Goal: show that the sum and the derivative for DSY algorithm commute by proving that the sum of derivatives converges.

By Fourier, $h(\theta) = \sum_{k\in\mathbb{Z}} \widehat{h}_k e^{2\pi i k\theta}$, so $Dh(\theta) = \sum_{k\in\mathbb{Z}} \widehat{h}_k 2\pi i k e^{2\pi i k\theta}$. Let $\theta = h^{-1}(y)$.

$$\frac{1}{A_L}\sum_{N=1}^{L} w\left(\frac{N}{L}\right) Dh\circ R^N\circ h^{-1}(y)\sum_{j=0}^{N-1}(Dh^{-1}\circ h\dot{f}\circ f^{-1}\circ h)\circ R^j(\theta)$$
$$= \frac{1}{A_L}\sum_{N=1}^{L} w\left(\frac{N}{L}\right) Dh(\theta+\omega N)\sum_{j=0}^{N-1}(Dh^{-1}\circ h\dot{f}\circ f^{-1}\circ h)\circ R^j(\theta)$$
$$= \frac{1}{A_L}\sum_{N=1}^{L} w\left(\frac{N}{L}\right) \sum_{k\in\mathbb{Z}}\widehat{h}_k 2\pi i k e^{2\pi i k(\theta+\omega N)}\sum_{j=0}^{N-1}(Dh^{-1}\circ h\dot{f}\circ f^{-1}\circ h)\circ R^j(\theta)$$

Let $s = \frac{t}{L}$. Then, $Lds = dt$, $t = sL$. By Poisson Lemma,

$$= \frac{1}{A_L}\sum_{N=1}^{L}\sum_{k\in\mathbb{Z}}\widehat{h}_k 2\pi i k e^{2\pi i k\theta}\int_{\mathbb{R}} w(s)e^{2\pi i(k\omega-N)sL}Lds\sum_{j=0}^{N-1}(Dh^{-1}\circ h\dot{f}\circ f^{-1}\circ h)\circ R^j(\theta)$$
$$= \frac{1}{A_L}\sum_{k\in\mathbb{Z}}\widehat{h}_k 2\pi i k e^{2\pi i k\theta}\sum_{N=1}^{L}\int_{\mathbb{R}} w(s)e^{2\pi i(k\omega-N)sL}Lds\sum_{j=0}^{N-1}(Dh^{-1}\circ h\dot{f}\circ f^{-1}\circ h)\circ R^j(\theta)$$

Let $\gamma = 2\pi(k\omega - N)L$.

Integrating by parts, $\left|\int_{\mathbb{R}} w(s)e^{i\gamma s}\right| = \left|\frac{1}{\gamma^m}\int_0^1 w^{(m)}(s)e^{i\gamma s}ds\right| \le |\gamma|^{-m}\|w^{(m)}\|_1$.

Also, $\left|\sum_{j=0}^{N-1}(Dh^{-1}\circ h\dot{f}\circ f^{-1}\circ h)\circ R^j(\theta)\right| \le aN$, for some constant a. Thus,

$$\left|\frac{1}{A_L}\sum_{N=1}^{L} w\left(\frac{N}{L}\right) Dh\circ R^N\circ h^{-1}(y)\sum_{j=0}^{N-1}(Dh^{-1}\circ h\dot{f}\circ f^{-1}\circ h)\circ R^j(\theta)\right|$$
$$\le \frac{L}{A_L}\sum_{k\in\mathbb{Z}}\left|\widehat{h}_k 2\pi i k e^{2\pi i k\theta}\right|\sum_{N=1}^{L}|\gamma|^{-m}\|w^{(m)}\|_1 aN$$

Note that $\frac{A_L}{L} \to \int_0^1 w(s)ds \neq 0$ as $L\to\infty$, so $\frac{L}{A_L}\le C_w$ for some constant C_w.

Plug back in γ.

$$\le 2\pi a C_w\|w^{(m)}\|_1\sum_{k\in\mathbb{Z}}|\widehat{h}_k||k|\sum_{N=1}^{L}|2\pi(k\omega-N)L|^{-m}N$$
$$= 2\pi a C_w\|w^{(m)}\|_1(2\pi L)^{-m}\sum_{k\in\mathbb{Z}}|\widehat{h}_k||k|\sum_{N=1}^{L}|k\omega-N|^{-m}N$$

Let $\tilde{C}_w = 2\pi a C_w\|w^{(m)}\|_1(2\pi)^{-m}$.

$$= \tilde{C}_w L^{-m}\sum_{k\in\mathbb{Z}}|\widehat{h}_k||k|\sum_{N=1}^{L}|k\omega-N|^{-m}N$$

Note that $\sum|k\omega-N|^{-m}N \le \sum|k\omega-N|^{-m}\left(|N-k\omega|+|k\omega|\right)\le C_m(1+|k\omega|)$ for some C_m.

$$\le \tilde{C}_w C_m L^{-m}\sum_k|\widehat{h}_k||k|(1+|k\omega|)$$

Remark. Finite regularity is enough. $|\widehat{h}_k|\le C_r|k|^{-r}\|h\|_{C^r}$ for some constant C_r.

$$\le \tilde{C}_w C_m L^{-m}\sum_k C_r|k|^{-r}\|h\|_{C^r}|k|(1+|k\omega|)$$
$$= \tilde{C}_w C_m C_r L^{-m}\|h\|_{C^r}\sum_k|k|^{-r+1}(1+|k\omega|)$$

This will converge if $r>3$. Q.E.D.