1 Introduction

Any real analysis class gives warnings that one cannot conclude from uniform convergence, the convergence of derivatives. Anybody knows how to construct examples.

The goal of this note is that in some problems in dynamical systems, these examples (notably the theory of rotation numbers) are the norm. Indeed, the cases where limits can be exchanged with derivatives are trivial. Theorem..

2 Set up

We will identify maps of the circle with nondecreasing maps of the line $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(t+1) = f(t) + 1$$

As usual in dynamics, we denote f^n, the nth iterate of f and f^{-1} denotes the inverse.

It was shown by Poincaré [] that

$$\rho_f = \lim \frac{1}{n}(f^n(x) - x)$$

exists and is reached uniformly. Indeed

$$\left| \frac{1}{n}(f^n(x) - x) - \rho \right| \leq \frac{1}{n}$$

so that ρ is a property of the map f.

When we consider families of mappings f_λ, $\lambda \in [0,1]$, we can consider the function ρ_λ that to λ associates ρ_{f_λ} is a very interesting function. For a trivial case $R_\lambda = x + \lambda$, we have $\rho_\lambda = x + \lambda$ and it is a differentiable function.

In a "typical" f, however, the function ρ_λ is a "devil's staircase". It is constant on open intervals which are dense, but the coplement has positive measure. Indeed in many points in this complement, ρ_λ is differentiable.

The paper [J. Moser 66] showed that when ρ_{λ_0} satisfies number theoretic properties, then ρ_λ is differentiable at $\lambda = \lambda_0$ and $f_{\lambda_0} = h \circ R_\rho \circ h^{-1}$ then ρ_λ is differentiable at $\lambda = \lambda_0$.

Somewhat later, [Herman, Yaccoz, K-S, K-O] showed that if ρ_{λ_0} satisfies arithmeic properties, then f_{λ_0} is indeed conjugate to a rotation.

3 Statement of Results

In view of the above results, it is natural to ask whether for the ?? numbers covered in the previous results

$$\frac{d}{d\lambda}\rho_\lambda\bigg|_{\lambda=0} = \lim\left(\frac{d}{d\lambda}\frac{1}{n}(f^n(x)-x)\bigg|_{\lambda=\lambda_0}\right) \tag{1}$$

The result we want to prove is this limit rarely exists.

Theorem 3.1 *Let ρ_{λ_0} be a Diophantine number. Assume that the limit in the RHS of (1) for some x. Then f_{λ_0} is a rotation.*

Applying the chain rule, we have

$$\frac{d}{d\lambda}f^N = \sum_{j=1}^{N}\left(Df^{N-j}\circ f^j\dot{f}\circ f^{j-1}\right).$$

We denote by D the derivative with respect to x and by $\cdot$ the derivative with respect to parameters.

For irrational rotations, we know that for any function $\frac{1}{N}\sum_{j=1}^{N}\varphi\circ R^j$ converges uniformly to $\int_0^{2\pi}\varphi$

If $f = h\circ R\circ h^{-1}$, then $f^j = h\circ R^j\circ h^{-1}$ and $Df^j = Dh\circ R^j\circ h^{-1}Dh^{-1}$. Therefore

$$\begin{aligned}
\frac{1}{N}\frac{d}{d\lambda}f^N &= \frac{1}{N}\sum_{j=1}^{N}Df^{N-j}\circ f^j\dot{f}\circ f^{-1}\circ f^j \\
&= \frac{1}{N}\sum_{j=1}^{N}(Dh)\circ R^N\circ h^{-1}(Dh^{-1})\circ h\circ R^j\circ h^{-1}(\dot{f}\circ f^{-1})\circ h\circ R^j\circ h^{-1} \\
&= Dh\circ R^N\circ h^{-1}\frac{1}{N}\left[\sum_{j=1}^{N}(Dh^{-1})\circ h\dot{f}\circ f^{-1}\circ h\right]\circ R^j\circ h^{-1}
\end{aligned}$$

We note that if Dh is not a constant (which, in such case has to be 1 if h is to satisfy (2)) then $Dh\circ R^N\circ h^{-1}$ oscillates quasiperiodically. So that if $Dh\not\equiv 1$, the limit in (**??**) is the product of a convergent sequence and a quasi-periodic one. Hence, the limit does not exist.

Moser's formula for the derivative of the rotation number $f_{\lambda_0}\circ h = h\circ R_\rho$, $Df_{\lambda_0}\circ hDh = Dh\circ R_\rho$, we try to find expansions $\dot{f}_{\lambda_0}\circ h + Df_{\lambda_0}\circ h\dot{h} = \dot{h}\circ R_\rho + Dh\circ R\dot{\rho}$. We write $\dot{h} = DhW$. Then the equation for the expansion becomes

$$\dot{f}_{\lambda_0}\circ h + Df_{\lambda_0}\circ hDhW = Dh\circ R_\rho W\circ R_\rho + Dh\circ\dot{R}_\rho,$$

$$[Dh\circ R]^{-1}\dot{f}_{\lambda_0}\circ h = W - W\circ R_\rho - \dot{\rho}.$$

We see that

$$\dot{\rho} = \int Dh \circ R^{-1} \dot{f}_{\lambda_0} \circ h.$$

In the paper [Luque], derivatives were computed using Moser's formula and some other methods from [Villanueva Luque] that are also rather more sophisticated than the discrete iteration.

Show that

$$\frac{d}{d\lambda}\rho_\lambda\bigg|_{\lambda=0} = \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \frac{d}{d\lambda} f^j\bigg|_{\lambda=0}$$

(so the limit does not exist, but the Cesaro sum exists).

Let $x \in \mathbb{T}$, $a_n(x) := Dh \circ R^n \circ h^{-1}(x)$, $b_n(x) := \frac{1}{n}\sum_{j=0}^{n-1}\Big[(Dh^{-1}) \circ h \dot{f} \circ f^{-1} \circ h\Big] \circ R^j \circ h^{-1}(x)$, and $b = \lim_{n\to\infty} b_n$. a_n is quasi-periodic so $a_n b_n$ will be oscillatory but its Cesàro sum converges. Let's how that $\frac{1}{N}\sum_{n=0}^{N-1} a_n b_n$ converges.

$$\left|\frac{1}{N}\sum_{n=0}^{N-1} a_n b_n\right| = \left|\frac{1}{N}\sum_{n=0}^{N-1}(a_n b_n - b + b)\right| \le \left|\frac{1}{N}\sum_{n=0}^{N-1} a_n(b_n - b)\right| + \left|\frac{1}{N}\sum_{n=0}^{N-1} a_n b\right|$$

$$\left|\frac{1}{N}\sum_{n=0}^{N-1} a_n(b_n - b)\right|$$

$$\left|\frac{1}{N}\sum_{n=0}^{N-1} a_n b\right|$$

Next, let's see if the derivative of the rotation number with the DSY algorithm converges or diverges.

Goal: show that the sum and the derivative for DSY algorithm commute by proving that the sum of derivatives converges.

By Fourier, $h(\theta) = \sum_{k\in\mathbb{Z}} \widehat{h}_k e^{2\pi ik\theta}$, so $Dh(\theta) = \sum_{k\in\mathbb{Z}} \widehat{h}_k 2\pi ike^{2\pi ik\theta}$. Let $\theta = h^{-1}(y)$.

$$\frac{1}{A_L}\sum_{N=1}^{L} w\left(\frac{N}{L}\right) Dh\circ R^N \circ h^{-1}(y) \sum_{j=0}^{N-1}(Dh^{-1}\circ h\dot{f}\circ f^{-1}\circ h)\circ R^j(\theta)$$
$$= \frac{1}{A_L}\sum_{N=1}^{L} w\left(\frac{N}{L}\right) Dh(\theta+\omega N) \sum_{j=0}^{N-1}(Dh^{-1}\circ h\dot{f}\circ f^{-1}\circ h)\circ R^j(\theta)$$
$$= \frac{1}{A_L}\sum_{N=1}^{L} w\left(\frac{N}{L}\right) \sum_{k\in\mathbb{Z}} \widehat{h}_k 2\pi ike^{2\pi ik(\theta+\omega N)} \sum_{j=0}^{N-1}(Dh^{-1}\circ h\dot{f}\circ f^{-1}\circ h)\circ R^j(\theta)$$

Let $s = \dfrac{t}{L}$. Then, $Lds = dt$, $t = sL$. By Poisson Lemma,

$$= \frac{1}{A_L}\sum_{N=1}^{L}\sum_{k\in\mathbb{Z}} \widehat{h}_k 2\pi ike^{2\pi ik\theta}\int_{\mathbb{R}} w(s)e^{2\pi i(k\omega-N)sL}Lds \sum_{j=0}^{N-1}(Dh^{-1}\circ h\dot{f}\circ f^{-1}\circ h)\circ R^j(\theta)$$
$$= \frac{1}{A_L}\sum_{k\in\mathbb{Z}} \widehat{h}_k 2\pi ike^{2\pi ik\theta}\sum_{N=1}^{L}\int_{\mathbb{R}} w(s)e^{2\pi i(k\omega-N)sL}Lds \sum_{j=0}^{N-1}(Dh^{-1}\circ h\dot{f}\circ f^{-1}\circ h)\circ R^j(\theta)$$

Let $\gamma = 2\pi(k\omega - N)L$.

Integrating by parts, $\left|\int_{\mathbb{R}} w(s)e^{i\gamma s}\right| = \left|\frac{1}{\gamma^m}\int_0^1 w^{(m)}(s)e^{i\gamma s}ds\right| \le |\gamma|^{-m}\|w^{(m)}\|_1$.

Also, $\left|\sum_{j=0}^{N-1}(Dh^{-1}\circ h\dot{f}\circ f^{-1}\circ h)\circ R^j(\theta)\right| \le aN$, for some constant a. Thus,

$$\left|\frac{1}{A_L}\sum_{N=1}^{L} w\left(\frac{N}{L}\right) Dh\circ R^N \circ h^{-1}(y) \sum_{j=0}^{N-1}(Dh^{-1}\circ h\dot{f}\circ f^{-1}\circ h)\circ R^j(\theta)\right|$$
$$\le \frac{L}{A_L}\sum_{k\in\mathbb{Z}}\left|\widehat{h}_k 2\pi ike^{2\pi ik\theta}\right|\sum_{N=1}^{L}|\gamma|^{-m}\|w^{(m)}\|_1 aN$$

Note that $\dfrac{A_L}{L} \to \int_0^1 w(s)ds \ne 0$ as $L\to\infty$, so $\dfrac{L}{A_L} \le C_w$ for some constant C_w.

Plug back in γ.

$$\le 2\pi aC_w\|w^{(m)}\|_1 \sum_{k\in\mathbb{Z}}|\widehat{h}_k||k|\sum_{N=1}^{L}|2\pi(k\omega-N)L|^{-m}N$$
$$= 2\pi aC_w\|w^{(m)}\|_1(2\pi L)^{-m} \sum_{k\in\mathbb{Z}}|\widehat{h}_k||k|\sum_{N=1}^{L}|k\omega-N|^{-m}N$$

Let $\tilde{C}_w = 2\pi aC_w\|w^{(m)}\|_1(2\pi)^{-m}$.

$$= \tilde{C}_w L^{-m}\sum_{k\in\mathbb{Z}}|\widehat{h}_k||k|\sum_{N=1}^{L}|k\omega-N|^{-m}N$$

Note that $\sum|k\omega-N|^{-m}N \le \sum|k\omega-N|^{-m}\left(|N-k\omega|+|k\omega|\right) \le C_m(1+|k\omega|)$ for some C_m.

$$\le \tilde{C}_w C_m L^{-m}\sum_{k}|\widehat{h}_k||k|(1+|k\omega|)$$

Remark. Finite regularity is enough. $|\widehat{h}_k| \le C_r|k|^{-r}\|h\|_{C^r}$ for some constant C_r.

$$\le \tilde{C}_w C_m L^{-m}\sum_{k} C_r|k|^{-r}\|h\|_{C^r}|k|(1+|k\omega|)$$
$$= \tilde{C}_w C_m C_r L^{-m}\|h\|_{C^r}\sum_{k}|k|^{-r+1}(1+|k\omega|)$$

This will converge if $r > 3$. Q.E.D.

Goal: show derivative and sum commute in Luque-Seara-Villanueva (LSV) method by proving that the sum of the derivatives converges.

Let's start by writing out the LSV method. This method approximates the rotation number by averaged sums and Richardson extrapolation.

$$\theta = \Theta_{q,p}(f) + \mathcal{O}(2^{-(p+1)q}),$$

where $\Theta_{q,p}$ is an *extrapolation operator* given by

$$\Theta_{q,p}(f) := \sum_{j=0}^{p} c_j^p \tilde{S}^p_{2^{q-p+j}}(f),$$

where p is the extrapolation order,
$\tilde{S}$ is the averaged sum given by

$$\tilde{S}^p_N = \binom{N+p}{p+1}^{-1} S^p_N = \theta + \sum_{l=1}^{p} \frac{\tilde{A}_l^{(p)}}{(N+p-l+1)\cdots(N+p)} + E(p,N),$$

$$\tilde{A}_l^{(p)} = (p-l+2)\cdots(p+1)A_l,\; A_l = (-1)^l \sum_{k\in\mathbb{Z}\setminus\{0\}} \xi_k \frac{e^{2(l-1)\pi i k\theta}}{(1-e^{2\pi i k\theta})^{l-1}},$$

$$S^p_N = \binom{N+p}{p+1}\theta + \sum_{l=1}^{p}\binom{N+p-l}{p+1-l}A_l + (-1)^{p+1}\sum_{k\in\mathbb{Z}\setminus\{0\}} \xi_k \frac{e^{2p\pi i k\theta}(1-e2\pi i k N\theta)}{(1-e^{2\pi i k\theta})^p},$$

$$S^0_N = f^N(x_0) - x_0,$$

and the coefficients $\{c_j^p\}_{j=0,\dots,p}$ are given by

$$c_l^p = (-1)^{p-l}\frac{2^{l(l+1)/2}}{\delta(l)\delta(p-l)},$$

where we define $\delta(n) := (2^n-1)(2^{n-1}-1)\cdots(2^1-1)$ for $n \ge 1$ and $\delta(0) := 1$.
Our goal is to show that the derivative commutes with the sum in the operator. We will take the derivative with respect to the parameter μ on which the rotation number depends in a regular way. Before taking the derivative, let's unravel the recursive sum S^p_N and write it as a simple sum of S^0_j's.

$$\begin{aligned}
S^p_N(f) &= \sum_{j=1}^{N} S_j^{p-1}(f) \\
&= \left(\sum_{n_1=1}^{N}\sum_{n_2=1}^{n_1}\sum_{n_3=1}^{n_2}\cdots\sum_{n_{p-2}=1}^{n_{p-3}} n_{p-2}\right) S^0_1 + \left(\sum_{n_1=1}^{N-1}\sum_{n_2=1}^{n_1}\sum_{n_3=1}^{n_2}\cdots\sum_{n_{p-2}=1}^{n_{p-3}} n_{p-2}\right)\alpha_2 S^0_2 \\
&\quad + \cdots + \left(\sum_{n_1=1}^{1}\sum_{n_2=1}^{n_1}\sum_{n_3=1}^{n_2}\cdots\sum_{n_{p-2}=1}^{n_{p-3}} n_{p-2}\right)\alpha_N S^0_N \\
&= \sum_{j=1}^{N}\left(\sum_{n_1=1}^{N-j+1}\sum_{n_2=1}^{n_1}\cdots\sum_{n_{p-2}=1}^{n_{p-3}} n_{p-2}\right) S^0_j \\
&= \sum_{j=1}^{N}\binom{N-j+p-1}{p-1} S^0_j
\end{aligned}$$

Let $N = 2^{q-p+j}$. Rewriting the operator,

$$
\begin{aligned}
\Theta_{q,p}(f) &= \sum_{j=0}^{p} c_j^p \tilde{S}_N^p(f) \\
&= \sum_{j=0}^{p} c_j^p \binom{N+p}{p+1}^{-1} S_N^p(f) \\
&= \sum_{j=0}^{p} c_j^p \binom{N+p}{p+1}^{-1} \sum_{n=1}^{N} \binom{N-n+p-1}{p-1} S_n^0 \\
&= \sum_{j=0}^{p} c_j^p \binom{N+p}{p+1}^{-1} \sum_{n=1}^{N} \binom{N-n+p-1}{p-1} (f^n(x_0)-x_0) \\
&= \sum_{j=0}^{p} \sum_{n=1}^{N} c_j^p \binom{N+p}{p+1}^{-1} \binom{N-n+p-1}{p-1} (f^n(x_0)-x_0) \\
&= \sum_{j=0}^{p} \frac{c_j^p}{A_N} \sum_{n=1}^{N} W_n^N (f^n(x_0)-x_0), \\
&= \sum_{j=0}^{p} \frac{p(p+1)c_j^p}{A_N} \sum_{n=1}^{N} W_n^N (f^n(x_0)-x_0),
\end{aligned}
$$

where

$$A_N := (N+p)(N+p-1)(N+p-2)\cdots N$$

and

$$W_n^N := (N-n+p-1)(N-n+p-2)\cdots(N-n+2).$$

Now, let's take the derivative of the operator. Let p be a fixed finite number. Need to show that for any given $p \leq q$, the sum of the derivative converges. Since p is a fixed finite number, it's enough to show the convergence of the inner sum. Let D denote the derivative with respect to x and $\cdot$ the derivative with respect to the parameter μ. Set $y := h^{-1}(x_0)$. Also, create a function $w(s) \in C^m(\mathbb{R})$ such that $w(\frac{n}{N}) = W_n^N$.

$$
\begin{aligned}
&\frac{1}{A_N} \sum_{n=1}^{N} W_n^N \frac{d}{d\mu}(f^n(x_0)-x_0) \\
&= \frac{1}{A_N} \sum_{n=1}^{N} W_n^N Dh \circ R^n \circ h^{-1}(x_0) \sum_{l=0}^{n-1} \left(Dh^{-1} \circ h\dot{f} \circ f^{-1} \circ h\right) \circ R^l \circ h^{-1}(x_0) \\
&= \frac{1}{A_N} \sum_{n=1}^{N} W_n^N Dh \circ R^n(y) \sum_{l=0}^{n-1} \left(Dh^{-1} \circ h\dot{f} \circ f^{-1} \circ h\right) \circ R^l(y) \\
&= \frac{1}{A_N} \sum_{n=1}^{N} W_n^N Dh(y+n\theta) \sum_{l=0}^{n-1} \left(Dh^{-1} \circ h\dot{f} \circ f^{-1} \circ h\right) \circ R^l(y) \\
&= \frac{1}{A_N} \sum_{n=1}^{N} W_n^N \sum_{k\in\mathbb{Z}} 2\pi i k \widehat{h}_k e^{2\pi i k(y+n\theta)} \sum_{l=0}^{n-1} \left(Dh^{-1} \circ h\dot{f} \circ f^{-1} \circ h\right) \circ R^l(y) \qquad \text{(by Fourier)} \\
&= \frac{1}{A_N} \sum_{n=1}^{N} W_n^N \sum_{k\in\mathbb{Z}} 2\pi i k \widehat{h}_k e^{2\pi i k y} e^{2\pi i k n\theta} \sum_{l=0}^{n-1} \left(Dh^{-1} \circ h\dot{f} \circ f^{-1} \circ h\right) \circ R^l(y) \\
&= \frac{1}{A_N} \sum_{k\in\mathbb{Z}} 2\pi i k \widehat{h}_k e^{2\pi i k y} \sum_{n=1}^{N} W_n^N e^{2\pi i k n\theta} \sum_{l=0}^{n-1} \left(Dh^{-1} \circ h\dot{f} \circ f^{-1} \circ h\right) \circ R^l(y) \\
&= \frac{1}{A_N} \sum_{k\in\mathbb{Z}} 2\pi i k \widehat{h}_k e^{2\pi i k y} \sum_{n=1}^{N} \left(\int_{\mathbb{R}} w(s) e^{2\pi i N(k\theta-n)s} N ds\right) \sum_{l=0}^{n-1} \left(Dh^{-1} \circ h\dot{f} \circ f^{-1} \circ h\right) \circ R^l(y) \qquad \text{(by Poisson)}
\end{aligned}
$$

Let $\gamma := 2\pi N(k\theta - n)$. Integrating by parts, we obtain

$$\left|\int_{\mathbb{R}} w(s) e^{2\pi i N(k\theta-n)s} N ds\right| = \left|\frac{1}{\gamma^m} \int_0^1 w^{(m)}(s) e^{i\gamma s} ds\right| \leq |\gamma|^{-m} \|w^{(m)}\|_1.$$

Also, note that

$$\left|\sum_{l=0}^{n-1} \left(Dh^{-1} \circ h\dot{f} \circ f^{-1} \circ h\right) \circ R^l(y)\right| \leq an$$

for some constant a. Hence,

$$\begin{aligned}\left|\frac{1}{A_N}\sum_{n=1}^{N} W_n^N \frac{d}{d\mu}(f^n(x_0)-x_0)\right| &\le \frac{a}{A_N}\sum_{k\in\mathbb{Z}}\left|2\pi k\widehat{h}_k e^{2\pi iky}\right|\sum_{n=1}^{N} n|\gamma|^{-m}\|w^{(m)}\|_1 \\ &\le \frac{2\pi a}{A_N}\|w^{(m)}\|_1\sum_{k\in\mathbb{Z}}|k|\left|\widehat{h}_k\right|\sum_{n=1}^{N} n|\gamma|^{-m} \\ &= \frac{a\|w^{(m)}\|_1}{A_N(2\pi N)^{m-1}}\sum_{k\in\mathbb{Z}}|k|\left|\widehat{h}_k\right|\sum_{n=1}^{N} n|k\theta-n|^{-m}\end{aligned}$$

Using the Diophantine property of θ,

$$\sum_{n=1}^{N} n|k\theta-n|^{-m} \le \sum_{n=1}^{N}|k\theta-n|^{-m}(|n-k\theta|+|k\theta|) \le C_m(1+k\theta)$$

for some constant C_m.

$$\begin{aligned}\frac{a\|w^{(m)}\|_1}{A_N(2\pi N)^{m-1}}\sum_{k\in\mathbb{Z}}|k|\left|\widehat{h}_k\right|\sum_{n=1}^{N} n|k\theta-n|^{-m} &\le \frac{a\|w^{(m)}\|_1}{A_N(2\pi N)^{m-1}}\sum_{k\in\mathbb{Z}}|k|\left|\widehat{h}_k\right| C_m(1+|k\theta|) \\ &\le \frac{aC_m\|w^{(m)}\|_1}{A_N(2\pi N)^{m-1}}\sum_{k\in\mathbb{Z}}|k|C_r|k|^{-r}\|h\|_{C^r}(1+|k\theta|) \qquad \text{(see Remark)} \\ &= \frac{aC_mC_r\|w^{(m)}\|_1\|h\|_{C^r}}{A_N(2\pi N)^{m-1}}\sum_{k\in\mathbb{Z}}|k|^{-r+1}(1+|k\theta|)\end{aligned}$$

Remark. Finite regularity is enough. $|\widehat{h}_k| \le C_r|k|^{-r}\|h\|_{C^r}$ for some constant C_r.
The sum will converge if $r > 3$. Q.E.D.