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THE TRANSITION FROM ZELDOVICH–VON NEUMANN–DORING TO CHAPMAN–JOUGUET THEORIES FOR A NONCONVEX SCALAR COMBUSTION MODEL*

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Abstract. We study the transition from the Zeldovich–von Neumann–Doring (ZND) theory to the Chapman–Jouguet (CJ) theory as the reaction rate tends to infinity for a nonconvex scalar combustion model. The Riemann solution of the nonconvex ZND combustion model is constructed, and the limit of solutions as the reaction rate goes to infinity is investigated. We classify the reaction solutions of the ZND combustion model as detonation and deflagration waves according to the essential difference that the former contains the von Neumann spike but the latter does not. Based on the analysis of this limit, we propose a set of entropy conditions for combustion and noncombustion waves to the nonconvex CJ combustion model, which is the indispensable preparation for the study of multidimensional combustion problems.

Key words. Zeldovich–von Neumann–Doring theory, Chapman–Jouguet theory, detonation, deflagration, von Neumann spike, reaction entropy condition

AMS subject classifications. Primary, 35L60, 35L67; Secondary, 76L05, 80A25

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1. Introduction.

The Zeldovich–von Neumann–Doring (ZND) theory and the Chapman–Jouguet (CJ) theory play an important role in gas dynamics combustion theory. The former describes the combustible gas with a finite reaction rate and the latter with an infinite reaction rate or, equivalently, the infinitely thin reaction region. Formally the CJ theory is regarded as the limit of the ZND theory as the reaction rate tends to infinity [1]. Interesting discussions on this transition can be found in [3, 16]. The Riemann problem for the CJ gas dynamic combustion is constructively solved in [22] to display rich combustion wave patterns satisfying the so-called geometrical entropy conditions. Yet, the study of combustion waves of gas dynamics based on the ZND theory is notoriously complex and difficult, and far from being complete [21]. This motivates us to consider simpler combustion models.

In [2, 14], Fickett and Majda independently proposed a simplified model to study combustion waves, as the Burgers equation models in gas dynamics,

$$
\begin{aligned}
&(u+qz)_t + f(u)_x = \mu u_{xx},\\
&z_t + k\phi(u)z = 0,
\end{aligned}
\tag{1.1}
$$

where $x \in \mathbb{R}$, $t > 0$. This model corresponds to the ZND model in gas dynamics. The first equation resembles the conservation law of energy in gas dynamics, leading to nonlinear phenomena such as shocks; the second is the reaction equation, which

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may give rise to a linear discontinuity in the solution. The dependent variable u is a lumped quantity, representing density, velocity, or temperature, but not the chemical binding energy, which is represented by q. z is the percentage of unburned gas, k is the reaction rate, and $\phi(u)$ is the standard Heaviside function. We take the ignition point $u = 0$ for simplicity. That is, the material begins to burn just as the temperature becomes higher than zero. Using this model, combustion problems were extensively investigated, such as the stability and asymptotic behavior of combustion waves (see [5, 9, 10, 11] and the references therein). The well-posedness of the general Cauchy problem and the zero viscosity limit of (1.1) was studied in [5]. Ying and Teng [17] studied the Riemann solution of (1.1) at $\mu = 0$ and obtained the limit of the solution as k tends to infinity and defined the limit function as the solution of the Riemann problem for the corresponding CJ model

$$
\begin{aligned}
&(u + qz)_t + f(u)_x = 0,\\
&z(x,t) = \begin{cases} z(x,0) & \text{if } \max\limits_{0\le\tau\le t} u(x,\tau) \le 0,\\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{1.2}
$$

In [13], Liu and Zhang extracted from the properties of CJ solutions in [17] to propose a set of entropy conditions, under which the existence and uniqueness of the Riemann solutions were shown constructively and the ignition and extinction problems were investigated as well. A slightly different model was considered in [7, 8, 12, 15]. All of these results were obtained under the assumption that $f(u)$ is strictly convex.

Based on these results in one dimension, one would naturally wish to study combustion phenomena in several dimensions. The generalization of (1.1) may be a reasonable trial in this aspect, for which the theory of well-posedness of a multidimensional scalar ZND model was established in [6]. As is well known, the fluxes must be nonconvex in some directions at this moment [4]. Therefore, it is interesting to investigate scalar combustion models with nonconvex fluxes $f(u)$ so as to make a good preparation for the study of structure of multidimensional combustion waves.

To this end, the Riemann problem for (1.2) with nonconvex fluxes $f(u)$ was solved constructively in [18] under the entropy restriction that mimics those in [13] and generalizes the classical Oleinik entropy condition for scalar conservation laws. A crucial issue here is just how to propose and justify entropy conditions to single out physically admissible solutions. There are some plausible ways: One is, most naturally, to consider the viscosity vanishing limit for corresponding models like (1.1) with nonconvex fluxes, which is found to be a very difficult issue, even only to discuss traveling wave solutions. Another is, as in the classical gas dynamics combustion theory [1], to investigate the limit of the nonviscous Fickett–Majda model (1.1) as the reaction rate k goes to infinity. This is what we do in this paper.

The self-similar combustion model corresponding to (1.1) with sufficiently large reaction rates reads

$$
\begin{aligned}
&(u + qz)_t + f(u)_x = 0,\\
&z_t + \tfrac{k}{t}\phi(u)z = 0,
\end{aligned} \tag{1.3}
$$

where $f(u)$ is nonconvex. The second author began to study this model and announced the very partial results in [19]; the Riemann problem was discussed there for some cases and the large reaction rate limit was taken into account to get the associated Riemann solutions of the corresponding CJ model (1.2). However, this result is far from understanding the entropy combustion solutions which we clarify below. As pointed out in [13], even the Riemann solution to (1.3) is not unique, provided that it is restricted by the classical Oleinik-type entropy condition (2.7). Motivated by the physical consideration that the temperature in the combustion wave front is as high as possible or, equivalently, the propagation speed of combustion wave front is as small as possible, we propose in this paper a reaction entropy condition to guarantee the uniqueness of solutions, under which the Riemann solution to (1.3) is uniquely constructed. Since we focus our attention on the transition from the ZND theory to the CJ theory as the reaction rate k goes to infinity, we mainly display the structure of solutions with the large reaction rate k. Our main contribution is to clarify the reaction solution as detonation and deflagration waves in accord with the essential difference that the former has a von Neumann spike in the reaction region while the latter does not, although it was already noticed earlier, e.g., in [1, 22, 16, 19]. Indeed, the temperature (the lumped variable u) is not monotone for the former but is monotone increasing for the latter. Then we study the limit of solutions by letting the reaction rate k tends to infinity. It is found that although both of these combustion waves in the limit become jump-ups, they still inherit the above intrinsic difference. Finally we formulate a set of entropy conditions based on the analysis of limit solutions, which enables us to greatly improve the result in [18] to get the unique entropy solution of (1.2) with nonconvex flux $f(u)$. Indeed, this entropy condition can be used to justifiably construct two-dimensional Riemann solutions for the CJ combustion model associated with (1.2); see [20].

The rest of this paper consists of three parts. In section 2, we give some preliminaries containing the general property of smooth solutions, the Rankine–Hugoniot jump conditions of combustion waves, and the reaction entropy condition. The Riemann solutions of (1.3) are constructed in section 3 for two typically distinct fluxes with just one inflection point. The limit behavior of solutions is also studied as the reaction rate goes to infinity. We propose the entropy condition from the limit behavior of solutions in the preceding sections for the CJ nonconvex combustion model (1.2) in section 4.

2. Self-similar solutions to the ZND model. This section serves as a preliminary for the forthcoming sections. We will discuss the general properties of smooth solutions, the Rankine–Hugoniot jump conditions of combustion waves, and the reaction entropy condition.

We begin by considering the Riemann problem for (1.3) with initial data

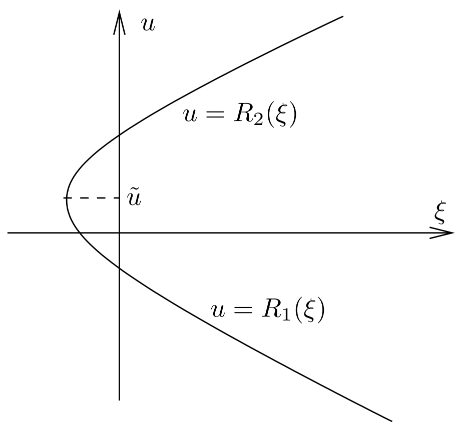
$$(u,z)|_{t=0} = \begin{cases} (u^-, z^-), & x < 0, \\ (u^+, z^+), & x > 0, \end{cases} \tag{2.1}$$

where (u^-, z^-) and (u^+, z^+) are two different states. For the self-similar solutions $(u(\xi), z(\xi))$, $\xi = x/t$, the Riemann problem for (1.3) becomes a boundary value problem with boundary values at infinity,

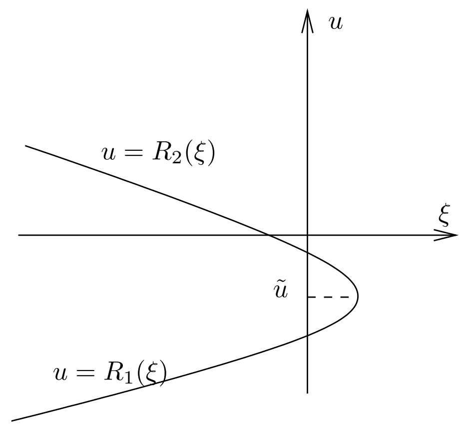
$$\begin{aligned} &(f'(u) - \xi)\frac{du}{d\xi} = qk\phi(u)z, \\ &\xi\frac{dz}{d\xi} = k\phi(u)z, \end{aligned} \tag{2.2}$$

and

$$(u,z)|_{\xi=\pm\infty} = (u^\pm, z^\pm). \tag{2.3}$$



(a) The branches with assumption (A_1).



(b) The branches with assumption (A_2).

FIG. 2.1. *The branches of the inverse function of* $\xi = f'(u)$.

Without loss of generality, we consider this Riemann problem with the data

$$z^- = 1,\ z^+ = 0; \qquad u^- \leq 0 < u^+. \tag{2.4}$$

The left state $(u^-, 1)$ is unburned and the right state $(u^+, 0)$ is burnt. For clarity of presentation, we restrict $f(u)$ to be the following two typical cases since the general case for $f(u)$ with a finite number of isolated inflection points is treated similarly without substantial difficulties.

ASSUMPTIONS.

(A_1) *$f(u)$ has one inflection point $\tilde{u}$ and* $f'(\pm\infty) = +\infty$;

(A_2) *$f(u)$ has one inflection point $\tilde{u}$ and* $f'(\pm\infty) = -\infty$.

These assumptions are reasonable in the application to the multidimensional generalization. Consider a two-dimensional counterpart of (1.1) (see [6]),

$$\begin{aligned} &(u + qz)_t + f(u)_x + g(u)_y = \epsilon \Delta u, \\ &z_t + k\phi(u)z = 0, \end{aligned} \tag{2.5}$$

where Δ is the Laplacian, the fluxes $f(u)$ and $g(u)$ satisfy that $f''(u) > 0$, $g''(u) > 0$, and $(f''(u)/g''(u))' > 0$. It is easily shown that for any given direction $(\mu, \nu) \in S^1$, the directional flux $F(u; \mu, \nu) = \mu f(u) + \nu g(u)$ has at most one inflection point.

For each case of these assumptions, the inverse function of $\xi = f'(u)$ has two branches, denoted by $R_1(\xi)$ and $R_2(\xi)$, where $R_1(\xi) < R_2(\xi)$. More specifically, with assumption (A_1), the flux $f(u)$ is convex when $u > \tilde{u}$, while it is concave when $u < \tilde{u}$. Therefore, $u = R_1(\xi)$ is defined via $\xi = f'(u)$ as $u < \tilde{u}$ and $u = R_2(\xi)$ via $\xi = f'(u)$ as $u > \tilde{u}$. Thus, $u = R_1(\xi)$ is decreasing while $u = R_2(\xi)$ is increasing. For (A_2), we have converse statements. See Figure 2.1 for the graphs of $R_1(\xi)$ and $R_2(\xi)$.

Thus, the smooth solutions of (2.2) are in the following:

(1) Constant states, $(u, z) = (\text{constant}, \text{constant})$.

(2) $(u, z) = (R_i(\xi), \text{constant})$, where $i = 1$ or 2.

(3) $z(\xi) = |\frac{\xi}{\eta}|^k$, where η is an arbitrary constant, $u(\xi) > 0$ satisfies

$$(f'(u) - \xi)\frac{du}{d\xi} = qk\left|\frac{\xi}{\eta}\right|^k.$$

We now turn to discuss discontinuous solutions. At this moment, we have to understand (2.2) in the sense of distributions. Let $(u(\xi), z(\xi))$ be a piecewise smooth solution with a discontinuity point at $\xi = \sigma$. Then we get the Rankine–Hugoniot jump condition

$$\begin{aligned} &-\sigma[u] + [f] = 0, \\ &\sigma[z] = 0. \end{aligned} \tag{2.6}$$

Throughout this paper, we fix the notation $[u]$ to be the jump of u across $\xi = \sigma$, etc. Then the Rankine–Hugoniot condition (2.6) provides two possibilities: (i) if $[z] \neq 0$, then $\sigma = 0$ and $[f] = 0$; (ii) if $[z] = 0$, then $\sigma = \frac{[f]}{[u]}$. The first corresponds to a slip line and the second a shock wave, provided that it satisfies the following Oleinik-type entropy condition:

$$\frac{f(u) - f(u_l)}{u - u_l} \geq \frac{f(u_r) - f(u_l)}{u_r - u_l} \quad \text{for } (u - u_l)(u - u_r) \leq 0. \tag{2.7}$$

The pair (u, z) is an entropy solution of (2.2) and (2.4) if the equations are satisfied at smooth points in the classical sense and the requirement of Oleinik-type entropy condition (2.7) is met at discontinuity points. This solution has the following property.

LEMMA 2.1. *Let $(u(\xi), z(\xi))$ be an entropy solution of* (2.2) *and* (2.4). *Then there exists $\eta \in (-\infty, 0]$ such that $z(\xi)$ has the structure*

$$z(\xi) = \begin{cases} 1, & \xi < \eta, \\ (\frac{\xi}{\eta})^k, & \eta \leq \xi \leq 0, \\ 0, & 0 < \xi. \end{cases} \tag{2.8}$$

Proof. With the above arguments and the Oleinik-type entropy condition (2.7), $z(\xi)$ consists of some constants and functions with the form $|\frac{\xi}{\eta}|^k$. The only possible discontinuity point of $z(\xi)$ is $\xi = 0$. Since $z(+\infty) = 0$, $z(\xi) \equiv 0$ for $\xi > 0$. Note that $z(-\infty) = 1$. We assert $z(\xi) \equiv 1$ in a neighborhood of negative infinity. If $z(\xi) \equiv 1$ for all $\xi < 0$, then $z(\xi)$ has the structure of (2.8) with $\eta = 0$. Otherwise, there exists a constant $\eta < 0$ satisfying the Rankine–Hugoniot condition $\sigma = \eta = (f(u(\eta - 0)) - f(u(\eta + 0))/(u(\eta - 0)) - u(\eta + 0))$, where u undergoes a jump (shock wave). Since z is continuous there, we conclude that $z(\eta) = 1$, and so $z(\xi) = (\xi/\eta)^k$ for $\xi \in (\eta, 0)$ by (2.2). Thus $z(\xi)$ has three different stages as expressed in (2.8). ∎

Lemma 2.1 gives the following corollary.

COROLLARY 2.2. *The entropy solution $(u(\xi), z(\xi))$ of* (2.2) *and* (2.4) *has the structure*

$$(u(\xi), z(\xi)) = \begin{cases} (A(\xi), 1), & -\infty < \xi < \eta, \\ (B(\xi), (\frac{\xi}{\eta})^k), & \eta \leq \xi \leq 0, \\ (C(\xi), 0), & 0 < \xi < +\infty, \end{cases} \tag{2.9}$$

where A, B, and C satisfy the following equations at smooth points:

$$\begin{aligned}&(f'(A)-\xi)\frac{dA}{d\xi}=0, \quad \xi\in(-\infty,\eta),\\ &A(-\infty)=u^-, \quad A(\eta-0)\le 0,\end{aligned} \tag{2.10}$$

$$\begin{aligned}&(f'(B)-\xi)\frac{dB}{d\xi}=qk\left(\frac{\xi}{\eta}\right)^k, \quad \xi\in[\eta,0],\\ &B(\xi)\ge 0,\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}&(f'(C)-\xi)\frac{dC}{d\xi}=0, \quad \xi\in(0,+\infty),\\ &C(+\infty)=u^+.\end{aligned} \tag{2.12}$$

The states $A(\xi)$, $B(\xi)$, and $C(\xi)$ are called the unburned, burning, and burnt parts of $u(\xi)$, respectively. The pair (u,z) is a combustion solution if it contains a burning part; otherwise, it is a noncombustion solution.

Analogous to [13], where $f(u)$ is convex (or concave), it can be shown that the entropy solution of (2.2) and (2.4) is not unique for some binding energy q and initial data (2.4) even though we impose the requirement of the entropy condition (2.7). Motivated by the physical consideration that the temperature in the wave front is as high as possible or, equivalently, the propagation speed of combustion wave front is as small as possible (cf. [22]), we propose the following *reaction entropy condition* to guarantee the uniqueness of solutions, in addition to the Oleinik-type entropy condition (2.7).

Reaction entropy condition (BE). *If the Riemann problem of* (2.2) *and* (2.4) *has several entropy solutions, we choose a solution so that its speed η in the wave front of the burning part achieves the absolute minimum value.*

A solution (u,z) is *admissible* if it satisfies both the Oleinik-type entropy condition (2.7) and the reaction entropy condition (BE). We call (2.7) and (BE) together as the entropy condition of the reaction solution of (2.2) and (2.4).

3. The entropy solution and the limit of the infinite reaction rate. In this section, we seek the entropy solution to (2.2) and (2.4) when the flux $f(u)$ satisfies the assumptions in the last section. We not only prove the solvability of this problem but display the explicit structure of solutions as well. Furthermore, we consider the limit behavior of entropy solutions as the reaction rate k goes to infinity. We achieve our goals through two cases according to the shape of flux $f(u)$ in the assumptions in the preceding section.

3.1. The solution of (2.2) and (2.4) when $f(u)$ satisfies (A_1). We consider the solution of (2.2) and (2.4) when $f(u)$ has only one inflection point and the slope at infinity is positive infinity. The main results are stated in Theorems 3.7–3.9. We investigate this problem using three cases depending on the position of u^+.

Let $\tilde{u}$ be the inflection point of $f(u)$. If $f'(\tilde{u})\ge 0$, then $f(u)$ is a monotone increasing function. It is easy to verify that the Riemann problem of (2.2) and (2.4) has a unique noncombustion solution (cf. [13]). Therefore, the admissible solution exists and is unique subject to the above entropy conditions. In the following, we

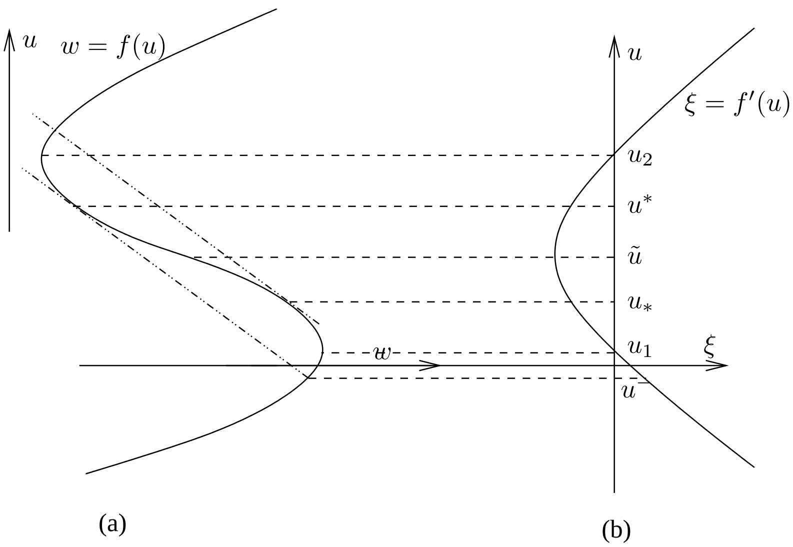


FIG. 3.1. *The graphs of $w = f(u)$ and $\xi = f'(u)$ when $f'(\pm\infty) = +\infty$.*

consider the case that $f'(\tilde{u}) < 0$. Then there are two critical points $u_1 < u_2$ such that $f'(u_1) = f'(u_2) = 0$ since $f'(\tilde{u}) < 0$ and $f'(\pm\infty) = +\infty$. For definiteness, we assume $u_1 > 0$. The graphs of $w = f(u)$ in the (w, u) plane and $\xi = f'(u)$ in the (ξ, u) plane are shown in Figures 3.1(a) and (b).

Let $u^- < 0$ be a given state. Since our attention in this paper is paid on nonconvex cases, we assume that $u^- < u_1$. If $f(u^-) \le f(u_2)$, then we have a unique admissible noncombustion solution consisting of a forward wave (a shock or a compound wave—a sonic shock plus a rarefaction wave),

$$(u(\xi), z(\xi)) = \begin{cases} (u^-, 1), & \xi < 0, \\ (u^-, 0), & \xi \in (0, \bar{\xi}), \\ (u^+, 0), & \xi > \bar{\xi}, \end{cases}$$

where $\bar{\xi} = \frac{f(v) - f(u^-)}{v - u^-} = f'(v)$. Therefore, it is sufficient to consider the case that $f(u^-) > f(u_2)$.

First, we fix the notation $\bar{u}$, u^*, u_*, $\bar{q}$, q_*, a_0, a_1, and a_2. Let $\bar{u}$, u^*, u_* be so defined that $u_1 < \bar{u} < u_2$, $u^* > u_*$, $f(\bar{u}) = f(u^-)$, and

$$f'(u^*) = \frac{f(u^*) - f(u^-)}{u^* - u^-} = f'(u_*). \tag{3.1}$$

Let $\bar{q}$, q_* be such that

$$\bar{q} = \bar{u} - u^-, \qquad \frac{f(u_*) - f(u^-)}{u_* - (u^- + q_*)} = f'(u_*). \tag{3.2}$$

Then we have $u^* > \bar{u}$ and $\bar{q} < q_*$. Note that q_* is the distance between the tangential lines of $f(u)$ at $(u^*, f(u^*))$ and at $(u_*, f(u_*))$ vertically. We restrict ourselves to dealing with the case where $u_* > \bar{u}$ and $0 < q < q_*$ since the other cases can be treated similarly.

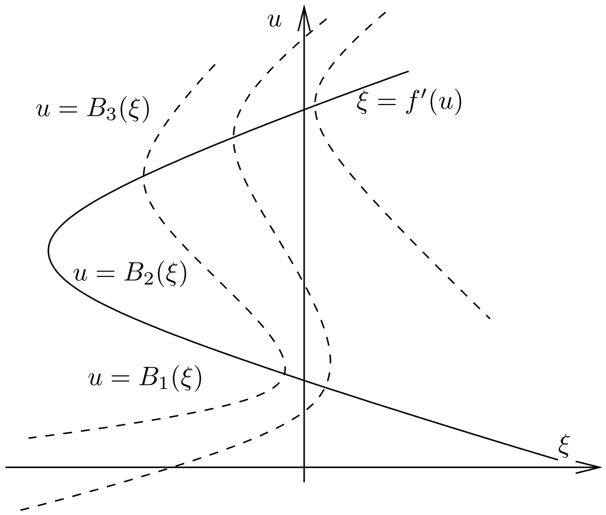


FIG. 3.2. *The branches of integral curve of* (3.3).

Draw a straight line $w - f(u^-) = f'(u^*)(u - (u^- + q))$ in the (u, w) plane, which intersects $w = f(u)$ at three points $(a_i, f(a_i)), i = 0, 1, 2$, $a_0 < a_1 < a_2$. Then we have the relation

$$u_2 > u^* > a_1 > u_* > a_0 > u_-.$$

We also denote $\eta^* = f'(u^*)$.

Fix u^- and q. Then the structure of solution will strongly depend on the value of u^+. Therefore we discuss the issue by the following three cases.

Case 3.1.1. $u^+ \in (a_1, +\infty)$.

Case 3.1.2. $u^+ \in (\bar{u}, a_1]$.

Case 3.1.3. $u^+ \in (0, \bar{u})$.

Before proceeding to discuss these cases, some lemmas are given in the following.

LEMMA 3.1. *All integral curves of the ordinary differential equation*

$$(f'(u) - \xi)\frac{du}{d\xi} = qk\left(\frac{\xi}{\eta}\right)^k \tag{3.3}$$

intersecting with $\xi = f'(u)$ in the (ξ, u) plane are shown in Figure 3.2. *Each of the integral curves consists of three branches: $u = B_i(\xi)$, $i = 1, 2, 3$, having the property that $B_3(\xi) > R_2(\xi) > B_2(\xi) > R_1(\xi) > B_1(\xi)$.*

Proof. Denote the integral curve of (3.3) by $u = B(\xi)$. Let (ξ_0, u_0) be the intersection point of $u = B(\xi)$ and $u = R_2(\xi)$. Then (ξ_0, u_0) is the singularity point of (3.3).

Denote by $u = B_3(\xi)$ the branch that lies in the upper side of $u = R_2(\xi)$ in the neighborhood of (ξ_0, u_0), i.e., $B_3(\xi) > R_2(\xi)$. Then it suffices to show that $B_3(\xi)$ will not intersect with $R_2(\xi)$ for the finite reaction rate $k < \infty$ as $\xi > \xi_0$. Indeed, using

(3.3), we obtain

$$\frac{d}{d\xi}(B_3(\xi) - R_2(\xi)) = \frac{kq(\frac{\xi}{\eta})^k}{f'(B_3(\xi)) - \xi} - \frac{1}{f''(R_2(\xi))}$$
$$= \frac{kq(\frac{\xi}{\eta})^k f''(R_2(\xi)) - (f'(B_3(\xi)) - \xi)}{(f'(B_3(\xi)) - \xi)f''(R_2(\xi))}.$$

Recall that $u = R_2(\xi)$ is the branch associated with the convex part of $f(u)$. Then $f''(R_2(\xi)) > 0$. Observe that once $u = B_3(\xi)$ is close to $u = R_2(\xi)$, $f'(B_3(\xi)) - \xi$ becomes small. Therefore, at this moment, the numerator is positive, which forces $u = B_3(\xi)$ to leave away $u = B_2(\xi)$ for the upper side of $R_2(\xi)$.

Similarly, we can prove the lemma for $u = B_2(\xi)$ and $u = B_1(\xi)$. ∎

The solution $u = B_i(\xi)$ $(i = 1, 2, 3)$ of (3.3) depends on the parameters η and k. When there is no risk of confusion, we suppress the dependence of B_i on η and k. Otherwise, we denote $u = B_i(\xi; \eta, k)$.

LEMMA 3.2. *Let $u = B_i(\xi)$, $i = 2, 3$, be the branches of integral curve of* (3.3) *through the point* (η^*, u^*). *Then*

(a) $\lim_{k\to+\infty} B_3(\xi) = \max\{a_2, R_2(\xi)\}$ *for all* $\xi \in (\eta^*, 0]$;

(b) $\lim_{k\to+\infty} B_2(\xi) = \max\{a_1, R_1(\xi)\}$ *for all* $\xi \in (\eta^*, 0]$.

Proof. We prove in two steps part (a) only. Part (b) can be treated in the same way. Note that $B_3(\xi)$ depends on k.

(i) The first step is to prove

$$\underline{\lim}_{k\to+\infty} B_3(\xi) \geq \max\{a_2, R_2(\xi)\}.$$

We write $\underline{\lim}_{k\to+\infty} =: \underline{\lim}$ for short. Since $B_3(\xi) \geq R_2(\xi)$, it suffices to prove $\underline{\lim} B_3(\xi) \geq a_2$. Assume to the contrary that this inequality is not true. Then there exists $\xi_0 \in (\eta^*, 0]$ such that $\underline{\lim} B_3(\xi_0) < a_2$. Setting $\eta = \eta^*$, $B_3(\eta^*) = u^*$ in (3.3), and integrating (3.3), from η^* to ξ_0, we get

(3.4)
$$f(B_3(\xi_0)) - f(u^*) - \xi_0 B_3(\xi_0) + \eta^* u^* + \int_{\eta^*}^{\xi_0} B_3(\xi)d\xi = \frac{qk\eta^*}{k+1}\left[\left(\frac{\xi_0}{\eta^*}\right)^{k+1} - 1\right].$$

Since $B_3(\xi)$ is increasing, we have

$$\int_{\eta^*}^{\xi_0} B_3(\xi)d\xi < B_3(\xi_0)(\xi_0 - \eta^*).$$

Letting $k \to +\infty$ in (3.4) gives

$$f(\underline{B}) - f(u^*) - \eta^*(\underline{B} - u^* - q) \geq 0, \quad (3.5)$$

where $\underline{B} = \underline{\lim} B_3(\xi_0)$. Since a_2 is so defined that

$$f(a_2) - f(u^-) = f'(u^*)(a_2 - (u^- + q)),$$

we get from (3.5)

$$f(a_2) - f(\underline{B}) - \eta^*(a_2 - \underline{B}) \leq 0, \quad (3.6)$$

where we use the definition of u^*, $f'(u^*)(u^*-u^-) = f(u^*) - f(u^-) < 0$.

On the other hand, it follows from $u^* \leq \underline{B} = \underline{\lim} B_3(\xi_0) < a_2$ and the convexity of $f(u)$ that

$$\frac{f(a_2) - f(\underline{B})}{a_2 - \underline{B}} > \eta^*, \tag{3.7}$$

which contradicts (3.6). Hence $\underline{\lim} B_3(\xi) \geq a_2$ for all $\xi \in (\eta^*, 0]$.

(ii) The second step is to verify

$$\overline{\lim_{k\to+\infty}} B_3(\xi) \leq \max\{a_2, R_2(\xi)\}.$$

For any $\xi_0 \in (\eta^*, 0]$, we get from (3.4) that

$$\begin{aligned} f(B_3(\xi_0)) &\leq f(u^*) + \xi_0 B_3(\xi_0) - \int_{\eta^*}^{\xi_0} B_3(\xi) d\xi - \eta^*(u^* + q) \\ &\leq f(u^*) + \xi_0 B_3(\xi_0) - u^*(\xi_0 - \eta^*) - \eta^*(u^* + q) \\ &< f(u^*) - \eta^* q, \end{aligned}$$

since $\xi_0 < 0$ and $B_3(\xi_0) > u^*$. This implies that $\{B_3(\xi_0)\}$ is bounded uniformly. Hence $\overline{\lim} B_3(\xi_0) := \overline{\lim}_{k_i\to+\infty} B_3(\xi_0)$ exists for any $\xi_0 \in (\eta^*, 0]$, denoted by $\overline{B}(\xi_0)$. Choose a subsequence $\{k_i\}$ from $\{k\}$ such that

$$\lim_{k_i\to+\infty} B_3(\xi_0) = \overline{B}(\xi_0).$$

Using Fatou's lemma, we get

$$\begin{aligned} \overline{\lim_{k_i\to+\infty}} \int_{\eta^*}^{\xi_0} B_3(\xi) d\xi &\geq \underline{\lim_{k_i\to+\infty}} \int_{\eta^*}^{\xi_0} B_3(\xi) d\xi \geq \int_{\eta^*}^{\xi_0} \underline{\lim_{k_i\to+\infty}} B_3(\xi) d\xi \\ &\geq \int_{\eta^*}^{\xi_0} \max(a_2, R_2(\xi)) d\xi. \end{aligned} \tag{3.8}$$

Setting $k_i \to +\infty$ in (3.4) and noting (3.8), we arrive at

$$f(\overline{B}(\xi_0)) - f(u^*) - \xi_0 \overline{B}(\xi_0) + \eta^* u^* + \int_{\eta^*}^{\xi_0} \max\{a_2, R_2(\xi)\} d\xi \leq -q\eta^*. \tag{3.9}$$

By the definition of a_2, we have $a_2 > u^*$. Let $\bar{\xi} \in (\eta^*, \xi_0]$ such that $R_2(\bar{\xi}) = a_2$. (If $a_2 > R_2(\xi)$ for all $\xi \in (\eta^*, \xi_0]$, then we take $\bar{\xi} = \xi_0$.) Then

$$\int_{\eta^*}^{\bar{\xi}} a_2 d\xi = (\bar{\xi} - \eta^*) a_2$$

and

$$\int_{\bar{\xi}}^{\xi_0} R_2(\xi) d\xi = f(a_2) - \bar{\xi} a_2 - (f(R_2(\xi_0)) - \xi_0 R_2(\xi_0)).$$

Note that

$$f(u^*) - f(u^-) = \eta^*(u^* - u^-), \qquad f(a_2) = f(u^-) + \eta^*(a_2 - (u^- + q)).$$

Then it is calculated that

$$(\bar{\xi} - \eta^*)a_2 = f(u^*) - f(a_2) + \bar{\xi}a_2 - \eta^*(u^* + q).$$

Consequently, we obtain

$$\begin{aligned}(3.10)\qquad &\int_{\eta^*}^{\xi_0} \max\{a_2, R_2(\xi)\}d\xi \\ &= f(u^*) - f(\max\{a_2, R_2(\xi_0)\}) + \xi_0 \max\{a_2, R_2(\xi_0)\} - \eta^*(u^* + q),\end{aligned}$$

which, combining with (3.9), provides

$$(3.11)\qquad f(\overline{B}(\xi_0)) - f(\max\{a_2, R_2(\xi_0)\}) - \xi_0(\overline{B}(\xi_0) - \max\{a_2, R_2(\xi_0)\}) \le 0.$$

Then there is $\theta \in (\max\{a_2, R_2(\xi_0)\}, \overline{B}(\xi_0))$ such that

$$f(\overline{B}(\xi_0)) - f(\max\{a_2, R_2(\xi_0)\}) = f'(\theta)(\overline{B}(\xi_0) - \max\{a_2, R_2(\xi_0)\}).$$

Note that $f''(u) > 0$ for all $u > \tilde{u}$ and $\overline{B}(\xi_0) > \theta > \max\{a_2, R_2(\xi_0)\} > \tilde{u}$. We have

$$f'(\theta) - \xi_0 = f'(\theta) - f'(R_2(\xi_0)) > 0.$$

Hence, (3.11) implies $\overline{B}(\xi_0) - \max\{a_2, R_2(\xi_0)\} < 0$. ∎

LEMMA 3.3. *Let $u = B_2(\xi)$ be the integral curve of (3.3) through $(0, u^+)$, where $u_1 < u^+ < u_2$. Then*

$$(3.12)\qquad \lim_{k\to+\infty} B_2(\xi) = \max\{u^+, R_1(\xi)\} \quad \textit{for all } \xi \in (f'(u^+), 0].$$

Proof. By Lemma 3.1, we have for a finite reaction rate k,

$$B_2(\xi) > u^+, \quad B_2(\xi) > R_1(\xi) \quad \text{for all } \xi \in (f'(u^+), 0],$$

and $B_2(\xi)$ is decreasing. Note that $u^+ > R_1(\xi)$ for all $\xi \in (f'(u^+), 0)$. Hence it suffices to prove that

$$(3.13)\qquad \lim_{r\to\infty} B_2(\xi) = u^+.$$

Assume to the contrary that there exists $\xi_0 \in (f'(u^+), 0)$ such that $\underline{\lim}_{k\to\infty} B_2(\xi_0) =: v > u^+$. Then we integrate (3.3) from ξ_0 to 0 and obtain

$$f(u^+) - f(B_2(\xi_0)) + \xi_0 B_2(\xi_0) + \int_{\xi_0}^0 B_2(\xi)d\xi = \frac{kq\xi_0}{k+1}\left(\frac{\xi_0}{\eta}\right)^k.$$

Since

$$\int_{\xi_0}^0 B_2(\xi)d\xi > -\xi_0 u^+,$$

we obtain that as k goes to infinity,

$$f(u^+) - f(v) + \xi_0(v - u^+) < 0,$$

i.e.,

$$\frac{f(u^+)-f(v)}{u^+-v} > \xi_0.$$

Therefore, there exists $\theta \in (u^+, v)$ such that

$$f'(\theta) = \frac{f(u^+)-f(v)}{u^+-v} > \xi_0. \tag{3.14}$$

(a) If $\theta \in (u_1, \tilde{u})$, then since $f''(u) < 0$ and $v > \theta > u^+$, we have $f'(v) < f'(\theta) < f'(u^+)$. This contradicts the fact that $f'(\theta) > \xi_0 > f'(u^+)$.

(b) If $\theta \in [\tilde{u}, u_2)$, then $f'(v) > f'(\theta)$ since $f''(u) > 0$ for all $u \in (\tilde{u}, u_2)$. Note that $\xi_0 > f'(v)$. We obtain the contradiction to (3.14). ∎

In the remainder of this section, we will suppress the subscript of B when doing so causes no confusion.

LEMMA 3.4. *Let $\eta \in (\eta^*, 0)$ and $B(0; \eta, k) = u^+ \in (\bar{u}, a_1)$. Then there exists a constant $k_0 > 0$ such that whenever $k > k_0$,*

$$B(\xi; \eta, k) < u^* \quad \textit{uniformly for all} \ \ \xi \in [\eta, 0], \ \eta \in (\eta^*, 0). \tag{3.15}$$

Proof. If (3.15) fails, then for any $n > 0$ there exist $k_n > n$, $\eta_n \in (\eta^*, 0)$, and $\xi_n \in [\eta_n, 0]$ such that $B(\xi; \eta_n, k_n) > u^*$ for all $\xi \in (\xi_n, 0]$ and $B(\xi_n; \eta_n, k_n) = u^*$.

Since $\eta_n \in (\eta^*, 0)$, we choose a convergent subsequence from $\{\eta_n\}$, still denoted by $\{\eta_n\}$. Set $\overline{\eta} = \lim \eta_n$, $\overline{\xi} = \lim \xi_n$. Substituting η, k, and u by η_n, k_n, and $B(\xi; \eta_n, k_n)$ in (3.3), respectively, and then integrating it from ξ_n to 0, we obtain

$$f(u^+) - f(u^*) + \xi_n u^* + \int_{\xi_n}^0 B(\xi; \eta_n, k_n) d\xi = -q\xi_n \frac{k_n}{k_n+1} \left(\frac{\xi_n}{\eta_n}\right)^{k_n}. \tag{3.16}$$

Letting $n \to +\infty$ and noting $B(\xi; \eta_n, k_n) \geq u^+$, we have from (3.16) that

$$\frac{f(u^*) - f(u^+)}{u^* + q - u^+} \geq \overline{\xi}. \tag{3.17}$$

On the other hand, when $u^+ \in (\bar{u}, a_1)$,

$$\frac{f(u^*) - f(u^+)}{u^* + q - u^+} < \eta^*. \tag{3.18}$$

Then $\eta^* > \overline{\xi}$, which contradicts $\overline{\xi} \geq \overline{\eta} \geq \eta^*$. Thus, we complete the proof. ∎

LEMMA 3.5. *Let $B(0; \eta_1, k) = B(0; \eta_2, k) = u^+ \in (\bar{u}, a_1)$. If $\eta_1 \geq \eta_2 > \eta^*$, then, for sufficiently large k,*

$$B(\xi; \eta_1, k) > B(\xi; \eta_2, k) \ \textit{for all} \ \xi \in [\eta_1, 0]. \tag{3.19}$$

Proof. Letting $w(\xi; \eta, k) = \frac{\partial B(\xi; \eta, k)}{\partial \eta}$, and differentiating (3.3) with respect to ξ, we obtain

$$\begin{cases} \dfrac{dw}{d\xi} + \dfrac{f''(B) qk (\frac{\xi}{\eta})^k}{(f'(B) - \xi)^2} w = \dfrac{-qk(\frac{\xi}{\eta})^k}{\eta(f'(B) - \xi)}, & \eta < \xi < 0, \\ w(0; \eta, k) = 0. \end{cases}$$

From Lemma 3.1, it follows that $f'(B(\xi;\eta,k)) - \xi < 0$. Then the fact that the right-hand side of the above equation is negative implies $w > 0$ for all $\xi \in (\eta, 0)$. Hence, $B(\xi;\eta,k)$ increases in η. Thus we obtain (3.19). ∎

LEMMA 3.6. *Let $B(0;\eta,k) = u^+ \in (\bar{u}, a_1)$. Then, for sufficiently large k, there exists $\eta_k \in (\eta^*, 0)$ such that*

$$\eta_k = \frac{f(B(\eta_k;\eta_k,k)) - f(u^-)}{B(\eta_k;\eta_k,k) - u^-}. \tag{3.20}$$

Proof. Let $H_k(\eta) = f(B(\eta;\eta,k)) - f(u^-) - \eta(B(\eta;\eta,k) - u^-)$. By Lemma 3.1, we have $u^+ < B(\eta^*;\eta^*,k) < u^*$. Then

$$H_k(\eta^*) = f(B(\eta^*;\eta^*,k)) - f(u^-) - \eta^*(B(\eta^*;\eta^*,k) - u^-) > 0$$

and

$$H_k(0) = f(u^+) - f(u^-) - 0(u^+ - u^-) < 0,$$

which implies the existence of η_k satisfying (3.20). ∎

Now we turn to consider the admissible solution of (2.2) and (2.4) corresponding to Cases 3.1.1–3.1.3. In the following arguments, we need to notice the dependence of solutions $(u(\xi), z(\xi))$ of (2.2) and (2.4) on the reaction rate k.

Case 3.1.1. $u^+ \in (a_1, +\infty)$.

THEOREM 3.7. *For Case* 3.1.1, *the Riemann problem* (2.2) *and* (2.4) *has a unique admissible solution* $(u(\xi), z(\xi))$,

$$(u(\xi), z(\xi)) = \begin{cases} (u^-, 1), & \xi < \eta^*, \\ (B(\xi;\eta^*,k), (\frac{\xi}{\eta^*})^k), & \xi \in [\eta^*, 0], \\ (C(\xi), 0), & \xi > 0, \end{cases} \tag{3.21}$$

where $C(\xi)$ is the solution of the following boundary value problem:

$$\begin{cases} (f'(C) - \xi)\frac{dC}{d\xi} = 0, C(\xi) \geq 0, \quad \xi \in (0, +\infty), \\ C(0) = B(0;\eta^*,k), \\ C(+\infty) = u^+. \end{cases} \tag{3.22}$$

The infinite reaction rate limit is

$$\lim_{k\to+\infty} (u(\xi), z(\xi)) = \begin{cases} (u^-, 1), & \xi < \eta^*, \\ (U(\xi), 0), & \xi \geq \eta^*, \end{cases} \tag{3.23}$$

where $U(\xi)$ is the solution of

$$\begin{cases} (f'(U) - \xi)\frac{dU}{d\xi} = 0, \eta^* < \xi < +\infty, \\ U(\eta^*) = a_2, \quad U(+\infty) = u^+. \end{cases} \tag{3.24}$$

Note that here $\eta^* = \frac{f(a_2) - f(u^-)}{a_2 - (u^- + q)} < f'(a_2)$.

Proof. Note that $B_3(0;\eta^*,k) > u_2$ and $\lim_{k\to\infty} B_3(\xi;\eta^*,k) = \max\{R_2(\xi), a_2\}$ for $\xi \in [\eta^*, 0]$. Also note that

$$\frac{f(u_2) - f(u)}{u_2 - u} \geq 0 \quad \text{for } u \in [u_2, \infty) \tag{3.25}$$

and

$$f(u_2) - f(u) \over u_2 - u < 0 \quad \text{for } u \in (a_1, u_2). \tag{3.26}$$

Then there exists $v \in (a_1, u_2)$ such that for the sufficiently large k, $f(B_3(0;\eta^*,k)) = f(v)$. Thus we have two subcases:

(i) $u^+ \in [v, \infty)$. Then

$$\frac{f(B_3(0;\eta^*,k)) - f(u)}{B_3(0;\eta^*,k)) - u} \geq 0 \quad \text{for all } u \in [v, \infty). \tag{3.27}$$

Hence the boundary problem (3.22) has the unique solution $C(\xi)$ for all $\xi > 0$ with $C(0+0) = B_3(0;\eta^*,k)$. Thus the solution $(u(\xi), z(\xi))$ can be expressed as in (3.21), where $u(\xi) = B_3(\xi;\eta^*,k)$ as $\xi \in [\eta^*, 0)$. This is obviously the admissible solution of (2.2) and (2.4), as shown in Figure 3.3(a).

Next we prove the uniqueness. Let $(u(\xi), z(\xi))$ be the solution of (2.2) and (2.4) for this case. According to Lemma 2.1, $z(\xi)$ has the structure (2.8). When $\xi < \eta$, u satisfies (2.10), of which the unique solution is $u = u^-$. When $\xi \geq 0$, u satisfies (2.12). When $u^+ > u_2$, there exists a unique solution of (2.12) for a fixed $u(0+0)$ if $u(0+0) > u_2$. So, we only need to prove $\eta = \eta^*$ and $u(\xi) = B(\xi;\eta^*,k)$, as $\xi \in (\eta^*, 0)$.

Indeed, we know from the Rankine–Hugoniot jump condition (2.6) that $\eta \geq \eta^*$. If $\eta > \eta^*$, then the Rankine–Hugoniot jump condition (2.6) and the Oleinik-type entropy condition (2.7) imply $u(\eta+0) < u^*$. Therefore the solution of (2.11) must be continuous and decreasing. Thus, $u(0-0) < u^*$. However, the discontinuity, which is the jump at $\xi = 0$ from $u(0-0) < u^*$ to $u(0+0) > u_2$, does not satisfy the Oleinik-type entropy condition (2.7). Therefore, $\eta = \eta^*$ and $u(\eta^*+0) = u^*$. Note from (2.9) that $u(\xi) = B_3(\xi;\eta^*,k)$ lies in the right-hand side of η^*. We conclude that $u(\xi)$ is continuous in $(\eta^*, 0)$. Otherwise, suppose ξ_0 to be a discontinuity point of $u(\xi)$. Then, by the Oleinik-type entropy condition (2.7) (observe Figure 3.1), it is easy to see that $u(\xi_0+0) < u^*$, which, combined with the entropy condition, implies that $u(\xi)$ is decreasing and continuous in $(\eta^*, 0)$. Consequently, $u(0-0) < u^*$. However, the discontinuity at $\xi = 0$ does not satisfies the Oleinik-type entropy condition (2.7). This is a contradiction.

By Lemma 3.2, we have

$$\lim_{k\to\infty} B_3(\xi;\eta^*,k) = \max\{a_2, R_2(\xi)\}, \quad \xi \in (\eta^*, 0),$$

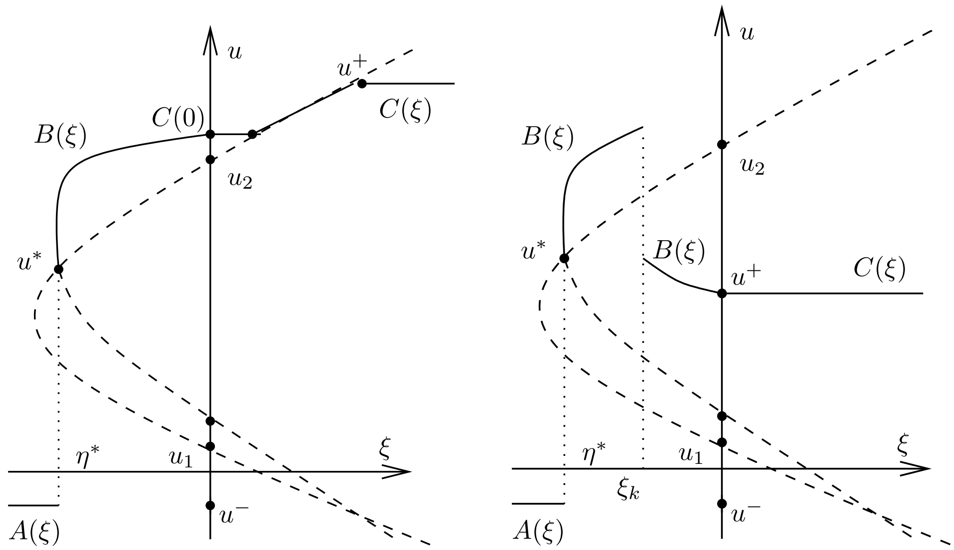
which is the same as the solution of (3.24) for $\xi \in (\eta^*, 0)$.

(ii) $u^+ \in [a_1, v)$. Note that

$$\frac{f(B_3(0;\eta^*,k)) - f(u)}{B_3(0;\eta^*,k)) - u} < 0 \quad \text{for all } u \in [a_1, v). \tag{3.28}$$

Then we cannot find a solution to (3.22) such that $C(0+0) = B_3(0;\eta^*,k)$. Instead, we construct a solution in the form of (3.21), in which $B(\xi;\eta^*,k)$, at this moment, becomes

$$B(\xi;\eta^*,k) = \begin{cases} B_3(\xi;\eta^*,k), & \xi \in [\eta^*, \xi_k), \\ B_2(\xi;\eta^*,k), & \xi \in [\xi_k, 0), \end{cases} \tag{3.29}$$



(a) The solution when $u^+ \in (v, +\infty)$.

(b) The solution when $u^+ \in (a_1, v)$.

FIG. 3.3. *The structure of the entropy solution to the ZND model when* $u^+ \in (a_1, \infty)$.

where $B_3(\xi;\eta^*,k) =: B_3(\xi)$ and $B_2(\xi;\eta^*,k) =: B_2(\xi)$ are defined in Lemmas 3.2 and 3.3, respectively, and $\xi_k \in [\eta^*, 0)$ is to be determined. Next we show that this is indeed the solution of (2.2) and (2.4) for a suitable ξ_k.

In fact, set $\eta = \eta^*$ in (3.3). Then we integrate (3.3) from η^* to ξ_k and from ξ_k to 0, respectively, to obtain

$$f(B_3(\xi_k)) - f(u^*) - \xi_k B_3(\xi_k) + \eta^* u^* + \int_{\eta^*}^{\xi_k} B_3(\xi)d\xi = \frac{kq\eta^*}{k+1}\left(\left(\frac{\xi_k}{\eta^*}\right)^{k+1} - 1\right) \tag{3.30}$$

and

$$f(u^+) - f(B_2(\xi_k)) + \xi_k B_2(\xi_k) + \int_{\xi_k}^{0} B_2(\xi)d\xi = -\frac{kq\eta^*}{k+1}\left(\frac{\xi}{\eta^*}\right)^{k+1}. \tag{3.31}$$

We add these two identities to get

$$\begin{aligned} &f(B_2(\xi_k)) - f(B_2(\xi_k)) - \xi_k(B_3(\xi_k) - B_2(\xi_k)) \\ &+ f(u^+) - f(u^*) + \eta^* u^* + \frac{kq\eta^*}{k+1} + \int_{\eta^*}^{\xi_k} B_3(\xi)d\xi + \int_{\xi_k}^{0} B_2(\xi)d\xi = 0. \end{aligned} \tag{3.32}$$

Set

$$H(\xi) = f(B_3(\xi)) - f(B_2(\xi)) - \xi(B_3(\xi) - B_2(\xi)). \tag{3.33}$$

Then

$$\frac{dH}{d\xi} = B_2(\xi) - B_3(\xi) < 0. \tag{3.34}$$

Note that

$$\frac{f(a_2) - f(u^+)}{a_2 - u^+} > \eta^*. \tag{3.35}$$

Hence, in light of Lemmas 3.2 and 3.3, for the sufficiently large k, there exists $\eta^* < \bar{\xi} < 0$ such that $H(\bar{\xi}) > 0$. Since $f(B_3(0)) = f(v) < f(u^+)$, $H(0) < 0$. Therefore, there exists a unique $\xi_k \in (\bar{\xi}, 0)$ such that $H(\xi_k) = 0$. Take

$$\xi_k = \frac{f(B_3(\xi_k)) - f(B_2(\xi_k))}{B_3(\xi_k) - B_2(\xi_k)}. \tag{3.36}$$

Then the solution $u = u(\xi)$ has a discontinuity at $\xi = \xi_k$, which satisfies the Oleinik entropy condition (2.7) since $B_3(\xi_k) > B_2(\xi_k)$ and $B_3(\xi_k) > u^*$.

In light of (3.32), the solution $(u(\xi), z(\xi))$ constructed above satisfies (2.2) and (2.4) in the sense of distributions and therefore is admissible. We display this solution in Figure 3.3(b).

The uniqueness can be proved similarly to that in (i).

By Lemmas 3.2 and 3.3, the limit of this solution as the reaction rate k goes to infinity is expressed in (3.23). ∎

Case 3.1.2. $u^+ \in (\bar{u}, a_1]$.

THEOREM 3.8. *When k is sufficiently large, then*

(1) *there exists $\eta_k \in (\eta^*, 0)$ such that the Riemann problem for Case* 3.1.2 *has the unique admissible solution*

$$(u(\xi), z(\xi)) = \begin{cases} (u^-, 1), & \xi \in (-\infty, \eta_k), \\ (B(\xi; \eta_k, k), (\frac{\xi}{\eta_k})^k), & \xi \in (\eta_k, 0), \\ (u^+, 0), & \xi \in (0, +\infty), \end{cases} \tag{3.37}$$

where $B(\xi; \eta_k, k)$ satisfies (3.3) *with $B(0; \eta_k, k) = u^+ \in (\bar{u}, a_1]$ and*

$$\eta_k = \frac{f(B(\eta_k; \eta_k, k)) - f(u^-)}{B(\eta_k; \eta_k, k) - u^-};$$

(2) *there holds*

$$\lim_{k \to +\infty} (u(\xi), z(\xi)) = \begin{cases} (u^-, 1), & \xi \in (-\infty, \eta_0), \\ (U(\xi), 0), & \xi \in (\eta_0, +\infty), \end{cases} \tag{3.38}$$

where $U(\xi) = \max\{R_1(\xi), u^+\}$ and η_0 satisfies

$$\eta_0 = \frac{f(U(\eta_0)) - f(u^-)}{U(\eta_0) - u^- - q} \geq f'(U(\eta_0)). \tag{3.39}$$

Proof. (1) If $(u(\xi), z(\xi))$ is the solution of (2.2) and (2.4), then $u(\xi)$ satisfies (2.10) and (2.12) when $\xi \in (-\infty, \eta)$ and $\xi \in (0, +\infty)$, respectively. By the Oleinik-type entropy condition (2.7) and under the assumption that $u^- < a_0$ and $u^+ \in (\bar{u}, a_1]$, we have $u(\xi) = u^-$ for $\xi \in (-\infty, \eta)$ and $u(\xi) = u^+$ for $\xi \in (0, +\infty)$. By Lemma 3.6, there exists $\eta_k \in (\eta^*, 0)$ such that

$$\eta_k = \frac{f(B(\eta_k; \eta_k, k)) - f(u^-)}{B(\eta_k; \eta_k, k) - u^-}.$$

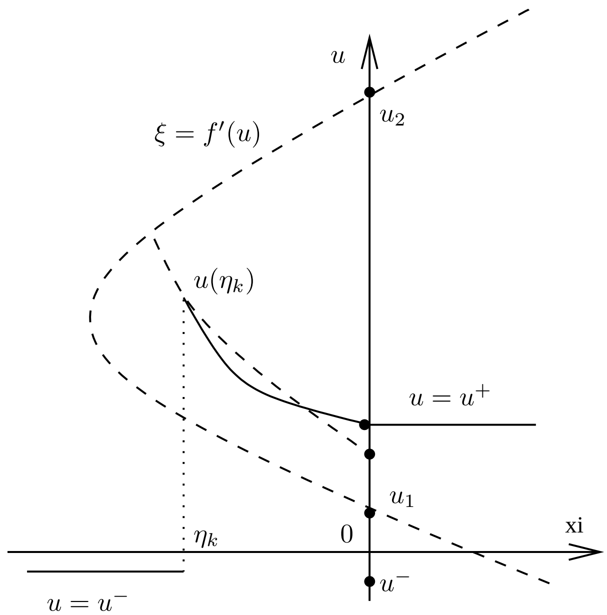


FIG. 3.4. *The structure of entropy solution to the ZND model when* $u^+ \in (\bar{u}, a_1)$.

It is evident that the entropy condition (2.7) is satisfied at the jump $\xi = \eta_k$. Thus, the admissible solution exists (see Figure 3.4). Arguments similar to those in Theorem 3.7 show that such a solution is unique.

(2) Integrating (3.3) from η_k to 0, we have

$$f(u^+) - f(B(\eta_k; \eta_k, k)) + \eta_k B(\eta_k; \eta_k, k) + \int_{\eta_k}^0 B(\xi; \eta_k, k) d\xi + q\eta_k \frac{k}{k+1} = 0. \tag{3.40}$$

Substituting $H(\eta_k) = 0$ (where $H(\eta_k)$ is defined in Lemma 3.6) into (3.40) gives

$$f(u^+) - f(u^-) + \eta_k u^- + q\eta_k \frac{k}{k+1} + \int_{\eta_k}^0 B(\xi; \eta_k, k) d\xi = 0. \tag{3.41}$$

Let $\bar{\eta}_0 = \overline{\lim}_{k\to+\infty} \eta_k$. Then, in light of Lemma 3.3, we get from (3.41)

$$f(u^+) - f(u^-) + \bar{\eta}_0 u^- + q\bar{\eta}_0 + \int_{\bar{\eta}_0}^0 U(\xi) d\xi = 0, \tag{3.42}$$

where $U(\xi) = \max\{u^+, R_1(\xi)\}$. Therefore, we have

$$\int_{\bar{\eta}_0}^0 U(\xi) d\xi = \begin{cases} -\bar{\eta}_0 u^+, & \bar{\eta}_0 > f'(u^+), \\ -\bar{\eta}_0 U(\bar{\eta}_0) - f(u^+) + f(U(\bar{\eta}_0)), & \bar{\eta}_0 \le f'(u^+). \end{cases} \tag{3.43}$$

Equations (3.42) and (3.43) imply that $\bar{\eta}_0$ satisfies (3.39).

Analogously, it is proved that $\underline{\eta}_0 = \underline{\lim}_{k\to+\infty} \eta_k$ satisfies (3.39). Therefore, $\eta_0 = \lim_{k\to+\infty} \eta_k = \bar{\eta}_0 = \underline{\eta}_0$ and satisfies (3.39).

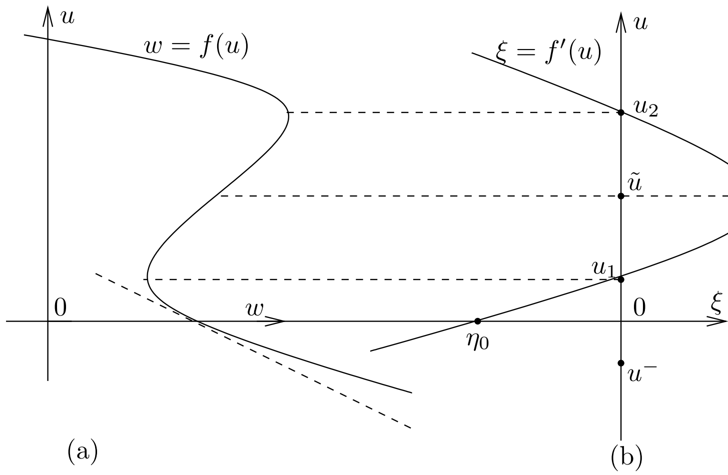


FIG. 3.5. *The graphs of $w = f(u)$ and $\xi = f'(u)$ when of $f'(\pm\infty) = -\infty$.*

For any given $\xi_0 \in (\eta_0, 0)$, choose a small $\epsilon > 0$ so that $\eta_0 + \epsilon < \xi_0$. Then we have $\eta_k < \eta_0 + \epsilon < \xi_0$ for sufficiently large k. From Lemma 3.6, we get

$$U(\xi_0) < B(\xi_0; \eta_k, k) < B(\xi_0; \eta_0 + \epsilon, k).$$

Thus, letting $k \to +\infty$, we obtain

$$\lim_{k\to\infty} B(\xi_0; \eta_k, k) = U(\xi_0).$$

Due to the arbitrariness of ξ_0, $\lim_{k\to\infty} B(\xi; \eta_k, k) = U(\xi)$ for all $\xi \in (\eta_0, 0)$. ☐

Case 3.1.3. $u^+ \in (0, \bar{u}]$. For this case, we have the following theorem.

THEOREM 3.9. *For Case* 3.1.3, *the unique admissible solution of* (2.2) *and* (2.4) *is the noncombustion solution*

$$(u(\xi), z(\xi)) = \begin{cases} (u^-, 1), & \xi < \frac{f(u^-)-f(u^+)}{u^- - u^+}, \\ (u^+, 0), & \xi > \frac{f(u^-)-f(u^+)}{u^- - u^+}. \end{cases} \tag{3.44}$$

3.2. The solution of (2.2) and (2.4) when $f(u)$satisfies (A_2). Parallel to section 3.1, we solve (2.2) and (2.4) as $f(u)$ has just one inflection point and the slope at infinity is negative infinity. The results are stated in Theorems 3.12 and 3.13. We omit some proofs because they are similar to those in section 3.1.

Let $\tilde{u}$ be the inflection point of $f(u)$. When $f'(\tilde{u}) \le 0$, there is no combustion solution (cf. [13]). Therefore, we only need to consider $f'(\tilde{u}) > 0$. Let u_1, u_2, and u_3 be such that $u_3 > u_2 > u_1 > 0$ and $f'(u_1) = f'(u_2) = 0$. Recall that $u = 0$ is assumed to be the ignition point and therefore plays an important role in the current context. We set $\hat{q}$ to satisfy $f'(0) = \frac{f(u_2)-f(0)}{u_2-\hat{q}}$. The value of the binding energy q is distinguished into two classes: $0 < q < \hat{q}$ and $q \ge \hat{q}$. We restrict our study to the solution to the case that $q \in (0, \hat{q})$. The case that $q \ge \hat{q}$ can be treated similarly.

Draw the straight line $w - f(0) = f'(0)(u - q)$ in the (u, w) plane. Then it intersects $w = f(u)$ at three points $(a_i, f(a_i))$, $i = 1, 2, 3$, where $a_2 > a_1 > a_0$. It is evident that $a_2 > u_2 > a_1 > 0 > a_0$.

For the fixed $q \in (0, \hat{q})$ and $u^- \in (-\infty, 0)$, the structure of the solution of the Riemann problem (2.2) and (2.4) will depend on the value of u^+. We proceed our discussion through two cases.

Case 3.2.1. $u^+ \in (0, a_2)$.

Case 3.2.2. $u^+ \in [a_2, \infty)$.

Case 3.2.1. $\boldsymbol{u^+ \in (0, a_2)}$**.** The main result is stated in Theorem 3.12 by following Lemmas 3.10 and 3.11.

LEMMA 3.10. *Let* $B(\xi; \eta_0, k)$ *be the smooth solution of the problem*

$$(f'(B) - \xi)\frac{dB}{d\xi} = qk\left(\frac{\xi}{\eta_0}\right)^k, \qquad \eta_0 < \xi \le 0, \tag{3.45}$$
$$B(\eta_0; \eta_0, k) = 0,$$

where $\eta_0 = f'(0)$*. Then* $u = B(\xi; \eta_0, k)$ *does not increase until it intersects with* $\xi = f'(u)$ *and* $\lim_{k\to\infty} B(\xi; \eta_0, k) = \max\{a_1, R_1(\xi)\}$, $\xi \in (\eta_0, 0]$.

LEMMA 3.11. *Let* $B(\xi; \eta_0, k)$ *be the smooth solution of the problem*

$$\begin{cases} (f'(B) - \xi)\dfrac{dB}{d\xi} = qk\left(\dfrac{\xi}{\eta_0}\right)^k, & \xi < 0, \\ B(0; \eta_0, k) = u^+. \end{cases} \tag{3.46}$$

Then for all $u^+ \in (u_2, a_2)$ *or* $u^+ \in (0, u_1)$, $f'(B(\xi; \eta_0, k) < \xi$, *and therefore* $B(\xi; \eta_0, k)$ *does not decrease until it intersects with* $\xi = f'(u)$*. Furthermore, the limit of* $B(\xi; \eta_0, k)$ *as the reaction rate goes to infinity is*

$$\lim_{k\to\infty} B(\xi; \eta_0, k) = u^+ \tag{3.47}$$

for all $\xi \in (f'(u^+), 0)$.

Note that $B(\eta_0; \eta_0, k)$ *does not need to be zero, which is different from Lemma* 3.10.

The proof of Lemmas 3.10 and 3.11 is similar to that of Lemmas 3.1 and 3.3. Using a method similar to that in Theorem 3.7, we can construct the solution of (2.2) and (2.4) for Case 3.2.1.

THEOREM 3.12. *When* k *is large enough, the Riemann problem of* (2.2) *and* (2.4) *for Case* 3.2.1 *has the unique admissible solution with the structure*

$$(u(\xi), z(\xi)) = \begin{cases} (\max\{u^-, R_1(\xi)\}, 1), & -\infty < \xi < \eta_0, \\ (B(\xi; \eta_0, k), (\frac{\xi}{\eta_0})^k), & \eta_0 < \xi < 0, \\ (C(\xi; \eta_0, k), 0), & 0 \le \xi < +\infty \end{cases} \tag{3.48}$$

and

$$\lim_{k\to\infty}(u(\xi), z(\xi)) = \begin{cases} (\max\{u^-, R_1(\xi)\}, 1), & -\infty < \xi \le \eta_0, \\ (C(\xi), 0), & \eta_0 < \xi < +\infty, \end{cases} \tag{3.49}$$

where $C(\xi)$ *is the solution of*

$$\begin{cases} (f'(C) - \xi)\frac{dC}{d\xi} = 0, \quad \eta_0 \le \xi < +\infty, \\ C(\eta_0) = a_1, \quad C(+\infty) = u^+. \end{cases} \tag{3.50}$$

Note here that $\eta_0 = \frac{f(a_1)-f(0)}{a_1-q} < f'(a_1)$.

Proof. Let $(u(\xi), z(\xi)) = (\max\{u^-, R_1(\xi)\}, 1)$ when $\xi \in (-\infty, \eta_0)$. Let $B_1(\xi;\eta_0,k)$ and $B_2(\xi;\eta_0,k)$ be the solutions of (3.45) and (3.46), respectively.

(i) If $\frac{f(u^+)-f(B_1(0;\eta_0,k))}{u^+-B_1(0;\eta_0,k)} \geq 0$, then we choose $u(\xi) = B_1(\xi;\eta_0,k)$ as the smooth solution of equation (3.45) when $\eta_0 < \xi < 0$. By Lemma 3.10, $\lim_{k\to\infty} B_1(\xi;\eta_0,k) = \max\{a_1, R_1(\xi)\}$, as $\xi \in (\eta_0, 0]$. Thus we have $u_1 < B(0;\eta_0,k) < u_2$, as k is large enough, due to $u_1 < a_1 < u_2$. Then the problem

$$\begin{cases} (f'(C)-\xi)\frac{dC}{d\xi} = 0, & 0 \leq \xi < +\infty, \\ C(0) = B_1(0;\eta_0,k), & C(+\infty;\eta_0,k) = u^+ \end{cases} \tag{3.51}$$

has a unique entropy solution since $\frac{f(u^+)-f(B_1(0;\eta_0,k))}{u^+-B_1(0;\eta_0,k)} \geq 0$.

(ii) If $\frac{f(u^+)-f(B_1(0;\eta_0,k))}{u^+-B_1(0;\eta_0,k)} < 0$, then we choose $u(\xi) = u^+$ as $\xi \geq 0$, $u(\xi) = B_1(\xi;\eta_0,k)$ as $\xi \in (\eta_0, \xi_k)$, and $u(\xi) = B_2(\xi;\eta_0,k)$ as $\xi \in (\xi_k, 0)$, where ξ_k is to be determined. Note that now $u^+ \notin (u_1,u_2)$ since $B_1(0;\eta_0,k) \in (u_1,u_2)$ and $\frac{f(B_1(0;\eta_0,k)-f(v)}{B_1(0;\eta_0,k)-v} > 0$ for all $v \in (u_1,u_2)$.

The summation of the integration of (3.45) from η_0 to ξ_k and the integration (3.46) from ξ_k to 0 results in

$$\begin{aligned} &[f(B_1(\xi_k;\eta_0,k)) - f(B_2(\xi_k;\eta_0,k))] \\ &- \xi_k[B_1(\xi_k;\eta_0,k) - B_2(\xi_k;\eta_0,k)] - [f(0) - f(u^+)] + q\eta_0\frac{k}{k+1} \\ &+ \int_{\eta_0}^{\xi_k} B_1(\xi;\eta_0,k)d\xi + \int_{\xi_k}^{0} B_2(\xi;\eta_0,k)d\xi = 0. \end{aligned} \tag{3.52}$$

Set

$$\begin{aligned} H(\xi) =\ &[f(B_1(\xi;\eta_0,k)) - f(B_2(\xi;\eta_0,k))] \\ &- \xi[B_1(\xi;\eta_0,k) - B_2(\xi;\eta_0,k)]. \end{aligned} \tag{3.53}$$

When $u^+ \in (0,u_1)$, we have

$$\frac{dH}{d\xi} = B_2(\xi;\eta_0,k) - B_1(\xi;\eta_0,k) < 0 \quad \text{for all } \xi \in (f'(u^+), 0) \tag{3.54}$$

and

$$H(0) = f(B_1(0;\eta_0,k)) - f(u^+) < 0. \tag{3.55}$$

In light of Lemmas 3.10 and 3.11, $\lim_{k\to\infty} B_1(\xi;\eta_0,k) = \max\{a_1, R_1(\xi)\}$ and $\lim_{k\to\infty} B_2(\xi;\eta_0,k) = u^+$ as $\xi \in (f'(u^+), 0)$. Since

$$\frac{f(a_1)-f(u^+)}{a_1-u^+} > f'(u^+),$$

$H(\xi) > 0$ for sufficiently large k as $\xi > f'(u^+)$, and ξ is close to $f'(u^+)$. Therefore, there exists $\xi_k \in (f'(u^+), 0)$ such that $H(\xi_k) = 0$.

When $u^+ \in (u_2, a_2)$, we have

$$H(0) = f(B_1(0;\eta_0,k)) - f(u^+) > 0$$

and

$$H(\eta_0) = f(0) - f(B_2(\eta_0;\eta_0,k)) - \eta_0(0 - B_2(\eta_0;\eta_0,k) < 0;$$

there also exists $\xi_k \in (\eta_0, 0)$ such that $H(\xi_k) = 0$. Thus, define

$$\xi_k = \frac{f(B_1(\xi_k;\eta_0,k)) - f(B_2(\xi_k;\eta_0,k))}{B_1(\xi_k;\eta_0,k) - B_2(\xi_k;\eta_0,k)}. \tag{3.56}$$

By Lemmas 3.10 and 3.11, we have

$$\lim_{k\to\infty} B_1(\xi;\eta_0,k) = \max\{a_1, R_1(\xi)\}, \quad \lim_{k\to\infty} B_2(\xi;\eta_0,k) = u^+.$$

Then, as k is sufficiently large, $B_1(\xi_k;\eta_0,k)$ is close to $\max\{a_1, R_1(\xi_k)\}$ and $B_2(\xi_k;\eta_0,k)$ is close to u^+. Therefore, the discontinuity of u at $\xi = \xi_k$ satisfies the Oleinik entropy condition (2.7). By (3.52), we conclude that this solution satisfies (2.2) and (2.4) in the sense of distributions. Thus we have constructed the admissible solution. The uniqueness is obvious.

Using Lemmas 3.10 and 3.11, the limit of the solution is obtained. □

Case 3.2.2. $u^+ \in [a_2, +\infty)$. Set

$$(u(\xi), z(\xi)) = \begin{cases} (\max\{u^-, R_1(\xi)\}, 1), & \xi \in (-\infty, \eta_k), \\ (B(\xi;\eta_k,k), (\frac{\xi}{\eta_k})^k), & \xi \in (\eta_k, 0], \\ (u^+, 0), & \xi \in (0, +\infty), \end{cases} \tag{3.57}$$

where η_k is to be determined, and $B(\xi;\eta_k,k)$ satisfies

$$\begin{aligned} &(f'(B) - \xi)\frac{dB}{d\xi} = qk\left(\frac{\xi}{\eta_k}\right)^k, \qquad \eta_k < \xi < 0, \\ &B(0;\eta_k,k) = u^+. \end{aligned} \tag{3.58}$$

Since $u^+ \geq a_2$, the Riemann problem of (2.2) and (2.4) does not have the same kind of solution as that in Case 3.2.1. This implies that $\eta_k < \eta_0$. Denote $\eta^- = f'(u^-)$. The summation of the integration of (3.3) from η^- to η_k when $q = 0$ and the integration from η_k to 0 when $q > 0$ is

$$\begin{aligned} &f(\max\{u^-, R_1(\eta_k)\}) - f(B(\eta_k;\eta_k,k)) - \eta_k(\max\{u^-, R_1(\eta_k)\} - B(\eta_k;\eta_k,k)) \\ &+ f(u^+) - f(u^-) + \eta^- u^- + \int_{\eta^-}^{\eta_k} \max\{u^-, R_1(\xi)\} d\xi + \int_{\eta_k}^{0} B(\xi;\eta_k,k) d\xi + \frac{kq\eta_k}{k+1} = 0. \end{aligned} \tag{3.59}$$

Set

$$\begin{aligned} H(\eta) = {}& f(B(\eta;\eta,k)) - f(\max\{u^-, R_1(\eta)\}) \\ &- \eta[B(\eta;\eta,k) - \max\{u^-, R_1(\eta)\}]. \end{aligned} \tag{3.60}$$

Then we claim that there exists $\eta_k < \eta_0$ such that $H(\eta_k) = 0$.

Draw a line $w - f(u^-) = f'(u^-)(u - (u^- + q))$. It has an intersection point, denoted by $(b_1, f(b_1))$, $b_1 > a_2$, with $w = f(u)$. Then we prove the above claim using two cases.

(i) $u^+ \in [a_2, b_1)$. Then we can find $u^* \in (u^-, 0]$ such that

$$f'(u^*) = \frac{f(u^+) - f(u^*)}{u^+ - (u^* + q)} =: \eta^*. \tag{3.61}$$

In light of Lemma 3.11, for sufficiently large k, the solution $B(\eta; \eta, k)$ of (3.58) is close to u^+. Therefore,

$$H(\eta^*) = f(u^*) - f(B(\eta^*; \eta^*, k)) - \eta^*(u^* - B(\eta^*; \eta^*, k)) > 0. \tag{3.62}$$

On the other hand,

$$H(\eta^-) = f(u^-) - f(B(\eta^-; \eta^-, k)) - \eta^-(u^- - B(\eta^-; \eta^-, k)) < 0. \tag{3.63}$$

Hence, there exists $\eta_k \in (\eta^-, \eta^*)$ such that $H(\eta_k) = 0$.

(ii) $u^+ \in [b_1, +\infty)$. Now we set

$$\eta^* = \frac{f(u^+) - f(u^-)}{u^+ - (u^- + q)}. \tag{3.64}$$

Then we assert

$$\eta^- > \eta^* > \eta^+ := f'(u^+). \tag{3.65}$$

Making use of (3.59), we can conclude that

$$H(\eta^+) < 0 \quad \text{and} \quad H(\eta^-) > 0. \tag{3.66}$$

Therefore, there also exists $\eta_k \in (\eta^+, \eta^-)$ such that $H(\eta_k) = 0$.

Thus we construct an admissible solution of (2.2) and (2.4) for Case 3.2.2. The uniqueness is obvious. Therefore, we summarize to obtain the following theorem.

THEOREM 3.13. *The Riemann problem for Case* 3.2.2 *has a unique admissible solution with the structure*

$$(u(\xi), z(\xi)) = \begin{cases} (\max\{u^-, R_1(\xi)\}, 1), & \xi \in (-\infty, \eta_k), \\ (B(\xi; \eta_k, k), (\frac{\xi}{\eta_k})^k), & \xi \in (\eta_k, 0), \\ (u^+, 0), & \xi \in [0, +\infty) \end{cases} \tag{3.67}$$

and

$$\lim_{k\to+\infty} (u(\xi), z(\xi)) = \begin{cases} (\max\{u^-, R_1(\xi)\}, 1), & \xi \in (-\infty, \eta^*), \\ (u^+, 0), & \xi \in (\eta^*, +\infty). \end{cases} \tag{3.68}$$

4. Entropy condition for combustion waves of the CJ model. In this section, we will propose the entropy condition for combustion waves of the CJ model (1.2) with nonconvex fluxes $f(u)$ by taking into account the limit behavior of the solutions $(u(\xi), z(\xi))$ of (2.2) and (2.4) as the reaction rate goes to infinity. As we discussed in section 3, the Riemann solutions are essentially classified into two kinds: noncombustion solutions and combustion solutions. The combustion solutions have two types.

(i) For Type 1, as shown in Cases 3.1.2 and 3.2.2, the gas is ignited through a shock at $\xi = \eta_k$. The position of the reaction wave front depends on the reaction rate

k. The solution u is increasing as $\xi < \eta_k$, while it is decreasing as $\xi > \eta_k$. There is a von Neumann spike $u = B(\eta_k; \eta_k, k)$ on the curve of $u = u(\xi)$ in a neighborhood of $\xi = \eta_k$. Denote $\bar{\eta} = \lim_{k\to+\infty} \eta_k$, $\bar{u}(\xi) = \lim_{k\to+\infty} B(\xi; \eta_k, k)$ for $\xi \in (\bar{\eta}, 0)$ and $\bar{u}_R = \lim_{k\to+\infty} B(\eta_k; \eta_k, k)$.

Then, at $\xi = \bar{\eta}$, we have the relation

$$\bar{\eta} = \frac{f(\bar{u}_r) - f(\bar{u}_l)}{\bar{u}_r - \bar{u}_l - q} < 0, \qquad f'(\bar{u}_l) \geq \bar{\eta} \geq f'(\bar{u}_r), \tag{4.1}$$

and $\bar{u}_R > \bar{u}_r$ such that

$$\frac{f(\bar{u}_R) - f(\bar{u}_l)}{\bar{u}_R - \bar{u}_l} = \frac{f(\bar{u}_r) - f(\bar{u}_l)}{\bar{u}_r - \bar{u}_l - q} \leq \frac{f(u) - f(\bar{u}_l)}{u - \bar{u}_l} \quad \text{for all } u \in (\bar{u}_l, \bar{u}_R), \tag{4.2}$$

where $\bar{u}_r = \bar{u}(\bar{\eta} + 0)$, $\bar{u}_l = \bar{u}(\bar{\eta} - 0)$.

(ii) For Type 2, as shown in Cases 3.1.1 and 3.2.1, the gas will burn when its temperature reaches the ignition point continuously at $\xi = \eta_0$ or jumps over the ignition point through a shock at $\xi = \eta^*$. The position of the reaction wave front $\xi = \eta_0$ (or $\xi = \eta^*$) does not depend on the reaction rate k. In the neighborhood of $\xi = \eta_0$ or $\xi = \eta^*$, u is increasing. There is no von Neumann spike on the curve of $u = u(\xi)$ in the neighborhood of $\xi = \eta_0$ or $\xi = \eta^*$. Denote $\bar{\eta} = \eta_0$ or η^*, $\bar{u}(\xi) = \lim_{k\to+\infty} B(\xi; \bar{\eta}, k)$ for $\xi \in (\bar{\eta}, 0)$, and $\bar{u}_R = \lim_{k\to+\infty} B(\bar{\eta}; \bar{\eta}, k)$. Then, at $\xi = \bar{\eta}$, we have the relation

$$\bar{\eta} = \frac{f(\bar{u}_r) - f(\bar{u}_l)}{\bar{u}_r - \bar{u}_l - q} < 0, \qquad \bar{\eta} \leq f'(\bar{u}_r), \tag{4.3}$$

and $\bar{u}_R \in [0, \bar{u}_r)$ such that

$$\frac{f(\bar{u}_R) - f(\bar{u}_l)}{\bar{u}_R - \bar{u}_l} = \frac{f(\bar{u}_r) - f(\bar{u}_l)}{\bar{u}_r - \bar{u}_l - q} \geq \frac{f(u) - f(\bar{u}_l)}{u - \bar{u}_l} \quad \text{for all } u \in (\bar{u}_l, \bar{u}_r), \tag{4.4}$$

where $\bar{u}_r = \bar{u}(\bar{\eta} + 0)$, $\bar{u}_l = \bar{u}(\bar{\eta} - 0)$.

DEFINITION 4.1. *For the CJ combustion model* (1.2), *we define the limit of the interface between the unburned and reaction states in Type* I *as a generalized detonation wave and in Type* II *as the generalized deflagration wave.*

From this definition, we extract the following entropy condition on combustion waves for the nonconvex CJ combustion model (1.2).

ENTROPY CONDITION. *Let $x = x(t)$ be a combustion wave of the CJ combustion model* (1.2). *Let $u_l = u(x(t) - 0, t)$ and $u_r = u(x(t) + 0, t)$ be the limit values of u in the wave front and the wave back, respectively. Then*

(1) *$x = x(t)$ is a generalized CJ deflagration wave if*

$$\frac{dx}{dt} = \frac{f(u_r) - f(u_l)}{u_r - u_l - q} < 0, \quad \frac{dx}{dt} \leq f'(u_r), \tag{4.5}$$

and there exists $u_R \in [0, u_r)$ such that

$$\frac{f(u_R) - f(u_l)}{u_R - u_l} = \frac{f(u_r) - f(u_l)}{u_r - u_l - q} \geq \frac{f(u) - f(u_l)}{u - u_l} \quad \text{for all } u \in (u_l, u_r); \tag{4.6}$$

(2) *$x = x(t)$ is a generalized detonation wave if*

$$\frac{dx}{dt} = \frac{f(u_r) - f(u_l)}{u_r - u_l - q} < 0, \qquad f'(u_l) \geq \frac{dx}{dt} \geq f'(u_r), \tag{4.7}$$

and there exists $u_R \in (u_r, +\infty)$ *such that*

$$\frac{f(u_R) - f(u_l)}{u_R - u_l} = \frac{f(u_r) - f(u_l)}{u_r - u_l - q} \leq \frac{f(u) - f(u_l)}{u - u_l} \quad \text{for all } u \in (u_l, u_R). \tag{4.8}$$

Furthermore the detonation wave is a CJ detonation wave if $\frac{dx}{dt} = f'(u_r)$*; otherwise, it is a strong detonation wave.*

As we have seen, this entropy condition for the nonconvex CJ combustion model (1.2) inherits the essential difference between detonation and deflagration waves in that the former contains a von Neumann spike in the finite reaction rate region but the latter does not, which reflects the intrinsic feature of combustion waves in gas dynamics (cf. [1, 22]). With this, we can improve the results in [18] greatly to justify the (entropy) solutions of the Riemann problem for (1.2). Actually this entropy condition can be used in the construction of two-dimensional Riemann problems for the counterpart of (1.2), the generalization of (1.2) in two-dimensions; see [20]. Therefore, to some extent, this entropy condition lays a foundation and gives insight into the study of the structure of multidimensional combustion solutions.

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