Goal: show derivative and sum commute in Luque-Seara-Villanueva (LSV) method by proving that the sum of the derivatives converges.

Let's start by writing out the LSV method. This method approximates the rotation number by averaged sums and Richardson extrapolation.

$$\theta = \Theta_{q,p}(f) + \mathcal{O}(2^{-(p+1)q}),$$

where $\Theta_{q,p}$ is an *extrapolation operator* given by

$$\Theta_{q,p}(f) := \sum_{j=0}^{p} c_j^p \tilde{S}^p_{2^{q-p+j}}(f),$$

where $p$ is the extrapolation order,
$\tilde{S}$ is the averaged sum given by

$$\tilde{S}_N^p = \binom{N+p}{p+1}^{-1} S_N^p = \theta + \sum_{l=1}^{p} \frac{\tilde{A}_l^{(p)}}{(N+p-l+1)\cdots(N+p)} + E(p,N),$$

$$\tilde{A}_l^{(p)} = (p-l+2)\cdots(p+1)A_l,\ A_l = (-1)^l \sum_{k\in\mathbb{Z}\setminus\{0\}} \xi_k \frac{e^{2(l-1)\pi i k\theta}}{(1-e^{2\pi i k\theta})^{l-1}},$$

$$S_N^p = \binom{N+p}{p+1}\theta + \sum_{l=1}^{p}\binom{N+p-l}{p+1-l}A_l + (-1)^{p+1}\sum_{k\in\mathbb{Z}\setminus\{0\}} \xi_k \frac{e^{2p\pi i k\theta}(1-e2\pi i k N\theta)}{(1-e^{2\pi i k\theta})^p},$$

$$S_N^0 = f^N(x_0) - x_0,$$

and the coefficients $\{c_j^p\}_{j=0,\dots,p}$ are given by

$$c_l^p = (-1)^{p-l}\frac{2^{l(l+1)/2}}{\delta(l)\delta(p-l)},$$

where we define $\delta(n) := (2^n-1)(2^{n-1}-1)\cdots(2^1-1)$ for $n\geq 1$ and $\delta(0) := 1$.
Our goal is to show that the derivative commutes with the sum in the operator. We will take the derivative with respect to the parameter $\mu$ on which the rotation number depends in a regular way. Before taking the derivative, let's unravel the recursive sum $S_N^p$ and write it as a simple sum of $S_j^0$'s.

$$\begin{aligned}
S_N^p(f) &= \sum_{j=1}^{N} S_j^{p-1}(f) \\
&= \left(\sum_{n_1=1}^{N}\sum_{n_2=1}^{n_1}\sum_{n_3=1}^{n_2}\cdots\sum_{n_{p-2}=1}^{n_{p-3}} n_{p-2}\right) S_1^0 + \left(\sum_{n_1=1}^{N-1}\sum_{n_2=1}^{n_1}\sum_{n_3=1}^{n_2}\cdots\sum_{n_{p-2}=1}^{n_{p-3}} n_{p-2}\right)\alpha_2 S_2^0 \\
&\quad + \cdots + \left(\sum_{n_1=1}^{1}\sum_{n_2=1}^{n_1}\sum_{n_3=1}^{n_2}\cdots\sum_{n_{p-2}=1}^{n_{p-3}} n_{p-2}\right)\alpha_N S_N^0 \\
&= \sum_{j=1}^{N}\left(\sum_{n_1=1}^{N-j+1}\sum_{n_2=1}^{n_1}\cdots\sum_{n_{p-2}=1}^{n_{p-3}} n_{p-2}\right) S_j^0 \\
&= \sum_{j=1}^{N}\binom{N-j+p-1}{p-1} S_j^0
\end{aligned}$$

Let $N = 2^{q-p+j}$. Rewriting the operator,

$$
\begin{aligned}
\Theta_{q,p}(f) &= \sum_{j=0}^{p} c_j^p \tilde{S}_N^p(f) \\
&= \sum_{j=0}^{p} c_j^p \binom{N+p}{p+1}^{-1} S_N^p(f) \\
&= \sum_{j=0}^{p} c_j^p \binom{N+p}{p+1}^{-1} \sum_{n=1}^{N} \binom{N-n+p-1}{p-1} S_n^0 \\
&= \sum_{j=0}^{p} c_j^p \binom{N+p}{p+1}^{-1} \sum_{n=1}^{N} \binom{N-n+p-1}{p-1} (f^n(x_0)-x_0) \\
&= \sum_{j=0}^{p} \sum_{n=1}^{N} c_j^p \binom{N+p}{p+1}^{-1} \binom{N-n+p-1}{p-1} (f^n(x_0)-x_0) \\
&= \sum_{j=0}^{p} \frac{c_j^p}{A_N} \sum_{n=1}^{N} W_n^N (f^n(x_0)-x_0), \\
&= \sum_{j=0}^{p} \frac{p(p+1)c_j^p}{A_N} \sum_{n=1}^{N} W_n^N (f^n(x_0)-x_0),
\end{aligned}
$$

where

$$A_N := (N+p)(N+p-1)(N+p-2)\cdots N$$

and

$$W_n^N := (N-n+p-1)(N-n+p-2)\cdots(N-n+2).$$

Now, let's take the derivative of the operator. Let $p$ be a fixed finite number. Need to show that for any given $p \le q$, the sum of the derivative converges. Since $p$ is a fixed finite number, it's enough to show the convergence of the inner sum. Let $D$ denote the derivative with respect to $x$ and $\cdot$ the derivative with respect to the parameter $\mu$. Set $y := h^{-1}(x_0)$. Also, create a function $w(s) \in C^m(\mathbb{R})$ such that $w(\frac{n}{N}) = W_n^N$.

$$
\begin{aligned}
&\frac{1}{A_N} \sum_{n=1}^{N} W_n^N \frac{d}{d\mu}(f^n(x_0)-x_0) \\
&= \frac{1}{A_N} \sum_{n=1}^{N} W_n^N Dh \circ R^n \circ h^{-1}(x_0) \sum_{l=0}^{n-1} \left(Dh^{-1} \circ h\dot{f} \circ f^{-1} \circ h\right) \circ R^l \circ h^{-1}(x_0) \\
&= \frac{1}{A_N} \sum_{n=1}^{N} W_n^N Dh \circ R^n(y) \sum_{l=0}^{n-1} \left(Dh^{-1} \circ h\dot{f} \circ f^{-1} \circ h\right) \circ R^l(y) \\
&= \frac{1}{A_N} \sum_{n=1}^{N} W_n^N Dh(y+n\theta) \sum_{l=0}^{n-1} \left(Dh^{-1} \circ h\dot{f} \circ f^{-1} \circ h\right) \circ R^l(y) \\
&= \frac{1}{A_N} \sum_{n=1}^{N} W_n^N \sum_{k\in\mathbb{Z}} 2\pi i k \widehat{h}_k e^{2\pi i k(y+n\theta)} \sum_{l=0}^{n-1} \left(Dh^{-1} \circ h\dot{f} \circ f^{-1} \circ h\right) \circ R^l(y) \qquad \text{(by Fourier)} \\
&= \frac{1}{A_N} \sum_{n=1}^{N} W_n^N \sum_{k\in\mathbb{Z}} 2\pi i k \widehat{h}_k e^{2\pi i k y} e^{2\pi i k n\theta} \sum_{l=0}^{n-1} \left(Dh^{-1} \circ h\dot{f} \circ f^{-1} \circ h\right) \circ R^l(y) \\
&= \frac{1}{A_N} \sum_{k\in\mathbb{Z}} 2\pi i k \widehat{h}_k e^{2\pi i k y} \sum_{n=1}^{N} W_n^N e^{2\pi i k n\theta} \sum_{l=0}^{n-1} \left(Dh^{-1} \circ h\dot{f} \circ f^{-1} \circ h\right) \circ R^l(y) \\
&= \frac{1}{A_N} \sum_{k\in\mathbb{Z}} 2\pi i k \widehat{h}_k e^{2\pi i k y} \sum_{n=1}^{N} \left(\int_{\mathbb{R}} w(s) e^{2\pi i N(k\theta-n)s} N ds\right) \sum_{l=0}^{n-1} \left(Dh^{-1} \circ h\dot{f} \circ f^{-1} \circ h\right) \circ R^l(y) \qquad \text{(by Poisson)}
\end{aligned}
$$

Let $\gamma := 2\pi N(k\theta - n)$. Integrating by parts, we obtain

$$\left|\int_{\mathbb{R}} w(s) e^{2\pi i N(k\theta-n)s} N ds\right| = \left|\frac{1}{\gamma^m} \int_0^1 w^{(m)}(s) e^{i\gamma s} ds\right| \le |\gamma|^{-m} \|w^{(m)}\|_1.$$

Also, note that

$$\left|\sum_{l=0}^{n-1} \left(Dh^{-1} \circ h\dot{f} \circ f^{-1} \circ h\right) \circ R^l(y)\right| \le an$$

for some constant $a$. Hence,

$$\left|\frac{1}{A_N}\sum_{n=1}^{N} W_n^N \frac{d}{d\mu}(f^n(x_0)-x_0)\right| \leq \frac{a}{A_N}\sum_{k\in\mathbb{Z}}\left|2\pi k\widehat{h}_k e^{2\pi iky}\right|\sum_{n=1}^{N} n|\gamma|^{-m}\|w^{(m)}\|_1$$
$$\leq \frac{2\pi a}{A_N}\|w^{(m)}\|_1\sum_{k\in\mathbb{Z}}|k|\left|\widehat{h}_k\right|\sum_{n=1}^{N} n|\gamma|^{-m}$$
$$= \frac{a\|w^{(m)}\|_1}{A_N(2\pi N)^{m-1}}\sum_{k\in\mathbb{Z}}|k|\left|\widehat{h}_k\right|\sum_{n=1}^{N} n|k\theta-n|^{-m}$$

Using the Diophantine property of $\theta$,

$$\sum_{n=1}^{N} n|k\theta-n|^{-m} \leq \sum_{n=1}^{N}|k\theta-n|^{-m}(|n-k\theta|+|k\theta|) \leq C_m(1+k\theta)$$

for some constant $C_m$.

$$\frac{a\|w^{(m)}\|_1}{A_N(2\pi N)^{m-1}}\sum_{k\in\mathbb{Z}}|k|\left|\widehat{h}_k\right|\sum_{n=1}^{N} n|k\theta-n|^{-m} \leq \frac{a\|w^{(m)}\|_1}{A_N(2\pi N)^{m-1}}\sum_{k\in\mathbb{Z}}|k|\left|\widehat{h}_k\right|C_m(1+|k\theta|)$$
$$\leq \frac{aC_m\|w^{(m)}\|_1}{A_N(2\pi N)^{m-1}}\sum_{k\in\mathbb{Z}}|k|C_r|k|^{-r}\|h\|_{C^r}(1+|k\theta|) \qquad \text{(see Remark)}$$
$$= \frac{aC_mC_r\|w^{(m)}\|_1\|h\|_{C^r}}{A_N(2\pi N)^{m-1}}\sum_{k\in\mathbb{Z}}|k|^{-r+1}(1+|k\theta|)$$

Remark. Finite regularity is enough. $|\widehat{h}_k| \leq C_r|k|^{-r}\|h\|_{C^r}$ for some constant $C_r$.
The sum will converge if $r > 3$. Q.E.D.