Interaction of Rarefaction Waves of the Two-Dimensional Self-Similar Euler Equations

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Abstract

Classical self-similar solutions to the interaction of two arbitrary planar rarefaction waves for the polytropic Euler equations in two space dimensions are constructed. The binary interaction represents a major type of interaction in the two-dimensional Riemann problems, and includes in particular the classical problem of the expansion of a wedge of gas into vacuum. Based on the hodograph transformation, the method involves the phase space analysis of a second-order equation and the inversion back to (or development onto) the physical space.

Keywords: 2-D Riemann problem, characteristic decomposition, compressible, gas dynamics, hodograph transformation, inclination angles of characteristics, Jacobian, simple waves.

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1 Introduction

Consider the two-dimensional isentropic compressible Euler system

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0, \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = 0, \end{cases} \tag{1.1}$$

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where ρ is the density, (u, v) is the velocity and p is the pressure given by $p(\rho) = K\rho^\gamma$ where $K > 0$ will be scaled to be one and $\gamma > 1$ is the gas constant. We are primarily interested in the so-called *pseudo–steady* case of (1.1); i.e., the solutions depend on the self–similar variables $(\xi, \eta) = (x/t, y/t)$. The expansion of a wedge of gas into vacuum is such a case. Assuming the flow is irrotational, a classical hodograph transformation (see [21]) can be used to eliminate the two self-similar variables, resulting in a partial differential equation of second order for the speed of sound c in the velocity variables (u, v). It has been known that the difficulties of the procedure are that the transformation is degenerate for common waves such as the constant states and some other types of waves resembling the simple waves of the steady Euler system, and the transformation of boundaries is difficult to handle. In 2001, Li ([13]) carried out an anaysis of the second order equation in the space (c, u, v), where he discovered a pair of variables resembling the well-known Riemann invariants together with their invariant regions and established the existence of a solution to the expansion of a wedge of gas into vacuum in the hodograph plane for wide ranges of the gas constant and the wedge angle. Recently in 2006, paper [15], in an attempt to establish the inversion of the hodograph mapping, clarified the concept of simple waves for (1.1). We show in this paper that the hodograph transformation is non-degenerate precisely for non-simple waves, and all the solutions constructed in [13] in the hodograph plane can now be transformed back to the self-similar plane. Thus we complete the procedural circle of construction of solutions.

We find that the circle of construction bears very interesting similarity to the construction of centered rarefaction waves in the one-dimensional systems of conservation laws. The self-similar variable(s) in both cases decouple from the phase space(s), and the equations in the phase space(s) are solved first. The development of the solutions from the phase space onto the physical space(s) requires genuine nonlinearity in the one-dimensional case and a non-degeneracy condition of the hodograph transformation in the two-dimensional case.

The key ingredient of the paper is the simplification of the form of the equations in the phase space brought about by the employment of the inclination angles of characteristics, and the discovery of the precise forms of the equations for the second-order derivatives. The approach now presents itself as a method of significant potential for the study of the pseudo-steady Euler system in hyperbolic regions. It yields structure of solutions in addition to existence. We use this approach for example to establish the Lipschitz continuity and monotonicity behavior of the vacuum boundary in the problem of a wedge of gas into vacuum and establish the dependence of the location of the boundary on the wedge angle and gas constant. The approach is particularly suitable for studying two-dimensional Riemann problems ([26]), since the apparent nature of the solutions of a Riemann problem is piece-wise smooth.

The expansion problem of gas into vacuum has been a favorite for a long time. The problem has been interpreted hydraulically as the collapse of a wedge–shaped dam containing water initially with a uniform velocity, see Levine [11]. In Suchkov [24], a set of interesting explicit solutions were found. Mackie [19] proposed a scalar equation of second order for a potential function, studied the interface of gas and vacuum by the method of unsteady Prandtl–Meyer expansions and related it to the PSI approach in [21], from which Li ([13]) started with new motivation from the success on the pressure gradient system ([6]). In the context of two-dimensional Riemann problems, the expansion problem of a wedge of gas into vacuum is the interaction of two two-dimensional planar rarefaction waves. We see it as one of two possible interactions of continuous waves in the hyperbolic region; this one expands without shocks but with a boundary degeneracy, while the other one forms shocks with a sonic boundary as well as a shock boundary. The method applies locally in both cases. A quick round-up of cases that involves hyperbolic regions of non-constant continuous waves [1, 2, 7, 8, 10, 14, 22, 27] show that the approach taken here has general applicability.

Our main results are the simple form of the equations in the phase space (c, u, v) (system (6.18)), the existence of solutions of the expansion of a wedge of gas into vacuum (Theorem 7.2), and the detailed properties of the expansion (Theorems 7.3–7.4). We provide some background information as well regarding the hodograph transformation and simple waves in Sections 2–5 for the convenience of nonexpert readers. Section 6 is for the phase space analysis, while Section 7 handles the gas expansion problem. We point out that the main work of this paper is the establishment of the invariance of various triangles and the validity of the inversion of the hodograph transformation with the associated trial-and-error process in finding the precise forms for the second-order equations suitable for bootstrapping (see formulae (6.27)(6.30)). The difficulty of the inversion manifests itself in the fact that the characteristics in the phase space do not have a fixed convexity type although the corresponding characteristics in the physical space do (in most cases), see Subsections 7.5 and 7.7. We mention additionally that hodograph transforms have been used in various forms, see [20, 3] and references therein.

Here is a list of our notations: ρ density, p pressure, (u, v) velocity, $c = \sqrt{\gamma p/\rho}$ speed of sound, $i = c^2/(\gamma - 1)$ enthalpy, γ gas constant, $(\xi, \eta) = (x/t, y/t)$ the self-similar (or pseudo-steady) variables, φ pseudo-velocity potential, θ wedge half-angle, and

$$U = u - \xi,\ V = v - \eta,\ \kappa = (\gamma - 1)/2,\ m = (3 - \gamma)/(\gamma + 1).$$

$$\alpha_1 = \frac{2m}{3 + m + \sqrt{(3 + m)^2 + 4m}}, \quad \tan\bar{n} = \sqrt{\alpha_1}, \quad \tan\bar{m} = \sqrt{m}.$$

$$\Lambda_+ = \tan\alpha, \quad \Lambda_- = \tan\beta, \quad \omega = \frac{\alpha - \beta}{2}, \quad \Lambda_\pm = \frac{UV \pm c\sqrt{U^2 + V^2 - c^2}}{U^2 - c^2}.$$

$$\partial^\pm = \partial_\xi + \Lambda_\pm\partial_\eta, \quad \partial_\pm = \partial_u + \lambda_\pm\partial_v, \quad \partial_0 = \partial_u, \quad \Lambda_\pm\lambda_\mp = -1.$$

$$\bar{\partial}_+ = (\sin\beta, -\cos\beta)\cdot(\partial_u, \partial_v), \quad \bar{\partial}_- = (\sin\alpha, -\cos\alpha)\cdot(\partial_u, \partial_v).$$

Letters C, C_1 and C_2 denote generic constants.

2 Primary system

Our primary system is system (1.1) in the self–similar variables $(\xi, \eta) = (x/t, y/t)$:

$$\begin{cases} (u-\xi)i_\xi + (v-\eta)i_\eta + 2\kappa\, i\,(u_\xi + v_\eta) = 0, \\ (u-\xi)u_\xi + (v-\eta)u_\eta + i_\xi = 0, \\ (u-\xi)v_\xi + (v-\eta)v_\eta + i_\eta = 0. \end{cases} \tag{2.1}$$

We assume further that the flow is ir-rotational:

$$u_\eta = v_\xi. \tag{2.2}$$

Then, we insert the second and third equations of (2.1) into the first one to deduce the system,

$$\begin{cases} (2\kappa\, i - (u-\xi)^2)u_\xi - (u-\xi)(v-\eta)(u_\eta + v_\xi) + (2\kappa\, i - (v-\eta)^2)v_\eta = 0, \\ u_\eta - v_\xi = 0, \end{cases} \tag{2.3}$$

supplemented by Bernoulli's law

$$i + \frac{1}{2}((u-\xi)^2 + (v-\eta)^2) = -\varphi, \quad \varphi_\xi = u - \xi, \ \varphi_\eta = v - \eta. \tag{2.4}$$

We remark that the difference between the pseudo-steady flow (2.3) and the steady case (3.27) (see Subsection 3.1) is that the latter is self-contained, since the sound speed c can be expressed by a pointwise function of the velocity explicitly through Bernoulli's law (3.28).

3 Concept of hodograph transformation

We introduce briefly the well-known hodograph transformation. The original form of a hodograph transformation is for a homogeneous quasi-linear system of two first-order equations for two known variables (u, v) in two independent variables (x, y). By regarding (x, y) as functions of (u, v) and assuming that the Jacobian does not vanish nor is infinity, one can re-write the system for the unknowns (x, y) in the variables (u, v), which is a linear system if the coefficients of the original system do not depend on (x, y). See the book of Courant and Friedrichs [4]. Specifically, consider the system of two equations of the form,

$$\begin{pmatrix} u \\ v \end{pmatrix}_x + A(u, v; x, y) \begin{pmatrix} u \\ v \end{pmatrix}_y = 0, \tag{3.1}$$

where the coefficient matrix $A(u, v; x, y)$ is

$$A(u, v; x, y) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{3.2}$$

The two eigenvalues, denoted by $\Lambda_\pm$, satisfies,

$$\Lambda_\pm^2 - (a_{11} + a_{22})\Lambda_\pm + |A| = 0. \tag{3.3}$$

We introduce the hodograph transformation,

$$T : (x, y) \to (u, v). \tag{3.4}$$

Then (3.1) is reduced to the system

$$\begin{pmatrix} y_v \\ -y_u \end{pmatrix} + A(u, v; x, y) \begin{pmatrix} -x_v \\ x_u \end{pmatrix} = 0. \tag{3.5}$$

Its eigenvalues, denoted by $\lambda_\pm$, satisfy

$$a_{12}\lambda_\pm^2 - (a_{22} - a_{11})\lambda_\pm - a_{21} = 0. \tag{3.6}$$

Obviously, if the coefficient matrix A does not depend on (x, y), (3.5) becomes a linear system for the unknowns (x, y).

The following proposition establishes the invariance of characteristics under the hodograph transformation.

Proposition 3.1 (Invariance of characteristics). *A characteristic of (3.1) in the (x, y) plane is mapped into a characteristic of (3.5) in the (u, v) plane by the hodograph transform T.*

Proof. Let $y = y(x)$ be a characteristic with $\frac{dy}{dx} = \Lambda_\pm$. Its image is $v = v(u)$ under the hodograph transform (3.4). Then we have

$$\frac{dy}{dx} = \frac{y_u + y_v \cdot \frac{dv}{du}}{x_u + x_v \cdot \frac{dv}{du}} = \Lambda_\pm, \tag{3.7}$$

i.e.,

$$\frac{dv}{du} = -\frac{\Lambda_\pm x_u - y_u}{\Lambda_\pm x_v - y_v}. \tag{3.8}$$

Using (3.5) we find

$$(\Lambda_\pm x_u - y_u) + \lambda_\pm(\Lambda_\pm x_v - y_v) = 0. \tag{3.9}$$

Therefore we have

$$\frac{dv}{du} = \lambda_\pm(\text{ or } \lambda_\mp), \tag{3.10}$$

which completes the proof of this proposition. □

The idea of hodograph transformation does not obviously generalize to other systems such as system (2.3) of more than two simple equations or for inhomogeneous systems.

For (2.3), one realizes that the three variables (i, u, v) are functions of (ξ, η), so one can still try to use (u, v) as the independent variables and regard (ξ, η) as functions of (u, v) and ultimately regard i as a function of (u, v). In this way we may obtain an equation for $i = i(u, v)$ in the plane (u, v) which eliminates (ξ, η). This was done in 1958 in a paper [21] by Pogodin, Suchkov and Ianenko, and has been referred to as the PSI approach. The implementation is as follows. Let the hodograph transformation be

$$T : (\xi, \eta) \to (u, v) \tag{3.11}$$

for (2.1), reverse the roles of (ξ, η) and (u, v) and regard i as a function of (u, v). Then i as the function of u and v satisfies

$$(u_\xi v_\eta - u_\eta v_\xi)di = (i_\xi v_\eta - i_\eta v_\xi)du + (-i_\xi u_\eta + i_\eta u_\xi)dv. \tag{3.12}$$

We insert this into the law of momentum conservation of (2.1) and use the ir-rotationality condition (2.2) to obtain

$$\begin{aligned} \xi - u &= i_u, \\ \eta - v &= i_v. \end{aligned} \tag{3.13}$$

These interesting identities provide an explicit correspondence between the physical plane and the hodograph plane provided that the transformation T is not degenerate.

Therefore, using (3.13), we convert (2.3) into a "linear" (in fact, linearly degenerate, see Section 6) system

$$\begin{cases} (2\kappa\, i(u, v) - i_u^2)\eta_v + i_u i_v(\xi_v + \eta_u) + (2\kappa\, i - i_v^2)\xi_u = 0, \\ \xi_v - \eta_u = 0. \end{cases} \tag{3.14}$$

for the unknowns (ξ, η). The difficulty here is that i, as a function of u and v, cannot be determined explicitly and point-wise. We will remark more later in contrast with the steady case.

We continue to differentiate (3.13) with respect to u and v:

$$\begin{array}{lllllll} \xi_u &=& 1 + i_{uu}, & \quad & \xi_v &=& i_{uv}, \\ \eta_u &=& i_{uv}, & & \eta_v &=& 1 + i_{vv}, \end{array} \tag{3.15}$$

and inserting these into the first equations of (3.14) to obtain,

$$(2\kappa\, i - i_u^2)i_{vv} + 2i_u i_v i_{uv} + (2\kappa\, i - i_v^2)i_{uu} = i_u^2 + i_v^2 - 4\kappa\, i. \tag{3.16}$$

This is a very interesting second order partial differential equation for i alone. So the study of ir-rotational, pseudo-steady and isentropic fluid flow can proceed along (3.16).

We point out for the case $\gamma = 1$ that the dependent variable $i = \ln \rho$, instead of $i = c^2/(\gamma - 1)$, is used [12]. Then we can obtain a similar equation for i,

$$(1 - i_u^2)i_{vv} + 2i_u i_v i_{uv} + (1 - i_v^2)i_{uu} = i_u^2 + i_v^2 - 2. \tag{3.17}$$

We will establish in the pseudo-steady case that the transform is not degenerate, i.e.,

$$J_T(u, v; \xi, \eta) = \frac{\partial(u, v)}{\partial(\xi, \eta)} = u_\xi v_\eta - u_\eta v_\xi \neq 0 \tag{3.18}$$

in regions of non-simple waves, to be detailed later. In the direction from (u, v) plane to the (ξ, η) plane, it is more direct to compute

$$J_T^{-1}(u, v; \xi, \eta) = \xi_u \eta_v - \xi_v \eta_u \neq 0. \tag{3.19}$$

Noting that (2.3) and (3.14) are all two by two systems, we use Proposition 3.1 to assert that the characteristics of (2.3) are mapped into the characteristics of (3.14) by the hodograph transformation (3.11). Moreover, the eigenvalues of (2.3) are

$$\Lambda_\pm = \frac{(u-\xi)(v-\eta) \pm c\sqrt{(u-\xi)^2 + (v-\eta)^2 - c^2}}{(u-\xi)^2 - c^2}, \tag{3.20}$$

while the eigenvalues of (3.14) are

$$\lambda_\pm = \frac{i_u i_v \pm c\sqrt{(i_u^2 + i_v^2 - c^2)}}{c^2 - i_v^2}. \tag{3.21}$$

By using (3.13), it is easy to see that

$$\lambda_\pm = -\frac{1}{\Lambda_\mp}. \tag{3.22}$$

Furthermore, there is a correspondence between $\Lambda_\pm$ and $\lambda_\pm$. Indeed, let $\eta = \eta(\xi)$ be a characteristic curve in the (ξ, η) plane with $\frac{d\eta}{d\xi} = \Lambda_+$ and be mapped onto a curve $v = v(u)$. Then, using (3.15) and (3.22), we have

$$\Lambda = \frac{d\eta}{d\xi} = \frac{\eta_u + \eta_v \frac{dv}{du}}{\xi_u + \xi_v \frac{dv}{du}}, \tag{3.23}$$

i.e.,

$$\frac{dv}{du} = -\frac{\xi_u \Lambda_+ - \eta_u}{\xi_v \Lambda_+ - \eta_v} = -\frac{(1 + i_{uu})\Lambda_+ - i_{uv}}{i_{uv}\Lambda_+ - (1 + i_{vv})} = -\frac{i_{uu} + 1 + \lambda_- i_{uv}}{i_{uv} + \lambda_-(i_{vv} + 1)}. \tag{3.24}$$

We rewrite (3.16) as

$$i_{uu} + 1 + (\lambda_- + \lambda_+) i_{uv} + \lambda_- \lambda_+ (i_{vv} + 1) = 0. \tag{3.25}$$

Then we conclude

$$\frac{dv}{du} = \lambda_+. \tag{3.26}$$

Similarly we obtain the correspondence between Λ_- and λ_-.

3.1 Steady Euler

The steady isentropic and irrotational Euler system of (1.1) has the form

$$\begin{cases} (c^2 - u^2)u_x - uv(u_y + v_x) + (c^2 - v^2)v_y = 0, \\ u_y - v_x = 0, \end{cases} \tag{3.27}$$

where c is the sound speed, given by Bernoulli's law

$$\frac{u^2 + v^2}{2} + \frac{c^2}{\gamma - 1} = \frac{k_0}{2}, \tag{3.28}$$

where k_0 is a constant. See [4]. Using the hodograph transform from (x, y) to (u, v), we obtain a linear system,

$$\begin{cases} (2\kappa\, i - u^2)y_v + uv(x_v + y_u) + (2\kappa\, i - v^2)x_u = 0, \\ x_v - y_u = 0. \end{cases} \tag{3.29}$$

The hodograph transform is valid in the region of non-simple waves. With the same set of steps in deriving (3.13), we obtain

$$-u = i_u, \qquad -v = i_v. \tag{3.30}$$

This can also be obtained formally from (3.13) by regarding the steady flow as the limit of unsteady flow (1.1) in $t \to \infty$. Comparing (3.30) with (3.13), we see that it is much more difficult to convert the hodograph plane of the steady case back into the physical plane than the pseudo-steady case. However, system (3.29) has more advantage over (3.14) of the pseudo–steady case because i is expressed in an explicit form by Bernoulli's law (3.28).

3.2 Similarity to one-dimensional problems

The current approach parallels the procedure that is used to find centered rarefaction waves to genuinely nonlinear strictly hyperbolic systems of conservation laws in one space dimension. Recall for a one-dimensional system $u_t + f(u)_x = 0$ of n equations, a centered rarefaction wave takes the form $\xi = \lambda_k(u)$ for a $k \in (1, n)$ and the state variable u satisfies the system of ordinary differential equations $(f'(u) - \lambda_k(u)I)u_\xi = 0$, whose solutions are rarefaction wave curves in the phase space. The development (or inversion) of the phase space solutions onto the $\xi-$axis requires the monotonicity of $\lambda_k(u)$ along the vector field of the $k-$th right eigenvector r_k; i.e., the genuine nonlinearity. For the self-similar 2-D Euler system, we have a pair $\xi = u + i_u, \eta = v + i_v$ from (3.13) in place of $\xi = \lambda_k(u)$; and the second-order partial differential equation (3.16) in place of the ordinary differential system. For inversion to the physical space, we show that the Jacobian J_T^{-1} of (3.19) does not vanish.

4 Simple waves

4.1 Concept of simple waves

Now we recall some facts about simple waves. Simple waves were systematically studied, e.g. in [9], for hyperbolic systems in two independent variables,

$$u_t + A(u)u_x = 0, \tag{4.1}$$

where $u = (u_1, \cdots, u_n)^\top$, the $n \times n$ matrix $A(u)$ has real and distinct eigenvalues $\lambda_1 < \cdots < \lambda_n$ for all u under consideration. They are defined as a special family of solutions of the form

$$u = U(\phi(x,t)). \tag{4.2}$$

The function $\phi = \phi(x,t)$ is scalar. Substituting (4.2) into (4.1) yields

$$U'(\phi)\phi_t + A(U(\phi))U'(\phi)\phi_x = 0, \tag{4.3}$$

which implies that $-\phi_t/\phi_x$ is an eigenvalue of $A(U(\phi))$ and $U'(\phi)$ is the associated eigenvector. This concludes that in the (x,t) plane a simple wave is associated with a kind of characteristic field, say, λ_k, and spans a domain in which characteristics of the k-kind are straight along which the solution is constant.

The property of simple waves can be analyzed by using *Riemann invariants*. A Riemann invariant is a scalar function $w = w(x,t)$ satisfying the following condition,

$$r_k \cdot \text{grad } w = 0, \tag{4.4}$$

for all values of u, where r_k is the k-th right eigenvector of A. Using the Riemann invariants, it can be shown that *a state in a domain adjacent to a domain of constant state is always a simple wave.*

In general, system (4.1) cannot be diagonalized because it does not have a *full coordinate system of Riemann invariants* [5]. Note that in (4.1) the coefficient matrix A depends on u only. Once A depends on x and t as well as u, the treatment in [9] and [5] breaks down. For example, we are unable to use the same techniques to show that it is a simple wave to be adjacent to a constant state.

4.2 Simple waves for pseudo-steady Euler equations

We introduce in a traditional manner a *simple wave* for (2.3) as a solution $(u,v) = (u,v)(\xi,\eta)$ that is constant along the level set $l : l(\xi,\eta) = C$ for some function $l(\xi,\eta)$, where C is constant. That is, this solution has the form,

$$(u,v)(\xi,\eta) = (F,G)(l(\xi,\eta)). \tag{4.5}$$

Inserting this into (2.3) gives

$$\begin{pmatrix} (2\kappa\, i - U^2)l_\xi - UVl_\eta, & -UVl_\xi + (2\kappa\, i - V^2)l_\eta \\ l_\eta & -l_\xi \end{pmatrix} \begin{pmatrix} F' \\ G' \end{pmatrix} = 0. \tag{4.6}$$

Here we use $U := u - \xi, V := v - \eta$ for short. It turns out that $(F', G') = (0, 0)$ or there exists a singular solution for which the coefficient matrix becomes singular. The former just gives a trivial constant solution. But for the latter, $l(\xi, \eta)$ satisfies

$$(2\kappa\, i - U^2)l_\xi^2 - 2UVl_\xi l_\eta + (2\kappa i - V^2)l_\eta^2 = 0; \tag{4.7}$$

i.e.,

$$-\frac{l_\xi}{l_\eta} = \frac{UV \pm \{2\kappa\, i(U^2 + V^2 - 2\kappa\, i)\}^{1/2}}{U^2 - 2\kappa\, i} =: \Lambda_\pm, \tag{4.8}$$

which implies that the level curves $l(\xi, \eta) = C$ are characteristic lines, and

$$F' + \Lambda_\pm G' = 0 \tag{4.9}$$

holds along each characteristic line $l(\xi, \eta) = C$ locally at least.

In a recent paper by Li, Zhang, Zheng [15], the pseudo-steady full Euler is shown to have a characteristic decomposition. Let us quote several identities from that paper. First, the flow will be isentropic and ir-rotational adjacent to a constant state. Then the pseudo-characteristics are defined as

$$\frac{d\eta}{d\xi} = \frac{UV \pm c\sqrt{U^2 + V^2 - c^2}}{U^2 - c^2} \equiv \Lambda_\pm. \tag{4.10}$$

Here c is the speed of sound $c^2 = \gamma p/\rho$. Regarding $\Lambda_\pm$ as simple straight functions of the three independent variables (U, V, c^2), we have

$$\partial_U \Lambda = \Lambda(U\Lambda - V)/\Theta,\ \partial_V \Lambda = (V - U\Lambda)/\Theta,\ \partial_{c^2}\Lambda = -(1 + \Lambda^2)/(2\Theta) \tag{4.11}$$

where $\Theta := \Lambda(c^2 - U^2) + UV$. Then we further obtain

$$\partial^\pm u + \Lambda_\mp \partial^\pm v = 0, \tag{4.12}$$

$$\partial^\pm c^2 = -2\kappa\left(U\partial^\pm u + V\partial^\pm v\right), \tag{4.13}$$

$$\partial^\pm \Lambda_\pm = [\partial_U \Lambda_\pm - \Lambda_\mp^{-1}\partial_V \Lambda_\pm - 2\kappa(U - V/\Lambda_\mp)\partial_{c^2}\Lambda_\pm]\partial^\pm u, \tag{4.14}$$

where $\partial^\pm = \partial_\xi + \Lambda_\pm \partial_\eta$. We keep $\partial_\pm$ for later use in the hodograph plane. Thus, if one of the quantities (u, v, c^2) is a constant along Λ_-, so are the remaining two and Λ_-. The same is true for the plus family Λ_+. Hence we have

Proposition 4.1 (Section 4, [15]). *For the irrotational and isentropic pseudo-steady flow (2.1) or (2.3), we have the following characteristic decomposition*

$$\partial^+\partial^- u = h\partial^- u, \quad \partial^-\partial^+ u = g\partial^+ u, \tag{4.15}$$

where $h = h(u, v, c)$ and $g = g(u, v, c)$ are some functions. Similar decompositions hold for v, c^2 and $\Lambda_\pm$. We further conclude that simple waves are waves such that one family of characteristic curves are straight along which the physical quantities (u, v, c^2) are constant.

5 Convertibility

We are now ready to discuss the non-degeneracy of hodograph transformation (3.11).

Theorem 5.1 (Sufficient and Necessary Condition). *Let the ir-rotational, isentropic and pseudo-steady fluid flow (2.1) be smooth at a point $(\xi, \eta) = (\xi_0, \eta_0)$. Then the Jacobian $J_T(u, v; \xi, \eta)$ of the hodograph transformation (3.11) vanishes in a neighborhood of the point if the flow is a simple wave in the neighborhood. Conversely, if the Jacobian $J_T(u, v; \xi, \eta)$ vanishes in a neighborhood of the point, then the flow is a simple wave in the neighborhood.*

Proof. Assume first that $c^2 - V^2 \neq 0$ at $(\xi, \eta) = (\xi_0, \eta_0)$. We compute

$$\begin{aligned} J_T(u, v; \xi, \eta) &= u_\xi v_\eta - u_\eta v_\xi \\ &= -\frac{1}{c^2 - V^2} \cdot [((c^2 - U^2)u_\xi - 2UV u_\eta)u_\xi] - u_\eta^2 \\ &= -\frac{1}{c^2 - V^2} \cdot [(c^2 - U^2)u_\xi^2 - 2UV u_\xi u_\eta + (c^2 - V^2)u_\eta^2]. \end{aligned} \tag{5.1}$$

Therefore, the degeneracy of the transformation implies

$$(c^2 - U^2)u_\xi^2 - 2UV u_\xi u_\eta + (c^2 - V^2)u_\eta^2 = 0. \tag{5.2}$$

It follows that

$$-\frac{u_\xi}{u_\eta} = \Lambda_\pm. \tag{5.3}$$

That is,

$$u_\xi + \Lambda_+ u_\eta = 0, \quad \text{or} \quad u_\xi + \Lambda_- u_\eta = 0, \tag{5.4}$$

at (ξ_0, η_0). For the former, we deduce that $\partial^+ u = 0$ along the whole Λ_--characteristic line through (ξ_0, η_0) in view of (4.15) in Proposition 4.1, and so do $\partial^+ v$ and $\partial^+ c$. Therefore, conclude that the wave is a simple wave associated with Λ_+.

Conversely, if a point $(\xi, \eta) = (\xi_0, \eta_0)$ is in the region of a simple wave, then equation (5.4) hold either for the plus or minus families. From there we go up the derivation to find that the Jacobian vanishes in the same neighborhood.

The case that $c^2 - (v - \eta)^2 = 0$ is a special planar simple wave. Therefore the conclusion follows naturally. □

We comment that the Jacobian $J_T(u, v; \xi, \eta)$ can be factorized as

$$J_T(u, v; \xi, \eta) = -\frac{1}{\Lambda_- \Lambda_+} \partial^+ u \cdot \partial^- u = -\partial^+ v \cdot \partial^- v. \tag{5.5}$$

6 Phase space system of equations

In this section we use the inclination angles of characteristics as useful variables to rewrite (3.16) in the hodograph plane. We proceed as follows. We first transform the second order equation (3.16) into a first-order system of equations as in [13]. Introduce

$$X = i_u, \qquad Y = i_v. \tag{6.1}$$

Then we deduce a 3×3 system of first order equations,

$$\begin{bmatrix} c^2 - Y^2 & XY & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ i \end{bmatrix}_u + \begin{bmatrix} XY & c^2 - X^2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ i \end{bmatrix}_v = \begin{bmatrix} X^2 + Y^2 - 2c^2 \\ 0 \\ X \end{bmatrix}, \tag{6.2}$$

where $c^2 = 2\kappa\, i$. This system is equivalent to (3.16) for C^1 solutions if the given datum for Y is compatible with the datum for i_v. The characteristic equation is

$$(c^2 - Y^2)\lambda^2 - 2XY\lambda + c^2 - X^2 = 0 \tag{6.3}$$

besides the trivial factor λ. This system has three eigenvalues

$$\begin{aligned} &\lambda_0 = 0, \\ &\frac{dv}{du} = \lambda_\pm = \frac{XY \pm \sqrt{c^2(X^2 + Y^2 - c^2)}}{c^2 - Y^2} = \frac{c^2 - X^2}{XY \mp \sqrt{c^2(X^2 + Y^2 - c^2)}}, \end{aligned} \tag{6.4}$$

from which we deduce that (6.2) is hyperbolic if $X^2 + Y^2 - c^2 > 0$ provided that $i > 0$ and $c^2 - Y^2 \neq 0$ (or $c^2 - X^2 \neq 0$). If $c^2 - Y^2 = 0$ or $c^2 - X^2 = 0$, we have planar rarefaction waves in the neighborhood. The three associated left eigenvectors with (6.4) are

$$l_0 = (0, 0, 1), \qquad l_\mp = (1, \lambda_\pm, 0). \tag{6.5}$$

We multiply (6.2) by the left eigen matrix $M = (l_+, l_-, l_0)^\top$ (here and onward the superscript $\top$ means transpose) from the left–hand side to obtain

$$\begin{cases} X_u + \lambda_- Y_u + \lambda_+(X_v + \lambda_- Y_v) = \dfrac{X^2 + Y^2 - 2c^2}{c^2 - Y^2}, \\ X_u + \lambda_+ Y_u + \lambda_-(X_v + \lambda_+ Y_v) = \dfrac{X^2 + Y^2 - 2c^2}{c^2 - Y^2}, \\ i_u = X. \end{cases} \tag{6.6}$$

Introduce the inclination angles α, β ($-\pi/2 < \alpha, \beta < \pi/2$) of Λ_+ and Λ_--characteristics by

$$\tan\alpha = \Lambda_+, \qquad \tan\beta = \Lambda_-. \tag{6.7}$$

Note that, see (3.22),

$$\Lambda_+ = -\frac{1}{\lambda_-}, \qquad \Lambda_- = -\frac{1}{\lambda_+}; \tag{6.8}$$

and denote

$$A := \tan(\alpha/2), \qquad B := \tan(\beta/2). \tag{6.9}$$

This explains the Riemann invariant introduced in [13]. Then we find that X, Y are related with A, B through the following identities,

$$\begin{aligned} A &= \frac{X - \sqrt{X^2 + Y^2 - c^2}}{c - Y}, \\ B &= -\frac{X - \sqrt{X^2 + Y^2 - c^2}}{c + Y}, \end{aligned} \tag{6.10}$$

or

$$X = \frac{c(1 - AB)}{A - B}, \qquad Y = \frac{c(A + B)}{A - B}. \tag{6.11}$$

In terms of α, β, we have

$$X = c\frac{\cos\frac{\alpha+\beta}{2}}{\sin\omega}, \qquad Y = c\frac{\sin\frac{\alpha+\beta}{2}}{\sin\omega}, \tag{6.12}$$

for

$$\omega := (\alpha - \beta)/2. \tag{6.13}$$

We observe that the variables α, β are Riemann invariants for (6.6). In fact, we can write (6.6) as

$$\begin{aligned} \partial_+\alpha &= \frac{1+\gamma}{4c} \cdot \frac{\sin(\alpha - \beta)}{\sin\beta} \cdot \left[m - \tan^2\omega\right], \\ \partial_-\beta &= \frac{1+\gamma}{4c} \cdot \frac{\sin(\alpha - \beta)}{\sin\alpha} \cdot \left[m - \tan^2\omega\right], \\ \partial_0 c &= \kappa\frac{\cos\frac{\alpha+\beta}{2}}{\sin\omega}, \end{aligned} \tag{6.14}$$

where we use the notations of directional derivatives,

$$\partial_+ = \frac{\partial}{\partial u} + \lambda_+\frac{\partial}{\partial v}, \qquad \partial_- = \frac{\partial}{\partial u} + \lambda_-\frac{\partial}{\partial v}, \qquad \partial_0 = \frac{\partial}{\partial u}, \tag{6.15}$$

and keep the letter m for

$$m = \frac{1-\kappa}{1+\kappa} = \frac{3-\gamma}{1+\gamma}. \tag{6.16}$$

We further introduce the normalized directional derivatives along characteristics,

$$\bar{\partial}_+ = (\sin\beta, -\cos\beta)\cdot(\partial_u, \partial_v), \quad \bar{\partial}_- = (\sin\alpha, -\cos\alpha)\cdot(\partial_u, \partial_v). \tag{6.17}$$

They are coordinate-free. Using them, we write (6.14) as,

$$
\begin{aligned}
\bar\partial_+\alpha &= \frac{1+\gamma}{4c}\cdot\sin(\alpha-\beta)\cdot\big[m-\tan^2\omega\big] =: G(\alpha,\beta,c),\\
\bar\partial_-\beta &= \frac{1+\gamma}{4c}\cdot\sin(\alpha-\beta)\cdot\big[m-\tan^2\omega\big] \equiv G(\alpha,\beta,c),\\
\partial_0 c &= \kappa\frac{\cos\frac{\alpha+\beta}{2}}{\sin\omega}.
\end{aligned}
\tag{6.18}
$$

In particular, we note that

$$\bar\partial_+ c = -\kappa, \quad \bar\partial_- c = \kappa. \tag{6.19}$$

This highlights that the first two equations of (6.18) are entirely decoupled from the third c-equation. In addition, each of the first two equations of (6.18) is actually a decomposition of the second-order equation (3.16) for c.

Note from (6.7) and (6.8) that $\lambda_+ = -\cot\beta$ and $\lambda_- = -\cot\alpha$. The system (6.14) is linearly degenerate in the sense of Lax [9]. For the particular case that $\tan((\alpha-\beta)/2) = m$ for $1<\gamma<3$, the first two equations become homogeneous equations

$$
\begin{aligned}
&\alpha_u + \lambda_+\alpha_v = 0,\\
&\beta_u + \lambda_-\beta_v = 0,
\end{aligned}
\tag{6.20}
$$

which always have a unique global continuous solution provided that the corresponding initial and/or boundary data have a uniform bound in C^1 norm (cf. [17]). In fact, the explicit solutions of Suchkov [24] in the expansion problem of a wedge of gas into a vacuum is such a case, see Remark 7.1 in Section 7.

The mapping $(X,Y)\to(\alpha,\beta)$ is bijective as long as system (6.6) is hyperbolic.

We summarizes the above as follows, which is similar to [13]:

Theorem 6.1. *The two–dimensional pseudo-steady, irrotational, isentropic flow (3.16) can be transformed into a linearly degenerate system of first order partial differential equations (6.14) or (6.18) provided that the transform $(X,Y)\to(\alpha,\beta)$ is invertible, i.e., system (6.6) is hyperbolic.*

Regarding $\bar\partial_-\alpha$ and $\bar\partial_+\beta$, we have second-order equations although we are unable to obtain explicit expressions for them like (6.18). By direct computations, we obtain

Lemma 6.1 (Commutator relation of $\partial_\pm$). *For any quantity $I = I(u,v)$, there holds*

$$\partial_-\partial_+ I - \partial_+\partial_- I = \frac{\partial_-\lambda_+ - \partial_+\lambda_-}{\lambda_- - \lambda_+}(\partial_- I - \partial_+ I). \tag{6.21}$$

Lemma 6.2 (Commutator relation of $\bar\partial_\pm$). *For any quantity $I = I(u, v)$, there holds,*

$$\bar\partial_-\bar\partial_+ I - \bar\partial_+\bar\partial_- I = \tan\omega(\bar\partial_- I + \bar\partial_+ I)\bar\partial_+\alpha, \tag{6.22}$$

where $\bar\partial_+\alpha$ is given in (6.18). Noting $\bar\partial_+\alpha = \bar\partial_-\beta$ in (6.18), we can also use $\bar\partial_-\beta$ in (6.22).

Using these commutator relations, we easily derive:

Theorem 6.2. *Assume that the solution of (6.18) $(\alpha, \beta) \in C^2$. Then we have*

$$\begin{aligned} \bar\partial_+\bar\partial_-\alpha + W\bar\partial_-\alpha \quad &= Q(\omega, c), \\ -\bar\partial_-\bar\partial_+\beta + W\bar\partial_+\beta \quad &= Q(\omega, c), \end{aligned} \tag{6.23}$$

where $W(\omega, c)$ and $Q(\omega, c)$ are

$$\begin{aligned} W(\omega, c) \quad &:= \frac{1+\gamma}{4c}\Big[\big(m - \tan^2\omega\big)\big(3\tan^2\omega - 1\big)\cos^2\omega + 2\tan^2\omega\Big], \\ Q(\omega, c) \quad &:= \frac{(1+\gamma)^2}{16c^2}\sin(2\omega)\big(m - \tan^2\omega\big)\big(3\tan^2\omega - 1\big). \end{aligned} \tag{6.24}$$

Proof. The proof is simple. Recall from (6.19) that

$$\bar\partial_+ c = -\kappa, \qquad \bar\partial_- c = \kappa. \tag{6.25}$$

Then we apply the commutator relation to obtain (setting $I = \alpha$ in (6.22))

$$\bar\partial_+\bar\partial_-\alpha = \bar\partial_-\bar\partial_+\alpha + \tan\omega(\bar\partial_-\alpha + \bar\partial_+\alpha)\bar\partial_-\beta. \tag{6.26}$$

Using the expressions of $\bar\partial_+\alpha$ and $\bar\partial_-\beta$ in (6.18), we compute directly to yield the result in (6.23) and the proof of Theorem 6.2 is complete. □

We can prove the next two theorems with straight computation.

Theorem 6.3. *Assume that the solution of (6.18) $(\alpha, \beta) \in C^2$. Then we have*

$$\begin{aligned} \bar\partial_+\bar\partial_-(\alpha+\beta) + W\bar\partial_-(\alpha+\beta) \quad &= \quad a(\omega, c)\bar\partial_+(\alpha+\beta) \\ -\bar\partial_-\bar\partial_+(\alpha+\beta) + W\bar\partial_+(\alpha+\beta) \quad &= \quad a(\omega, c)\bar\partial_-(\alpha+\beta), \end{aligned} \tag{6.27}$$

where

$$\begin{aligned} a(\omega, c) \quad &:= \quad \frac{\gamma+1}{4c}[2\tan^2\omega - (m - \tan^2\omega)\cos(2\omega)] \\ &= \quad \frac{\gamma+1}{4c}\cos^2\omega(\tan^2\omega + \alpha_2)(\tan^2\omega - \alpha_1), \end{aligned} \tag{6.28}$$

where

$$\alpha_2 := \frac{1}{2}[3 + m + \sqrt{(3+m)^2 + 4m}], \quad \alpha_1 := \frac{2m}{3 + m + \sqrt{(3+m)^2 + 4m}}. \tag{6.29}$$

Theorem 6.4. *Assume that the solution of (6.18)* $(\alpha, \beta) \in C^2$*. Then we have*

$$\begin{aligned} (\bar{\partial}_+ + W)(Z - \bar{\partial}_-\alpha) &= \frac{\gamma+1}{4c}(\tan^2\omega + 1)(Z - \bar{\partial}_+\beta) \\ (-\bar{\partial}_- + W)(Z - \bar{\partial}_+\beta) &= \frac{\gamma+1}{4c}(\tan^2\omega + 1)(Z - \bar{\partial}_-\alpha), \end{aligned} \tag{6.30}$$

where

$$Z := \frac{\gamma+1}{2c}\tan\omega. \tag{6.31}$$

We need Theorem 6.3 for lower bound and Theorem 6.4 for upper bound of the derivatives $\bar{\partial}_-\alpha$ and $\bar{\partial}_+\beta$.

7 The gas expansion problem

We now use the hodograph transformation and the decomposition of the previous section to study the expansion of a wedge of gas into vacuum. The problem was studied earlier in [24, 19, 11, 13], and especially by Li in [12, 13], but the solution of Li is in the hodograph plane for $1 \le \gamma < 3$, and the behavior of the vacuum boundary was left open. We continue the effort of Li and prove that the solution in the hodograph plane can be transformed back to the physical self-similar plane for all $\gamma > 1$ and the vacuum boundary is a Lipschitz continuous curve which is monotone in the upper and lower parts of the wedge respectively. We also determine explicitly the relative location of the vacuum boundary with respect to the vertical position of the explicit solution of Suchkov [24]. Moreover, we can draw a clear picture of the distribution of characteristics. For notational simplicity in this section, we use $\bar{m}$, m_0, and $\bar{n}$ defined by

$$\tan^2\bar{m} = m, \qquad m_0 = 1/\sqrt{m}, \quad \tan^2\bar{n} = \alpha_1 \tag{7.1}$$

for $1 < \gamma < 3$; and $\bar{m} \equiv 0, \bar{n} \equiv 0$ for $\gamma \ge 3$. Note that $\bar{n} < \bar{m}$ for $1 \le \gamma < 3$.

7.1 The planar rarefaction waves.

First we prepare our planar rarefaction waves. Assume that the initial data for (1.1) is

$$(\rho, u, v)(x, y, 0) = \begin{cases} (\rho_1, 0, 0), & \text{for } \ n_1x + n_2y > 0, \\ \text{vacuum}, & \text{for } \ n_1x + n_2y < 0, \end{cases} \tag{7.2}$$

where $n_1^2 + n_2^2 = 1$, and ρ_1 is a constant. The solution of (1.1) and (7.2) takes the form, see [14],

$$(\rho, u, v)(x, y, t) = \begin{cases} (\rho_1, 0, 0), & \zeta > 1, \\ (\rho, u, v)(\zeta), & -1/\kappa \le \zeta \le 1, \\ \text{vacuum}, & \zeta < -1/\kappa, \end{cases} \tag{7.3}$$

where $\zeta = n_1\xi + n_2\eta$, $(\xi,\eta) = (x/t, y/t)$, and the solution (c,u,v) has been normalized so that $c_1 = 1$. The rarefaction wave solution $(\rho,u,v)(\zeta)$ satisfies

$$\zeta = n_1 u + n_2 v + c, \qquad \frac{n_1}{\kappa}c - u = \frac{n_1}{\kappa}, \quad \frac{n_2}{\kappa}c - v = \frac{n_2}{\kappa}. \tag{7.4}$$

Note that this rarefaction wave corresponds to a segment in the hodograph plane, $n_2 u - n_1 v = 0$, $-n_1/\kappa \le u \le 0$.

In particular, when we consider the rarefaction wave propagates in the x-direction, i.e., $(n_1,n_2) = (1,0)$, this wave can be expressed as

$$x/t = u + c, \quad c = \kappa u + 1, \quad v \equiv 0, \quad -1/\kappa \le u \le 0. \tag{7.5}$$

That is, in the hodograph (u,v) plane, this rarefaction wave is mapped onto a segment $v \equiv 0$, $-1/\kappa \le u \le 0$, on which we have

$$i = \frac{1}{2\kappa}(\kappa u + 1)^2, \qquad i_u = \kappa u + 1, \quad i_{uu} = \kappa. \tag{7.6}$$

When we consider the expansion problem of a wedge of gas in the next subsection, we need to know not only the derivatives of i with respect to u in (7.6), but also the derivatives with respect to v, on the segment $v \equiv 0$, $-1/\kappa \le u \le 0$. For this purpose, we insert (7.6) into (3.16) to obtain

$$(i_v^2)_u - \frac{\kappa+1}{\kappa u + 1} i_v^2 = -(\kappa+1)(\kappa u + 1). \tag{7.7}$$

Solving this equation in terms of i_v^2 yields,

$$i_v^2 = \begin{cases} (\kappa u + 1)^2 \left[\dfrac{1}{m} + \left(C^2 - \dfrac{1}{m} \right) (\kappa u + 1)^{\frac{1-\kappa}{\kappa}} \right], & \text{for } \gamma \neq 3, \\ (1+u)^2 [C^2 - 2\ln(1+u)], & \text{for } \gamma = 3, \end{cases} \tag{7.8}$$

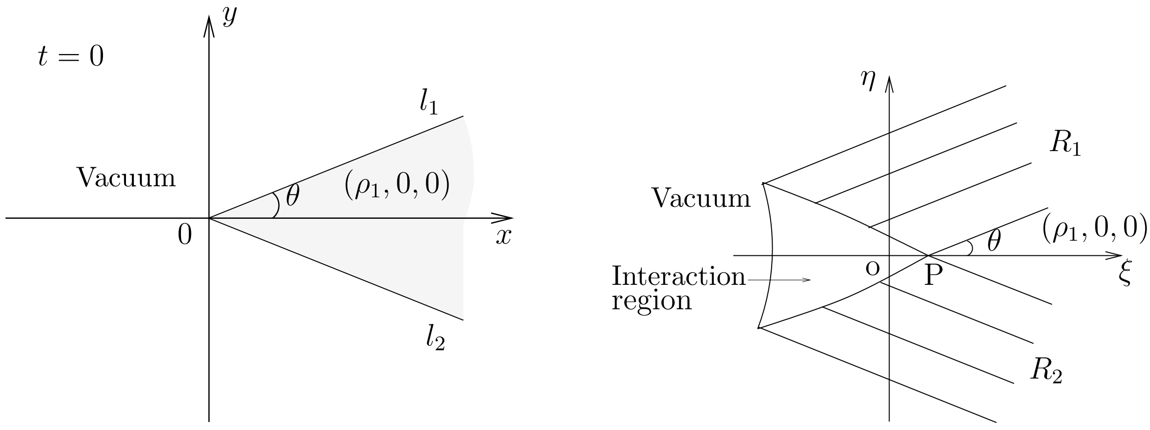
where C is an integral constant. This was obtained in [11].

7.2 A wedge of gas

We place the wedge symmetrically with respect to the x–axis and the sharp corner at the origin, as in Figure 7.1(a). This problem is then formulated mathematically as seeking the solution of (1.1) with the initial data,

$$(i,u,v)(t=0,x,y) = \begin{cases} (i_0, u_0, v_0), & -\theta < \delta < \theta, \\ (0, \bar{u}, \bar{v}), & \text{otherwise}, \end{cases} \tag{7.9}$$

where $i_0 > 0$, u_0 and v_0 are constant, $(\bar{u},\bar{v})$ is the velocity of the wave front, not being specified in the state of vacuum, $\delta = \arctan y/x$ is the polar angle, and θ is the half-angle



(a) Initial Data

(b) Interaction of rarefaction waves

Figure 7.1: The expansion of a wedge of gas

of the wedge restricted between 0 and $\pi/2$. This can be considered as a two–dimensional Riemann problem for (1.1) with two pieces of initial data (7.9). As we will see below, this problem is actually the interaction of two whole planar rarefaction waves. See Figure 7.1(b). We note that the solution we construct is valid for any "portions" of (7.9) as the solutions are hyperbolic.

The gas away from the sharp corner expands into the vacuum as planar rarefaction waves R_1 and R_2 of the form $(i,u,v)(t,x,y) = (i,u,v)(\zeta)$ ($\zeta = (n_1x + n_2y)/t$) where (n_1, n_2) is the propagation direction of waves. We assume that initially the gas is at rest, i.e., $(u_0, v_0) = (0, 0)$. Otherwise, we replace (u, v) by $(u - u_0, v - v_0)$ and (ξ, η) by $(\xi - u_0, \eta - v_0)$ in the following computations (see also (2.1)). We further assume that the initial sound speed is unit since the transformation $(u, v, c, \xi, \eta) \to c_0(u, v, c, \xi, \eta)$ with $c_0 > 0$ can make all variables dimensionless. Then the rarefaction waves R_1, R_2 emitting from the initial discontinuities l_1, l_2 are expressed in (7.4) with $(n_1, n_2) = (\sin\theta, -\cos\theta)$ and $(n_1, n_2) = (\sin\theta, \cos\theta)$, respectively. These two waves begin to interact at $P = (1/\sin\theta, 0)$ in the (ξ, η) plane due to the presence of the sharp corner and a wave interaction region, called the *wave interaction region* $\mathcal{D}$, is formed to separate from the planar rarefaction waves by k_1, k_2,

$$
\begin{aligned}
&k_1 : (1-\kappa^2)\xi_1^2 - (\kappa\eta_1 + 1)^2 = 2^{(1-\kappa)/\kappa} C(\kappa\eta_1 + 1)^{(\kappa+1)/\kappa}, \quad (\xi_1 > 0, -1 \le \eta_1 \le 1/\kappa),\\
&k_2 : (1-\kappa^2)\xi_2^2 - (\kappa\eta_2 + 1)^2 = 2^{(1-\kappa)/\kappa} C(\kappa\eta_2 + 1)^{(\kappa+1)/\kappa}, \quad (\xi_2 > 0, -1/\kappa \le \eta_2 \le 1),
\end{aligned}
\tag{7.10}
$$

where k_1 and k_2 are two characteristics from P, associated with the nonlinear eigenvalues of system (2.1), see [14, 26], and the constant C is

$$
\begin{aligned}
C = \;& (\gamma+1)\left[\frac{1}{\sqrt{\gamma}(\gamma+1)}\right]^{(\gamma+1)/(\gamma-1)}\\
&\cdot\left[(3-\gamma)(\gamma)^{-(\gamma+1)/(2(\gamma-1))} + (\gamma+1)(\gamma)^{(\gamma-3)/(2(\gamma-1))}\right],
\end{aligned}
\tag{7.11}
$$

and

$$\begin{cases} \xi_1 = \xi\cos\theta + \eta\sin\theta, \\ \eta_1 = -\xi\sin\theta + \eta\cos\theta, \end{cases} \qquad \begin{cases} \xi_2 = \xi\cos\theta - \eta\sin\theta, \\ \eta_2 = \xi\sin\theta + \eta\cos\theta. \end{cases} \tag{7.12}$$

So, the wave interaction region $\mathcal{D}$ is bounded by k_1, k_2 and the interface of gas with vacuum, connecting D and E, see Figure 7.1(b). The solution outside $\mathcal{D}$ consists of the constant state (i_0, u_0, v_0), the vacuum, and the planar rarefaction waves R_1 and R_2.

Problem A. *Find a solution of (2.1) inside the wave interaction region $\mathcal{D}$, subject to the boundary values on k_1 and k_2, which are determined continuously from the rarefaction waves R_1 and R_2.*

This problem is a Goursat–type problem for (2.1) since k_1 and k_2 are characteristics. Our strategy to solve this problem is to use the hodograph transform, solve the associated problem in the hodograph plane, and show that the hodograph transformation is invertible.

Note that initial data (7.9) is ir-rotational, we conclude that the flow is always irrotational provided that it is continuous. So the irrotationality condition (2.2) holds and all results about the hodograph transformation can be used to treat this problem. Then Problem A can be converted into a problem in the hodograph plane.

For this purpose, we need to map the wave interaction region $\mathcal{D}$ in the (ξ, η) plane into a region Ω in the (u, v) plane. Notice that the mapping of the planar rarefaction waves R_1 and R_2 into (u, v) plane are exactly two segments

$$\begin{aligned} &H_1 : u\cos\theta + v\sin\theta = 0, \quad (-\sin\theta/\kappa \le u \le 0) \text{ and} \\ &H_2 : u\cos\theta - v\sin\theta = 0, \quad (-\sin\theta/\kappa \le u \le 0). \end{aligned}$$

The boundary values of c on H_1, H_2, are

$$c|_{H_1} = 1 + \kappa v' =: c_0^1, \qquad c|_{H_2} = 1 + \kappa v'' =: c_0^2, \tag{7.13}$$

where $v' = u\sin\theta - v\cos\theta$ and $v'' = u\sin\theta + v\cos\theta$. Obviously,

$$0 \le c_0^1, c_0^2 \le 1. \tag{7.14}$$

Thus the wave interaction region Ω is bounded by H_1, H_2 and the interface of vacuum connecting D and E in the hodograph (u, v)-plane, see Figure 7.2. We define Ω more precisely to contain the boundaries H_1 and H_2, but not the vacuum boundary $c = 0$.

Boundary conditions. We need to derive the necessary boundary conditions on H_1 and H_2, respectively. This can be done simply by using coordinate transformations for (7.6) and (7.8). Indeed, denote temporarily (cf. (7.8))

$$\Gamma(u, C) := \begin{cases} \left[\frac{1}{m} + \left(C^2 - \frac{1}{m}\right)(\kappa u + 1)^{\frac{1-\kappa}{\kappa}}\right]^{\frac{1}{2}}, & \text{for } \gamma \ne 3, \\ \left[C^2 - 2\ln(1+u)\right]^{\frac{1}{2}}, & \text{for } \gamma = 3, \end{cases} \tag{7.15}$$

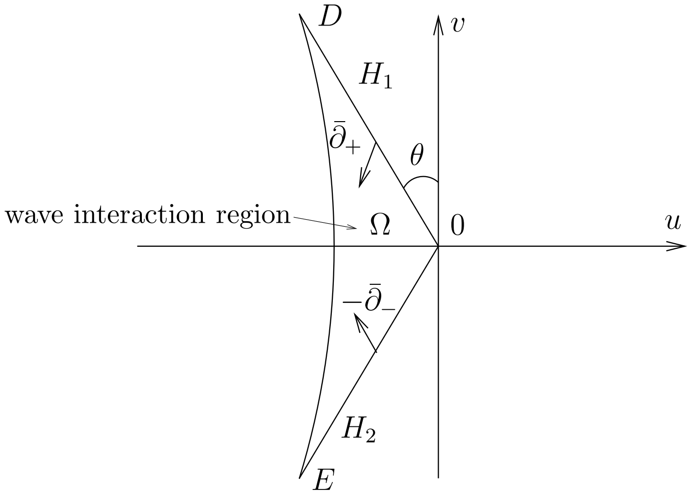


Figure 7.2: Wave interaction region in the hodograph plane

Then we have

$$\left.\begin{aligned} i_u &= (1-\kappa v')\left\{\Gamma(v',C_1)\cos\theta+\sin\theta\right\}, \\ i_v &= (1-\kappa v')\left\{(\Gamma(v',C_1)\sin\theta-\cos\theta\right\}, \end{aligned}\right\} \quad \text{on } H_1, \tag{7.16}$$

and

$$\left.\begin{aligned} i_u &= (1+\kappa v'')\left\{\Gamma(v'',C_2)\cos\theta+\sin\theta\right\}, \\ i_v &= (1+\kappa v'')\left\{-\Gamma(v'',C_2)\sin\theta+\cos\theta\right\}, \end{aligned}\right\} \quad \text{on } H_2, \tag{7.17}$$

where C_1 and C_2 are two constants. Applying the compatibility condition that i_u, i_v are continuous at $(u,v)=(0,0)$, we obtain

$$C_1 = -C_2 = \cot\theta. \tag{7.18}$$

Thus we obtain the boundary conditions as in (7.16) and (7.17).

In order to evaluate the boundary values of α, β, we substitute (7.13), (7.16)–(7.17) into (6.10) to deduce

$$\begin{aligned} A|_{H_1} &= \frac{\sin\theta}{1+\cos\theta} = \tan(\theta/2), \\ B|_{H_1} &= -\frac{-\Gamma(v',\cot\theta)\sin\theta+(1+\cos\theta)}{\Gamma(v',\cot\theta)(1+\cos\theta)+\sin\theta} =: B_1, \\ A|_{H_2} &= \frac{-\Gamma(v'',\cot\theta)\sin\theta+(1+\cos\theta)}{\Gamma(v'',\cot\theta)(1+\cos\theta)+\sin\theta} =: A_2, \\ B|_{H_2} &= -\frac{1+\cos\theta}{\sin\theta} = -\tan(\theta/2). \end{aligned} \tag{7.19}$$

Thus the boundary values for α, β on H_1 and H_2 are

$$\begin{aligned} \alpha|_{H_1} &= \theta, & \beta|_{H_1} &= 2\arctan(-B_1), \\ \alpha|_{H_2} &= 2\arctan(A_2), & \beta|_{H_2} &= -\theta. \end{aligned} \tag{7.20}$$

The boundary values of c on H_1 and H_2 are given in (7.13). Now Problem A becomes:

Problem B. *Find a solution (α, β, c) of (6.14) with boundary values (7.20) and (7.13), in the wave interaction region Ω in the hodograph plane.*

In order to solve Problem B, we estimate the boundary values (7.20) and (7.13).

Lemma 7.1 (Boundary data estimate). *For the boundary data (7.20) on the boundaries H_i, $i = 1, 2$, we have the following estimates:*

(i) If $\theta < \bar{m}$, there holds

$$2\theta \le (\alpha - \beta)|_{H_i} \le 2\bar{m}. \tag{7.21}$$

(ii) If $\theta > \bar{m}$, there holds

$$2\bar{m} \le (\alpha - \beta)|_{H_i} \le 2\theta. \tag{7.22}$$

Proof. For the first case, i.e., $\theta < \bar{m}$, by noting $0 \le 1 + \kappa v', 1 + \kappa v'' \le 1$, we estimate to get

$$\tan(\theta/2) \le B|_{H_1} \le \frac{-m_0 \tan(\theta/2) + 1}{m_0 + \tan(\theta/2)} =: m_\theta, \tag{7.23}$$

where $m_0 = 1/\sqrt{m}$. It is easy to check that

$$\tan(\theta/2 + \arctan m_\theta) = \sqrt{m} = \tan \bar{m}. \tag{7.24}$$

Therefore

$$2\theta \le (\alpha - \beta)|_{H_1} \le 2\bar{m}. \tag{7.25}$$

Similarly we can prove the second inequality on H_2 in (7.21).

For the second case that $\theta > \bar{m}$, the proof is also similar if $1 < \gamma < 3$. If $\gamma \ge 3$, it is evident that

$$-\tan(\theta/2) \le A|_{H_2}, B|_{H_1} \le \tan(\theta/2). \tag{7.26}$$

Then the proof is complete. $\square$

The local existence of solutions at the origin $(u, v) = (0, 0)$ follows routinely from the idea [18, Chapter 2] or [25]. We need only to check the compatibility condition to this problem, i.e.,

$$\frac{1}{\lambda_+}\left[l^0 \cdot \partial_+ K - \kappa \cos\frac{\alpha + \beta}{2} \sin^{-1}\omega\right] = \frac{1}{\lambda_-}\left[l^0 \cdot \partial_- K - \kappa \cos\frac{\alpha + \beta}{2} \sin^{-1}\omega\right] \tag{7.27}$$

at $(u,v)=(0,0)$, where $K=(\alpha,\beta,c)^\top$ and $l^0=(0,0,1)$. That is, we need to check if there holds

$$\frac{1}{\lambda_+}\left[\partial_+ c-\kappa\cos\frac{\alpha+\beta}{2}\sin^{-1}\omega\right]=\frac{1}{\lambda_-}\left[\partial_- c-\kappa\cos\frac{\alpha+\beta}{2}\sin^{-1}\omega\right]. \tag{7.28}$$

This is obviously true by using (6.19). Hence we have

Lemma 7.2 (Local existence). *There is a $\delta>0$ such that the C^1–solution of (6.14) and (7.13), (7.20) exists uniquely in the region $\bar{\Omega}=\{(u,v)\in\Omega;-\delta<u<0\}$, where δ depends only on the C^0 and C^1 norms of α, β on the boundaries H_1 and H_2.*

We do not give the proof. For details, see [18, Chapter 2] or [25].

Next we will extend the local solution to the whole region Ω. Therefore some *a priori* estimates on the C^0 and C^1 norms of α, β and i, are needed. The norm of i comes from the norms of α and β, see the third equation of (6.14). Therefore we need only the estimate on α and β. Recall that the derivation of (6.14) is based on the strict hyperbolicity of the flow, $i>0$. These will be achieved when we estimate the C^0 norms of α and β, see Subsection 7.3. The main existence theorem is stated as follows. Let l be the interface of the gas with the vacuum.

Theorem 7.1 (Global existence in the hodograph plane). *There exists a solution $(\alpha,\beta,i)\in C^1$ to the boundary value problem (6.14) with boundary values (7.13) and (7.20)(Problem B) in Ω. The vacuum interface l exists and is Lipschitz continuous.*

We prove this theorem by two steps. We estimate the solution itself in Subsection 7.3 and then proceed with estimates on the gradients in Subsection 7.4. The proof of Theorem 7.1 is also given in Subsection 7.4.

After we solve Problem B, we show the inversion of hodograph transformation in Subsection 7.5, which establishes the existence of the gas expansion problem, Problem A.

Theorem 7.2 (Global existence in the physical plane). *There exists a solution $(c,u,v)\in C^1$ of (2.1) for the gas expansion problem (Problem A) in the wave interaction region $\mathcal{D}$ in the self-similar (ξ,η)-plane for all $\gamma\geq 1$ and all wedge half-angle $\theta\in(0,\pi/2)$.*

7.3 The maximum norm estimate on (α,β,c)

We estimate the solution (α,β,c) itself, i.e, the C^0 norm of α, β and c. We adopt the method of invariant regions [23]. The notations $\bar{m}$ and $\bar{n}$ are given in the beginning of this section.

Lemma 7.3. *Suppose that there exists a C^1 solution $(\alpha(u,v), \beta(u,v), c(u,v))$ to problem (6.14), (7.13) and (7.20) in Ω. Then the C^0–norms of α and β have uniform bounds:*

(i) If $\theta \leq \bar{n}$, then $\theta \leq \alpha \leq 2\bar{m} - \theta$ and $-2\bar{m} + \theta \leq \beta \leq -\theta$;

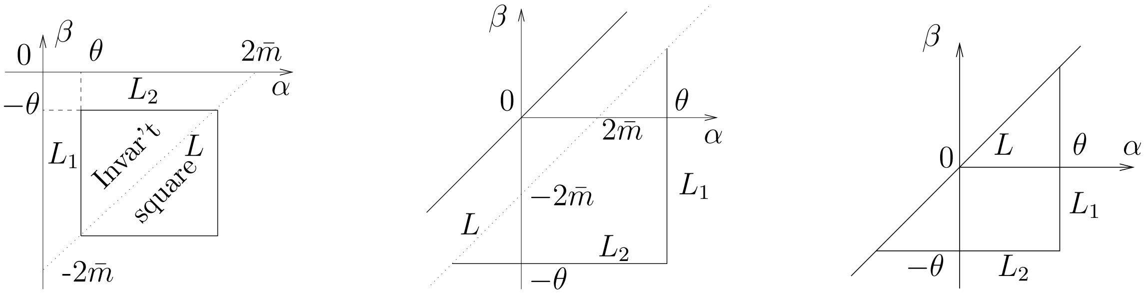
(ii) If $\bar{n} < \theta < \bar{m}$, then $2\theta \leq \alpha - \beta \leq 2\bar{m}$, $\theta \leq \alpha$, and $\beta \leq -\theta$;

(iii) If $\theta > \bar{m}$, then $2\bar{m} \leq \alpha - \beta \leq 2\theta, \alpha \leq \theta, \beta \geq -\theta$.

Proof. Case (i). Consider using system (6.18). We construct a square, shown in Figure 7.3(a), with left and upper sides denoted by L_1 and L_2, respectively. By Lemma 7.1, we know that the data belong to this square. Note that L_1 corresponds to H_1 and L_2 to H_2. We have

$$G(\alpha, \beta, c) > 0, \qquad \text{on } L_1, L_2. \tag{7.29}$$

We have the opposite on the other two sides. Note that the vector $(\sin\beta, -\cos\beta)$ on H_1 points toward the interior of Ω, and the vector $(\sin\alpha, -\cos\alpha)$ on H_2 point towards outside of Ω, see Figure 7.2. Thus, such a square bounded by L_1 and L_2 is invariant. (The square exists and remains in the fourth quadrant and is invariant for all the cases $\theta \leq 2\bar{m}$, but we have smaller invariant regions (i.e., invariant triangles) for all $\theta > \bar{n}$.)



(a) Case $\theta < \bar{n}$ (or $2\bar{m}$) (b) Case $\theta > \bar{m} > 0$ (c) Case $\bar{m} = 0$

Figure 7.3: Invariant regions

Case (ii). We show in this case that the upper triangle in 7.3(a) is invariant. Let L denote the diagonal line $\alpha - \beta = 2\bar{m}$. We show that the solution remains above L.

Consider using equation (6.27) to establish $\bar{\partial}_{\pm}(\alpha+\beta) \geq 0$. We note that $\bar{\partial}_+\alpha > 0$ and $\bar{\partial}_-\beta > 0$ before the solution hits L, so $\alpha \geq \theta$ and $\beta \leq -\theta$, thus $\omega > \bar{n}$ and $a(\omega, c) > 0$ for the solution, recalling that $\bar{n}$ is such that $a = 0$ at $\theta = \bar{n}$. In addition, there hold $\bar{\partial}_+(\alpha+\beta) > 0$ on H_2 and $\bar{\partial}_-(\alpha+\beta) > 0$ on H_1. A simple bootstrapping argument on equation (6.27) implies that both $\bar{\partial}_{\pm}(\alpha+\beta) > 0$ inside the domain.

Thus, we find

$$\bar{\partial}_+(\alpha - \beta) = -\bar{\partial}_+(\alpha+\beta) + 2\bar{\partial}_+\alpha \leq 2\bar{\partial}_+\alpha = \frac{\gamma+1}{2c}\sin(2\omega)(m - \tan^2\omega) \tag{7.30}$$

or

$$c\bar{\partial}_+\psi \geq -(\gamma+1)(\tan^2\omega)\psi \geq -(3-\gamma)\psi \tag{7.31}$$

for

$$\psi := m - \tan^2\omega. \tag{7.32}$$

Inequality (7.31) implies that $\psi > 0$ in the domain $c > 0$. Thus the upper triangle is invariant.

Case (iii). First let $\gamma \in [1,3)$ so that $\theta > \bar{m} > 0$. We show that the lower triangle is invariant, see Figure 7.3(b). The proof is similar. Again L denotes the line $\alpha - \beta = 2\bar{m}$, which is below the line $\alpha - \beta = 0$. Then we have

$$G(\alpha,\beta,c) < 0, \qquad \text{on } L_1, L_2$$

as in the case for the invariant square. We need to show that the solution does not go above the line L.

Consider using equation (6.27) to establish $\bar{\partial}_\pm(\alpha+\beta) \leq 0$. We note that $\bar{\partial}_+\alpha < 0$ and $\bar{\partial}_-\beta < 0$ before the solution hits L, so $\alpha \leq \theta$ and $\beta \geq -\theta$, thus $\omega < \theta$ and $\omega > \bar{m} > \bar{n}$ and $a(\omega, c) > 0$ for the solution before the solution hits L. In addition, there hold $\bar{\partial}_+(\alpha+\beta) < 0$ on H_2 and $\bar{\partial}_-(\alpha+\beta) < 0$ on H_1. A simple bootstrapping argument on equation (6.27) implies that both $\bar{\partial}_\pm(\alpha+\beta) < 0$ inside the domain.

Thus, we find

$$\bar{\partial}_+(\alpha-\beta) = -\bar{\partial}_+(\alpha+\beta) + 2\bar{\partial}_+\alpha \geq 2\bar{\partial}_+\alpha = \frac{\gamma+1}{2c}\sin(2\omega)(m-\tan^2\omega) \tag{7.33}$$

or

$$c\bar{\partial}_+\psi \leq -(\gamma+1)(\tan^2\omega)\psi \leq -(\gamma+1)(\tan^2\theta)\psi \tag{7.34}$$

for

$$\psi := m - \tan^2\omega. \tag{7.35}$$

Inequality (7.34) implies that $\psi < 0$ in the domain $c > 0$. Thus the upper triangle is invariant.

Lastly, we consider $\gamma \geq 3$ and $\bar{m} = 0$. We show that the lower triangle is invariant and so $\alpha - \beta > 0$, see Figure 7.3(c). Let L denote the line $\alpha - \beta = 0$. Similar to the previous case, we need to show that the solution does not cross L. We have similarly obtained $\bar{\partial}_\pm(\alpha+\beta) < 0$ inside the domain. Thus, we find

$$\bar{\partial}_+(\alpha-\beta) = -\bar{\partial}_+(\alpha+\beta) + 2\bar{\partial}_+\alpha \geq 2\bar{\partial}_+\alpha = \frac{\gamma+1}{2c}\sin(2\omega)(m-\tan^2\omega) \tag{7.36}$$

or

$$c\bar{\partial}_+(\alpha-\beta) \geq -(\gamma+1)\sin\omega\cos\omega(\tan^2\omega - m), \tag{7.37}$$

or

$$c\bar{\partial}_+\sin\omega \geq -(\gamma+1)\sin\omega(\tan^2\theta + |m|). \tag{7.38}$$

Inequality (7.38) implies that $\alpha - \beta > 0$ in the domain $c > 0$. Thus the upper triangle is invariant.

$\square$

Corollary 7.1. *For solutions (α, β, c) of (6.18),(7.20) and (7.13), we have:*

(i) If $\bar{n} < \theta < \bar{m}$, then $G(\alpha, \beta, c) > 0$ and $\bar{\partial}_+\alpha > 0$, $\bar{\partial}_-\beta > 0$ for all $(u, v) \in \Omega$.

(ii) If $\theta > \bar{m}$, then $G(\alpha, \beta, c) < 0$ and $\bar{\partial}_+\alpha < 0$, $\bar{\partial}_-\beta < 0$ for all $(u, v) \in \Omega$.

Proof. They follow trivially from the invariant triangles. $\square$

Remark 7.1. *If the angle of the wedge θ and the adiabatic index γ are related by*

$$\tan^2\theta = \frac{3-\gamma}{\gamma+1}, \tag{7.39}$$

for $1 < \gamma < 3$, i.e., $\theta = \bar{m}$, then boundary value (7.20) becomes constant $(\alpha, \beta)|_{H_j} = (\theta, -\theta)$, $j = 1, 2$. In this case the invariant region shrinks to a point $(\theta, -\theta)$ on the line $\alpha - \beta = 2\bar{m}$. Note that the source terms of (6.18) vanish on the boundaries H_1, H_2. We can use (6.20) to get an explicit solution,

$$c = 1 + \frac{\kappa}{\sin\theta}u, \tag{7.40}$$

where $-\sin\theta/\kappa \le u \le 0$. We further use (3.13) to get an explicit solution for the original gas expansion problem,

$$\begin{aligned} c &= 1 + \frac{\kappa(\xi\sin\theta - 1)}{\kappa + \sin^2\theta}, \\ u &= \frac{\sin\theta(\xi\sin\theta - 1)}{\kappa + \sin^2\theta}, \\ v &= \eta. \end{aligned} \tag{7.41}$$

This solution was first observed in [24].

Remark 7.2. *In the proof of Lemma 7.3, we observe that*

$$\frac{\cos((\alpha+\beta)/2)}{\sin\omega} > \delta \tag{7.42}$$

for some constant $\delta > 0$. It follows from the third equation of (6.18) that

$$c < 1 + \delta u \tag{7.43}$$

for $u < 0$ and thus c vanishes at $u > -1/\delta$ (assuming $\gamma > 1$). Therefore there exists a curve $u = u(v)$ such that $c(u(v), v) = 0$ where $u = u(v)$ is well-defined in the (u, v) plane. This is the interface of gas and vacuum.

Corollary 7.2. *For the gas expansion problem, the mappings $(X, Y) \to (\alpha, \beta)$ and $(X, Y) \to (A, B)$ are all bijective in the whole region Ω.*

Proof. It suffices to check the non-degeneracy of the Jacobian, say from $(X, Y) \to (\alpha, \beta)$,

$$J(X, Y; \alpha, \beta) = -\frac{1}{\sin^2 \omega} \cdot \cot \omega. \tag{7.44}$$

In view of Lemma 7.3, we obtain the conclusion. $\square$

Corollary 7.2 show that we can convert system (6.2) into system (6.14) and therefore use system (6.14) or (6.18) to discuss Problem B in the hodograph plane.

7.4 Gradient estimates and the proof of Theorem 7.1

In order to establish the existence of smooth solutions in the whole wave interaction region Ω, we need to establish gradient estimates for system (6.14) or (6.18). Due to the degeneracy of interface l, we cut off a sufficient thin strip between the interface l and the level set of $c = \epsilon$, $\epsilon > 0$. The remaining sub-domain is denoted by Ω_ϵ, in which $c > \epsilon$. We first show that there is a unique solution on Ω_ϵ. Then we extend the solution to Ω by using the argument of the arbitrariness of $\epsilon > 0$.

Lemma 7.4 (Gradient estimate). *Consider system (6.14) or (6.18) with boundary values (7.20) and (7.13). Assume that there is a C^1 solution (α, β) in Ω_ϵ, then the C^1 norm of α and β has a uniform bound, which only depends on the C^0 and C^1 norms of boundary values (7.20). That is, there is a constant $C > 0$, depending only on the boundary data (7.20) and (7.13), but not on ϵ, such that*

$$\|(\alpha, \beta)\|_{C^1(\Omega_\epsilon)} \le C/\epsilon^2, \tag{7.45}$$

where $\| \cdot \|_{C^1(\Omega_\epsilon)}$ represents the C^1-norm.

Proof. We use (6.23) to integrate $\bar{\partial}_- \alpha$ and $\bar{\partial}_+ \beta$ along λ_+ and λ_-–characteristics, respectively. Noting (6.19), we know that the integral path has a limited length. Also we note that Q has a uniform bound C/ϵ^2 in Ω_ϵ. Then we deduce that $\bar{\partial}_- \alpha$ and $\bar{\partial}_+ \beta$ are uniformly bounded in Ω_ϵ,

$$|\bar{\partial}_- \alpha| < C/\epsilon^2, \quad |\bar{\partial}_+ \beta| < C/\epsilon^2. \tag{7.46}$$

On the other hand, since G has a bound C/ϵ in Ω_ϵ (see (6.18)), so are $\bar{\partial}_+ \alpha$ and $\bar{\partial}_- \beta$,

$$|\bar{\partial}_+ \alpha| < C/\epsilon, \quad |\bar{\partial}_- \beta| < C/\epsilon. \tag{7.47}$$

Hence using the identities,

$$\partial_u = -\sin^{-1}(2\omega)(\cos\alpha\ \bar{\partial}_+ - \cos\beta\ \bar{\partial}_-),\quad \partial_v = -\sin^{-1}(2\omega)(\sin\alpha\ \bar{\partial}_+ - \sin\beta\ \bar{\partial}_-),\ (7.48)$$

and using the hyperbolicity $\alpha \neq \beta$ in Ω_ϵ, we conclude that $\partial_u\alpha$, $\partial_v\alpha$, $\partial_u\beta$ and $\partial_v\beta$ are uniformly bounded in Ω_ϵ, as expressed in (7.45). □

Lemma 7.5 (Modulus estimate). *Assume that the solution $(\alpha, \beta) \in C^2(\Omega_\epsilon)$. Then we have the following modulus estimate,*

$$\|(\alpha,\beta)\|_{C^{1,1}(\Omega_\epsilon)} < C/\epsilon^2, \tag{7.49}$$

where $\|\cdot\|_{C^{1,1}(\Omega_\epsilon)}$ represents the $C^{1,1}$-norm, and $C^{1,1}(\Omega_\epsilon)$ is the space of functions whose C^1-derivatives are Lipschitz continuous.

Proof. We mainly follow [6, Lemma 3.6, Page 291] to obtain the estimate (7.49) by using (6.23). The verification process is specified as follows. For any point $(\bar{u}, \bar{v})$ inside Ω_ϵ, we denote $v = v(u; \bar{u}, \bar{v})$ the λ_+-characteristic curve from $(\bar{u}, \bar{v})$. Then we draw the characteristics $v = v(u; u_1, v_1)$ and $v = v(u; u_2, v_2)$ from two points (u_1, v_1) and (u_2, v_2), $u_1 \le u_2$, and they intersect H_1 at two points $(\bar{u}_1, \bar{v}_1)$ and $(\bar{u}_2, \bar{v}_2)$ (cf. Figure 7.2). We want to use (6.23) to show the Lipschitz continuity of $\bar{\partial}_-\alpha$. The same is true for $\bar{\partial}_+\beta$. Denote

$$\Theta(u, v(u;\bar{u},\bar{v})) = \exp\left(\int_u^{\bar{u}} W(q, v(q;\bar{u},\bar{v}))dq\right) Q(u, v(u;\bar{u},\bar{v})). \tag{7.50}$$

Recall that on H_1, $\bar{\partial}_-\alpha(u, v) \equiv 0$. Then we obtain by integrating (6.23) along the λ_+-characteristics,

$$\begin{aligned} \bar{\partial}_-\alpha(u_1, v_1) &= \int_{\bar{u}_1}^{u_1} \Theta(u, v(u; u_1, v_1))du, \\ \bar{\partial}_-\alpha(u_2, v_2) &= \int_{\bar{u}_2}^{u_2} \Theta(u, v(u; u_2, v_2))du. \end{aligned} \tag{7.51}$$

Therefore, we proceed to obtain

$$\begin{aligned} & |\bar{\partial}_-\alpha(u_1, v_1) - \bar{\partial}_-\alpha(u_2, v_2)| \\ = & \left|\int_{\bar{u}_1}^{u_1} \Theta(u, v(u; u_1, v_1))du - \int_{\bar{u}_2}^{u_2} \Theta(u, v(u; u_2, v_2))du\right| \\ \le & \left|\int_{\bar{u}_2}^{u_1} [\Theta(u, v(u; u_1, v_1)) - \Theta(u, v(u; u_2, v_2))]du\right| \\ & + \left|\int_{\bar{u}_2}^{\bar{u}_1} \Theta(u, v(u; u_1, v_1))du\right| + \left|\int_{u_2}^{u_1} \Theta(u, v(u; u_2, v_2))du\right| \\ =: & \ T_1 + T_2 + T_3. \end{aligned} \tag{7.52}$$

Obviously, since $|\Theta(u, v(u; u_2, v_2))| \le C/\epsilon^2$ for some constant C, we have

$$T_3 \le C|u_1 - u_2|. \tag{7.53}$$

To estimate T_1, we use the definition of $v = v(u; \bar{u}, \bar{v})$: $\frac{dv(u;\bar{u},\bar{v})}{du} = \lambda_+(u, v(u; \bar{u}, \bar{v}))$ and obtain

$$\frac{d}{du}\frac{\partial v(u; \bar{u}, \bar{v})}{\partial \bar{v}} = \frac{\partial \lambda_+}{\partial v}(u, v(u; \bar{u}, \bar{v})) \cdot \frac{\partial v(u; \bar{u}, \bar{v})}{\partial \bar{v}}. \tag{7.54}$$

Integration along $v = v(u; \bar{u}, \bar{v})$ yields,

$$\frac{\partial v(u; \bar{u}, \bar{v})}{\partial \bar{v}} = \exp\left(\int_{\bar{u}}^{u} \frac{\partial \lambda_+}{\partial v}(q, v(q; \bar{u}, \bar{v}))dq\right). \tag{7.55}$$

Recall again that $\lambda_+ = -\cot\beta$ and apply Lemma 7.4. Then we deduce:

$$\left|\frac{\partial v(u; \bar{u}, \bar{v})}{\partial \bar{v}}\right| \leq C/\epsilon^2. \tag{7.56}$$

Noting that

$$\frac{\partial v(u; \bar{u}, \bar{v})}{\partial \bar{v}} = -\lambda_+(u, v(u; \bar{u}, \bar{v})) \cdot \frac{\partial v(u; \bar{u}, \bar{v})}{\partial \bar{u}}, \tag{7.57}$$

we obtain

$$\left|\frac{\partial v(u; \bar{u}, \bar{v})}{\partial \bar{u}}\right| \leq C/\epsilon^2. \tag{7.58}$$

Therefore we have

$$T_1 \leq \left|\int_{\bar{u}_2}^{u_1} \left[\frac{\partial \Theta}{\partial \bar{u}}(u_1 - u_2) + \frac{\partial \Theta}{\partial \bar{v}}(v_1 - v_2)\right] du\right| \leq C/\epsilon^2(|u_1 - u_2| + |v_1 - v_2|), \tag{7.59}$$

where we use (7.56) and (7.58) as well as the property of W and Q.

It remains to estimate T_2. By the definition of $v = v(u; \bar{u}, \bar{v})$, we can show, by the Gronwall inequality, that

$$|\bar{u}_1 - \bar{u}_2| \leq C(|u_1 - u_2| + |v_1 - v_2|). \tag{7.60}$$

So we have

$$T_2 \leq C/\epsilon^2(|u_1 - u_2| + |v_1 - v_2|). \tag{7.61}$$

Hence we conclude that

$$|\bar{\partial}_-\alpha(u_1, v_1) - \bar{\partial}_-\alpha(u_2, v_2)| \leq C/\epsilon^2(|u_1 - u_2| + |v_1 - v_2|) \tag{7.62}$$

for some C independent of ϵ.

Besides, we can use (6.18) to obtain the Lipschitzian property of $\bar{\partial}_+\alpha$ and $\bar{\partial}_-\beta$ directly. Thus we complete the proof. $\square$

Proof of Theorem 7.1. With the classical technique in [17] or [6], we obtain the "global" solution in Ω_ϵ by the extension from the local solution.

In view of Lemma 7.2, we obain a local solution (α, β, c) in $\Omega_\delta = \{(u, v) \in \Omega_\epsilon; -\delta < u < 0\}$. We take a level set of c, denoted by Υ_c, in Ω_δ. On this curve, (α, β, c) is known from the local solution and $(\alpha, \beta) \in C^1(\Upsilon_c)$ in view of Lemma 7.5. Then our problem becomes to find a solution of (6.14) in the remaining region, subject to the data on H_1, H_2 and Υ_c.

Denote the slope of Υ_c by s_0,

$$s_0 := \frac{dv}{du} = -\frac{c_u}{c_v} = -\cot\frac{\alpha+\beta}{2}. \tag{7.63}$$

Then we have

$$\frac{1}{s_0} - \frac{1}{\lambda_-} = \frac{\sin\omega}{\cos\frac{\alpha+\beta}{2}\cos\alpha} > 0, \quad \frac{1}{s_0} - \frac{1}{\lambda_+} = -\frac{\sin\omega}{\cos\frac{\alpha+\beta}{2}\cos\beta} < 0. \tag{7.64}$$

This shows that the level set Υ_c is not a characteristic and $\lambda_\pm$-characteristics always points toward the right hand side of Υ_c. Thus, we follow the proof of Lemma 4.1 in [6, Page 294], using Lemmas 7.4 and 7.5, to finish the proof of the existence of solutions in Ω_ϵ.

Owing to the arbitrariness of width $\epsilon > 0$, we use the contradiction argument to show that the C^1 solution (α, β, c) can be extend to the whole region Ω.

The discussion of vacuum boundary is left in Subsection 7.7.2. □

7.5 Inversion

We now establish the global one-to-one inversion of the hodograph transform. Consider the hodograph transformation $T : (\xi, \eta) \to (u, v)$. The mapping (3.13) defines a domain via $\xi = u + i_u,\ \eta = v + i_v$. We need to show that the Jacobian $J_T^{-1}(u, v; \xi, \eta)$ in (3.19) does not vanish:

$$J_T^{-1}(u, v; \xi, \eta) = \xi_u\eta_v - \xi_v\eta_u = (1 + i_{uu})(1 + i_{vv}) - i_{uv}^2 \neq 0. \tag{7.65}$$

We calculate, on the one hand, multiplying (3.16) with $(1 + i_{uu})$,

$$\begin{aligned}&(2\kappa i - i_u^2)i_{uv}^2 + 2i_ui_vi_{uv}(1 + i_{uu}) + (2\kappa i - i_v^2)(1 + i_{uu})^2\\ =\ &(2\kappa i - i_u^2)[i_{uv}^2 - (1 + i_{uu})(1 + i_{vv})].\end{aligned} \tag{7.66}$$

On the other hand, from (3.21) and (6.3) we have

$$(2\kappa i-i_u^2)i_{uv}^2+2i_ui_vi_{uv}(1+i_{uu})+(2\kappa i-i_v^2)(1+i_{uu})^2 = (2\kappa i-i_v^2)(\partial_+i_u+1)(\partial_-i_u+1). \tag{7.67}$$

Then we obtain

$$J_T^{-1}(u,v;\xi,\eta) = -\frac{(\partial_+ X+1)(\partial_- X+1)}{\lambda_-\lambda_+} = -\frac{(\bar\partial_+ X+\sin\beta)(\bar\partial_- X+\sin\alpha)}{\cos\alpha\cos\beta}, \tag{7.68}$$

by using the definition of $\bar\partial_\pm$, see (6.17). This is parallel to (5.5). Therefore, in order to show that $J_T^{-1}(u,v;\xi,\eta)$ does not vanish, it is equivalent to prove that:

Lemma 7.6. *The non-degeneracy of the Jacobian $J_T^{-1}(u,v;\xi,\eta)$ is equivalent to*

$$\bar\partial_+ X+\sin\beta \neq 0 \quad \textit{and} \quad \bar\partial_- X+\sin\alpha\neq 0. \tag{7.69}$$

Recall the expression of X in terms of α, β in (6.12). Then we compute

$$\begin{aligned} \bar\partial_+ X+\sin\beta &= -\kappa\frac{\cos\frac{\alpha+\beta}{2}}{\sin\omega} - \frac{1+\kappa}{2}\cot\omega\cos\beta[m-\tan^2\omega] \\ &\quad +\sin\beta+\frac{c}{2}\frac{\cos\alpha}{\sin^2\omega}\bar\partial_+\beta, \\ \bar\partial_- X+\sin\alpha &= \kappa\frac{\cos\frac{\alpha+\beta}{2}}{\sin\omega} + \frac{1+\kappa}{2}\cot\omega\cos\alpha[m-\tan^2\omega] \\ &\quad +\sin\alpha-\frac{c}{2}\frac{\cos\beta}{\sin^2\omega}\bar\partial_-\alpha. \end{aligned} \tag{7.70}$$

They are easily simplified to be

$$\begin{aligned} \bar\partial_+ X+\sin\beta &= -\frac{1+\kappa}{\sin(\alpha-\beta)}\cos\alpha+\frac{c}{2}\frac{\cos\alpha}{\sin^2\omega}\bar\partial_+\beta &= \frac{c}{2}\frac{\cos\alpha}{\sin^2\omega}[\bar\partial_+\beta-Z], \\ \bar\partial_- X+\sin\alpha &= \frac{1+\kappa}{\sin(\alpha-\beta)}\cos\beta-\frac{c}{2}\frac{\cos\beta}{\sin^2\omega}\bar\partial_-\alpha &= -\frac{c}{2}\frac{\cos\beta}{\sin^2\omega}[\bar\partial_-\alpha-Z], \end{aligned} \tag{7.71}$$

where $Z=(1+\gamma)\tan\omega/(2c)$, as denoted in (6.31) before. Note that on the boundaries H_1 and H_2, the values $\bar\partial_+\beta$ and $\bar\partial_-\alpha$ are, respectively,

$$\bar\partial_+\beta|_{H_2}\equiv 0, \qquad \bar\partial_-\alpha|_{H_1}\equiv 0. \tag{7.72}$$

Therefore (7.69) follows from the following Lemma.

Lemma 7.7. *There holds*

$$\bar\partial_+\beta<Z, \qquad \bar\partial_-\alpha<Z \tag{7.73}$$

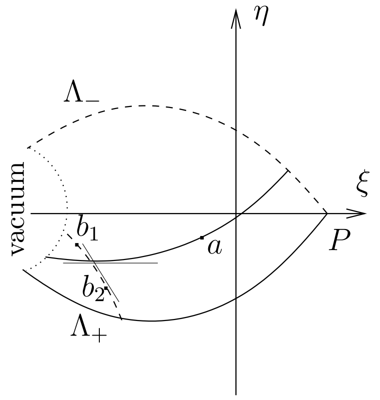
in the region Ω, see Figure 7.2.

Proof. By identities (6.30), the boundary condition on Z, and (7.72) we obtain the inequalities (7.73). More precisely, we note that both inequalities hold at the origin, and thus they hold in a neighborhood of the origin. Let $c=\epsilon\in(0,1)$ be the first level curve on which at least one of the two strict inequalities becomes equality. Note that the level curves of c are transversal to the characteristics. We integrate identities (6.30) in the domain $\epsilon<c<1$ along characteristics to yield strict inequalities (7.73), resulting in a contradiction. Thus, both strict inequalities must hold up to $c=0$. □

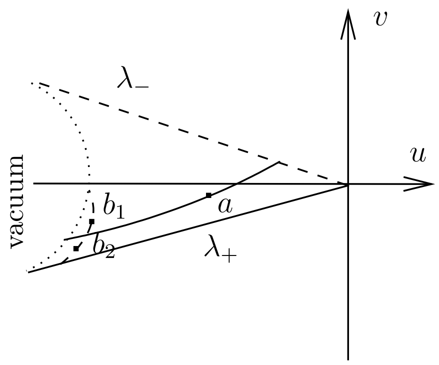
While non-vanishing Jacobian guarantees local one-to-one, we need global one-to-one, which is guaranteed by the monotonicity of ξ and η along characteristics for the cases $\theta \in (0, 2\bar{m})$. In fact, $\partial_\pm \xi = 1 + \partial_\pm X \neq 0$, or $\pm\bar{\partial}_\pm \xi < 0$ more precisely, following the above lemma. From (3.15)(3.25) we have $\partial_+\xi = -\lambda_-\partial_+\eta, \partial_-\xi = -\lambda_+\partial_-\eta$. Thus ξ and η have the same monotonicity along a plus characteristic curve since $-\lambda_- > 0$, but opposite monotonicity along a minus characteristics since $-\lambda_+ < 0$ for the cases $\theta \in (0, 2\bar{m})$ for which $\alpha > 0$ and $\beta < 0$. For any two points in the interaction zone in the (u, v) plane, there exist two characteristic curves connecting the two points. Either ξ or η is monotone along the connecting path. Thus, no two points from the (u, v) domain maps to one point in the (ξ, η) plane when $\theta \in (0, 2\bar{m})$.

For $\theta \geq 2\bar{m}$, the angles α and β may become negative or positive respectively, thus the monotonicity of η ceases along characteristics. However, the variable ξ remains monotone decreasing along both characteristics regardless of the signs of α or β because of $\pm\bar{\partial}_\pm\xi < 0$. See Figure 7.4(b) where we have indicated three points a, b_1, and b_2. If b_1 is located above the plus characteristic curve passing a, the variable ξ is monotone decreasing from a to b_1 because it is so along both characteristic curves and the minus characteristic curve is oriented toward the vacuum, see Figure 7.4(a). Let b_2 be located below the plus characteristic curve passing a, then we use the fact from next subsections (that the two characteristic curves in the (ξ, η) plane are convex and concave respectively) so that points a and b_2 will not give the same value of η. (We have drawn two tangent-lines at the intersection point to show that a and b_2 indeed have different η values) Thus no two different points on the (u, v) plane map to a single point of the (ξ, η) plane.

We have therefore established the global one-to-one property.



(a) (ξ, η) plane



(b) (u, v) plane

Figure 7.4: Global one-to-one between (ξ, η) and (u, v).

7.6 Proof of Theorem 7.2

The above estimates are sufficient for the proof of Theorem 7.2. For completeness, we sum it as follows. First we use the hodograph transformation (3.11) to convert Problem A into Problem B. Since the region $\mathcal{D}$ in Figure 7.1(b) is a wave interaction region, the Jacobian $J_T(u,v;\xi,\eta)$ does not vanish in view of Theorem 5.1, so the hodograph transformation (3.11) is valid. Then we solve Problem B in Theorem 7.1. In subsection 7.5, we have shown that the hodograph transformation is invertible in the entire domain of interaction, using properties to be established in the next subsection 7.7. Thus the proof of Theorem 7.2 is complete, once we finish subsection 7.7.

7.7 Properties of the solutions

7.7.1 Convexity of characteristics in the physical plane

Now we discuss the convexity of $\Lambda_\pm$-characteristics in the mixed wave region $\mathcal{D}$, in the (ξ,η) plane. It is a rather simple way to look at this from the correspondence between the (ξ,η) plane and the (u,v) plane.

Consider the hodograph transformation T of (3.11). We note, by using the chain rule, that,

$$\partial_u + \lambda_+\partial_v = \left(\frac{\partial\xi}{\partial u} + \lambda_+\frac{\partial\xi}{\partial v}\right)\partial_\xi + \left(\frac{\partial\eta}{\partial u} + \lambda_+\frac{\partial\eta}{\partial v}\right)\partial_\eta. \tag{7.74}$$

We rewrite (3.14) as

$$\frac{\partial\xi}{\partial u} + \lambda_+\frac{\partial\xi}{\partial v} = -\lambda_-\left(\frac{\partial\eta}{\partial u} + \lambda_+\frac{\partial\eta}{\partial v}\right). \tag{7.75}$$

Using (3.13), we have

$$\frac{\partial\xi}{\partial u} + \lambda_+\frac{\partial\xi}{\partial v} = \partial_+X + 1. \tag{7.76}$$

Thus we derive a differential relation from (7.74), by noting $\Lambda_+ = -1/\lambda_-$,

$$\bar\partial_+ = \left(\bar\partial_+X + \sin\beta\right)\left(\partial_\xi + \Lambda_+\partial_\eta\right). \tag{7.77}$$

Similarly, we have

$$\bar\partial_- = \left(\bar\partial_-X + \sin\alpha\right)\left(\partial_\xi + \Lambda_-\partial_\eta\right). \tag{7.78}$$

Acting (7.77) on Λ_+ and (7.78) on Λ_- as well as using the definition of α, β (i.e., $\Lambda_+ = \tan\alpha$, $\Lambda_- = \tan\beta$), we obtain

$$\begin{aligned}
\left(\partial_\xi + \Lambda_+\partial_\eta\right)\Lambda_+ &= (1+\tan^2\alpha)\cdot\left(\bar\partial_+X + \sin\beta\right)^{-1}\cdot\bar\partial_+\alpha, \\
\left(\partial_\xi + \Lambda_-\partial_\eta\right)\Lambda_- &= (1+\tan^2\beta)\cdot\left(\bar\partial_-X + \sin\alpha\right)^{-1}\cdot\bar\partial_-\beta.
\end{aligned} \tag{7.79}$$

By Corollary 7.1, the signs of $\bar{\partial}_+\alpha$ and $\bar{\partial}_-\beta$ are

$$\begin{aligned} &\bar{\partial}_+\alpha < 0, \quad \bar{\partial}_-\beta < 0, \qquad \text{if } \theta > \bar{m};\\ &\bar{\partial}_+\alpha > 0, \quad \bar{\partial}_-\beta > 0, \qquad \text{if } \theta \in (\bar{n}, \bar{m}). \end{aligned} \tag{7.80}$$

In view of Lemma 7.7, we have

$$\bar{\partial}_+ X + \sin\beta < 0, \quad \bar{\partial}_- X + \sin\alpha > 0. \tag{7.81}$$

Hence we conclude,

$$\begin{aligned} &(\partial_\xi + \Lambda_+\partial_\eta)\,\Lambda_+ > 0, \quad (\partial_\xi + \Lambda_-\partial_\eta)\,\Lambda_- < 0, \quad \text{for } \theta > \bar{m},\\ &(\partial_\xi + \Lambda_+\partial_\eta)\,\Lambda_+ < 0, \quad (\partial_\xi + \Lambda_-\partial_\eta)\,\Lambda_- > 0, \quad \text{for } \theta \in (\bar{n}, \bar{m}). \end{aligned} \tag{7.82}$$

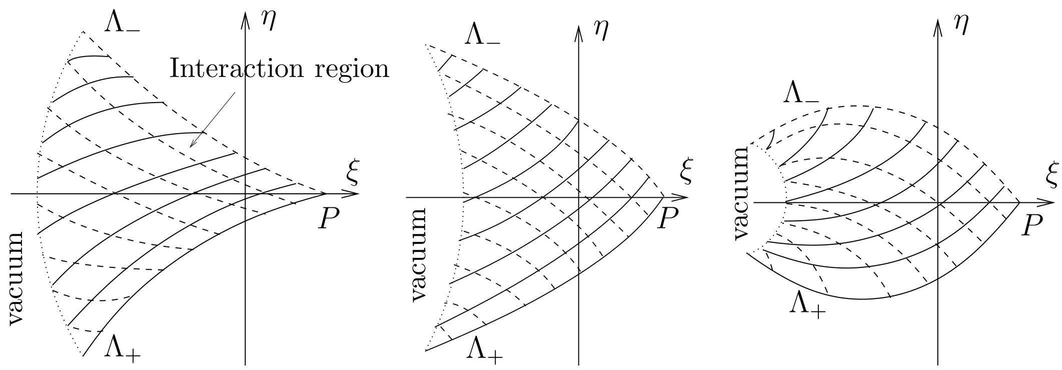
In sum, we have

Theorem 7.3. *The $\Lambda_\pm$-characteristics in the wave interaction region $\mathcal{D}$ of the (ξ, η) plane have fixed convexity types, see Figure 7.5:*

(i) If $\theta > \bar{m}$, the $\Lambda_\pm$-characteristics are convex and concave, respectively.

(ii) If $\theta \in (\bar{n}, \bar{m})$, the $\Lambda_\pm$-characteristics are concave and convex, respectively.

(iii) If $\theta = \bar{m}$, the solution has the explicit form (7.40) with straight characteristics.



(a) Case $\theta \in (\bar{n}, \bar{m})$ (b) Case $\theta \in (\bar{m}, 2\bar{m})$ (c) Case $\theta \in (2\bar{m}, \pi/2)$

Figure 7.5: Convexity types of the characteristics and the vacuum boundaries. Dashed curves are of minus family.

7.7.2 Regularity of the vacuum boundary.

Recall that formulae (3.13) transform the solution (α, β, c) in the (u, v) plane, back into the (ξ, η)-plane. Note that (α, β), and thus c_u, c_v, are uniformly bounded for $1 < \gamma < 3$, and that c tends to zero with a rate greater than c_u, c_v for $\gamma \geq 3$. Thus, by using (3.13), we have

$$\xi = u + i_u = u + \frac{c}{\kappa} c_u = u, \qquad \eta = v + i_v = v + \frac{c}{\kappa} c_v = v. \tag{7.83}$$

So we conclude that on the vacuum boundary, the (u, v) coordinates coincide with the (ξ, η) coordinates.

Next, we prove that the vacuum boundary is Lipschitz continuous. Let us consider the curve $\{(u, v) \mid i(u, v) = \epsilon > 0\}$ for all small positive ϵ. Differentiating the equation $i(u(v), v) = \epsilon$ with respect to v, we find

$$\frac{du}{dv} = -\frac{Y}{X} = -\tan\frac{\alpha + \beta}{2}. \tag{7.84}$$

Since $|\alpha + \beta| < \pi/2$ uniformly with respect to $\epsilon > 0$, the level curve $i(u, v) = \epsilon$ has a bounded derivative and in the limit as $\epsilon \to 0+$ converges to a Lipschitz continuous vacuum boundary.

7.7.3 Relative location

For the explicit solution with $\theta = \bar{m}$, the vacuum boundary is a vertical segment. Now we hold θ fixed and consider decreasing γ so that $\theta < \bar{m}$ (but $\theta > \bar{n}$). Then we find α and β lies on the left-hand side of the line $\alpha - \beta = 2\bar{m}$ in the (α, β) phase plane. By the formula $i_v = Y$ and the location of the boundary data, we have $Y < 0$ on the upper half of the wedge, thus i is monotone decreasing in v on the upper half, hence the vacuum boundary is on the left of the Suchkov boundary and of a convex type (i.e., bulging outward). Similarly, the other case $\theta > \bar{m}$ has the opposite result.

Theorem 7.4. *Let the vacuum boundary be represented as $\xi = \xi(\eta)$. Then it is Lipschitz continuous. It is less than the boundary of the Suchkov solution and is convex if $\theta \in (\bar{n}, \bar{m})$, but it is concave and greater than that of the Suchkov solution for $\theta > \bar{m}$.*

7.7.4 Characteristics on the vacuum boundary

We already know that the sound speed c attains zero in a finite range of u assuming $\gamma > 1$. Conversely, we deduce from (6.19) that the length of $\lambda_\pm$-characteristics is finite. Further, in view of (6.18) for $\gamma \in (1, 3)$ it can be seen that on the vacuum boundary,

$$\alpha - \beta = 2\bar{m}. \tag{7.85}$$

In fact, on the one hand, if (7.85) were not true, then $\bar{\partial}_+\alpha$ and $\bar{\partial}_-\beta$ would become infinite as (u, v) approaches the vacuum boundary, which would force (α, β) to reach the line

$\alpha-\beta=2\bar{m}$ in the (α,β)-plane. See Figure 7.3. On the other hand, the line $\alpha-\beta=2\bar{m}$ is the set of stationary points of (α,β). Thus once (7.85) holds, we have $c=0$.

Hence it is clear how the characteristics behave at the vacuum boundary. That is, for $1<\gamma<3$, a Λ_+-characteristic line has a non-zero intersection angle with a Λ_--characteristic line; however, if $\gamma\geq 3$, then there must be $\alpha=\beta$ on the vacuum boundary.

8 Summary remarks

We have considered the phase space equation (3.15)

$$(2\kappa\, i-i_u^2)i_{vv}+2i_ui_vi_{uv}+(2\kappa\, i-i_v^2)i_{uu}=i_u^2+i_v^2-4\kappa\, i \tag{8.1}$$

known from 1958 for the enthalpy i with the inverse of the hodograph transformation (3.13)

$$\xi=u+i_u,\qquad \eta=v+i_v \tag{8.2}$$

for the two-dimensional self-similar isentropic ir-rotational Euler system. Upon introducing the variables of inclination angles of characteristics and normalized characteristic derivatives, we have changed the second-order phase space equation to a first order system (6.18)

$$\bar{\partial}_+\alpha=G(\alpha,\beta,c),\quad \bar{\partial}_-\beta=G(\alpha,\beta,c),\quad \partial_0 c=\frac{\gamma-1}{2}\cdot\cos\frac{\alpha+\beta}{2}/\sin\frac{\alpha-\beta}{2}, \tag{8.3}$$

where

$$G(\alpha,\beta,c)=\frac{1+\gamma}{4c}\cdot\sin(\alpha-\beta)\cdot\left[\frac{3-\gamma}{\gamma+1}-\tan^2\frac{\alpha-\beta}{2}\right].$$

Derivatives of the variables α and β along directions not represented in (8.3) are provided by the higher-order systems (6.23)(6.27)(6.30). We use these infrastructure to construct solutions to binary interactions of planar waves in the phase space and show that the Jacobian of the inverse of the hodograph transform does not vanish, so we obtain in particular a global solution to the gas expansion problem with detailed shapes and positions of the vacuum boundaries and characteristics.

The invariant regions in the phase space revealed in the process have more potential than what has been utilized here. For example, we will use them to handle binary interactions of simple waves, which will lead to the eventual construction of global solutions to some four-wave Riemann problems that will not have vacuum in their data, see a forthcoming paper [16]. However, being a well-known difficult problem, the Euler system does not give in easily, which manifests in our inability to establish an invariant triangle (rather than the loose square) for the case $\theta<\bar{n}$, a techincal blemish of the paper.

We have made a comparison of the pair (8.1)(8.2) to the pair of eigenvalue $\xi = \lambda(u)$ and wave curve system $(\lambda - f'(u))u' = 0$ for the one-dimensional system $u_t + f(u)_x = 0$ from Lax [9]. The wave curves of the one-dimensional case correspond to surfaces in the phase space (i, u, v). It will be a very interesting next step to find out the phase space structure that involves subsonic domains and shock waves as well as the hyperbolic surfaces.

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