Social Networks 32 (2010) 221–233

Contents lists available at ScienceDirect

Social Networks

journal homepage: www.elsevier.com/locate/socnet

A puzzle concerning triads in social networks: Graph constraints and the triad census

Katherine Faust*

Department of Sociology and Institute for Mathematical Behavioral Sciences, University of California, Irvine, Irvine, CA 92697-5100, USA

ARTICLE INFO

Keywords:
Triads
Triad census
Dyad census
Graph constraints
Social network

ABSTRACT

Evidence from many sources shows that triadic tendencies are important structural features of social networks (e.g. transitivity or triadic closure) and triadic configurations are the basis for both theoretical claims and substantive outcomes (e.g. strength of weak ties, tie stability, or trust). A contrasting line of research demonstrates that triads in empirical social networks are well predicted by lower order graph features (density and dyads), accounting for around 90% of the variability in triad distributions when comparing different social networks (Faust, 2006, 2007, 2008). These two sets of results present a puzzle: how can substantial triadic tendencies occur when triads in empirical social networks are largely explained by lower order graph features? This paper provides insight into the puzzle by considering constraints that lower order graph features place on the triad census. Taking a comparative perspective, it shows that triad censuses from 159 social networks of diverse species and social relations are largely explained by their lower order graph features (the dyad census) through formal constraints that force triads to occur in narrow range of configurations. Nevertheless, within these constraints, a majority of networks exhibit significant triadic patterning by departing from expectation.

1. Triads

Triads, triples of entities and the ties between them, are theoretically, substantively, and formally important features of social networks. Both the overall structure of a network (for example, whether it is polarized into factions or configured as a hierarchy) and its small scale characteristics (density, degree distributions, and dyadic arrangements) affect, indeed to great extent determine, possible triadic patterns that can occur.

Triads are theoretically important because social arrangements are possible with three entities that are not possible with individuals or pairs. For example, as Simmel (1950) observed, mediation, *tertius gaudens*, and divide and rule strategies are only possible with three (or more) actors. In addition, triads are critical for social configurations such as brokerage (Gould and Fernandez, 1989), network closure (Coleman, 1988), hierarchy (Krackhardt, 1994; Landau, 1951), the possibility of exclusion in exchange networks (Skvoretz and Willer, 1993), structural holes (Burt, 1992), forbidden triads and bridging weak ties in the strength of weak ties argument (Granovetter, 1973), the distinction between direct and generalized exchange (Ekeh, 1974; Bearman, 1997), ridge structures in networks (Friedkin, 1998), and bystander effects in dynamic network formation (Chase et al., 2002; Skvoretz et al., 1996).

Extensive research over the last four decades has studied triads in social networks, primarily focusing on positive interpersonal sentiments between humans (Brewer and Webster, 1999; Davis and Leinhardt, 1972; Friedkin, 1998; Hallinan, 1974a,b; Hallinan and Felmlee, 1975; Holland and Leinhardt, 1970, 1973, 1976, 1979; Johnsen, 1985, 1986, 1989a,b, 1998; Leinhardt, 1972). Much of the early research in this vein related observed triadic patterns to specific global structural forms, such as clusterability, the tendency for ties to occur within rather than between subgroups (Cartwright and Harary, 1956; Davis, 1967, 1970), ranked clusters, the tendency for directed ties to be consistently oriented between subgroups, or transitivity, the tendency for presence of $i \rightarrow j$ and $j \rightarrow k$ ties to be accompanied by the $i \rightarrow k$ tie (Holland and Leinhardt, 1971).

Empirical research has consistently found substantial triadic effects in human's interpersonal relations, including transitivity in friendships (Hallinan, 1974a; Holland and Leinhardt, 1971), developmental tendencies toward triadic transitivity of friendships (Leinhardt, 1972), increasing transitivity of interpersonal relations over time in groups of newly acquainted individuals (Doreian et al., 1996; Schaefer et al., 2009), relatively higher agreement between pairs of individuals embedded in complete triads (Krackhardt and Kilduff, 2002), more suicide ideation by adolescents with intransitive friendships (Bearman and Moody, 2004), and a relationship between local triad patterns and psychological dispositions (Kalish and Robins, 2006), to name a few.

* Tel.: +1 949 824 9383; fax: +1 949 824 4717.
E-mail address: kfaust@uci.edu.

0378-8733/$ – see front matter
doi:10.1016/j.socnet.2010.03.004

The importance of triads extends beyond positive interpersonal sentiments to quite different kinds of social relations, such as win-loss encounters in tournaments, in which relations are generally asymmetric, or observations of co-presence or association, in which relations are necessarily mutual or null. Given the variety of social relations that occur between individuals, it is reasonable to expect that different triadic patterns will be found in different kinds of social relations. For example, it seems reasonable to expect that triadic patterns in friendships between humans will differ from patterns of wins and losses between fighting patas monkeys, and that both of these might differ from patterns of associations among dolphins. Although these seem sensible expectations, perhaps outright obvious ones, contrasts among triadic patterns for varieties of social relations and species have yet to be subjected to extended systematic empirical investigation. This is so even though associations among triples are clearly important for many social processes, such as emergence of dominance hierarchies (Chase et al., 2003; Skvoretz et al., 1996) and patterned associations and subgrouping of individuals (Freeman, 1992).

A different line of investigation, concerning the formal, graph theoretic, properties of networks, has demonstrated that many network features, such as degree centralization, connectivity, and subgrouping are necessarily bounded or limited by lower order graph features, such as size, density, and degree distributions (Anderson et al., 1999; Butts, 2006; Holland and Leinhardt, 1973). However, comparatively little attention has been devoted to the formal properties of triads, such as the constraining effects that lower order graph features place on triadic outcomes. Faust (2007, 2008) demonstrated empirically that network density and the dyad census "explain" a large percent of the distribution of the triad census, and those results provide one point of departure for the current paper.

These two lines of investigation, substantive importance of triadic configurations and graph theoretic bounding effects in networks, present a puzzle: how can substantial triadic tendencies occur when triad distributions are largely explained by lower order graph properties? The general argument pursued in this paper is that the triad census observed in a social network is the result of both triadic substantive tendencies, which might differ among kinds of social relations, and the constraints that formal properties of the graph (such as its size, density, and dyad census) place on possible triadic outcomes. Together, these two forces, substantive and formal, combine to result in the triad patterns we observe in empirical social networks and thus explain the seemingly inconsistent results noted above and documented in the results presented below.

These observations suggest that the empirically realized range of triadic outcomes and the bounding effects of lower order graph features will be most apparent when triads can be compared across quite different kinds of social relations that vary in their graph theoretic properties. Such comparisons require both an empirical base with a variety of social relations (such as interpersonal sentiments, agonistic displays, or co-participation) and a means for comparing their triad structures. Thus, in contrast to Hallinan's (1972) argument that a substantively homogeneous pool of relations is required to detect common structural tendencies in a particular kind of relation, for the current problem a heterogeneous collection is needed. Analyses presented in this paper use the triad census (defined below) as the means for comparison and examine the triad censuses for a sample of 159 social networks from a variety of social relations and animal species (described in Section 3).

Following introduction of basic concepts and data, the paper is organized into two sections of analysis and results. The first analysis section focuses on triad censuses from a variety of social networks and directly assesses both the extent to which the triad censuses can be accounted for by the dyad census, and whether or not specific triad censuses depart from expectation given their dyad censuses. It also introduces the idea of constraint on triad census outcomes. The second analysis section broadens perspective to consider the same issues with respect to a theoretical space spanning expected outcomes for the triad census. In both analysis sections the motivating theme is the interplay between substantive tendencies for triadic interdependence, on the one hand, and formal graph constraints on triad outcomes, on the other. The paper concludes with a general discussion of graph constraints, variability in the triad census across different kinds of social relations, and open areas for further investigation.

2. Social network notation and concepts

A binary relation, representing a social network, consists of set of entities (actors, individuals) and a relation measured between pairs of entities. The relation can be represented as a graph or directed graph $G(N,E)$ consisting of a set of g nodes N and a set of arcs E between pairs of nodes, or as a sociomatrix $\mathbf{X}$ where the entry x_{ij} records the state of the tie from actor i to actor j. Analyses in this paper are restricted to relations where x_{ij} is dichotomous, taking on values of 0 (absent) or 1 (present) and where self-ties (diagonal entries) are undefined.

2.1. Network density and mean degree

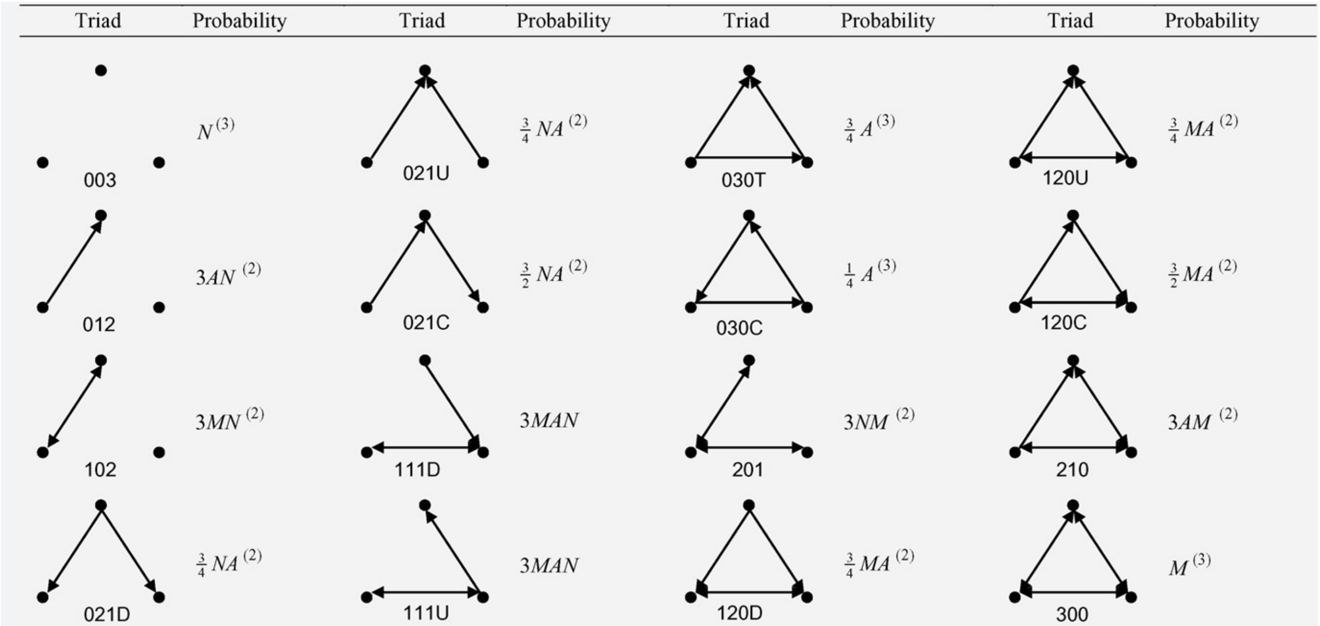
Several network properties are useful for understanding triads. First, network density $\Delta = \sum_{\substack{i=1 \\ i \neq j}}^{g} \sum_{\substack{j=1 \\ j \neq i}}^{g} x_{ij}/g(g-1)$ is the proportion of possible arcs that are present in the graph. Importantly, density is mathematically related to the mean in- and outdegree as $\Delta = \left(\sum_{i=1}^{g} x_{i+}/g\right)/(g-1) = \left(\sum_{j=1}^{g} x_{+j}/g\right)/(g-1)$, where x_{i+} indicates the outdegree of node i and x_{+j} indicates the indegree of node j.

2.2. Subgraphs

A dyad in a directed graph is a subgraph consisting of a pair of nodes and the states of the possible arcs between them. In a directed graph of g nodes there are $\binom{g}{2} = g(g-1)/2$ dyads, each of which can be in one of three states or isomorphism classes: mutual (M), asymmetric ignoring arc direction (A), or null (N). The dyad census for a network records the number of dyads in each of the three isomorphism classes and is referred to as its MAN count.

A triad in a directed graph is a subgraph of three nodes and the states of the possible arcs between the nodes. In a directed graph there are $\binom{g}{3} = g(g-1)(g-2)/6$ triads, each of which must be in one of 16 isomorphism classes, as shown in Table 1. In this table triads are labeled in the standard way, giving the number of mutual (M), asymmetric (A), and null (N) dyads they contain, along with a letter indicating directionality of arcs when there are two or more non-isomorphic triads with the same MAN count (Holland and Leinhardt, 1970). The triad census for a network reports the frequency of each of the 16 triad isomorphism classes and is summarized in a 16 element vector, $\mathbf{t} = (t_1, t_2, \ldots, t_{16})$, where t_i is the count of the number of triads in isomorphism class i. The triad census is an especially useful summary of a social network since numerous properties of the network can be calculated from it, including network density, the dyad census, and indices of transitivity or intransitivity (Holland and Leinhardt, 1976).

Table 1
Sixteen isomorphism classes for triads in a directed graph with notation indicating the number of mutual, asymmetric, and null dyads and equations for calculating triad expectations under U|MAN.



Triad	Probability	Triad	Probability	Triad	Probability	Triad	Probability
003	$N^{(3)}$	021U	$\frac{3}{4}NA^{(2)}$	030T	$\frac{3}{4}A^{(3)}$	120U	$\frac{3}{4}MA^{(2)}$
012	$3AN^{(2)}$	021C	$\frac{3}{2}NA^{(2)}$	030C	$\frac{1}{4}A^{(3)}$	120C	$\frac{3}{2}MA^{(2)}$
102	$3MN^{(2)}$	111D	$3MAN$	201	$3NM^{(2)}$	210	$3AM^{(2)}$
021D	$\frac{3}{4}NA^{(2)}$	111U	$3MAN$	120D	$\frac{3}{4}MA^{(2)}$	300	$M^{(3)}$

Numerators for triad expectation given dyad census (MAN). The denominators are $\binom{g}{2}^{(3)}$ where $z^{(k)} = z(z-1)\cdots(z-k+1)$ (Holland and Leinhardt, 1970, 1976).

2.3. Conditional uniform graph distributions

Evaluating the presence or absence of triadic tendencies in a network requires a suitable baseline for comparison, and conditional uniform graphs are often used for that purpose. Conditional uniform graphs both establish an expectation for a network index and provide a basis for simulating random graphs to give a sense of the variability in outcomes, conditional on specified graph features. The following analyses use the conditional uniform graph distribution, conditioning on the dyad census MAN. This distribution, referred to as U|MAN, has as its sample space all graphs with a specified count of M, A, and N dyads (Holland and Leinhardt, 1970, 1976; Wasserman and Faust, 1994).

The U|MAN distribution also necessarily conditions on network density since $\Delta = p_M + (1/2)p_A$, where p_M and p_A are the proportions of mutual and asymmetric dyads, respectively. Although other conditional distributions could be used, for example the uniform distribution conditional on the in- or outdegrees (Wasserman, 1977) or conditional on the in- and outdegrees plus the dyad census (Snijders, 1991), U|MAN is useful because triad census expectations can be calculated directly from it and it is relatively straightforward to simulate random graphs using U|MAN. In addition, prior research has shown that triad censuses for several different samples of networks closely resemble what is expected given the dyad census (Faust, 2006, 2007).

Equations for triad expectations under U|MAN were derived by Holland and Leinhardt (1970, 1976) and are presented in Table 1. Under U|MAN, the probability of a particular triad isomorphism class is a function of the probability of each of the three dyads in the triad, taking into account the number of ways that the dyads can be arranged to form the triad. For a given network, these expectations are summarized in a 16 element vector, $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_{16})$ where μ_i is the expected number of triads in isomorphism class i.

The U|MAN distribution can also be used to simulate random graphs, as described by Holland and Leinhardt (1973). For a particular network, random graphs are generated by randomly assigning dyads to be mutual, asymmetric, or null, in numbers equal to the MAN count for the network. Direction of the arc in an asymmetric dyad is randomly determined with equal probability of being $i \to j$ or $j \to i$. The network statistic of interest (for example the triad census) is found for each random graph and the distribution of these statistics provides a reference distribution for the observed quantity. In the following applications, 100 random graphs were simulated for each observed network.

There are several advantages of simulated random graphs. First, results provide a distribution of outcomes which gives a sense of variability in the statistic of interest. Second, random graphs do not require assumptions about the shape of the distribution of the statistic of interest. Moreover, distributions from simulated random graphs are useful in comparative perspective, especially when variability occurs along multiple dimensions (as shown in Section 5.4).

3. Data

To investigate triadic patterns in different kinds of social relations a sample of 159 social networks representing a variety of social relations and animal species is used. A heterogeneous sample is intended to survey both a range of formal graph arrangements (dyad and triad distributions) and substantive tendencies (kinds of triadic interdependencies). Data were compiled from published sources (journals and books) and standard social network data archives (for example UCINET). To be included in the sample a network had to be measured on a complete group or population (not a sample from a group or a collection of ego-centered networks). The network also had to be measured on individuals rather than collective entities.

The animal species in the sample include ants, baboons, bighorn sheep, bison, caribou, cattle, chimpanzees, colobus monkeys, dolphins, finches, goats, hens, howler monkeys, humans, kangaroos, macaques, orangutans, patas monkeys, ponies, red deer, rhesus monkeys, silvereye birds, sparrows, vervet monkeys, wasps, and wolves. The social relations include both sociometric measure-

Table 2
Descriptive statistics for 159 social networks.

		Network size	Density	Proportion mutual	Proportion asymmetric	Proportion null
Mean		27.04	0.30	0.13	0.34	0.53
Median		21.00	0.29	0.07	0.26	0.56
Std. Deviation		28.55	0.19	0.15	0.25	0.28
Minimum		4	0.01	0.00	0.00	0.00
Maximum		217	0.86	0.81	1.00	0.98
Percentiles	25	14.00	0.14	0.03	0.16	0.33
	50	21.00	0.29	0.07	0.26	0.56
	75	28.00	0.44	0.19	0.50	0.77
N		159	159	159	159	159

ments (reports from humans on friendship, advice, satisfying interactions, workmate preferences, rejection, esteem), cognitive social structure data (human reports of their perceptions of advice relations among members of an office), records of transactions (email messages sent between humans) and observations of interactions (outcomes of agonistic bouts, grooming, non-agonistic social acts, threats, winner and loser in encounters, and observed associations). The sample combines networks used in earlier papers by Faust (2006, 2007), and thus should not be considered an independent replication of results presented in those papers.

Descriptive statistics for the 159 networks are presented in Table 2. The networks range in size from 4 to 217 individuals and in density from 0.01 to 0.86. There also is considerable variability in the dyadic properties of the networks. Two networks have only asymmetric dyads (victories in fights between finches and peck dominance between hens) and three networks have only mutual or null dyads (co-observations of kangaroos, dolphins, or howler monkeys).

More detailed information for a subset of eight example networks is presented in Table 3. These networks were deliberately selected to represent extreme local structural properties (density and dyad census proportions) and distinctive triadic patterns, and provide points of reference for interpreting results using the entire sample. Of particular interest are: victories in fights between patas monkeys, which has mostly asymmetric dyads; co-observation of howler monkeys, with only mutual and null dyads; social grazing between cows, which forms a low density network; and grooming between chimpanzees in a dense network.

4. Analyses part one: empirical triad censuses, expectations, and rough bounds

As noted above, this paper presents puzzling results along with a possible explanation. Thus, analyses center on three issues: (1) showing that triad censuses from empirical networks are close to expectation given the dyad census; (2) demonstrating that many social networks have substantial triadic patterning, as seen in departure from expectation given the dyad census; and (3) establishing the extent of constraint on triad census outcomes and arguing that this constraint can account for the apparently inconsistent results. This first section of analyses focuses on triad censuses from 159 social networks.

4.1. Empirical triad censuses

The 16 element triad census was found for each of the 159 social networks in the sample. Since networks vary greatly in size, and therefore in the number of triads they contain, each triad census is expressed in relative frequencies in further analyses. Triad censuses for eight example networks are presented in Table 3. In this table it is clear that triad censuses vary considerably across these networks. The triad census for victories in fights between patas monkeys is largely composed of the all transitive 030T triad (in this network 81% of triads are 030T), whereas social grazing between cows has an abundance of all null (003) triads (this is a low density network with many null dyads). Initiation of grooming between chimpanzees is notable in that many of its triads are all mutual (300) or missing a single arc (210) (this network is among the densest in the sample). The network of observed associations among howler monkeys has only four kinds of triads, 003, 102, 201, and 300, consistent with the fact that it has only mutual and null dyads.

4.2. Resemblance of empirical and expected triad censuses

We turn now to the question of resemblance of triads in empirical social networks to their expectations under U|MAN. For each of the 159 empirical networks, the expected triad census was calculated from its dyad census using the U|MAN distribution (equations in Table 1). Fig. 1 shows graphs of the observed and expected triad proportions for eight example networks. Overall, the proximity of observed points (filled circles) to their expectations (solid lines) shows that observed triad proportions closely track their expectations under U|MAN.

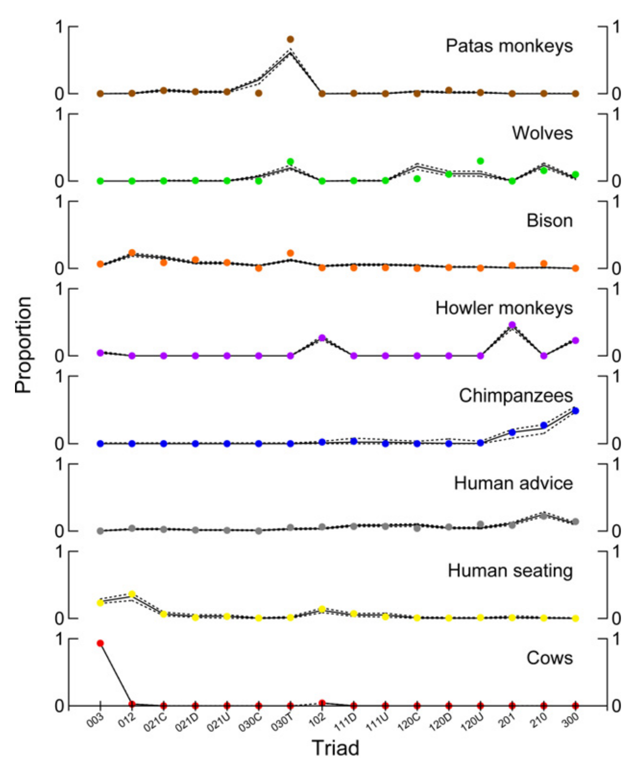


Fig. 1. Triad proportions for eight example social networks. Empirical points are in filled circles, solid lines show triad expectation under U|MAN, dashed lines show minimum and maximum triad proportions in conditional random graphs.

Table 3
Size, density, dyad and triad censuses (in percents) for eight example networks[a].

Species	Relation	Size	Density	M	A	N	003	012	102	021D	021U	021C	111D	111U	030T	030C	201	120D	120U	120C	210	300
Patas monkeys	Win fights	18	0.49	3	93	4		0		3	3	5	0		81	1		5	2	0	0	
Wolves	Deference (low posture)	16	0.67	35	64	1				1	0	0	1	1	29			10	30	3	16	10
Bison	Aggressive behaviors	25	0.37	10	55	35	6	23	1	13	9	9	1	1	23	0	5	1	0	0	7	0
Howler monkeys	Co-observation	17	0.63	63		38	4		27								46					23
Chimps	Initiates grooming	9	0.86	81	11	8			2				4				17		1		27	49
Humans	Perceived advice	21	0.66	48	36	16	0	4	6	2	1	2	7	7	5	0	9	6	10	4	22	14
Humans	Seating partner choice	14	0.23	10	26	64	23	36	14	1	3	6	7	2	1	1	1	1	1	1	1	
Cows	Social grazing	29	0.02	1	1	98	93	2	4		0		0	0			0				0	

[a] True zeros are indicated by blanks.

The resemblance of empirical and expected triad censuses for the entire sample of 159 social networks is quantified using canonical redundancy (Lambert et al., 1988; Stewart and Love, 1968). In general, canonical redundancy $\mathbf{R}^2_{Y \cdot X}$ expresses the proportion of variance in linear combinations of a set of dependent variables, **Y**, explained by linear combinations of a set of independent variables, **X**. $\mathbf{R}^2_{Y \cdot X}$ is calculated from matrices of correlations between columns of **X** and columns of **Y**, $\mathbf{R}_{XY}$ and $\mathbf{R}_{YX}$, and correlations between columns of **X**, $\mathbf{R}_{XX}$, as:

$$\mathbf{R}^2_{Y \cdot X} = \frac{1}{k} trace(\mathbf{R}^{-1}_{XX} \mathbf{R}_{XY} \mathbf{R}_{YY}) \tag{1}$$

where k is the number of variables in **Y** and $\mathbf{R}^{-1}_{XX}$ is the inverse of $\mathbf{R}_{YX}$.

In the current application, a matrix of the empirical triad censuses,

$$\underset{159 \times 16}{\mathbf{T}} = \begin{bmatrix} \mathbf{t}_1 \\ \vdots \\ \mathbf{t}_{159} \end{bmatrix},$$

is the dependent variable set and a matrix of the expected triad censuses,

$$\underset{159 \times 16}{\boldsymbol{M}} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \vdots \\ \boldsymbol{\mu}_{159} \end{bmatrix},$$

is the independent variable set. The variables are the triad proportions, in columns of the matrices. Canonical redundancy is then interpreted as the proportion of variance in linear combinations of the empirical triad censuses explained by linear combinations of the corresponding expected triad censuses. However, since triads with the same MAN count have identical expectations under U|MAN, $\boldsymbol{M}$ is singular, and the set of triads for comparison must be reduced to ten triads with unique MAN counts.

Canonical redundancy comparing the empirical triad censuses with their expectations under U|MAN yields $\mathbf{R}^2_{T \cdot \boldsymbol{M}} = 0.849$ for the sample of 159 networks. This result demonstrates that triad censuses from empirical networks fairly closely resemble triad censuses expected from a network's distribution of mutual, asymmetric, and null dyads. The expectation under U|MAN accounts for around 85% of the variance in the empirical triad censuses.

This resemblance of empirical triad censuses to their expectations under U|MAN appears to suggest that the dyad censuses provide a satisfactory explanation of triad census distributions for these 159 social networks, with little room left for substantial triadic patterning beyond that expectation. But this conclusion needs to be confronted by directly assessing whether or not substantial triadic patterning occurs, above and beyond what is expected from the dyad censuses.

4.3. Departure from expectation

The presence of triadic tendencies in a social network means that there are interdependencies among triples of individuals, beyond dyadic tendencies for mutuality or asymmetry (or individual differences in sending or receiving ties). One way that triadic tendencies can be detected is by departure of an empirical triad census from its expectation under a reasonable null model. An overall index of this departure, τ^2, was proposed by Holland and Leinhardt (1978) as an "omnibus" test for presence of triadic patterning. The τ^2 statistic compares the 16 element triad census vector, **t**, for a social network with its expectation, $\boldsymbol{\mu}$, under a specific model (for example U|MAN) and provides a summary measure of the magnitude of departure from expectation. The τ^2 statistic is defined

Table 4
Singular values and percent sum of squares from singular value decomposition of 1326 triad censuses expected under U|MAN.

Dimension	Singular value	Squared singular value	Percent	Cumulative percent
1	10.534	110.965	38.865	38.865
2	8.230	67.733	23.723	62.589
3	6.706	44.970	15.751	78.340
4	4.618	21.326	7.469	85.809
5	4.013	16.104	5.640	91.449
6	3.991	15.928	5.579	97.028
7	1.780	3.168	1.110	98.138
8	1.462	2.137	0.748	98.886
9	1.309	1.713	0.600	99.486
10	1.211	1.467	0.514	100.000
Total			100.000	

as

$$\tau^2 = (\mathbf{t} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{t} - \boldsymbol{\mu}) \qquad (2)$$

where $\boldsymbol{\Sigma}$ is the covariance matrix of $\mathbf{t}$ and $\boldsymbol{\Sigma}^{-1}$ is the (pseudo)inverse of $\boldsymbol{\Sigma}$ (Holland and Leinhardt, 1978, p. 241). The τ^2 statistic for each network was calculated using the expectation from U|MAN (Table 1).

Holland and Leinhardt (1978, p. 242) observe that if $\mathbf{t}$ is normally distributed, τ^2 follows a chi square distribution with degrees of freedom equal to the rank of $\boldsymbol{\Sigma}$. This assumption is not met in the sample of 159 networks, thus referencing τ^2 to a chi square distribution would be inappropriate. Instead, τ^2 values are referenced to distributions from conditional uniform random graphs. For each of the 159 networks, 100 random graphs were generated using the U|MAN distribution (as described in Section 2.3). For each random graph the triad census was found and the τ^2 statistic calculated, yielding 100 comparison τ^2 values for each empirical triad census.

For 133 of the 159 networks (84%) the τ^2 value for the empirical network exceeded all τ^2 values for the 100 random graphs, and an additional eight networks (5%) had a τ^2 statistic that was exceeded by only one of the τ^2 values from the random graphs. Thus, between 84% and 89% of the social networks exhibit substantial triadic patterning, above and beyond what is expected from their dyad censuses. Among the eight example networks, no significant triadic tendencies were found in co-observation of howler monkeys, grooming between chimpanzees, and seating partner choices of humans.

These results show that a substantial majority of social networks in the sample possess triadic tendencies, beyond what is expected from their dyads. This finding has both substantive and methodological implications. Substantively, considerable triadic patterning in social relations indicates interdependencies among triples of individuals, beyond dyadic tendencies for mutuality or asymmetry. However, from a methodological perspective, presence of substantial triadic patterning presents a puzzle, in light of the earlier result demonstrating that triad censuses for the sample of networks are well predicted by the dyad census, accounting for around 85% of the variance in the observed triad censuses. Thus, we are confronted with a problem in reconciling these two results. A key to the puzzle lies in recognizing the amount of constraint that lower order graph features place on triadic outcomes, and how, in comparative perspective, this constraint explains or accounts for differences in triad censuses across networks of different social relations. These ideas are elaborated in the following sections.

4.4. Constraint on the triad census

Results in Table 3, presenting descriptive statistics and triad censuses for the eight example networks, hint at an important point concerning the constraint that lower order graph features (density and the dyad census) place on the triad census. Networks that have extreme density or dyad distributions have distinctive triad censuses. Since triads are composed of dyads, which in turn are made up of arcs or their absence, the distribution of triads across the 16 isomorphism classes largely depends on these lower order graph features as building blocks. In other words, the dyad census and other properties of a graph *constrain* triads that can occur: completely mutual 300 triads are rare in sparse networks; all null 003 triads are unlikely in dense networks; and networks with dyads that are overwhelmingly asymmetric will mostly have 030T and 030C triads.

For a given social network, the amount of constraint on the triad census is inversely related to the amount of variability possible in the distribution of triads across isomorphism classes in the census: the greater the constraint, the lower the variability in possible outcomes. For a given network, this constraint can be approximated using conditional uniform random graphs. Variability is then seen in the range of triad census outcomes in the random graphs. In the following analyses 100 random graphs were generated for each network in the sample, conditional on the network's dyad census MAN (as described in Section 2.3). For each random graph, the triad census was found and the proportion of triads in each isomorphism class recorded.

The dashed lines in Fig. 1 show the minimum and maximum triad proportions in the 100 random graphs generated for each of the eight example networks. On the scale from 0 to 1, it is clear that for any particular network the range of values for triad proportions is extremely restricted. The minimum and maximum are often quite close to each other. This means that for a network with a particular MAN count, triad probabilities are concentrated in narrow bands of values. Moreover, these bands differ markedly between networks, depending on the network's MAN.

A seen by locations of the filled circles on the graphs in Fig. 1, with a few exceptions, the empirical points closely track the bands of triad proportions from the random networks. Thus, comparing across networks, the dyad census "explains" or "accounts for" the triad census proportions. This is consistent with the canonical redundancy of $\mathbf{R}^2_{T\cdot\boldsymbol{M}} = 0.849$ (reported above) for the 159 observed and expected triad censuses. Moreover, without exception, the expectations for the triad census under U|MAN (solid lines in Fig. 1) fall within the ranges from the random graphs.

5. Analyses part two: a space for triad census expectations

Triad censuses for the empirical networks are obviously a sample from a larger universe of possible triad censuses. Considering this universe, and the locations of empirical triad censuses within it, provides a comparative perspective on the three intertwined issues of resemblance of triad censuses to expectation, triadic departure from expectation, and formal constraints on triad census outcomes. To pursue these issues, the following analyses construct a space encompassing expected outcomes for the triad census. This space then provides a backdrop against which to examine locations of empirical triad censuses, their expectations, and the range of triad outcomes in random graphs conditional on the dyad census.

These analyses generalize and broaden the perspective provided by the line graphs in Fig. 1, which show intervals for separate types of triads. The space described in this section is a multidimensional representation, within which the overall configuration of a triad census is located. In this space, empirical points represent the triad census as a whole (the distribution across the 16 isomorphism classes). Locations of different networks reveal contrasts between triad censuses from different social relations. In addition, outcomes from random graphs convey a sense of overall constraint in terms of the overall range of outcomes in each network's triad census.

5.1. A space of triad census expectations

A fully encompassing space for the triad census would cover the entire range of possible triad censuses outcomes. A more tractable and conceptually useful space conditions on specific lower order graph features, particularly the dyad census, and maps triad census expectations under this conditional distribution.

A space for triad census expectations under U|MAN was constructed by varying the proportions of mutual, asymmetric, and null dyads from 0.0 to 1.0, in increments of 0.02, and using all legitimate combinations (that is, $p_M + p_A + p_N = 1.0$). For each of the 1326 legitimate dyad combinations, expectation for the triad census, $\boldsymbol{\mu}$, was found using the equations in Table 1. This collection of 1326 triad censuses covers the full range of triadic outcomes, conditional on the dyad proportions, and can be viewed as spanning the space or universe of expectations for triad censuses under U|MAN. Triad census expectations were aggregated into a 1326×16 matrix, $\underset{1326\times16}{\tilde{\boldsymbol{M}}}$, in which each row represents a legitimate combination of dyad proportions and each column is one of the 16 triad isomorphism classes. Entries give the expected proportion of triads of a given type for the specific combination of dyad proportions.

Singular value decomposition is used to graphically represent the space in low dimensionality. Singular value decomposition of $\tilde{\boldsymbol{M}}$ is defined as:

$$\underset{1326\times16}{\tilde{\boldsymbol{M}}} = \underset{1,326\times16}{\mathbf{U}}\underset{16\times16}{\mathbf{D}}\underset{16\times16}{\mathbf{V}'} \tag{3}$$

where $\mathbf{U}$ is an orthogonal matrix of left singular vectors pertaining to the rows of $\tilde{\boldsymbol{M}}$ (triad census expectation from a specific dyad census), $\mathbf{V}$ is an orthogonal matrix of right singular vectors pertaining to the columns of $\tilde{\boldsymbol{M}}$ (16 triad isomorphism classes), and $\mathbf{D} = diag(\lambda_1, \lambda_2, \ldots, \lambda_K)$ is a diagonal matrix of singular values, in non-increasing order (Ben-Israel and Greville, 1974; Digby and Kempton, 1987). This decomposition yields a low rank approximation to $\tilde{\boldsymbol{M}}$, with $\mathbf{U}$ and $\mathbf{V}$ providing coordinates for k-dimensional display. A full rank solution requires K sets of singular values and singular vectors, with K equal to the rank of $\tilde{\boldsymbol{M}}$. The fit of a lower rank solution ($k<K$) is assessed as the percent sum of squares of the data explained by k sets of singular values and singular vector pairs: $100 \times \left(\sum_{l=1}^{k}\lambda_l^2 / \sum_{l=1}^{K}\lambda_l^2\right)$.

Table 4 presents the singular values, $\mathbf{D}$, and percent sum of squares from singular value decomposition of $\tilde{\boldsymbol{M}}$. This result shows that the space for triad census expectations is of relatively high dimensionality. There are 10 non-zero singular values. The first three dimensions account for 78.3% of the sum of squares.

The left or right singular vectors, weighted by the square roots of their respective singular values, are used as coordinates to display the space graphically. Fig. 2 displays the first three left singular vectors in the left three panels of the figure. In these plots each point represents a triad census expectation generated by a specific combination of dyad proportions. The figure shows that, in three dimensions, the space is roughly a triangular manifold (a surface without volume). Regions of the space are characterized by distinctive dyad proportions (as labeled in the figure), which is to be expected since dyads generate the triad expectations under U|MAN. The first and third dimensions are related, in that they both contrast triad censuses generated mostly by asymmetric dyads from those generated by null or mutual dyads. The second dimension contrasts triad censuses primarily composed of null dyads from those with mutual dyads, and thus is related to network density.

Fig. 3 displays the first three right singular vectors from singular value decomposition of $\tilde{\mathbf{M}}$. Each point in this plot represents a triad isomorphism class. The right singular vector space echoes the left singular vector space in that a roughly triangular configuration is characterized by triads with distinctive dyadic properties. Notably, the all mutual 300 triad all null 003 triad anchor opposite ends of the second right singular vector, distinguishing triads with many mutual dyads from those with many null dyads. The first and third right singular vectors contrast the all asymmetric transitive 030T triad with triads having no asymmetric dyads (triads 003, 102, 201, and 300).

These theoretical spaces provide a “skeleton” showing expectation for the triad census, conditional on the distribution mutual, asymmetric, and null dyads. However, these expectations may or may not be realized in empirical social networks, a point examined in the following sections.

5.2. Empirical triad censuses in the theoretical space of expectations

To examine where triad censuses from the 159 empirical networks fall within the space of expectations, the censuses in the aggregated matrix $\underset{159\times16}{\mathbf{T}}$ are projected into the space as supplementary points (Lebart et al., 1984). Projection of the empirical triad censuses into the theoretical space uses the equation

$$\hat{\mathbf{U}}_{(\mathbf{t})} = \mathbf{TDV}' \tag{4}$$

where $\hat{\mathbf{U}}_{(\mathbf{t})}$ gives projected left singular vectors in the space of right singular vectors $\mathbf{V}'$ defined by the 1326 expected triad censuses in $\tilde{\boldsymbol{M}}$. Thus, each empirical triad census is located as a point within the space defined by the full range of triad censuses expected under U|MAN.

The right three panels of Fig. 2 display projections of the 159 empirical triad censuses using the first three vectors of $\hat{\mathbf{U}}_{(\mathbf{t})}$. These panels parallel the left panels showing the corresponding space of expectations. To facilitate interpretation, the eight example networks are labeled in the right panels of Fig. 2.

First, notice the locations of specific empirical networks. In the upper right panel of Fig. 2, the triad census from initiation of grooming between chimpanzees is at the top of the figure. This is the densest network in the sample (density = 0.86) and it has the highest percent of mutual dyads (81%). Consistent with these graph properties, the triad census for chimpanzee grooming has a high percent of all mutual 300 triads (49%). In contrast, social grazing of cows (density = 0.02) is at the bottom of the upper right panel of Fig. 2. This network has many null dyads (98%) and thus many all null 003 triads (93%). The triad census for victories in fights between patas monkeys is on the low end of the first and third vectors (middle panel on the right of Fig. 2). This network has a high percent of asymmetric dyads and 030T transitive triads. The first and third left singular vectors contrast triad censuses of networks with many asymmetric dyads (for example victories in fights between patas monkeys) from those with many mutual or null dyads (cows grazing or observed associations between howler monkeys).

Next, notice that empirical triad censuses are not evenly spread throughout the space of triad census expectations. Most notably, the region characterized by null dyads (low end of the second vector) is well populated whereas the region characterized by mutual dyads is not. The unoccupied region of the space would contain triad censuses from dense networks, which consequently have many mutual dyads and many triads constructed from mutual dyads (such as the completely mutual 300 triad). The scarcity of such networks raises important questions about why they are rare, a topic that is pursued in the discussion.

From these results it can be seen that different kinds of social relations occupy different regions in the space of triad expectations. Notably, networks of agonistic or win/loss encounters have many asymmetric dyads and consequently reside in the region associated with asymmetry. In contrast, networks of co-observation, which necessarily have only mutual or null dyads, reside in the portion of the space characterized by triad censuses without asymmetry. A multidimensional space of triad expectations facilitates observa-

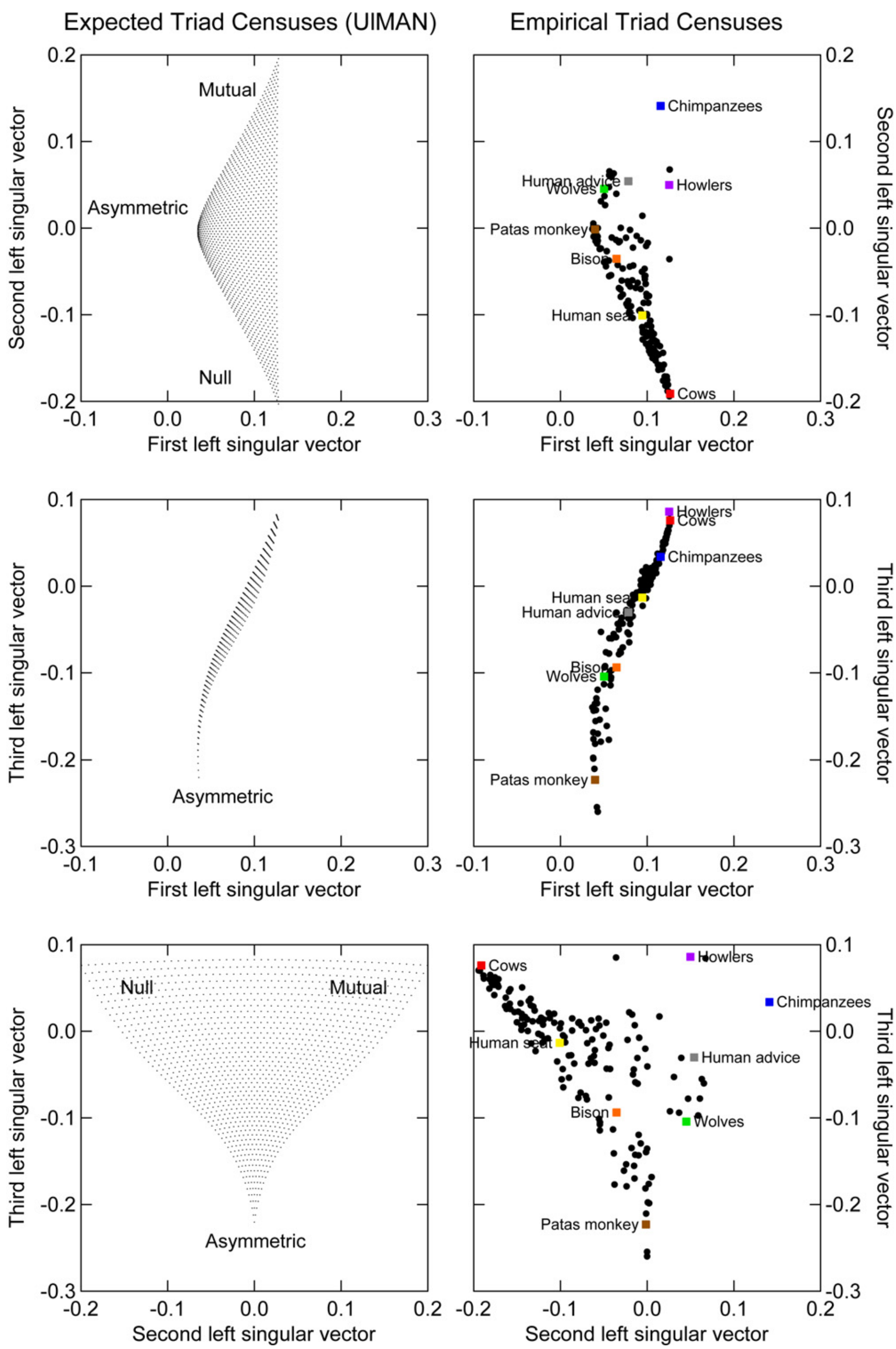


Fig. 2. First three left singular vectors from singular value decomposition of triads: 1326 triad censuses expected under U|MAN (left panels), projections of triad censuses from 159 empirical social networks (right panels). Eight example networks are labeled.

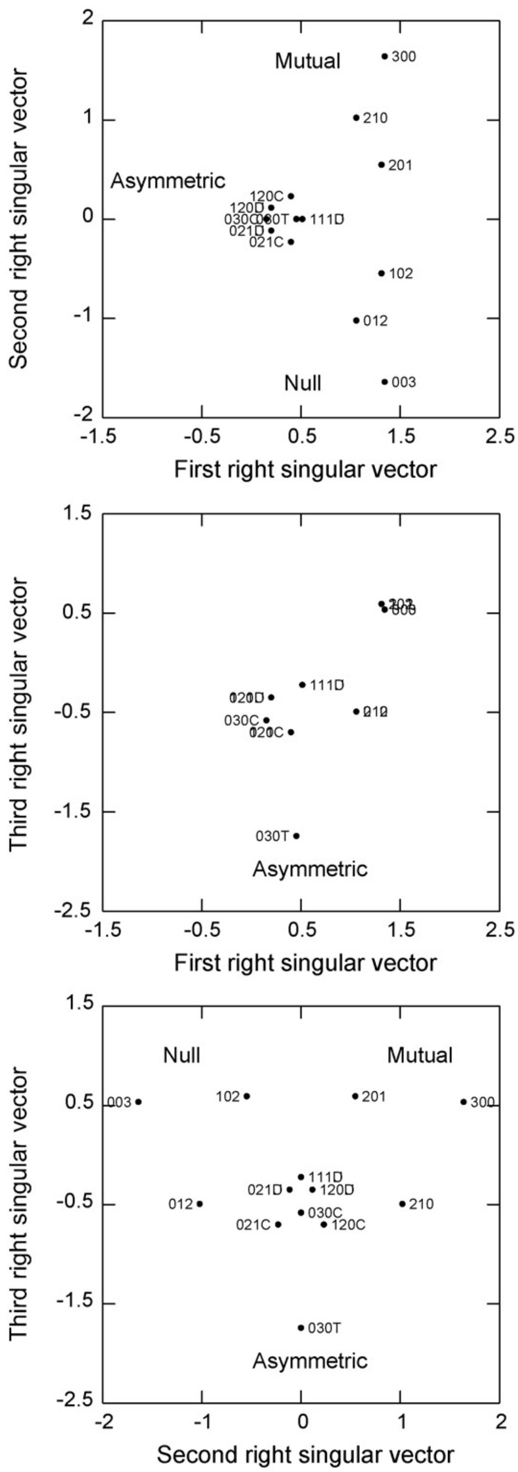


Fig. 3. First three right singular vectors from singular value decomposition of 1326 triad censuses expected under U|MAN.

tions about distribution of, and contrasts between, social relations. These observations are more difficult if made separately for the 16 types of triads (e.g., the line graphs in Fig. 1).

5.3. Resemblance of empirical and expected triad censuses, revisited

The next analysis again considers the resemblance between empirical and expected triad censuses, this time by examining how well the empirical triad censuses fit within the space of triad expectations. This fit depends on how close the point for an empirical triad census is to a point corresponding to its expectation. Therefore it is necessary to explicitly link each empirical triad census with an expectation generated by the appropriate dyad proportions. As described in Section 4.2, the 159 triad census expectations were calculated and aggregated into the matrix $\boldsymbol{M}$. Each of the 159 triad census expectations is then projected into the space as $\hat{\mathbf{U}}_{(\boldsymbol{\mu})} = \boldsymbol{M}\mathbf{D}\mathbf{V}'$ (Eq. (4)) to give a specific expectation point for each of the empirical triad censuses.

The left panels of Fig. 4 show both the empirical triad censuses and their expectations, projected into the first three dimensions of the larger space of triad expectations. Each pair of empirical and expected points is connected by a line, with a filled circle indicating the location of the empirical triad census. Since most lines are quite short, most empirical triad censuses are very close to their expectations. The largest distances occur toward the low end of the third left singular vector, indicating the arrangement of asymmetric dyads into the transitive 030T triad at levels greater than expected.

Canonical redundancy, $\mathbf{R}^2_{\hat{\mathbf{U}}_{(\mathbf{t})}\cdot\hat{\mathbf{U}}_{(\boldsymbol{\mu})}}$, (Eq. (1)) quantifies the resemblance of the observed and expected triad censuses, this time comparing locations of projected points for the observed and expected censuses, $\hat{\mathbf{U}}_{(\mathbf{t})}$ and $\hat{\mathbf{U}}_{(\boldsymbol{\mu})}$. Results for projections into 1- through 9-dimensional representations are reported in Table 5. Comparing locations of the 159 empirical triad censuses to their corresponding expected triad censuses in solutions up to four dimensions, shows that the expected triad censuses account for around 99% of the variance in locations of the empirical points. (The percent of variance falls off at higher dimensionality.) Again, this confirms the resemblance of observed triad censuses to their expectation under U|MAN, though from a more abstract perspective than the result presented earlier.

Table 5
Canonical redundancy comparing projections of empirical and expected triad censuses.

Number of dimensions	Canonical redundancy
1	0.992
2	0.996
3	0.994
4	0.991
5	0.988
6	0.983
7	0.955
8	0.939
9	0.909

5.4. Constraint on the triad census, revisited

A final analysis again examines constraint on the triad census, this time within the general space of triad expectations. To do so, triad censuses from the 159×100 random graphs are projected into the space of triad expectations (Eq. (4)). This gives 100 points corresponding to each of the 159 social networks. These points cover a distribution of locations for the triad census, given a network's MAN. The right panels of Fig. 4 show locations of the 100 points for each network as an ellipse covering 99% of the area occupied by the projected points along with a point for the mean of the random locations. Viewing the ellipses in this general space shows that, for any particular network, triad censuses from its corresponding random graphs occupy only a limited portion of the possible space and some ellipses are extremely small.

Fig. 5 presents expanded plots of ellipses for eight example networks, using the second and third left singular vectors. These plots also show the point for the empirical triad census with a line linking the empirical point to the triad census expected under U|MAN. Substantial magnification is required to display each ellipse in detail, as can be seen by the scales of the axes on the expanded plots. At this resolution, it can be seen that the empirical points for many of the example triad censuses fall outside their corresponding ellipses (labels A, B, C, E, F, and G in Fig. 5), while one is toward the edge of its ellipse (D, chimpanzee grooming), and one falls roughly in the center of its ellipse (H, deference between wolves).

Three related points are conveyed in Fig. 5. First, the location of an empirical triad census in the space of expectations is largely determined by its dyad census MAN: empirical points are close both to their respective expected points and to triad censuses from random graphs conditioning on MAN. Second, many of the example networks have triad censuses that depart from expectation: they fall outside their corresponding ellipses. Third, in comparative perspective, the dyad census determines locations for an triad census in the wider space of expectations, accounting for differences between triad censuses with different dyad distributions: most ellipses covering locations of triad censuses from U|MAN random graphs are found in different regions of the space of expectations.

As an aside to the main argument, notice that the amount of constraint on the triad census varies across networks in the sample, as seen in the variability of ellipse sizes in the right panels of Fig. 4. The larger the ellipse covering locations of triad censuses from random graphs, the less constraint on triad census outcomes for a network. Thus, ellipse area can serve as an inverse index of constraint. In general, the overall amount of variability in a set of variables is quantified using the generalized variance, defined as the determinant of the covariance matrix for the variables, det($\boldsymbol{\Sigma}$) (Tatsuoka, 1971). This approach can also be used to approximate the area of an ellipse, using the coordinates for the points in k-dimensional space as the variables (Koeppl et al.,

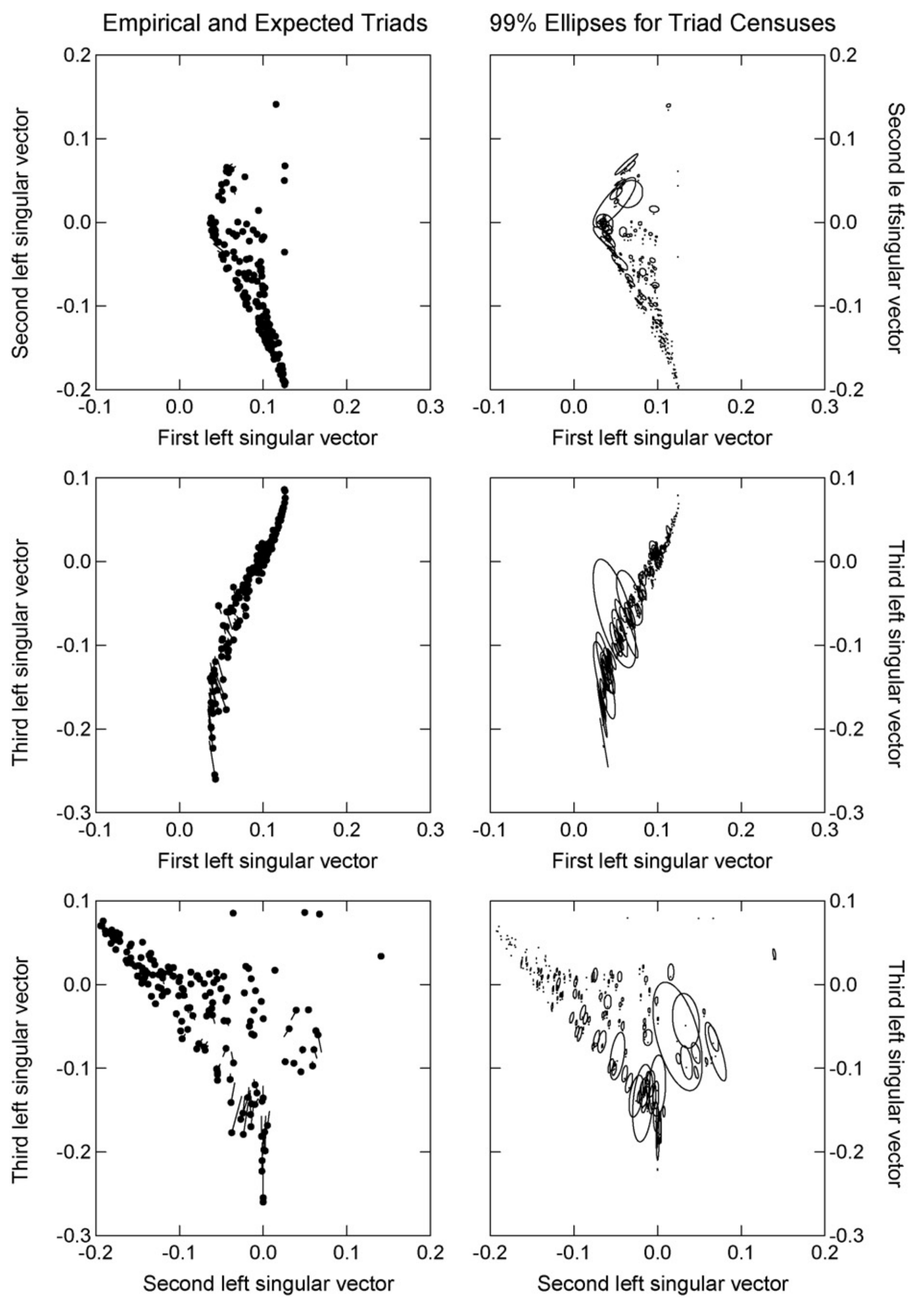


Fig. 4. First three left singular vectors from singular value decomposition of triad censuses: projections of triad censuses from 159 empirical social networks (solid points) linked to their corresponding expected triad censuses under U|MAN (left panels), 99% ellipses for triads from 100 random graphs under U|MAN for 159 social networks (right panels).

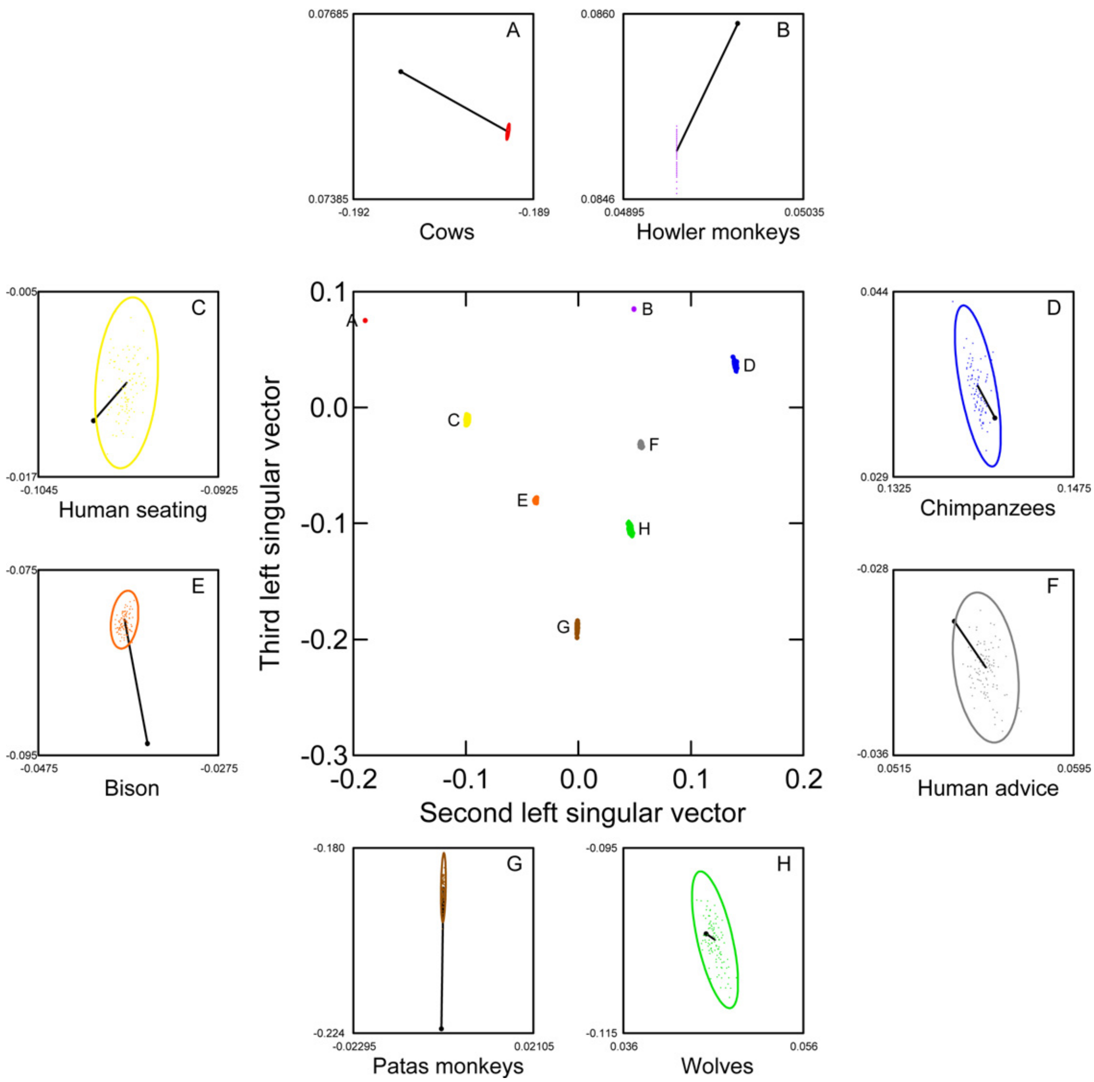


Fig. 5. Second and third left singular vectors from singular value decomposition of triad censuses. Expanded 99% ellipses for projected triad censuses from 100 random graphs under U|MAN, for eight example networks. Empirical triad census (filled point) linked to the expectation under U|MAN.

1975; Sokal and Rohlf, 1981). Thus, the determinant of the covariance matrix from coordinates for projected points in k-dimensional space, $\det(\Sigma_{\hat{U}})$, provides an inverse measure of constraint. For the sample of 159 networks, constraint on the triad census is directly related to network size and inversely related to network density, though the relationship is clearly not linear. (Of course, size and density are related to each other.) Using the area of ellipses in the 2-dimensional plot of the second and third left singular vectors (bottom right panel of Fig. 4), Fig. 6 shows scatter plots of $\log_{10}(\det(\Sigma_{\hat{U}}))$ (the inverse measure of constraint), $\log_{10}$(network size), and $\log_{10}$(density). The $\log_{10}$(network size) accounts for 86% of the variance in $\log_{10}(\det(\Sigma_{\hat{U}}))$ and $\log_{10}$(density) accounts for 64%. Together, (log) network size and (log) density account for 97% of (log) constraint on the triad censuses. (The largest ellipse is for the smallest network, social acts between four colobus monkeys.)

6. Discussion

We are now in a position to shed light on the puzzling results showing, on the one hand, that triad censuses for social networks are largely determined by the dyad census, while, on the other hand, many social networks exhibit triadic tendencies, above and beyond what is expected from the dyad census. Insight builds on the fact that lower order graph features, such as the dyad census, constrain possible triadic outcomes. The constraint imposed by a particular dyad census limits triadic configurations that can occur; some triad censuses become impossible and probability for the triad census is concentrated on a narrow range of outcomes. Thus, when considered against a universe of expected triad censuses, constraints imposed by a dyad census determine a relatively small region of probability for the triad census. These regions differ across networks to the extent that their dyad censuses differ. When triad censuses are viewed comparatively across networks with different dyad censuses, the triad censuses look very different, and they do so in ways that are largely explained by the dyad censuses. Nevertheless, for many dyad censuses a number of different triad censuses are possible. This provides the opportunity for an empirical triad census to depart from expectation by aligning triads in ways that show triadic structural biases, such as a tendency toward transitivity rather than intransitivity, or toward triadic closure and away from the "forbidden" triad. Thus, when compared with expectation from the dyad census, many empirical triad censuses differ substantially from that expectation. Empirical triad censuses can stray from expectation, but they must do so within fairly narrow confines.

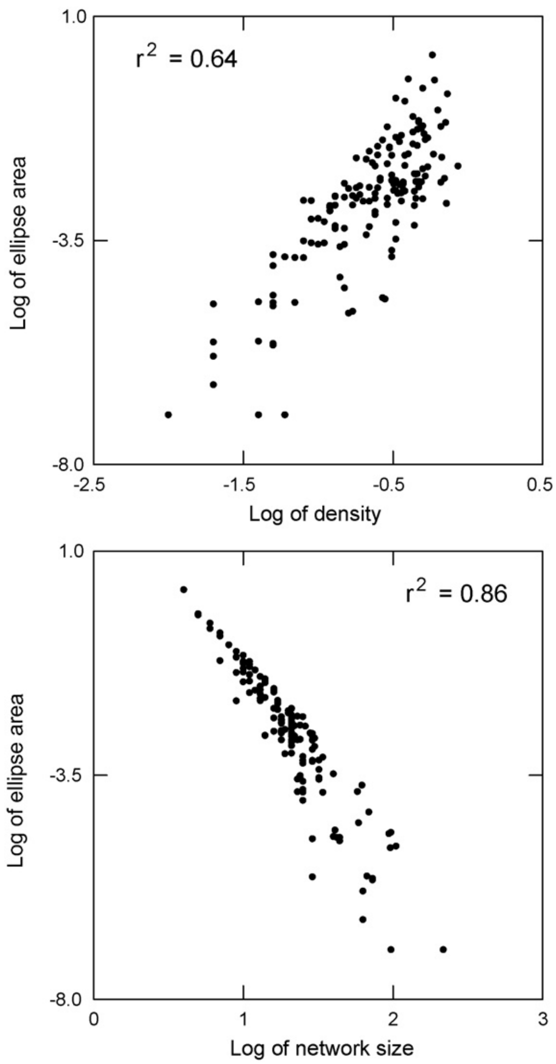


Fig. 6. Scatter plots of constraint on the triad census (log of area of 99% ellipse on second and third left singular vectors) versus network size and network density.

These insights should moderate earlier observations (Faust, 2007, 2008) that might lead one to mistakenly conclude that the full story of triadic patterning is contained in lower order graph properties, such as network density or the distribution of dyads, since they "explain" a large percent of the variability among empirical triad censuses. The message of the current paper is more nuanced. In comparative perspective, a large "between network" effect due to different dyad censuses accounts for a substantial proportion of the overall variability among triad censuses. Triadic structural tendencies then operate to align dyads into triads in ways that can depart from expectation. However, due to graph theoretic constraints, the "within network" variance is generally quite small relative to the between network variance.[1]

Results presented in this paper also provide insights into contrasting triadic patterns that are found in a variety of empirical relations and point to fundamental contrasts among them. Though not unexpected, it is clear that different kinds of social relations, such as win-loss encounters (that result in asymmetric dyads) and observed associations among individuals (that are required to be mutual or null) must have different triad censuses. This is so because they have different dyadic building blocks. Characterizing the triadic (or other) structural signatures of different social relations remains a fertile area for investigation.

In addition, results reveal regions of the universe of triad census expectations where empirical observations are relatively rare and regions where there is relatively good empirical coverage. In the sample of 159 social networks, triad censuses composed of triads with many mutual dyads (such as the 300 triad) are rare (right panels of Fig. 2). Triad censuses that require many mutual dyads occur in networks that are dense, and dense networks are rare. In the sample, the small network of grooming between nine chimpanzees is most notable, with density = 0.86. The rarity of dense social networks could be due to peculiarities of the sample used in this paper. It could also result from network measurement protocols that limit outdegree and thus limit network density (Faust, 2008; Holland and Leinhardt, 1973). Or, it could be a fundamental property of social systems that individuals are only capable of maintaining a limited number of contacts with others (Mayhew et al., 1995), thus restricting network density, especially as network size increases. These issues raise questions for further research on the intersection of social structural processes, actor capacities, network measurement, and social network structure.

Two more general issues are suggested by results in this paper, though they point to rather different avenues for further research. First, observations about constraints on triad outcomes have implications for the linkage between micro-structural configurations and macro-structural models. A fruitful body of research has proposed a series of structural models (such as clusterability, transitivity, or hierarchical cliques) corresponding to particular permitted or forbidden triads at a micro-level (Davis and Leinhardt, 1972; Johnsen, 1985, 1989b). For example, a clusterable network contains only 300, 201, and 003 triads (other triads are forbidden) and generates a macro-structure in which all (mutual) ties are within rather than between subgroups. Results presented in the current paper suggest a different role for "forbidden" triads, in which lower order graph features, including the dyad census, determine which triads are disallowed, by construction. Substantive structural tendencies then operate within constraints determined by graph theoretic and measurement properties to produce observed triad distributions, which (might) give rise to macro-structural models. That different models are required depending on graph theoretic properties is foreshadowed in work that posits separate models for "large" networks (Davis, 1970; Johnsen, 1985).

A second general observation points to limitations of the current paper with respect to process models generating triadic configurations. Process models examine hypothesized interdependencies in dynamically changing networks. Often hypothesized effects operate on triples, for example, by positing that a (triadic) network environment induces dyadic tendencies (such as orientation of triples of arcs consistent with transitivity or closure of a null dyad in an otherwise "forbidden" triad). In such dynamic processes, triads constrain dyads, rather than the other way around. Data in this paper show triadic tendencies in cross-sectional sentiments (e.g., friendships) or time-aggregated dyadic interactions (e.g., agonistic encounters). Although these configurations arise from interactions that unfold through time, the current paper is limited in its implications regarding processes leading to these configurations. Rather, it is more narrowly focused on graph theoretic results that hold in static representation of networks.

Finally, observations about constraints of lower order graph properties are not confined to the triad census. There is a growing body of research on the bounding effects that network size, density, and other graph properties place on graph level indices (Anderson et al., 1999; Butts, 2006; Faust, 2008; Friedkin, 1981). The current paper provides illustrative results showing the range of outcomes for triad censuses with specific dyad distributions but

[1] I am grateful to a reviewer for suggesting this analogy.

does not attempt to calculate bounds for triad isomorphism classes. Indeed, interdependencies among the triads in the census lead us to expect that finding exact bounds will be challenging.

Acknowledgements

I am grateful to John Boyd, Carter Butts, Lin Freeman, Kim Romney and members of the UCI Social Network Research Group for discussions of this research, and to *Social Networks* editor Pat Doreian and three anonymous reviewers for insightful comments.

References

Anderson, B.S., Butts, C.T., Carley, K., 1999. The interaction of size and density with graph-level indices. Social Networks 21 (3), 239–267.
Bearman, P.S., 1997. Generalized exchange. American Journal of Sociology 102 (5), 1383–1415.
Bearman, P.S., Moody, J., 2004. Suicide and friendships among American adolescents. American Journal of Public Health 94 (1), 89–95.
Ben-Israel, A., Greville, T., 1974. Generalized Inverses: Theory and Applications. Wiley, New York.
Brewer, D.D., Webster, C.M., 1999. Forgetting of friends and its effects on measuring friendship networks. Social Networks 21, 361–373.
Burt, R.S., 1992. Structural Holes: The Social Structure of Competition. Harvard University Press, Cambridge.
Butts, C.T., 2006. Exact bounds for degree centralization. Social Networks 28, 283–296.
Cartwright, D., Harary, F., 1956. Structural balance: a generalization of Heider's theory. Psychological Review 63, 277–293.
Chase, I.D., Tovey, C., Spangler-Martin, D., Manfredonia, M., 2002. Individual differences versus social dynamics formation of animal dominance hierarchies. Proceedings of the National Academy of Sciences 99 (8), 5744–5749.
Chase, I.D., Tovey, C., Murch, P., 2003. Two's company, three's a crowd: differences in dominance relationships in isolated versus socially embedded pairs of fish. Behaviour 140 (10), 1193–1217.
Coleman, J., 1988. Social capital in the creation of human capital. American Journal of Sociology 94, S95–S120.
Davis, J.A., 1967. Clustering and structural balance in graphs. Human Relations 20, 181–187.
Davis, J.A., 1970. Clustering and hierarchy in interpersonal relations: testing two graph theoretical models on 742 sociomatrices. American Sociological Review 35 (5), 843–851.
Davis, J.A., Leinhardt, S., 1972. The structure of positive interpersonal relations in small groups. In: Joseph Berger, Morris Zeldith Jr., Bo Anderson (Eds.), Sociological Theories in Progress, vol. 2. Houghton Mifflin, Boston, pp. 218–251.
Digby, P.G.N., Kempton, R.A., 1987. Multivariate Analysis of Ecological Communities. Chapman and Hall, London.
Doreian, P., Kapuscinski, R., Krackhardt, D., Szczypula, J., 1996. A brief history of balance through time. Journal of Mathematical Sociology 21 (1–2), 113–131.
Ekeh, P., 1974. Social Exchange Theory: Two Traditions. Harvard University Press, Cambridge, MA.
Faust, K., 2006. Comparing social networks: size, density, and local structure. Metodološki Zvezki (Advances in Methodology and Statistics) 3 (2), 185–216.
Faust, K., 2007. Very local structure in social networks. In: Yu Xie (Ed.), Sociological Methodology, vol. 32. Basil Blackwell, Cambridge, MA, pp. 209–256.
Faust, K., 2008. Triadic configurations in limited choice sociometric networks: empirical and theoretical results. Social Networks 30, 273–282.
Freeman, Linton C., 1992. The sociological concept of "Group": an empirical test of two models. American Journal of Sociology 98 (1), 152–166.
Friedkin, N.E., 1998. A Structural Theory of Social Influence. Cambridge University Press, Cambridge.
Friedkin, N.E., 1981. The development of structure in random networks: an analysis of the effects of increasing network density on five measures of structure. Social Networks 3 (1), 41–52.
Gould, R.V., Fernandez, R.M., 1989. Structures of mediation: a formal approach to brokerage in transaction networks. Sociological Methodology 19, 89–126.
Granovetter, M.S., 1973. The strength of weak ties. American Journal of Sociology 78, 1360–1380.
Hallinan, M.T., Felmlee, D., 1975. An analysis of intransitivity in sociometric data. Sociometry 38 (2), 195–212.
Hallinan, M.T., 1972. Comment on Holland and Leinhardt. American Journal of Sociology 77 (6), 1201–1205.
Hallinan, M.T., 1974a. The Structure of Positive Sentiment. Elsevier, New York.
Hallinan, M.T., 1974b. Structural model of sentiment relations. American Journal of Sociology 80, 364–378.
Holland, P.W., Leinhardt, S., 1970. A method for detecting structure in sociometric data. American Journal of Sociology 76 (3), 492–513.
Holland, P.W., Leinhardt, S., 1971. Transitivity in structural models of small groups. Comparative Group Studies 2 (2), 107–124.
Holland, P.W., Leinhardt, S., 1973. The structural implications of measurement error in sociometry. Journal of Mathematical Sociology 3, 85–111.
Holland, P.W., Leinhardt, S., 1976. Local structure in social networks. Sociological Methodology 7, 1–45.
Holland, P.W., Leinhardt, S., 1978. An omnibus test for social structure using triads. Sociological Methods and Research 7, 227–256.
Holland, P.W., Leinhardt, S., 1979. Structural sociometry. In: Holland, P.W., Leinhardt, S. (Eds.), Perspectives on Social Network Research. Academic Press, New York, pp. 63–83.
Johnsen, E.C., 1985. Network macrostructure models for the Davis-Leinhardt set of empirical sociomatrices. Social Networks 7, 203–224.
Johnsen, E.C., 1986. Structure and process: agreement models for friendship formation. Social Networks 8, 257–306.
Johnsen, E.C., 1989a. Agreement-friendship processes related to empirical social macrostructures. In: Manfred Kochen (Ed.), The Small World. Ablex, Norwood, NJ, pp. 239–279.
Johnsen, E.C., 1989b. The micro-macro connection: exact structure and process. In: Fred Roberts (Ed.), Applications of Combinatorics and Graph Theory to the Biological and Social Sciences. Springer-Verlag, New York, pp. 169–201.
Johnsen, E.C., 1998. Structures and processes of solidarity: an initial formalization. In: Patrick Doreian, Thomas Fararo (Eds.), The Problem of Solidarity: Theories and Models. Gordon Breech, Amsterdam, pp. 263–302.
Kalish, Y., Robins, G., 2006. Psychological predispositions and network structure: the relationship between individual predispositions, structural holes and network closure. Social Networks 28, 56–84.
Koeppl, J.W., Slade, N.A., Hoffman, R.S., 1975. A bivariate home range model with possible application to ethological data analysis. Journal of Mammalogy 56 (1), 81–90.
Krackhardt, D., Kilduff, M., 2002. Structure, culture and Simmelian ties in entrepreneurial firms. Social Networks 24 (3), 279–290.
Krackhardt, D., 1994. Graph theoretical dimensions of informal organizations. In: Carley, K.M., Prietula, M.J. (Eds.), Computational Organization Theory, Lawrence Erlbaum, Hillsdale, NJ, pp. 89–112.
Lambert, Z.V., Wildt, A.R., Durand, R.M., 1988. Redundancy analysis: an alternative to canonical correlation and multivariate multiple regression in exploring interset associations. Psychological Bulletin 104, 282–289.
Landau, H.G., 1951. On dominance relations and the structure of animal societies. I. Effect of inherent characteristics. Bulletin of Mathematical Biophysics 13, 1–19.
Lebart, L., Morineau, A., Warwick, K.M., 1984. Multivariate Descriptive Statistical Analysis: Correspondence Analysis and Related Techniques for Large Matrices. Wiley, New York.
Leinhardt, S., 1972. Developmental change in the sentiment structure of children's groups. American Sociological Review 37, 202–212.
Mayhew, B.H., McPherson, J., Rotolo, T., Smith-Lovin, L., 1995. Sex and race homogeneity in naturally occurring groups. Social Forces 74 (1), 15–52.
Schaefer, D.R., Light, J.M., Fabes, R.A., Hanish, L.D., Martin, C.L., 2009. Fundamental principles of network formation among preschool children. Social Networks, doi:10.1016/j.socnet.2009.04.003.
Simmel, G., 1950. The triad. In: Wolff, K.H. (Ed.), The Sociology of Georg Simmel. Free Press, New York, NY, pp. 145–169.
Skvoretz, J., Willer, D., 1993. Exclusion and power: a test of four theories of power in exchange networks. American Sociological Review 58 (6), 801–818.
Skvoretz, J., Faust, K., Fararo, T.J., 1996. Social structure, networks, and e-state structuralism models. Journal of Mathematical Sociology 21, 57–76.
Snijders, T.A.B., 1991. Enumeration and simulation methods for 0–1 matrices with given marginals. Psychometrika 56, 397–417.
Sokal, R.R., Rohlf, F.J., 1981. Biometry: Principles and Practice of Statistics in Biological Research. W.H. Freeman, San Francisco.
Stewart, D., Love, W., 1968. A general canonical correlation index. Psychological Bulletin 70, 160–163.
Tatsuoka, M.M., 1971. Multivariate Analysis: Techniques for Educational and Psychological Research. John Wiley, New York, NY.
Wasserman, S., 1977. Random directed graph distributions and the triad census in social networks. Journal of Mathematical Sociology 5, 61–86.
Wasserman, S., Faust, K., 1994. Social Network Analysis: Methods and Applications. Cambridge University Press, New York.