpha}{\cos(2\delta)} + f_2\frac{\bar{\partial}^-\beta}{\cos(2\delta)}\right\}, \\ \bar{\partial}^+\left(\frac{\bar{\partial}^-\beta}{\cos(2\delta)}\right) = \frac{\bar{\partial}^-\beta}{\cos(2\delta)}\left\{(-2-\cos(2\delta))\tan\delta + f_1\frac{\bar{\partial}^-\beta}{\cos(2\delta)} + f_2\frac{-\bar{\partial}^+\alpha}{\cos(2\delta)}\right\}, \end{cases} \tag{4.20}$$

where

$$f_1 = \frac{1}{\sin(2\delta)}, \qquad f_2 = \frac{\cos^2(2\delta) + 2\sin^2\delta}{\sin(2\delta)}.$$

Proposition 4.8. *For the variables α and β satisfying $\alpha - \beta \neq \pi/2, \pi$ and any real number n, we have the following characteristic decompositions:*

$$\begin{cases} \bar{\partial}^-\left(\frac{\bar{\partial}^+\alpha}{(-\cot(2\delta))^n}\right) = \frac{\bar{\partial}^+\alpha}{(-\cot(2\delta))^n}\left\{g\tan\delta + g_1\frac{\bar{\partial}^+\alpha}{(-\cot(2\delta))^n} + g_2\frac{\bar{\partial}^-\beta}{(-\cot(2\delta))^n}\right\}, \\ -\bar{\partial}^+\left(\frac{\bar{\partial}^-\beta}{(-\cot(2\delta))^n}\right) = \frac{\bar{\partial}^-\beta}{(-\cot(2\delta))^n}\left\{-g\tan\delta + g_1\frac{\bar{\partial}^-\beta}{(-\cot(2\delta))^n} + g_2\frac{\bar{\partial}^+\alpha}{(-\cot(2\delta))^n}\right\}, \end{cases} \tag{4.21}$$

where

$$\begin{cases} g = \frac{n-2+\cos^2(2\delta)+4\cos^4\delta}{-\cos(2\delta)}, \\ g_1 = \frac{(-\cot(2\delta))^n}{-\cos(2\delta)\sin(2\delta)}, \qquad g_2 = \frac{(-\cot(2\delta))^n[\cos^2\delta+(2n-3)\sin^2\delta]}{\cos^2(2\delta)\sin(2\delta)}. \end{cases} \tag{4.22}$$

5. The gas expansion problem

In this section, we use the characteristic decompositions of the previous section to study the expansion of a wedge of gas into vacuum directly in the self-similar plane and obtain a unique global smooth solution in the interaction region.

5.1. Boundary data estimates and local existence

For the boundary data

$$\alpha\mid_{k_1}=\theta, \qquad \beta\mid_{k_2}=-\theta, \tag{5.1}$$

we have the following estimates:

Lemma 5.1. (Boundary data estimate)

(i) *If* $0<\theta<\pi/4$, *there holds*

$$2\theta\leq(\alpha-\beta)\mid_{k_i}<\frac{\pi}{2}, \qquad (i=1,2).$$

(ii) *If* $\pi/4<\theta<\pi/2$, *there holds*

$$\frac{\pi}{2}<(\alpha-\beta)\mid_{k_i}\leq 2\theta, \qquad (i=1,2).$$

Proof. We only consider the estimate on k_1 when $0<\theta<\pi/4$, and the other case can be proved in a similar way. Since $\alpha\equiv\theta$ on k_1, we restrict (3.15) on k_1 to obtain

$$\begin{cases}-\bar{\partial}^-(\theta-\beta)=2\sin^2\frac{\theta-\beta}{2}\cos(\theta-\beta),\\ \theta-\beta=2\theta, \qquad \text{at } P=k_1\cap k_2.\end{cases}$$

Integrating along k_1 and noting the direction of $\bar{\partial}^-$, it is easy to deduce that $2\theta\leq\theta-\beta<\pi/2$ (note that at point $P=k_1\cap k_2,\beta=-\theta$). Therefore, we have

$$2\theta\leq(\alpha-\beta)\mid_{k_1}<\frac{\pi}{2}.$$

The proof is complete. □

We rewrite the equation (3.18) into

$$\begin{cases}(-\beta+\cot\delta)_\xi+\tan\alpha(-\beta+\cot\delta)_\eta=\frac{\cos(2\delta)}{\cos\alpha},\\ (\alpha+\cot\delta)_\xi+\tan\beta(\alpha+\cot\delta)_\eta=\frac{\cos(2\delta)}{\cos\beta}.\end{cases} \tag{5.2}$$

It is easy to obtain the local existence of this system, see [22]. Let $(\xi_{\bar{\delta}},\eta_{\bar{\delta}})=\{\delta=\bar{\delta}\}\cap k_1(\text{or } \mathrm{k}_2)$, and $D_{\bar{\delta}}(\bar{\delta}$ is in between θ and $\pi/4)$ be the region enclosed by the curves k_1, k_2 and $\xi=\xi_{\bar{\delta}}$. Then, we have

Lemma 5.2. (Local existence) *There is a* $\delta_0>0$ *such that a* C^1*-solution of* (5.2) *and* (5.1) *exists uniquely in the region* D_{δ_0}, *where* δ_0 *depends only on the* C^1 *norms of* α, β *on the boundaries* k_1 *and* k_2.

5.2. The signs of $\bar{\partial}^+\alpha$, $\bar{\partial}^-\beta$ and the invariant triangle

Theorem 5.3. *Assume that the solution* $(\alpha,\beta)\in C^1$. *Then, we have*

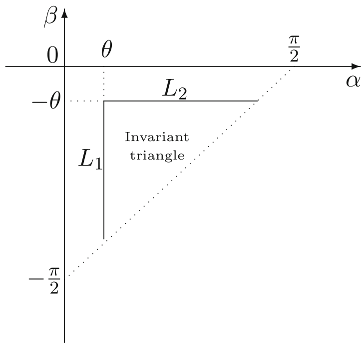
(i) *If* $\bar{\theta}<\theta<\pi/4$, *there holds*

$$\alpha\geq\theta, \quad \beta\leq-\theta, \quad 2\theta\leq\alpha-\beta\leq\frac{\pi}{2}, \quad \bar{\partial}^+\alpha<0, \quad \bar{\partial}^-\beta>0.$$

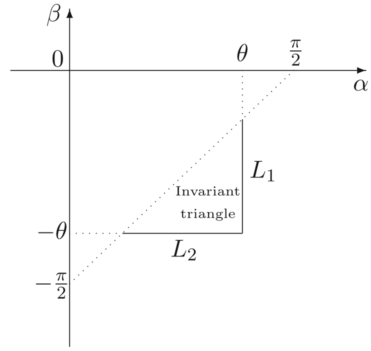
(ii) *If* $\pi/4<\theta<\pi/2$, *there holds*

$$\alpha\leq\theta, \quad \beta\geq-\theta, \quad \frac{\pi}{2}\leq\alpha-\beta\leq 2\theta, \quad \bar{\partial}^+\alpha>0, \quad \bar{\partial}^-\beta<0.$$

See Fig. 3. *Recall that* $\bar{\theta}$ *satisfies* $\tan^2\bar{\theta}=1/(2+\sqrt{5})$ *such that* $a(\bar{\theta})=0$ *(see Remark* 4.5*).*



(a) Case $\bar{\theta} < \theta < \pi/4$



(b) Case $\frac{\pi}{4} < \theta < \pi/2$

FIG. 3. Invariant triangles

Proof. Case (i). Noting $\bar{\partial}^-\alpha = 0$ and $\bar{\partial}^+\beta = 0$ on k_1 and k_2, respectively, we then obtain $\bar{\partial}^-\beta > 0$ on k_1 and $\bar{\partial}^+\alpha < 0$ on k_2 by (3.15). Using the characteristic decompositions (4.19), we obtain that $\bar{\partial}^-\beta > 0$ and $\bar{\partial}^+\alpha < 0$ in the interaction region before the solution hits the line $\delta = \pi/4$. Noting the direction of $\bar{\partial}^-$ and $\bar{\partial}^+$, we have $\alpha > \theta$ and $\beta < -\theta$. Thus, we have $\delta > \bar{\theta}$ and $a(\delta) > 0$ for the solution.

Next we show that $\delta < \pi/4$, that is, $\cos(2\delta) > 0$ in the interaction region. Since $\bar{\partial}^-\sigma > 0$ on k_1 and $\bar{\partial}^+\sigma < 0$ on k_2, the bootstrapping argument on (4.11) shows that $\bar{\partial}^-\sigma > 0$ and $\bar{\partial}^+\sigma < 0$ inside the interaction region. Hence, we have

$$\bar{\partial}^+(\alpha - \beta) = -\bar{\partial}^+(\alpha + \beta) + 2\bar{\partial}^+\alpha > 2\bar{\partial}^+\alpha = 2\cos(2\delta)(-2\sin^2\delta + \bar{\partial}^+\beta),$$

or

$$-\bar{\partial}^+\cos(2\delta) = \sin(2\delta)\bar{\partial}^+(\alpha - \beta) > 2\cos(2\delta)\sin(2\delta)(-2\sin^2\delta + \bar{\partial}^+\beta). \tag{5.3}$$

This implies that $\cos(2\delta) > 0$ in the interaction region. Thus, the upper triangle is invariant.

Case (ii). We show that the lower triangle is invariant. We can obtain $\bar{\partial}^+\alpha > 0$ and $\bar{\partial}^-\beta < 0$ in the interaction region in a similar way with Case (i). Then, we have $\alpha < \theta$ and $\beta > -\theta$, thus $\delta < \theta$ and $\delta > \pi/4 > \bar{\theta}$ and $a(\delta) > 0$ for the solution before the solution hits the line $\delta = \pi/4$. Using the equation (4.11) and noting $\bar{\partial}^-\sigma < 0$ on k_1 and $\bar{\partial}^+\sigma > 0$ on k_2, we can establish $\bar{\partial}^+\sigma > 0$ and $\bar{\partial}^-\sigma < 0$ by the bootstrapping method. Thus, we have

$$\bar{\partial}^+(\alpha - \beta) = -\bar{\partial}^+(\alpha + \beta) + 2\bar{\partial}^+\alpha < 2\bar{\partial}^+\alpha = 2\cos(2\delta)(-2\sin^2\delta + \bar{\partial}^+\beta),$$

or

$$-\bar{\partial}^+\cos(2\delta) = \sin(2\delta)\bar{\partial}^+(\alpha - \beta) < 2\cos(2\delta)\sin(2\delta)(-2\sin^2\delta + \bar{\partial}^+\beta), \tag{5.4}$$

which implies $\cos(2\delta) < 0$, that is, $\delta > \pi/4$ in the interaction region. Thus, the lower triangle is invariant. □

5.3. Gradient estimates

In this subsection, we provide a priori gradient estimates of the solutions (u, v) or (α, β). We use the characteristic decompositions (4.20) to establish the gradient estimate. Let

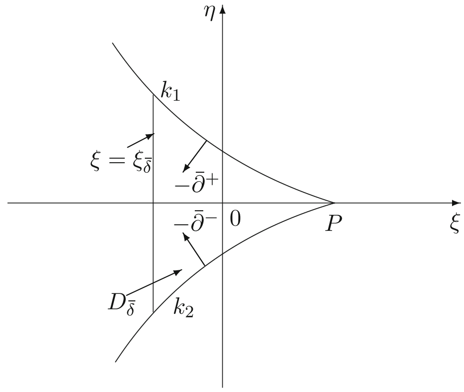


Fig. 4. Interaction domain

$$X = \max\left\{ \max_D \left[(2+\cos(2\delta)\sin(2\delta)\cos(2\delta)\tan\delta), \frac{2+\cos(2\delta)\cos(2\delta)\tan\delta}{\cos^2(2\delta)+2\sin^2\delta}\right], \right.$$
$$\left. \max_{k_2} \mid \bar{\partial}^+\alpha \mid, \ \max_{k_1} \mid \bar{\partial}^-\beta \mid \right\}. \tag{5.5}$$

Then, X is independent of δ. Denote $D_{\bar{\delta}}$ to be the domain enclosed by the two characteristics from P and the line $\xi = \xi_{\bar{\delta}}$. See Fig. 4. Then, we have

Lemma 5.4. *$\bar{\partial}^{\pm}u$ and $\bar{\partial}^{\pm}v$, $\bar{\partial}^{\pm}\alpha$ and $\bar{\partial}^{\pm}\beta$ are all uniformly bounded for C^1 solutions in $D_{\bar{\delta}}$.*

Proof. We only prove $\bar{\partial}^+\alpha$ and $\bar{\partial}^-\beta$ are uniformly bounded in $D_{\bar{\delta}}$ for the case $0 < \bar{\theta} < \theta < \pi/4$, and then the uniformly boundedness of $\bar{\partial}^{\pm}u$ and $\bar{\partial}^{\pm}v$, $\bar{\partial}^-\alpha$ and $\bar{\partial}^+\beta$ can be obtained by the relation formulas (3.11), (3.12) and (3.15).

We next prove that $-\bar{\partial}^+\alpha/\cos(2\delta)$ and $\bar{\partial}^-\beta/\cos(2\delta)$ satisfy

$$\left(\frac{-\bar{\partial}^+\alpha}{\cos(2\delta)}, \frac{\bar{\partial}^-\beta}{\cos(2\delta)}\right) \in \mathbb{I} := \left(0, \frac{X}{\cos(2\bar{\delta})}\right) \times \left(0, \frac{X}{\cos(2\bar{\delta})}\right).$$

By Theorem 5.3, we obtain that $-\bar{\partial}^+\alpha/\cos(2\delta)$ and $\bar{\partial}^-\beta/\cos(2\delta)$ are positive in the region under consideration. Suppose the line $\xi = \xi_{\delta_1}$(δ_1 is in between $\bar{\delta}$ and θ) is the first time that one of $-\bar{\partial}^+\alpha/\cos(2\delta)$ and $\bar{\partial}^-\beta/\cos(2\delta)$ touches the boundary of $\mathbb{I}$ for the local solution in $D_{\bar{\delta}}$. Without the loss of generality, we assume that $-\bar{\partial}^+\alpha/\cos(2\delta) = X/\cos(2\bar{\delta})$ at a point on the line $\xi = \xi_{\delta_1}$. Putting $-\bar{\partial}^+\alpha/\cos(2\delta) = X/\cos(2\bar{\delta})$ into the first equation of (4.20) and noticing both of f_1 and f_2 are positive functions, we obtain that $\bar{\partial}^-(-\bar{\partial}^+\alpha/\cos(2\delta)) > 0$. Noting the direction of $\bar{\partial}^-$, we know that it is impossible. Then, we have $\mid \bar{\partial}^+\alpha \mid< X/\cos(2\bar{\delta})$ and $\mid \bar{\partial}^-\beta \mid< X/\cos(2\bar{\delta})$. Therefore, there exists some constant $C = C(\theta)$ such that

$$\mid \bar{\partial}^{\pm}(\alpha,\beta,u,v) \mid\leq \frac{C}{\cos(2\bar{\delta})}.$$

□

Lemma 5.5. (Gradient estimate) *Assume that there is a C^1 solution in $D_{\bar{\delta}}$($\bar{\delta}$ is in between θ and $\pi/4$), where the system is hyperbolic($\alpha - \beta \neq 0, \pi$). Then the C^1 norm of α and β, u and v have a uniform bound*

$$\parallel (\alpha,\beta,u,v) \parallel_{C^1(D_{\bar{\delta}})}\leq \frac{C}{\cos(2\bar{\delta})}, \tag{5.6}$$

for some constant C, independent of $\bar{\delta}$.

Proof. Using the previous lemma and the identities

$$\partial_\xi = -\frac{\sin\beta\bar{\partial}^+ - \sin\alpha\bar{\partial}^-}{\sin(\alpha-\beta)}, \quad \partial_\eta = \frac{\cos\beta\bar{\partial}^+ - \cos\alpha\bar{\partial}^-}{\sin(\alpha-\beta)}, \tag{5.7}$$

we conclude that the gradient estimates are expressed in (5.6). □

5.4. Global solutions

We are now at the position to obtain the global solution in the whole interaction region. At first, we need $C^{1,1}$ estimates on the solutions.

Lemma 5.6. ($C^{1,1}$ estimate) *Assume that there exists a C^1 solution in the domain $D_{\bar{\delta}}$. Then, for some constant C independent of $\bar{\delta}$, we have*

$$\| (\alpha,\beta) \|_{C^{1,1}(D_{\bar{\delta}})} \leq \frac{C}{\cos^2(2\bar{\delta})}. \tag{5.8}$$

Proof. We use the diagonal form (3.18) to prove it. For convenience, we denote $R_1 = \cot\delta - \beta$ and $R_2 = \cot\delta + \alpha$. In view of Lemma 5.5 and the relation between $(\partial_\xi, \partial_\eta)$ and $(\bar{\partial}^+, \bar{\partial}^-)$, it suffices to prove that R_1 and R_2 satisfy

$$\overline{\nabla}^2 R_i = W(\alpha, \beta, \overline{\nabla} R_1, \overline{\nabla} R_2), \quad i = 1, 2, \tag{5.9}$$

where $\overline{\nabla} = (\bar{\partial}^+, \bar{\partial}^-)$ and $\overline{\nabla}^2 = (\bar{\partial}^+\bar{\partial}^+, \bar{\partial}^+\bar{\partial}^-, \bar{\partial}^-\bar{\partial}^+, \bar{\partial}^-\bar{\partial}^-)$. The equation (5.9) implies all second-order derivatives can be expressed with lower-order terms.

We just prove for R_1 that the results of R_2 can be obtained in a similar way. Differentiating (3.18) yields,

$$\bar{\partial}^+\bar{\partial}^+ R_1 = -2\sin(2\delta)\bar{\partial}^+\delta, \quad \bar{\partial}^-\bar{\partial}^+ R_1 = -2\sin(2\delta)\bar{\partial}^-\delta. \tag{5.10}$$

Using the commutator relation (4.2) for $I = R_1$, we obtain

$$\bar{\partial}^+\bar{\partial}^- R_1 = \bar{\partial}^-\bar{\partial}^+ R_1 + \text{lower-order terms}. \tag{5.11}$$

Combining (5.10) and (5.11), we find that the term $\bar{\partial}^+\bar{\partial}^- R_1$ can be rewritten in the form of (5.9).

For the term $\bar{\partial}^-\bar{\partial}^- R_1$, we use again the commutator relation (4.2) for $I = \bar{\partial}^- R_1$,

$$\bar{\partial}^+\bar{\partial}^-\bar{\partial}^- R_1 = \bar{\partial}^-\bar{\partial}^+\bar{\partial}^- R_1 + \text{lower-order terms}. \tag{5.12}$$

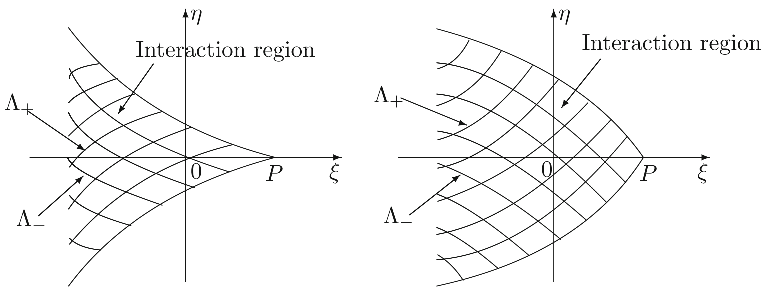
Then, we can obtain a first-order equation of $\bar{\partial}^-\bar{\partial}^-$,

$$\bar{\partial}^+(\bar{\partial}^-\bar{\partial}^- R_1) + P\bar{\partial}^-\bar{\partial}^- R_1 = Q \tag{5.13}$$

for some $P = P(\alpha, \beta, \overline{\nabla} R_1, \overline{\nabla} R_2)$, $Q = Q(\alpha, \beta, \overline{\nabla} R_1, \overline{\nabla} R_2)$. Therefore, the term $\bar{\partial}^-\bar{\partial}^- R_1$ can be rewritten in the form of (5.9) by integrating along the direction $\bar{\partial}^+$. The proof is complete. □

Lemma 5.7. *The line $\xi = \xi_{\bar{\delta}}$ is non-characteristic and δ attains $\pi/4$ only at infinity.*

Proof. It is easy to obtain that the line $\xi = \xi_{\bar{\delta}}$ is non-characteristic by Theorem 5.3. Next we prove δ attains $\pi/4$ only possible at infinity. We just prove the case $\theta < \pi/4$, and the other case is similar. We use the contradiction argument to prove it. Suppose that there is a point T that is the first time such that $\delta = \pi/4$. From the point T, we draw the negative characteristic, called l^-, up to the boundary k_2, then obtain $\delta < \pi/4$ on $l^- \backslash \{T\}$. By Theorem 5.3 again, we have $\bar{\partial}^+\alpha < 0$ on $l^- \backslash \{T\}$, which implies $\bar{\partial}^-\bar{\partial}^+\alpha < 0$ at the point T. However, from Theorem 5.3, we also have $\bar{\partial}^+\sigma < 0$ and $\bar{\partial}^-\sigma > 0$. Then, we



(a) Case $\bar{\theta} < \theta < \frac{\pi}{4}$ **(b)** Case $\frac{\pi}{4} < \theta < \frac{\pi}{2}$

Fig. 5. Convexity of characteristic

find $\bar{\partial}^+\beta < 0$ and $\bar{\partial}^-\alpha > 0$ at T by the equation (3.17). Noting $\bar{\partial}^+\alpha = \bar{\partial}^-\beta = 0$ at T and using the first equation of (4.19), we obtain

$$\begin{aligned}\bar{\partial}^-\bar{\partial}^+\alpha &= \left[\frac{\sin(2\delta)}{2}(3\tan^2\delta - 1) - \frac{\bar{\partial}^+\beta}{\sin(2\delta)}\right]\bar{\partial}^+\alpha \\ &\qquad +(-2\sin^2\delta + \bar{\partial}^+\beta)\frac{\cos(2\delta) + 2\sin^2\delta\cos(2\delta) - \sin^2(2\delta)}{\sin(2\delta)}\bar{\partial}^-\alpha \\ &= -\sin(2\delta)\bar{\partial}^-\alpha[-2\sin^2(2\delta) + \bar{\partial}^+\beta] > 0\end{aligned}$$

at T, which contradicts to our earlier assertion $\bar{\partial}^-\bar{\partial}^+\alpha < 0$ at the point T. □

Theorem 5.8. (Global existence) *There exists a unique global smooth solution to the interaction of two rarefaction waves with interaction(half) angle $\bar{\theta} < \theta < \pi/2$. The characteristics are either convex or concave depending on whether the interaction angle is larger or smaller than $\pi/4$. See Fig. 5.*

Proof. The proof follows from the previous lemmas, as sketched below. We can first show that there is a C^1 solution (α, β) in $D_{\bar{\delta}}$ for any given $\bar{\delta}$. Then, we can use almost the same argument as in Lemma 6.8 in the next section to prove that this solution can be extended to the domain $D_{\bar{\delta}-\delta_0}$ for some small $\delta_0 > 0$ if $\theta > \pi/4$ (resp.$D_{\bar{\delta}+\delta_0}$ if $\theta < \pi/4$). Thus, the solution can be extended up to $\delta = \pi/4$ (equivalently, $\xi = \infty$). □

5.5. Global Hölder continuity of solutions

We will prove that the solution of the gas expansion problem is globally Hölder continuous.

Theorem 5.9. *The solution in Theorem 5.8 is globally Hölder continuous with exponent* $1/2$.

Proof. From the characteristic decompositions (4.20), we mimic the estimates deriving (5.6) to obtain

$$|\cos(2\delta)\bar{\partial}^{\pm}\alpha| \le C, \qquad |\cos(2\delta)\bar{\partial}^{\pm}\beta| \le C$$

for a generic constant C only depending on the boundary data. Then from (3.16), we get

$$|\cos(2\delta)\bar{\partial}^{\pm}\delta| \le C,$$

which implies the function $\sin(2\delta)$ is Lipschitz continuous in terms of ξ, η. That is, for any two points $T_1 = (\xi_1, \eta_1)$ and $T_2 = (\xi_2, \eta_2)$ in the interaction region, there holds

$$|\sin(2\delta)(T_1) - \sin(2\delta)(T_2)| \le C|T_1 - T_2|.$$

Let

$$\hat{\delta} := \begin{cases} \frac{\pi}{2} - 2\delta, & \text{if } \bar{\theta} < \theta < \frac{\pi}{4}, \\ 2\delta - \frac{\pi}{2}, & \text{if } \frac{\pi}{4} < \theta < \frac{\pi}{2}. \end{cases}$$

Then, we have $\hat{\delta} > 0$ and

$$|\cos \hat{\delta}(T_1) - \cos \hat{\delta}(T_2)| \leq C|T_1 - T_2|.$$

We now consider δ near $\pi/4$, that is, $\hat{\delta}$ near zero, and

$$\cos \hat{\delta} = 1 - \frac{1}{2}\hat{\delta}^2 + o(\hat{\delta}^3).$$

Then, we find

$$|\hat{\delta}^2(T_1) - \hat{\delta}^2(T_2)| \leq C|T_1 - T_2|.$$

Notice that the fact $|\hat{\delta}(T_1) - \hat{\delta}(T_2)| \leq \hat{\delta}(T_1) + \hat{\delta}(T_2)$, we get

$$|\hat{\delta}(T_1) - \hat{\delta}(T_2)| \leq C|T_1 - T_2|^{\frac{1}{2}},$$

which implies the function $\delta \in C^{\frac{1}{2}}$.

To verify the Hölder continuity of α and β, we use diagonal form (3.18)

$$\begin{cases} \partial^+(-\beta + \cot \delta) = \frac{\cos(2\delta)}{\cos \alpha}, \\ \partial^-(\alpha + \cot \delta) = \frac{\cos(2\delta)}{\cos \beta}. \end{cases} \tag{5.14}$$

Integrating (5.14) along the characteristic curves $\Gamma^{\pm}$, respectively, we have

$$\begin{cases} \alpha = -\cot \delta + \displaystyle\int_{\Gamma^-} \frac{\cos(2\delta)}{\cos \beta}\, \mathrm{d}\xi, \\ \beta = \cot \delta - \displaystyle\int_{\Gamma^+} \frac{\cos(2\delta)}{\cos \alpha}\, \mathrm{d}\xi, \end{cases} \tag{5.15}$$

or

$$\begin{cases} \alpha = 2\delta + \cot \delta - \displaystyle\int_{\Gamma^+} \frac{\cos(2\delta)}{\cos \alpha}\, \mathrm{d}\xi, \\ \beta = -2\delta - \cot \delta + \displaystyle\int_{\Gamma^-} \frac{\cos(2\delta)}{\cos \beta}\, \mathrm{d}\xi. \end{cases} \tag{5.16}$$

We now use (5.15) and (5.16) to prove $(\alpha, \beta) \in C^{\frac{1}{2}}$. For any two points $T_1 = (\xi_1, \eta_1)$ and $T_2 = (\xi_2, \eta_2)$, if T_1, T_2 on a Γ^+-characteristic or Γ^--characteristic, it is easy to make the conclusion. Otherwise, there exists a unique point $T_3 = (\xi_3, \eta_3)$ such that T_1, T_3 on a Γ^+-characteristic and T_2, T_3 on a Γ^--characteristic. Furthermore, by the monotonicity of the characteristics, we get

$$\min\{\xi_1, \xi_2\} < \xi_3 < \max\{\xi_1, \xi_2\},$$

that is,

$$|\xi_i - \xi_3| < |\xi_1 - \xi_2|, \quad i = 1, 2.$$

We notice that

$$|\eta_3 - \eta_1| = \left| \int_{\xi_1}^{\xi_3} \tan \alpha\, \mathrm{d}\xi \right| \leq \max\{1, \tan \theta\}|\xi_1 - \xi_3|,$$

and

$$|\eta_3 - \eta_2| = \left| \int_{\xi_2}^{\xi_3} \tan \beta\, \mathrm{d}\xi \right| \leq \max\{1, \tan \theta\}|\xi_2 - \xi_3|.$$

Then, we obtain

$$|T_1 - T_3| < C|T_1 - T_2|, \qquad |T_2 - T_3| < C|T_1 - T_2|. \tag{5.17}$$

Since $|\cos\alpha|$, $|\cos\beta| > \min\{\cos\frac{\pi}{4}, \cos\theta\}$, then using (5.16) we have

$$\begin{aligned}|\alpha(T_1) - \alpha(T_3)| &\le |(2\delta + \cot\delta)(T_1) - (2\delta + \cot\delta)(T_3)| + \left|\int_{\xi_3}^{\xi_1} \frac{\cos(2\delta)}{\cos\alpha}\,\mathrm{d}\xi\right| \\ &\le C|T_1 - T_3|^{\frac{1}{2}} + C|\xi_1 - \xi_3| \\ &\le C|T_1 - T_3|^{\frac{1}{2}}.\end{aligned} \tag{5.18}$$

Similarly, using (5.15), we get

$$\begin{aligned}|\alpha(T_3) - \alpha(T_2)| &\le |\cot\delta(T_2) - \cot\delta(T_3)| + \left|\int_{\xi_2}^{\xi_3} \frac{\cos(2\delta)}{\cos\beta}\,\mathrm{d}\xi\right| \\ &\le C|T_2 - T_3|^{\frac{1}{2}} + C|\xi_2 - \xi_3| \\ &\le C|T_2 - T_3|^{\frac{1}{2}}.\end{aligned} \tag{5.19}$$

Therefore, combining (5.17)–(5.19), we obtain

$$\begin{aligned}|\alpha(T_1) - \alpha(T_2)| &\le |\alpha(T_1) - \alpha(T_3)| + |\alpha(T_3) - \alpha(T_2)| \\ &\le C|T_1 - T_3|^{\frac{1}{2}} + C|T_2 - T_3|^{\frac{1}{2}} \\ &\le C|T_1 - T_2|^{\frac{1}{2}},\end{aligned}$$

which implies the function $\alpha \in C^{\frac{1}{2}}$. The function $\beta \in C^{\frac{1}{2}}$ can be treated similarly. $\square$

6. The semi-hyperbolic patch problem

In this section, we use the characteristic decompositions to study the semi-hyperbolic patch problem for the isothermal Euler equations (3.1) and obtain a global smooth solution. The method is similar to Sect. 5.

6.1. Boundary data and local existence of solutions

We specify boundary conditions on $\widehat{AB}$ and $\widehat{BC}$. We require the position of point B to be close to point A, so that the circular arc $\widehat{AB}$ is a quarter of a circle or less. Denote by β_C the inclination angle of the convex negative characteristic $\widehat{BC}$ at point C. We consider only $-\pi/2 < \beta_C < 0$.

The given boundary values for α, β on $\widehat{AB}$ and $\widehat{BC}$ are

$$\beta\,|_{\widehat{AB}} = 0, \qquad \frac{\pi}{2} \le \alpha\,|_{\widehat{BC}} \le \pi + \beta_C. \tag{6.1}$$

From the convexity of the characteristic, we have the following estimates.

Lemma 6.1. (Boundary data estimate) *For the boundary data (6.1) on the boundaries $\widehat{AB}$ and $\widehat{BC}$, we have the following estimates:*

$$\frac{\pi}{2} \le \alpha\,|_{\widehat{AB}} \le \pi, \qquad \beta_C \le \beta\,|_{\widehat{BC}} \le 0. \tag{6.2}$$

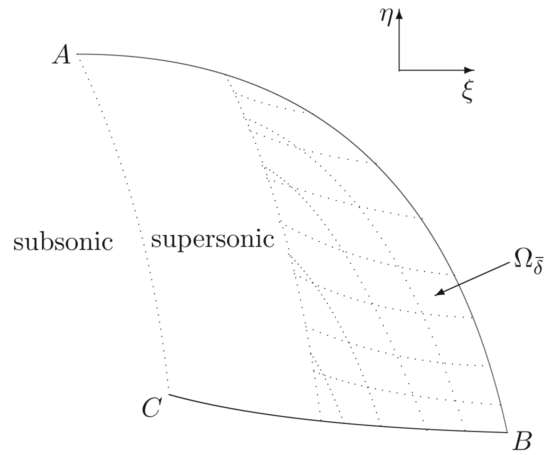


Fig. 6. Local solution region

Thus, we obtain

$$\frac{\pi}{4} \leq \delta \mid_{\widehat{AB},\widehat{BC}} \leq \frac{\pi}{2}. \tag{6.3}$$

It is easy to obtain local existence to the Goursat problem near the point B for the 2×2 system [22], see Fig. 6. Let $\Omega_{\bar{\delta}}(\pi/4 < \bar{\delta} < \pi/2)$ be the region enclosed by curves $\widehat{AB}$, $\widehat{BC}$ and $\delta = \bar{\delta}$.

Lemma 6.2. *There is a $\bar{\delta}^* > 0$ such that a C^1 solution of (3.18) and (6.1) exists in the region $\Omega_{\bar{\delta}^*}$, where $\bar{\delta}^*$ depends only on the C^1 norms of α, β on the boundaries $\widehat{AB}$ and $\widehat{BC}$.*

6.2. The signs of $\bar{\partial}^{\pm}\alpha, \bar{\partial}^{\pm}\beta$ and the invariant triangle

We estimate C^0 norm of the solution (α, β) by the method of invariant regions as the gas expansion problem.

Lemma 6.3. (Invariant triangle) *For any local C^1-solutions, we have*

$$\alpha \geq \frac{\pi}{2}, \quad \beta \leq 0, \quad \frac{\pi}{2} \leq \alpha - \beta \leq \pi, \quad \pm\bar{\partial}^{\pm}\alpha > 0, \quad \pm\bar{\partial}^{\pm}\beta < 0. \tag{6.4}$$

See Fig. 7.

Proof. By the local existence of solutions, we deduce that there exists a $\delta_0 \in (\pi/4, \pi/2)$ such that the solution exists in the region Ω_{δ_0}. Recall the characteristic decompositions (4.19) and the boundary conditions $\bar{\partial}^{+}\alpha \mid_{\widehat{AB}} > 0$, $\bar{\partial}^{-}\beta \mid_{\widehat{BC}} > 0$, we find $\bar{\partial}^{+}\alpha > 0$ and $\bar{\partial}^{-}\beta > 0$ in Ω_{δ_0}. For the sign of $\bar{\partial}^{+}\beta$, we use the second equation of (4.3) to obtain

$$-\bar{\partial}^{-}\bar{\partial}^{+}\beta - M_2\bar{\partial}^{+}\beta = -\frac{1}{2}\sin(2\delta)(3\tan^2\delta - 1)\bar{\partial}^{-}\beta < 0. \tag{6.5}$$

Combining (6.5) with the boundary condition $\bar{\partial}^{+}\beta = 0$ on $\widehat{AB}$, we find $\bar{\partial}^{+}\beta < 0$ in Ω_{δ_0}. For the sign $\bar{\partial}^{-}\alpha$, we recall the equation (3.15) to find

$$\cos(2\delta)\bar{\partial}^{-}\alpha = \bar{\partial}^{-}\beta - 2\cos(2\delta)\sin^2\delta > 0.$$

Thus, we have $\bar{\partial}^{-}\alpha < 0$ in Ω_{δ_0}.

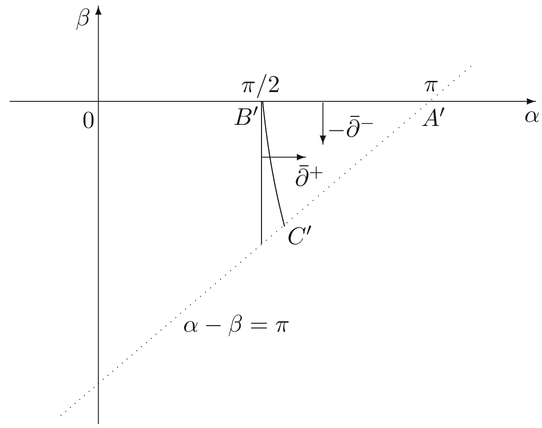


Fig. 7. Invariant region

Take care of the signs $\bar\partial^+\alpha > 0$ and $\bar\partial^-\beta > 0$ and the direction of $(\bar\partial^+, \bar\partial^-)$, then we obtain the invariant triangle

$$\alpha \geq \frac{\pi}{2}, \quad \beta \leq 0, \quad \frac{\pi}{2} \leq \alpha - \beta \leq \pi$$

for the local solutions. □

6.3. Gradient estimates and global solutions

Firstly, we derive the properties of solutions from the previous lemmas.

Lemma 6.4. (Properties) *For the smooth solution (α, β) in the domain ABC, we have the following properties:*

(1) *δ is monotone increasing along the negative characteristic curve starting on a point of the boundary $\widehat{AB}$ or along the positive one starting on a point of $\widehat{BC}$.*
(2) *$\delta = \frac{\pi}{2}$ at the interior points of the curve $\widehat{AC}$.*
(3) *The curve $\delta(\xi, \eta) = \bar\delta(\pi/4 < \bar\delta < \pi/2)$ is C^1.*
(4) *Level curves of δ are non-characteristic.*
(5) *The Λ_+-characteristics are concave and Λ_--characteristics are convex in the domain ABC.*

Proof. (1) This follows directly from Lemma 6.3: $\bar\partial^+\delta > 0$, $\bar\partial^-\delta < 0$.
(2) From the equation (3.9), we find the sonic curve to be

$$0 = (u - \xi)^2 + (v - \eta)^2 - 1 = \frac{1}{\sin^2\delta} - 1 = \frac{\cos^2\delta}{\sin^2\delta}.$$

Then, we obtain $\delta = \pi/2$ on the sonic curve by the invariant triangle in Lemma 6.3.
(3) Combine the second equation of (3.15) with the second equation of (3.16) to yield

$$\bar\partial^-\delta + \sin^2\delta = \frac{\sin^2\delta}{\cos(2\delta)}\bar\partial^-\beta < 0.$$

Hence, we have

$$\bar\partial^-\delta < -\sin^2\delta \leq -\frac{1}{2} < 0, \tag{6.6}$$

that is, $|\bar{\partial}^-\delta| \geq 1/2 > 0$. Hence by the implicit function theorem, the curves $\{(\xi,\eta) \mid \delta(\xi,\eta) = \bar{\delta}\}$ are C^1 in the (ξ,η) plane.

(4) This property follows from Lemma 6.3: $\bar{\partial}^+\delta > 0$, $\bar{\partial}^-\delta < 0$.

(5) This property follows from Lemma 6.3: $\bar{\partial}^+\alpha > 0, -\bar{\partial}^-\beta < 0$. $\square$

Next, we use the characteristic decompositions (4.21) to establish the gradient estimates of α and β.

Lemma 6.5. (Gradient estimate) *Assume that there exists a C^1 solution in the domain $\Omega_{\bar{\delta}}$, $\bar{\delta} < \pi/2$. Then, the C^1 norm of (α,β) and (u,v) has a uniform bound*

$$\| (\alpha,\beta,u,v) \|_{C^1(\Omega_{\bar{\delta}})} \leq \frac{C}{\sin^{n+1}(2\bar{\delta})} \tag{6.7}$$

for some constant C, independent of $\bar{\delta}$ and any fixed real number $n > 3/2$.

Proof. Noting the identities

$$\partial_\xi = -\frac{\sin\beta\bar{\partial}^+ - \sin\alpha\bar{\partial}^-}{\sin(2\delta)}, \qquad \partial_\eta = \frac{\cos\beta\bar{\partial}^+ - \cos\alpha\bar{\partial}^-}{\sin(2\delta)},$$

and the relation formulas (3.11), (3.12) and (3.15), it suffices to prove that α and β satisfy

$$|\bar{\partial}^+\alpha|,\ |\bar{\partial}^-\beta| \leq \frac{C}{\sin^n(2\bar{\delta})} \tag{6.8}$$

for some constant C, independent of $\bar{\delta}$, in the domain $\Omega_{\bar{\delta}}$. Recalling the local existence lemma 6.2, we only need prove (6.8) holds in the domain $\hat{\Omega} := \Omega_{\bar{\delta}} \backslash \Omega_{\tilde{\delta}}$, where $\tilde{\delta}$ is a constant between $\pi/4$ and $\bar{\delta}^*$. Let

$$Y = \max\left\{ \max_{\hat{\Omega}} \Phi(\delta),\ \max_{\widehat{AB}\backslash\overline{\Omega}_{\tilde{\delta}}} \frac{\bar{\partial}^+\alpha}{(-\cot(2\delta))^n},\ \max_{\widehat{BC}\backslash\overline{\Omega}_{\tilde{\delta}}} \frac{\bar{\partial}^-\beta}{(-\cot(2\delta))^n} \right\}, \tag{6.9}$$

where

$$\Phi(\delta) = \frac{4n(-\tan(2\delta))^n[n - 2 + \cos^2(2\delta) + 4\cos^4\delta]}{\cos^2\delta + (2n-3)\sin^2\delta},$$

and $\overline{\Omega}_{\tilde{\delta}}$ denotes the closure of $\Omega_{\tilde{\delta}}$. Note that the characteristic decompositions (4.21) and both of g_1 and g_2 are positive functions for any fixed real number $n > 3/2$. Then, we use the same method of Lemma 5.4 to complete the proof. $\square$

Using the same method of Lemma 5.6 yields the following $C^{1,1}$ estimates.

Lemma 6.6. ($C^{1,1}$ estimate) *Assume that there exists a C^1 solution in the domain $\Omega_{\bar{\delta}}$. Then we have*

$$\| (\alpha,\beta) \|_{C^{1,1}(\Omega_{\bar{\delta}})} \leq \frac{C}{\sin^{n+2}(2\bar{\delta})} \tag{6.10}$$

for some constant C, independent of $\bar{\delta}$ and any fixed real number $n > 3/2$.

To extend the solution up to the sonic boundary, we introduce the definition of strong determinate domain.

Definition 6.7. *Let Ω be a closed domain bounded by $\widehat{AB}$, $\widehat{BC}$ and ℓ, where the curve ℓ is a level set of δ intersecting with $\widehat{AB}$, $\widehat{BC}$ and stays in the domain ABC. Let further $\delta \in C^1(\Omega)$ and $\pi/4 < \delta < \pi/2$. We call Ω is a strong determinate domain of δ if and only if for all $(\xi_0,\eta_0) \in \Omega$ the two characteristic curves defined by*

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \tan\alpha,\ \tan\beta, \quad \xi \geq \xi_0$$

intersect only with $\widehat{AB}$, $\widehat{BC}$.

Now we prove the following lemma, from which it is easy to obtain the global existence theorem.

Lemma 6.8. *For any $\bar{\xi} : 0 < \bar{\xi} < \xi_B$ on the curve $\widehat{AB} : \eta = \eta_{\widehat{AB}}(\xi)$, there exists a smooth curve $\ell_{\bar{\xi}} : \delta(\xi,\eta) = \delta(\bar{\xi}, \eta_{\widehat{AB}}(\bar{\xi}))$ is C^1 inside the domain ABC to encircle a domain $\Omega_{\bar{\delta}}$ with $\widehat{AB}$ and $\widehat{BC}$ such that*

(1) *$\Omega_{\bar{\delta}}$ is a strong determinate domain of δ;*
(2) *the Goursat problem (3.18) and (6.1) has a supersonic solution $(\alpha,\beta) \in C^1(\Omega_{\bar{\delta}})$.*

Proof. We use the same method as that in papers [7,19] to prove this lemma. Denote by (ξ_B, η_B) the coordinate of the point B in Fig. 2. We designate by S the set consisting of all elements of $(0, \xi_B)$ which satisfies the above assertions listed in this lemma. In view of Lemma 6.4, we have $\pm\bar{\partial}^{\pm}\delta > 0$, which implies that if $\xi_0 \in S$, then $[\xi_0, \xi_B] \subset S$. So it suffices to prove that the set S is not empty and $\inf S = 0$. The local existence lemma 6.2 shows that S is not empty.

Next, we use the contradiction argument to prove $\inf S = 0$. Now suppose $\inf S = \hat{\xi} > 0$. The following argument is divided into two steps. In the first step, we will prove $\hat{\xi} \in S$. In the second step, we will show that there exists a small $\epsilon > 0$ such that $[\hat{\xi} - \epsilon, \hat{\xi}] \subset S$.

Step 1. In view of the definition of $\hat{\xi}$, there exists a monotone decreasing sequence $\{\xi_i\}_{i=1}^{\infty} \subset S$ satisfying $\lim\limits_{i\to\infty} \xi_i = \hat{\xi}$. Then for every ξ_i, there exists a smooth curve $\ell_i : \delta(\xi,\eta) = \delta(\xi_i, \eta_{\widehat{AB}}(\xi_i))$ is C^1 inside the domain ABC to encircle Ω_i with $\widehat{AB}$, $\widehat{BC}$ such that

(1) Ω_i is a strong determinate domain of δ;
(2) the Goursat problem (3.18) and (6.1) has a supersonic smooth solution $(\alpha,\beta) \in C^1(\Omega_i)$.

By the uniqueness of the supersonic solution of (3.13) and (6.1), and Lemma 6.4, we deduce that the level set ℓ_j is below ℓ_i for $i < j$ along the directions $-\bar{\partial}^+$ and $\bar{\partial}^-$. Then, we obtain $\{\ell_i\}_{i=1}^{\infty}$ is a monotone decreasing sequence along the directions $-\bar{\partial}^+$ and $\bar{\partial}^-$. Hence, there exists a curve $\hat{\ell} : \delta(\xi,\eta) = \delta(\hat{\xi}, \eta_{\widehat{AB}}(\hat{\xi}))$ defined by the limit of ℓ_i along the characteristic directions.

Denote by $\hat{\Omega}$ the closed domain bounded by $\widehat{AB}$, $\widehat{BC}$ and $\hat{\ell}$. Then, the Goursat problem (3.18) and (6.1) has a C^1 solution in $\hat{\Omega}\backslash\hat{\ell}$, and furthermore, $\hat{\Omega}\backslash\hat{\ell}$ is a strong determinate domain of δ.

Using Lemma 6.6, we know that for all $i : 1 \le i < \infty$, there exists a positive constant C independent of i such that

$$\| (\alpha,\beta) \|_{C^{1,1}(\Omega_i)} \le C,$$

and the same is true for

$$\| (\alpha,\beta) \|_{C^{1,1}(\hat{\Omega}\backslash\hat{\ell})} \le C.$$

In view of Lemmas 6.4 and 6.6, we obtain the sequence ℓ_i is $C^{1,1}$ and the bounds of δ_ξ and δ_η are independent of i. Then, using the Arzela-Ascoli theorem to deduce the curve $\hat{\ell}$ is C^1.

Define $(\alpha_i, \beta_i) = (\alpha,\beta)|_{\ell_i}$. Then, we obtain

$$\| (\alpha_i,\beta_i) \|_{C^{1,1}(\Omega_i)} \le C,$$

where the constant C is independent of i. By the Arzela-Ascoli theorem, there exists $(\hat{\alpha}, \hat{\beta}) \in C^1$ such that

$$\lim_{i\to\infty} (\alpha_i, \beta_i) = (\hat{\alpha}, \hat{\beta}).$$

We further have

$$\lim_{i\to\infty} \delta_i = \hat{\delta}.$$

Now we extend $(\alpha,\beta)(\xi,\eta)$ to $\hat{\Omega}$ by letting $(\alpha,\beta)|_{\hat{\ell}} = (\hat{\alpha}, \hat{\beta})$. Thus, the Goursat problem (3.18) and (6.1) has a C^1 solution in $\hat{\Omega}$. Then, we can easily find $\hat{\xi} \in S$ by the fact $\pm\bar{\partial}^{\pm}\delta > 0$ in Lemma 6.4.

Step 2. Since $\hat{\xi} \in S$, we obtain a C^1 solution in a closed domain near $\hat{\ell}$, denoted by $\mathcal{E}_{\hat{\ell}}$. For any point $(\tilde{\xi}, \tilde{\eta}) \in \hat{\ell}$, without the loss of generality, we can assume $|\delta_\xi(\tilde{\xi}, \tilde{\eta})| \geq 1/4$ by (6.6). Then, the curve $\hat{\ell}$ can be written as $\xi = \hat{\xi}(\eta)$ in a small neighborhood of $(\tilde{\xi}, \tilde{\eta})$. In this neighborhood, let us denote the limits of $(\alpha_\xi, \beta_\xi, \alpha_\eta, \beta_\eta)$ on the upper and the lower sides of the curve $\xi = \hat{\xi}(\eta)$ by $(\alpha_\xi^u, \beta_\xi^u, \alpha_\eta^u, \beta_\eta^u)$ and $(\alpha_\xi^\ell, \beta_\xi^\ell, \alpha_\eta^\ell, \beta_\eta^\ell)$, respectively. Then, both of these vector-valued functions are solutions to the system

$$M(\eta)u(\eta) = N(\eta),$$

where $u(\eta) = ((u_1, u_2, u_3, u_4)(\eta))^\top$ is a unknown function,

$$M(\eta) = \begin{pmatrix} 1 & -\cos(2\delta) & \tan\alpha & -\cos(2\delta)\tan\alpha \\ -\cos(2\delta) & 1 & -\cos(2\delta)\tan\beta & \tan\beta \\ \hat{\xi}'(\eta) & 0 & 1 & 0 \\ 0 & \hat{\xi}'(\eta) & 0 & 1 \end{pmatrix},$$

$$N(\eta) = \begin{pmatrix} -\frac{2\cos(2\delta)\sin^2\delta}{\cos\alpha} \\ \frac{2\cos(2\delta)\sin^2\delta}{\cos\beta} \\ \hat{\alpha}'(\eta) \\ \hat{\beta}'(\eta) \end{pmatrix},$$

and $\hat{\alpha}(\eta) = \alpha(\hat{\xi}(\eta), \eta)$, $\hat{\beta}(\eta) = \beta(\hat{\xi}(\eta), \eta)$. By direct calculation, we have

$$\det(M(\eta)) = \sin^2(2\delta)(\hat{\xi}'(\eta) - \cot\alpha)(\hat{\xi}'(\eta) - \cot\beta).$$

Then, we obtain $\det M(\eta) \neq 0$ along $\hat{\ell}$ by Lemma 6.4. Thus, the system has only a unique solution in the neighborhood of $(\tilde{\xi}, \tilde{\eta})$ and so we obtain

$$(\alpha_\xi^u, \beta_\xi^u, \alpha_\eta^u, \beta_\eta^u)|_{\hat{\ell}} = (\alpha_\xi^\ell, \beta_\xi^\ell, \alpha_\eta^\ell, \beta_\eta^\ell)|_{\hat{\ell}}$$

Thus, the Goursat problem (3.18) and (6.1) has a C^1 solution in $\hat{\Omega} \cup \mathcal{E}_{\hat{\ell}}$ and $\hat{\Omega} \cup \mathcal{E}_{\hat{\ell}}$ is a strong determinate domain of δ. Recalling Lemma 6.4, the equation $\delta(\xi, \eta) = \delta(\hat{\xi} - \epsilon, \eta_{\widehat{AB}}(\hat{\xi} - \epsilon))$ defines a C^1 curve whose graph lies in $\mathcal{E}_{\hat{\ell}}$ if $\epsilon > 0$ is small enough. With the same method as for $\hat{\xi}$, we can prove that $\hat{\xi} - \epsilon \in S$, which leads to a contradiction. Thus, $\inf S = 0$ and this lemma is valid. □

Finally, we obtain the main existence theorem.

Theorem 6.9. (Global existence) *There exists a smooth solution to the semi-hyperbolic patch problem. The characteristics are either convex or concave.*

Proof. The proof follows from Lemma 6.8. □

6.4. Global Hölder continuity of solutions

In order to depict the sonic boundary, or piece together this patch with a subsonic patch in the future study, we need to know more properties of solutions in the global sense, for example, the global Hölder continuity we will investigate below.

Theorem 6.10. (Hölder continuity) *The global solution in the semi-hyperbolic patch established in Theorem 6.9 is uniformly Hölder continuous with the exponent up to $2/7$, and the sonic boundary is of order up to $2/7$ locally.*

Proof. The proof is very similar to the case of gas expansion in the last section. For the completeness, we prove it in detail.

From the characteristic decompositions (4.21), we mimic the estimates deriving (6.8) to obtain

$$|\cos^n\delta\bar{\partial}^+\alpha|\leq C,\quad |\cos^n\delta\bar{\partial}^-\beta|\leq C$$

for any fixed real number $n>3/2$, where C is a generic constant only depending on the boundary data. Using the relation (3.16), we have

$$|\cos^n\delta\bar{\partial}^\pm\delta|\leq C.$$

Then, we obtain

$$|\cos^{n+1}\delta\partial_\xi\delta|\leq C,\quad |\cos^{n+1}\delta\partial_\eta\delta|\leq C.$$

Denote by $F(\delta)$ the primitive function of $\cos^{n+1}\delta$, that is, $F(\delta):=\int\cos^{n+1}\delta\,\mathrm{d}\delta$. The above relation implies

$$|F_\xi(\delta)|\leq C,\quad |F_\eta(\delta)|\leq C.$$

Then obviously the function $F(\delta(\xi,\eta))$ is Lipschitz continuous in terms of ξ,η, that is, for any two points $T_1=(\xi_1,\eta_1)$ and $T_2=(\xi_2,\eta_2)$ in the domain ABC, there holds

$$|F(\delta(T_1))-F(\delta(T_2))|\leq C|T_1-T_2|.$$

We now consider the solution near the sonic boundary, where δ is close to $\pi/2$, and

$$\cos\delta=\frac{\pi}{2}-\delta+o\left(\frac{\pi}{2}-\delta\right),\qquad \delta\sim\pi/2,$$

or

$$F(\delta)=\left(\frac{\pi}{2}-\delta\right)^{n+2}+o\left(\left(\frac{\pi}{2}-\delta\right)^{n+2}\right).$$

Then, we obtain

$$|(\delta(T_1)-\delta(T_2))^{n+2}+o((\delta(T_1)-\delta(T_2))^{n+2})|\leq C\,|T_1-T_2|$$

if T_1 is on the sonic curve, that is, $\delta(T_1)=\pi/2$. Thus, we verify the global Hölder continuity of δ with the exponent $1/(n+2)$ for any fixed $n>3/2$,

$$|\delta(T_1)-\delta(T_2)|\leq C\,|T_1-T_2|^{\frac{1}{n+2}}.$$

We emphasize here that the constant C is uniformly bounded.

To verify the Hölder continuity of α and β, we use diagonal form (3.18)

$$\begin{cases}\partial^+(-\beta+\cot\delta)=\frac{\cos(2\delta)}{\cos\alpha},\\ \partial^-(\alpha+\cot\delta)=\frac{\cos(2\delta)}{\cos\beta}.\end{cases}\tag{6.11}$$

Integrating (6.11) along the characteristic curves $\Gamma^\pm$, respectively, we have

$$\begin{cases}\alpha=-\cot\delta+\int_{\Gamma^-}\frac{\cos(2\delta)}{\cos\beta}\,\mathrm{d}\xi,\\ \beta=\cot\delta-\int_{\Gamma^+}\frac{\cos(2\delta)}{\cos\alpha}\,\mathrm{d}\xi,\end{cases}\tag{6.12}$$

or

$$\begin{cases}\alpha=2\delta+\cot\delta-\int_{\Gamma^+}\frac{\cos(2\delta)}{\cos\alpha}\,\mathrm{d}\xi,\\ \beta=-2\delta-\cot\delta+\int_{\Gamma^-}\frac{\cos(2\delta)}{\cos\beta}\,\mathrm{d}\xi.\end{cases}\tag{6.13}$$

We now use (6.12) and (6.13) to prove $(\alpha,\beta)\in C^{\frac{1}{n+2}}$. For any two points $T_1=(\xi_1,\eta_1)$ and $T_2=(\xi_2,\eta_2)$, if T_1, T_2 on a Γ^+-characteristic or Γ^--characteristic, it is easy to obtain the argument. Otherwise, there exists a unique point $T_3=(\xi_3,\eta_3)$ such that T_1,T_3 on a Γ^+-characteristic and T_2,T_3 on a Γ^--characteristic. Furthermore, by the monotonicity and convexity of the characteristics, we get

$$\min\{\xi_1,\xi_2\}<\xi_3<\max\{\xi_1,\xi_2\},\qquad \min\{\eta_1,\eta_2\}<\eta_3<\max\{\eta_1,\eta_2\},$$

that is,

$$|\xi_i - \xi_3| < |\xi_1 - \xi_2|, \qquad |\eta_i - \eta_3| < |\eta_1 - \eta_2|, \ i = 1, 2.$$

Then, we obtain

$$|T_1 - T_3| < |T_1 - T_2|, \qquad |T_2 - T_3| < |T_1 - T_2|. \tag{6.14}$$

In view of Lemma 6.3, there exists a uniform constant C_0 such that $|\cos\alpha| > C_0$ and $|\cos\beta| > C_0$ near the sonic curve. Then using (6.13), we have

$$\begin{aligned}|\alpha(T_1) - \alpha(T_3)| &\leq |(2\delta + \cot\delta)(T_1) - (2\delta + \cot\delta)(T_3)| + \left|\int_{\xi_3}^{\xi_1} \frac{\cos(2\delta)}{\cos\alpha}\,\mathrm{d}\xi\right| \\ &\leq C|T_1 - T_3|^{\frac{1}{n+2}} + C|\xi_1 - \xi_3| \\ &\leq C|T_1 - T_3|^{\frac{1}{n+2}}. \end{aligned} \tag{6.15}$$

Similarly, using (6.12), we get

$$\begin{aligned}|\alpha(T_3) - \alpha(T_2)| &\leq |\cot\delta(T_2) - \cot\delta(T_3)| + \left|\int_{\xi^2}^{\xi_3} \frac{\cos(2\delta)}{\cos\beta}\,\mathrm{d}\xi\right| \\ &\leq C|T_2 - T_3|^{\frac{1}{n+2}} + C|\xi_2 - \xi_3| \\ &\leq C|T_2 - T_3|^{\frac{1}{n+2}}. \end{aligned} \tag{6.16}$$

Therefore, combining (6.14)–(6.16), we obtain

$$\begin{aligned}|\alpha(T_1) - \alpha(T_2)| &\leq |\alpha(T_1) - \alpha(T_3)| + |\alpha(T_3) - \alpha(T_2)| \\ &\leq C|T_1 - T_3|^{\frac{1}{n+2}} + C|T_2 - T_3|^{\frac{1}{n+2}} \\ &\leq C|T_1 - T_2|^{\frac{1}{n+2}}, \end{aligned}$$

which implies the function $\alpha \in C^{\frac{1}{n+2}}$ uniformly for $n > 3/2$. The function $\beta \in C^{\frac{1}{n+2}}$ can be proved in a similar way.

However, the isoline of Mach number, or equivalently the isoline of δ, is just $C^{\frac{1}{n+2}}$ locally. In fact, for any two point (ξ_1, η_1) and (ξ_2, η_2) on the isoline $\delta = \text{const.}$, we have

$$\delta(\xi_1, \eta_1) = \delta(\xi_2, \eta_2).$$

Then, we obtain

$$\delta(\xi_2, \eta_2) - \delta(\xi_2, \eta_1) = \delta(\xi_1, \eta_1) - \delta(\xi_2, \eta_1),$$

and

$$\delta(\xi_2, \eta_2) - \delta(\xi_1, \eta_2) = \delta(\xi_1, \eta_1) - \delta(\xi_1, \eta_2),$$

which imply that

$$|\delta_\eta(\xi_2, \bar{\eta})||\eta_2 - \eta_1| = |\delta(\xi_2, \eta_1) - \delta(\xi_1, \eta_1)| \leq C|\xi_2 - \xi_1|^{\frac{1}{n+2}},$$

and

$$|\delta_\xi(\bar{\xi}, \eta_2)||\xi_2 - \xi_1| = |\delta(\xi_1, \eta_1) - \delta(\xi_1, \eta_2)| \leq C|\eta_2 - \eta_1|^{\frac{1}{n+2}},$$

for some $\bar{\eta}, \bar{\xi}$, respectively. By the fact $|\bar{\partial}^-\delta| \geq 1/2$, we have $|\delta_\xi| \geq \tilde{C}$ or $|\delta_\eta| \geq \tilde{C}$ for a positive constant $\tilde{C}$. Thus, we obtain

$$|\eta_2 - \eta_1| \leq C|\xi_2 - \xi_1|^{\frac{1}{n+2}}, \tag{6.17}$$

or

$$|\xi_2 - \xi_1| \leq C|\eta_2 - \eta_1|^{\frac{1}{n+2}}. \tag{6.18}$$

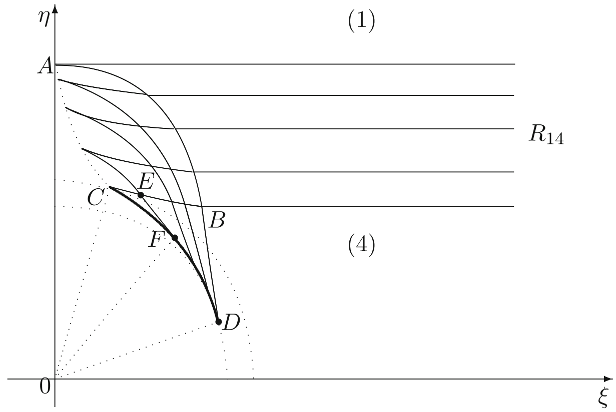


FIG. 8. Envelope formation in the lower half

Since $|\delta_\xi|$ and $|\delta_\eta|$ cannot be guaranteed to be bounded from below, just either (6.17) or (6.18) holds. Hence, the sonic boundary is just locally Hölder continuous.

Moreover, the above estimates are true for all $n > 3/2$, the Hölder exponent can be up to 2/7. □

6.5. Shock formation

In the subsection, we confirm that an envelope of positive characteristic forms before sonic points, for a strictly convex curve $\widehat{BC}$.

Theorem 6.11. (Envelope formation). *For a given monotone and strictly convex curve $\widehat{BC}$, we draw the positive characteristic starting on the curve $\widehat{BC}$, downward, as shown in Fig. 8. Then the positive characteristics form an envelope before their sonic points.*

Proof. First of all, let us establish a second-order equation for u. Letting $I = u$ in the commutator relation (4.2) and using the characteristic form

$$\bar{\partial}^{\pm} u + \Lambda_{\mp} \bar{\partial}^{\pm} v = 0, \tag{6.19}$$

by a similar procedure of getting characteristic decomposition in [18], we have

$$\bar{\partial}^{+}\bar{\partial}^{-} u = \frac{\bar{\partial}^{-} u}{\sin(2\delta)}\left[-4\sin^4\delta + \frac{2\sin^2\delta}{\sin(2\delta)\sin\alpha}\bar{\partial}^{-} u - \frac{2\sin^2\delta\cos(2\delta)}{\sin(2\delta)\sin\alpha}\bar{\partial}^{+} u\right]. \tag{6.20}$$

Since $\bar{\partial}^{+} u = 0$ in the simple wave region with the positive characteristics, we obtain

$$\bar{\partial}^{+}\bar{\partial}^{-} u = \frac{-2\sin^3\delta}{\cos\delta}\bar{\partial}^{-} u + \frac{1}{2\sin\alpha\cos^2\delta}(\bar{\partial}^{-} u)^2, \tag{6.21}$$

which is also expressed as

$$-\bar{\partial}^{+}\left(\frac{1}{-\bar{\partial}^{-} u}\right) = -\frac{2\sin^3\delta}{\cos\delta}\frac{1}{-\bar{\partial}^{-} u} - \frac{1}{2\sin\alpha\cos^2\delta}. \tag{6.22}$$

Combining the relation (3.11), (3.12) and (6.19), we find

$$\begin{aligned}\bar{\partial}^{-}\beta &= \frac{2\sin^2\delta\sin\beta\bar{\partial}^{-} u - 2\sin^2\delta\cos\beta\bar{\partial}^{-} v}{\sin(2\delta)}\\ &= \frac{2\sin^2\delta\cos(2\delta)}{\sin(2\delta)\sin\alpha}\bar{\partial}^{-} u.\end{aligned} \tag{6.23}$$

Notice that $\bar{\partial}^+\alpha = 0$ in this simple wave area, we obtain $-\bar{\partial}^+\delta > 0$, which implies $\cos(2\delta) < 0$, and then obtain $\bar{\partial}^- u < 0$. So we obtain the right-hand side of (6.22) is always negative, while on the boundary

$$\frac{1}{-\bar{\partial}^- u} \,|_{\widehat{BC}} > 0. \tag{6.24}$$

Then, $1/(-\bar{\partial}^- u)$ is decreasing along the positive characteristic in the direction from B to D. Furthermore, we notice that the coefficient $-2\sin^3\delta/\cos\delta$ and the inhomogeneous term $-1/2\sin\alpha\cos^2\delta$ in (6.22) will become infinite when the characteristic goes to the sonic point. So $1/(-\bar{\partial}^- u)$ reaches zero before the characteristic gets to its sonic point. The blow-up of $\bar{\partial}^- u$ yields that the positive characteristics form an envelope before their sonic points. □

References

1. Canic, S., Keyfitz, B.L., Kim, E.H.: A free boundary problem for a quasi-linear degenerate elliptic equation: regular reflection of weak shocks. Commun. Pure Appl. Math. **55**(1), 71–92 (2002)
2. Chen, G.Q., Feldman, M.: Global solutions of shock reflection by large-angle wedges for potential flow. Ann. Math **171**(2), 1067–1182 (2010)
3. Chen, S.X.: Mach configuration in pseudo-stationary compressible flow. J. Am. Math. Soc. **21**(1), 63–100 (2008)
4. Chen, X., Zheng, Y.X.: The direct approach to the interaction of rarefaction waves of the two-dimensional Euler equations. Indian J Math **59**(1), 231–256 (2010)
5. Cole, J.D., Cook, L.P.: Transonic aerodynamics. Elsevier, Amsterdam (1986)
6. Courant, R., Friedrichs, K.O.: Supersonic Flow and Shock Waves. Interscience, New York (1948)
7. Dai, Z., Zhang, T.: Existence of a global smooth solution for a degenerate Goursat problem of gas dynamics. Arch. Ration. Mech. Anal. **155**(4), 277–298 (2000)
8. Elling, V., Liu, T.P.: Supersonic flow onto a solid wedge. Commun. Pure Appl. Math. **61**(10), 1347–1448 (2008)
9. Glimm, G., Ji, X., Li, J., Li, X., Zhang, P., Zhang, T., Zheng, Y.: Transonic shock formation in a rarefaction Riemann problem for the 2-D compressible Euler equations. SIAM J. Appl. Math. **69**(3), 720–742 (2008)
10. Tesdall, A., Hunter, J.K.: Self-similar solutions for weak shock reflection. SIAM J. Appl. Math. **63**(1), 42–61 (2002)
11. Lei, Z., Zheng, Y.X.: A complete global solution to the pressure gradient equation. J. Differ. Equ. **236**(1), 280–292 (2007)
12. Levine, L.E.: The expansion of a wedge of gas into a vacuum. Proc. Camb. Philol. Soc. **64**, 1151–1163 (1968)
13. Li, J.Q.: On the two-dimensional gas expansion for compressible Euler equations. SIAM J. Appl. Math. **62**(3), 831–852 (2001/2002)
14. Li, J.Q.: Global solution of an initial-value problem for two-dimensional compressible Euler equations. J. Differ. Equ. **179**(1), 178–194 (2002)
15. Li, J.Q., Sheng, W.C., Zhang, T., Zheng, Y.X.: Two-dimensional Riemann problems: from scalar conservation laws to compressible Euler equations. Acta Math. Sci. Ser. B Eng. Ed. **29**(4), 777–802 (2009)
16. Li, J.Q., Yang, Z.C., Zheng, Y.X.: Characteristic decompositions and interactions of rarefaction waves of 2-D Euler equations. J. Differ. Equ. **250**, 782–798 (2011)
17. Li, J.Q., Zhang, T., Yang, S.L.: The Two-dimensional Riemann Problem in Gas Dynamics. Pitman Monographs and Surveys in Pure and Applied Mathematics 98, Longman (1998)
18. Li, J.Q., Zhang, T., Zheng, Y.X.: Simple waves and a characteristic decomposition of the two dimensional compressible Euler equations. Commun. Math. Phys. **267**(1), 1–12 (2006)
19. Li, J.Q., Zheng, Y.X.: Interaction of rarefaction waves of the two-dimensional self-similar Euler equations. Arch. Rat. Mech. Anal. **193**, 623–657 (2009)
20. Li, J.Q., Zheng, Y.: Interaction of four rarefaction waves in the bi-symmetric class of the two-dimensional Euler equations. Commun. Math. Phys. **296**, 303–321 (2010)
21. Li, M.J., Zheng, Y.X.: Semi-hyperbolic patches of solutions of the two-dimensional Euler equations. Arch. Ration. Mech. Anal. **201**, 1069–1096 (2011)
22. Li, T.T., Yu, W.C.: Boundary Value Problems of Hyperbolic System. Duke University, Durham (1985)
23. Song, K., Zheng, Y.X.: Semi-hyperbolic patches of solutions of the pressure gradient system. Disc. Cont. Dyna. Syst. **24**(4), 1365–1380 (2009)
24. Suchkov, V.A.: Flow into a vacuum along an oblique wall. J. Appl. Math. Mech. **27**, 1132–1134 (1963)
25. Zhang, T., Zheng, Y.X.: Conjecture on the structure of solution of the Riemann problem for two-dimensional gas dynamics systems. SIAM J. Math. Anal. **21**(3), 593–630 (1990)
26. Zheng, Y.X.: Systems of Conservation Laws: Two-dimensional Riemann Problems. In the series of Progress in Nonlinear Differential Equations. Birkhauser (2001)

27. Zheng, Y.X.: Two-dimensional regular shock reflection for the pressure gradient system of conservation laws. Acta Math. Appl. Sin. Engl. Ser. **22**(2), 177–210 (2006)

Yanbo Hu and Wancheng Sheng
Department of Mathematics
Shanghai University
Shanghai 200444
China
e-mail: yanbo.hu@hotmail.com

Wancheng Sheng
e-mail: mathwcsheng@shu.edu.cn

Jiequan Li
School of Mathematical Sciences
Beijing Normal University
Beijing 100875
China
e-mail: jiequan@bnu.edu.cn

(Received: April 21, 2011; revised: February 18, 2012)

View publication stats