

MATH70056 Algebraic geometry :: Lecture notes

Lecturer: Pierre Descombes

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References:

1. *Undergraduate algebraic geometry*, Miles Reid
2. *Basic algebraic geometry*, Igor R. Shafarevich
3. Chapter 1 of *Algebraic geometry*, Robin Hartshorne

k by default is an algebraically closed field.

1 Affine

1.1 Introduction

Definition 1.1.1. The *affine algebraic space* of dimension n is $\mathbb{A}_k^n := \{(c_1, \dots, c_n) : c_i \in k\} \cong k^n$.

For $S \subset k[x_1, \dots, x_n]$, an *affine algebraic subvariety* of \mathbb{A}_k^n is the vanishing set $V(S) := \{x \in \mathbb{A}_k^n : f(x) = 0 \forall f \in S\}$ of S .

Example 1.1.2. Any finite set of points in \mathbb{A}_k^1 is an affine algebraic subvariety: $\{a_1, \dots, a_n\} = V((x - a_1) \cdots (x - a_n))$. The whole space \mathbb{A}_k^1 is $V(\{0\})$. Any algebraic curve (as studied before in Algebraic curves) is an example.

A nonexample would be an infinite proper subset of \mathbb{A}_k^1 .

Any point set $\{(a_1, \dots, a_n)\} \subset \mathbb{A}_k^n$ is an affine algebraic subvariety: $V(x_1 - a_1, \dots, x_n - a_n)$.

For $n > m$, the image of the linear map $\varphi : \mathbb{A}_k^m \rightarrow \mathbb{A}_k^n$ via $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, 0, \dots, 0)$ is $V(x_{m+1}, \dots, x_n)$, another example. However this is not true for any linear map; consider $\varphi : \mathbb{A}_k^2 \rightarrow \mathbb{A}_k^2 : (x, y) \mapsto (x, xy)$, then $\text{im } \varphi = \mathbb{A}_k^2 \setminus \{(0, y) : y \in k \setminus \{0\}\}$. If $\text{im } \varphi = V(S)$ and $f \in S$, then $f(x, y) = 0 \forall x \neq 0$, so $f = 0$ and $V(S) = \mathbb{A}_k^2$, a contradiction.

Remark 1.1.3. Note that $S_1 \subset S_2 \implies V(S_2) \subset V(S_1)$.

We can then assume S is an ideal since $V(S) = V((S))$. If an ideal is generated by f_1, \dots, f_n , then denote the subvariety of this ideal by $V(f_1, \dots, f_n) := V((f_1, \dots, f_n)) = V(\{f_1, \dots, f_n\})$.

If f is a nonconstant polynomial, then $V(f) \subset \mathbb{A}_k^n$ is called a *hypersurface*.

Theorem 1.1.4 (Hilbert basis). $k[x_1, \dots, x_n]$ is Noetherian for any field k .

Lemma 1.1.5. If I, J are ideals of $k[x_1, \dots, x_n]$, then $V(I) \cup V(J) = V(I \cdot J)$.

Proof. Since $I \cdot J \subset I, J$, one has $V(I) \cup V(J) \subset V(I \cdot J)$. Now suppose $x \in V(I \cdot J)$ but $x \notin V(I)$. Then $\exists f \in I : f(x) \neq 0$. But $fg \in I \cdot J \forall g \in J$, so $fg(x) = f(x)g(x) = 0$, so $g(x) = 0$ (since k is a domain) and $x \in V(J)$. \square

Lemma 1.1.6. If I_k is a (possibly infinite) set of ideals of $k[x_1, \dots, x_n]$, then

$$\bigcap_k V(I_k) = V\left(\sum_k I_k\right).$$

Proof. Since $I_k \subset \sum_k I_k$, $V(\sum_k I_k) \subset V(I_k) \forall k$, i.e. $V(\sum_k I_k) \subset \bigcap_k V(I_k)$. Now suppose $x \in \bigcap_k V(I_k)$ and $f \in \sum_k I_k$, then $f(x) = \sum_k f_k(x) = 0$. \square

Definition 1.1.7. The *Zariski topology* on \mathbb{A}_k^n is the topology whose closed subsets are the affine algebraic subvarieties, and for $V \subset \mathbb{A}_k^n$ the Zariski topology on V is the topology induced by the Zariski topology on \mathbb{A}_k^n . This topology is well-defined by previous two lemmas.

Example 1.1.8. The Zariski topology on \mathbb{A}_k^1 is the cofinite topology. Note that this is not Hausdorff.

Let $k = \mathbb{C}$, then the Zariski topology on $\mathbb{A}_k^n = \mathbb{C}^n = \mathbb{R}^{2n}$ is coarser than the Euclidean topology:

Lemma 1.1.9. Any Zariski closed subset is closed under the Euclidean topology.

Proof. Write $V = V(f_1, \dots, f_n)$ where f_1, \dots, f_n are polynomials, so in particular continuous, hence $V = \bigcap_k f_k^{-1}(\{0\})$ is closed Euclidean-wise. \square

Lemma 1.1.10. If $V \subset \mathbb{A}_k^n$ and $W \subset \mathbb{A}_k^m$ are affine algebraic subvarieties, then $V \times W \subset \mathbb{A}_k^{n+m} = \mathbb{A}_k^n \times \mathbb{A}_k^m$ is an affine algebraic subvariety.

Proof. Give coordinates (x, y) to $\mathbb{A}_k^n \times \mathbb{A}_k^m$. Write $V = V(f_1, \dots, f_r)$ and $W = V(g_1, \dots, g_s)$, then $V \times W = V(f_1(x, 0), \dots, f_r(x, 0), g_1(0, y), \dots, g_s(0, y))$. \square

Remark 1.1.11. The Zariski topology on $V \times W$ is not the product of the Zariski topologies on V and W ; e.g. for $V = W = \mathbb{A}_k^1$, then $V(x - y) \subset \mathbb{A}_k^2$ is closed.

Definition 1.1.12. For $A \subset \mathbb{A}_k^n$, the *vanishing ideal* is $I(A) := \{f \in k[x_1, \dots, x_n] : f(x) = 0 \ \forall x \in A\} \subset k[x_1, \dots, x_n]$.

One has $A_1 \subset A_2 \implies I(A_2) \subset I(A_1)$.

Lemma 1.1.13. If V is an affine algebraic subvariety, then $V(I(V)) = V$.

Proof. Clearly $V \subset V(I(V))$. Now suppose $x \notin V$ and write $V = V(J)$. Then $\exists f \in J : f(x) \neq 0$, so $f \notin I(V)$ and hence $x \notin V(I(V))$. \square

In general $I \subset I(V(I))$ but the converse is not true: consider $I = (x^2) \subset k[x]$.

Definition 1.1.14. For $I \subset R$ an ideal, the *radical* of I is $\sqrt{I} = \{f \in R : f^n \in I \text{ for some } n\}$.

Theorem 1.1.15 (Hilbert Nullstellensatz). If k is algebraically closed, then \forall ideal $I \subset k[x_1, \dots, x_n]$ one has $I(V(I)) = \sqrt{I}$.

Week 1, lecture 2, 10th January

Example 1.1.16 (Decomposition). $\mathbb{A}_k^2 \supset V(x(x-1)) = V(x) \sqcup V(x-1)$.

$\mathbb{A}_k^2 \supset V(xy) = V(x) \cup V(y)$. Note that it's easy to break a closed set into union of two closed subsets in the Euclidean topology, but harder in Zariski.

$\mathbb{A}_k^2 \supset V(xy-1)$ is a hyperbola, but it's connected in Zariski topology.

1.1.1 Irreducibles

Definition 1.1.17. A topological space is *irreducible* if can't be written as the union of two proper closed subsets. Equivalently, any two nonempty open subsets have nonempty intersection, or any nonempty open subset is dense.

Lemma 1.1.18. An affine algebraic subvariety $V \subset \mathbb{A}_k^n$ is irreducible iff $I(V)$ is a prime ideal.

Proof. Suppose V is irreducible and $fg \in I(V)$ where $f, g \in k[x_1, \dots, x_n]$. Then one can write $V = \{x \in V : f(x) = 0\} \cup \{x \in V : g(x) = 0\}$. But then V is equal to one of its components, so either f or $g \in I(V)$, as desired.

Now suppose V is reducible and write $V = V_1 \cup V_2$ where $V_1, V_2 \subsetneq V$. This allows $f \in I(V_1)$ but $f \notin I(V_2)$, and similarly $g \in I(V_2)$ but $g \notin I(V_1)$. But then $fg \in I(V)$, so $I(V)$ is not prime. \square

Remark 1.1.19. So far we have

ideals of $k[x_1, \dots, x_n]$	subvarieties of \mathbb{A}_k^n
ideal	closed subset
maximal ideal	point
prime ideal	irreducible subvariety

Now note that $k[x_1, \dots, x_n]$ is Noetherian, which means any descending chain of closed subsets of \mathbb{A}_k^n stabilises, or equivalently any ascending chain of open subsets of \mathbb{A}_k^n stabilises. This implies any affine variety is quasicompact.

Also, primary decomposition of ideals gives decomposition of varieties into unique irreducible decompositions $V = \bigcup_{i=1}^n V_i$ where $V_i \subset V$ are irreducible.

1.2 Regular functions

Definition 1.2.1. A function $f : V \rightarrow k$ is *regular* if it is the restriction of some polynomial $F : \mathbb{A}_k^n \rightarrow k$. Denote by $k(V)$ the ring of regular functions.

If F, G define the same regular function, then $F = G$ on V , so $F - G \in I(V)$.

Proposition 1.2.2. The ring of regular functions is identified with $k[V] \cong k[x_1, \dots, x_n]/I(V)$.

Example 1.2.3. $k[\mathbb{A}_k^n]$ is just the whole polynomial ring $k[x_1, \dots, x_n]$.

For a point $V = (c_1, \dots, c_n) = V(x_1 - c_1, \dots, x_n - c_n)$, one has $k[V] = k[x_1, \dots, x_n]/(x_1 - c_1, \dots, x_n - c_n) \cong k$.

One has $k[V(x(x-1))] = k[x]/x(x-1) \cong k \times k$.

$k[V(xy)] = k[x, y]/xy = \{a + \sum_{i \geq 1} b_i x^i + \sum_{i \geq 1} c_i y^i\}$.

$k[V] = k[x, y]/xy - 1 = k[x, x^{-1}] = \{\sum_{i \in \mathbb{Z}} a_i x^i\}$.

Corollary 1.2.4. V is irreducible iff $k[V]$ is an integral domain.

Definition 1.2.5. For $V \subset \mathbb{A}_k^n, W \subset \mathbb{A}_k^m$, a map $f : V \rightarrow W$ is *regular* if there are polynomials $F_1, \dots, F_m \in k[x_1, \dots, x_n]$ such that $f(x_1, \dots, x_n) = (F_1(x_1, \dots, x_n), \dots, F_m(x_1, \dots, x_n))$.

Example 1.2.6. For $m \geq n$ and $V \subset \mathbb{A}_k^n$, the projection map $\pi : V \rightarrow \mathbb{A}_k^n : (x_1, \dots, x_m) \mapsto (x_1, \dots, x_n)$ is regular.

A regular function defined before is equivalent to a regular map $V \rightarrow \mathbb{A}_k^1$.

Consider $V = \mathrm{SL}(k^n)$ as $V(\det - 1) \subset \mathbb{A}_k^{n \times n}$, then the map $V \rightarrow V : A \mapsto A^{-1}$ is regular.

Lemma 1.2.7. A regular map is continuous with respect to the Zariski topology.

Proof. Let $f : V \rightarrow W$ where $V \subset \mathbb{A}_k^n, W \subset \mathbb{A}_k^m$ be regular and expressed by $F_1, \dots, F_m \in k[x_1, \dots, x_n]$, and take a closed subset $W' \subset W$. Since it's closed, write $W' = \{x \in W : g_1(x) = \dots = g_l(x) = 0\}$. Then $f^{-1}(W') = \{x \in V : g_1 f(x) = \dots = g_l f(x)\} = \{x \in V : g_1(F_1, \dots, F_m)(x) = \dots = g_l(F_1, \dots, F_m)(x) = 0\}$, so closed. \square

NB there are more continuous maps than regular maps. In fact, it's easy to be continuous for Zariski: any map $f : \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ is continuous as long as it's finite-to-one.

Lemma 1.2.8. Let $\varphi, \psi : V \rightarrow W$ be two regular maps. If there is a dense open subset $A \subset V$ such that $\varphi|_A = \psi|_A$ then $\varphi = \psi$.

Proof. Write $\varphi = (F_1, \dots, F_m)$ and $\psi = (G_1, \dots, G_m)$. Consider the set

$$V_{\mathrm{eq}} = \{x \in V : F_1(x) = G_1(x) = \dots = F_m(x) = G_m(x) = 0\} \supset A,$$

which is closed, but the smallest closed set containing A is V by definition of dense, we have $V_{\mathrm{eq}} = V$, i.e. φ and ψ are equal on the whole V as desired. \square

Definition 1.2.9. If $\varphi : V \rightarrow W$ is a regular map, then define $\varphi^* : k[W] \rightarrow k[V] : g \mapsto g \circ \varphi$.

Proposition 1.2.10. φ^* gives a bijection between ring homomorphisms $k[W] \rightarrow k[V]$ and regular maps $V \rightarrow W$.

Week 2, lecture 1, 13th January

Definition 1.2.11. A regular map $\varphi : V \rightarrow W$ is an *isomorphism* if it has an inverse which is a regular map $\psi : W \rightarrow V$ with $\varphi \circ \psi = \mathrm{id}$, $\psi \circ \varphi = \mathrm{id}$.

Example 1.2.12. Consider the parabola $V(y - x^2) \subset \mathbb{A}_k^2$ and the map $\varphi : V(y - x^2) \rightarrow \mathbb{A}_k^1 : (x, y) \mapsto x$ has an inverse $\psi : \mathbb{A}_k^1 \rightarrow V(y - x^2) : t \mapsto (t, t^2)$ (parameterisation).

Now $\varphi : V(xy - 1) \rightarrow \mathbb{A}_k^1 : (x, y) \mapsto x$ is not surjective: the image is $\mathbb{A}_k^1 \setminus \{0\}$, so it cannot be an isomorphism.

$V(y^2 - x^3)$ has a cusp at the origin. We can try to parameterise the variety by $\mathbb{A}_k^1 \rightarrow V(y^2 - x^3) : t \mapsto (t^2, t^3)$, but this is not an isomorphism; suppose $\psi : V(y^2 - x^3) \rightarrow \mathbb{A}_k^1$ is an inverse, so in particular it's regular and $\exists p \in k[x, y]$ with $p(t^2, t^3) = t$, which is impossible.

A corollary of 1.2.10 is

Corollary 1.2.13. Two affine varieties V, W are isomorphic iff $k[V] \cong k[W]$.

Example 1.2.14. $V = V(xy - 1)$ is not isomorphic to \mathbb{A}_k^1 : indeed, $k[\mathbb{A}_k^1] = k[x]$ and $k[V] = k[x, y]/xy - 1 = k[x, x^{-1}]$.

Proof of 1.2.10. Let $V = V(I) \subset \mathbb{A}_k^n$ and $W = V(J) \subset \mathbb{A}_k^m$. Consider a k -algebra morphism $\phi : k[W] = k[x_1, \dots, x_m]/J \rightarrow k[V] = k[x_1, \dots, x_n]/I$. Take polynomials $F_1, \dots, F_m \in k[x_1, \dots, x_n]$ which lift $\phi(x_1), \dots, \phi(x_m) \in k[x_1, \dots, x_n]/I$. Then the map $\varphi : \mathbb{A}_k^m \rightarrow \mathbb{A}_k^n : x \mapsto (F_1(x), \dots, F_m(x))$ is regular which does not depend on choice of F_i , and one can check φ^* is ϕ . \square

1.2.1 Locally defined regular functions

Definition 1.2.15. Let V be an affine variety and $f \in k[V]$. The *distinguished open subset* $D_V(f)$ is the open subset $V \setminus V(f)$, i.e. $\{x \in V : f(x) \neq 0\}$.

Note that $V(fg) = V(f) \cup V(g)$, so $D_V(fg) = D_V(f) \cap D_V(g)$.

Example 1.2.16. $\mathbb{A}_k^2 \setminus \{(0, 0)\} = D_V(x) \cup D_V(y)$ is not distinguished.

Lemma 1.2.17. Let U be open in an affine variety V . Then $\forall p \in U, \exists f \in k[V] : p \in D_V(f) \subset U$.

Proof. Since U is open, it's a complement of some closed set, i.e. variety, so write $U = V \setminus V(I)$. In particular, $p \in V$ but there is some $f \in I$ such that $f(p) \neq 0$. Then $p \in D_V(f)$, and since $V(I) \subset V(f)$, one has $D_V(f) \subset U$. \square

Corollary 1.2.18. Any open subset $U \subset V$ is a finite union of distinguished open subsets.

Proof. By lemma above, one can cover U by distinguished open subsets, and the desired follows from quasicompactness of varieties. \square

Proposition 1.2.19. Consider $f \in k[V]$ and $g_1, \dots, g_n \in k[V]$ such that $D_V(g_i) \subset D_V(f) \forall i$. Then $D_V(g_i)$'s cover $D_V(f)$, i.e. $D_V(f) = \bigcup_{i=1}^n D_V(g_i)$ iff $f \in \sqrt{(g_1, \dots, g_n)}$. In particular, take $f = 1$, then $D_V(g_i)$'s cover V iff $1 \in (g_1, \dots, g_n)$.

Proof. Suppose $D_V(f) \subset \bigcup_{i=1}^n D_V(g_i)$. then $V(f) \supset \bigcap_{i=1}^n V(g_i) = V(g_1, \dots, g_n)$, so $I(V(f)) \subset I(V(g_1, \dots, g_n))$, i.e. $\sqrt{(f)} \subset \sqrt{(g_1, \dots, g_n)}$, so in particular $f \in \sqrt{(g_1, \dots, g_n)}$. The argument goes backwards as well. \square

Given two regular functions $f, g \in k[V]$ on $D_V(g)$, we would want that $x \mapsto \frac{f(x)}{g(x)}$ is regular. We also want a local theory with which one can check if near a point a function is regular.

Definition 1.2.20. Let V be an affine variety and $U \subset V$ an open subset. A function $f : U \rightarrow k$ is *regular* if there is a open cover (U_i) of U and $g_i, h_i \in k[V]$ with $h_i \neq 0$ on U_i (i.e. $U_i \subset D_V(h_i)$) and for $x \in U_i$, $f(x) = \frac{g_i(x)}{h_i(x)}$. Denote by $\mathcal{O}_V(U)$ the ring of regular function on U .

Remark 1.2.21 (not a part of the course). $U \rightarrow \mathcal{O}_U$ defines a sheaf of regular function. The idea is if we have an open subset $U' \subset U$, we have a restriction $\mathcal{O}_V(U) \rightarrow \mathcal{O}_V(U')$, and if $U = \bigcup_{i=1}^n U_i$, then $f \in \mathcal{O}_V(U) \iff f|_{U_i} \in \mathcal{O}_V(U_i) \forall i$.

We have to check that it is consistent with our previous definition with $U = V$, i.e. when $\mathcal{O}_V(U) = k[V]$. This is not trivial.

Proposition 1.2.22. Let V be an affine variety and $h \in k[V]$. Then every regular function on $D_V(h)$ is of the form $\frac{g}{h}$ with $g \in k[V]$ and $n \in \mathbb{N}$, i.e. the ring morphism $k[V] \left[\frac{1}{h} \right] \xrightarrow{\sim} \mathcal{O}_V(D_V(h))$ is an isomorphism. In particular, fixing $h = 1$, then $k[V] \xrightarrow{\sim} \mathcal{O}_V(V)$ is an isomorphism.

Proof. Let f be regular on $D_V(h)$. By definition, take (U_i) covering $D_V(h)$ and $g_i, h_i \neq 0 \in k[V]$ such that $U_i \subset D_V(h_i)$ and $f(x) = \frac{g_i(x)}{h_i(x)} \forall x \in U_i$. Cover each U_i by distinguished open subsets and write $U_i = \bigcup_j D_V(h_{ij})$, in particular $D_V(h_{ij}) \subset D_V(h_i)$. Then by 1.2.19, $h_{ij} \in \sqrt{(h_i)}$, so $h_{ij}^r = g'_{ij} h_i$ for some $g'_{ij} \in k[V]$ and r . Hence $\forall x \in D_V(h_{ij})$, one has $f(x) = \frac{g_i(x)}{h_i(x)} = \frac{g_i(x) g'_{ij}(x)}{h_{ij}^r(x)}$. Now since $D_V(h_{ij}) = D_V(h_{ij}^r)$, set $h'_{ij} = h_{ij}^r$, then a cover of U_i by $D_V(h'_{ij})$ and an identity $f(x) = \frac{p}{h'_{ij}}$ on U_i are obtained where p is some polynomial.

If we do this for each U_i , we get an open cover of U by distinguished open subsets of the form $D_V(h_i)$ (not the h_i before, an abuse of notation), and $f(x) = \frac{g_i(x)}{h_i(x)}$ on U_i for some polynomial g_i .

Now, on $D_V(h_i) \cap D_V(h_j)$ one has $f(x) = \frac{g_i(x)}{f_i(x)} = \frac{g_j(x)}{f_j(x)}$, so $g_i h_j - g_j h_i = 0$. But now $h_i h_j = 0$ on $(D_V(h_i) \cap D_V(h_j))^C$, so $g_i h_i h_j^2 - g_j h_i^2 h_j = 0$ globally $\forall i, j$.

Since the $D_V(h_i)$'s cover $D_V(h)$ and $D_V(h_i) = D_V(h_i^2) \forall i$, the sets $D_V(h_i^2)$ still cover $D_V(h)$. Then again by 1.2.19, $h^r \in (h_i^2)$ for some r , i.e. $h^r = a_1 h_1^2 + \dots + a_s h_s^2$ for some $a_i \in k[V]$. We claim

$$f(x) = \frac{1}{h^r(x)} \sum_{i=1}^s a_i(x) g_i(x) h_i(x) \quad \forall x \in D_V(h),$$

and thus proving the desired. Indeed, write $x \in D_V(h_i)$, then $f(x) = \frac{g_i(x)}{h_i(x)}$, so

$$f(x) h_i(x)^2 h(x)^r = g_i(x) h_i(x) \sum_{j=1}^s a_j(x) h_j(x)^2 = h_i(x)^2 \sum_{j=1}^s a_j(x) g_j(x) h_j(x),$$

which by dividing both sides by $h_i(x)^2 h(x)^r$ proves the claim. \square

Week 2, lecture 2, 16th January: problem class (sheet 1)

Exercise 1.2.23. 1. Let V be an affine subvariety of \mathbb{A}_k^n . Prove that $V(I(V)) = V$. Moreover, if A is any subset of \mathbb{A}_k^n , show that $V(I(A))$ is the Zariski closure of A .

Solution. Obviously $V \subset V(I(V))$, since if a point vanishes on all polynomials that vanishes on the point. Now if $x \notin V$, then write $V = V(J)$ where $J \subset k[x_1, \dots, x_n]$ is an ideal. so that $\exists f \in J : f(x) \neq 0$, and $J \subset I(V)$ so in particular $f \in I(V)$.

Now obviously $V(I(A))$ is closed. Take $A \subset V$ where V is a subvariety, then $I(V) \subset I(A)$, so $V(I(A)) \subset V(I(V)) = V$.

2. Determine all closed subsets of $V(xy) \subset \mathbb{A}_k^2$. Deduce that $V(xy)$ is connected.

Solution. \emptyset , $V(xy)$, $V(x)$, $V(y)$ are closed. Now note that $V(x) = V(y) = \mathbb{A}_k^1$, and we know that closed subsets of \mathbb{A}_1 are precisely finite sets of points, so we have countable unions of above sets with finite point sets as well.

$V(xy)$ is connected means $V(xy)$ cannot be written as a disjoint union of two open subsets, which is equivalent to not able to be written as a disjoint union of two closed subsets by passing to the complementary. But any pair of closed subsets we listed above either share the origin so not disjoint, or don't have enough points (countable unions of finite point sets).

3. Let $V = V(x^2 + y^2 - 1, x - 1)$. Determine V and $I(V)$. Is $I(V) = (f, g)$?

Solution. Drawing a picture helps us to see that $V = \{(1, 0)\}$, so $I(V) = (x - 1, y)$ and $I(V) \neq (f, g)$.

4. Prove that $\{x \in \mathbb{A}_k^n : f(x) = 0\}$ is irreducible iff f is a power of an irreducible polynomial.

Solution. Suppose $V(f)$ is irreducible. Then (f) is prime by 1.1.18, so f is irreducible. Now if f is a power of an irreducible polynomial g , then $V(f) = V(g)$ with (g) prime, so $V(f)$ is irreducible again by 1.1.18.

5. Using 4, show that $V(y - x^2), V(x^2 + y^2 - 1) \subset \mathbb{A}_k^2$ are irreducible.

Solution. Suppose we have a decomposition $y - x^2 = (y + f(x))g(x) = yg(x) + f(x)g(x)$. Then $g(x) = 1$ and $f(x) = -x^2$, so the decomposition is trivial.

Now write $x^2 + y^2 - 1 = (y + f(x))(y + g(x)) = y^2 + yg(x) + yf(x) + f(x)g(x)$, so $f(x) + g(x) = 0$ and $f(x)g(x) = x^2 - 1$, i.e. we need to find a polynomial h such that $h(x)^2 = 1 - x^2$, which is impossible.

6. Decompose $V(xy, z) \subset \mathbb{A}_k^3$ and $V(xy, xz) \subset \mathbb{A}_k^3$.

Solution. $V(xy, z) = V(xy) \cap V(z) = (V(x) \cup V(y)) \cap V(z) = (V(x) \cap V(z)) \cup (V(y) \cap V(z)) = V(x, z) \cup V(y, z)$.

$$\begin{aligned} V(xy, xz) &= V(xy) \cap V(xz) = (V(x) \cup V(y)) \cap (V(x) \cup V(z)) \\ &= V(x) \cup V(x, z) \cup V(y, x) \cup V(y, z) \\ &= V(x) \cup V(y, z) \end{aligned}$$

Week 2, lecture 3, 17th January: problem class (sheet 1 continued)

7. Assume $\text{char } k \neq 2$. Decompose $V(x^2 + y^2 + z^2, x^2 - y^2 - z^2 + 2) \subset \mathbb{A}_k^3$.

Solution.

$$\begin{aligned} V(x^2 + y^2 + z^2, x^2 - y^2 - z^2 + 2) &= V(x^2 + y^2 + z^2, 2x^2 + 2) = V(x^2 + y^2 + z^2, x^2 + 1) \\ &= V(y^2 + z^2 - 1, x^2 + 1) = V(y^2 + z^2 - 1, (x + i)(x - i)) \\ &= V(y^2 + z^2 - 1, x + i) \cup V(y^2 + z^2 - 1, x - i) \end{aligned}$$

8. Decompose $V(y^2 - xz, z^2 - y^3) = V(f, g)$. Start by factoring $yf + g$.

Solution. We have $yf + g = y^3 - xyz + z^2 - y^3 = z^2 - xyz = z(z - xy)$, so

$$\begin{aligned} V &= V \cap V(z(z - xy)) = V \cap (V(z) \cup V(z - xy)) \\ &= (V \cap V(z)) \cup (V \cap V(z - xy)) \end{aligned}$$

where

$$V \cap V(z) = V(y^2, y^3, z) = V(y, z)$$

and

$$\begin{aligned} V \cap V(z - xy) &= V(y^2 - xz, z - xy) = V(y(y - x^2), z - xy) \\ &= V(y, z - xy) \cup V(y - x^2, z - xy) \\ &= V(y, z) \cup V(y - x^2, z - x^3), \end{aligned}$$

so we conclude $V = V(y, z) \cup V(y - x^2, z - x^3)$.

9. Show that $V = V(y^2 - xz, x^3 - yz) \subset \mathbb{A}_k^3$ has two irreducible components, one of which is $C = \{(t^3, t^4, t^5) : t \in k\}$.

Solution. If $x = 0$ then $y = 0$, so $V(x, y) \subset V$.

If $x \neq 0$ then $y, z \neq 0$. Let $t = \frac{y}{x}$. Then $y = tx$, so $t^2x^2 - xz = x^3 - txz = 0$, and since $x \neq 0$, we have $z = t^2x$ and $x^2 = tz = t^3x$, hence $x = t^3, y = t^4, z = t^5$.

10. Let $V \subset \mathbb{A}_k^m$ and $W \subset \mathbb{A}_k^n$ be irreducible affine subvarieties. Prove $V \times W$ is irreducible.

Solution. Write $V \times W = F_1 \cup F_2$ where F_1, F_2 are closed. We want to prove $V \times W = F_1$ or F_2 . Fix $x \in V$, then we claim for any $y \in W$, either $(x, y) \in F_1$ or F_2 . Write

$$W \cong \{x\} \times W = \{y : (x, y) \in F_1\} \cup \{y : (x, y) \in F_2\},$$

and since $W \rightarrow V \times W : y \mapsto (x, y)$ is regular, it's continuous, and by irreducibility of W the claim is true. Now

$$V = \{x \in V : (x, y) \in F_1 \ \forall y\} \cup \{x \in V : (x, y) \in F_2 \ \forall y\},$$

and again by continuity and irreducibility of V , one has the desired follows.

Week 3, lecture 1, 23rd January

Lemma 1.2.24. Given U open in an affine variety V and $f : U \rightarrow k$ regular, $f^{-1}(0)$ is closed in the Zariski topology.

Proof. Choose U_i open in U with $U = \bigcup_{i=1}^s U_i$ and $g_i, h_i \in k[V]$ with $h_i(x) \neq 0$ and $f(x) = \frac{g_i(x)}{h_i(x)} \ \forall x \in U_i$. Then $U_i \cap f^{-1}(0) = U_i \cap g_i^{-1}(0)$ where $g_i^{-1}(0)$ is closed in U since $g_i \in k[V]$, so $f^{-1}(0)$ is closed in U as well. \square

Definition 1.2.25. Let $V \subset \mathbb{A}_k^n, W \subset \mathbb{A}_k^m$ be affine algebraic subvarieties and $U \subset V, U' \subset W$ be open. A function $f : U \rightarrow U'$ is *regular* if there are regular functions f_1, \dots, f_m on U such that $f(x) = (f_1(x), \dots, f_m(x)) \ \forall x \in U$.

Lemma 1.2.26. If $f : U \rightarrow U'$ and $g : U' \rightarrow k$ are regular, then $g \circ f : U \rightarrow k$ is regular.

Proof. For $x \in U$, we can write $g(y) = \frac{t(y)}{s(y)}$ on a neighbourhood U'' of $y = f(x) \in U'$, so for $x \in f^{-1}(U'')$ one has

$$g \circ f(x) = \frac{t(f_1(x), \dots, f_m(x))}{s(f_1(x), \dots, f_m(x))} = \frac{\text{some polynomial}}{\text{some other polynomial}},$$

so $g \circ f$ is regular. \square

In particular, given $U \subset V$ and $U' \subset W$ open in affine subvarieties and $f : U \rightarrow U'$, we have a pullback map $f^* : \mathcal{O}_{W'}(U') \rightarrow \mathcal{O}_V(U) : g \mapsto g \circ f$.

Corollary 1.2.27. Any regular map $f : U \rightarrow U'$ between open subsets of affine subvarieties is continuous.

Proof. Let $Z \subset W$ be closed, then $Z = \bigcap_i g_i^{-1}(0)$ for some $g_i \in k[V]$, so $f^{-1}(Z \cap U') = \bigcap_i (g_i \circ f)^{-1}(0)$ where $g_i \circ f$ is regular by above. \square

Proposition 1.2.28. The composition of two regular maps is regular.

Proof. Let $U \subset V, U' \subset V'', U'' \subset V'' \subset \mathbb{A}_k^m$ be open subsets of affine subvarieties, and suppose $f : U \rightarrow U', g : U' \rightarrow U''$ are regular. But then by definition, $g(x) = (g_1(x), \dots, g_m(x))$ for some g_i 's regular, so by 1.2.26, each $g_i \circ f$ is regular, so again by definition, $g \circ f$ is regular. \square

Definition 1.2.29. Given $U \subset V, U' \subset W$ open subsets of affine subvarieties,

- A regular map $f : U \rightarrow U'$ is an *isomorphism* if $\exists g : U' \rightarrow U : fg = gf = \text{id}$.
- A regular map $f : U \rightarrow U'$ is an *open/closed embedding* if $f(U)$ is open/closed in W , and $f : U \rightarrow f(U)$ is an isomorphism.
- A regular map $f : U \rightarrow U'$ is *dominant* if $f(U)$ is dense in V . Then the pullback $\mathcal{O}_W(U') \rightarrow \mathcal{O}_V(U)$ is injective (indeed, if $g_1, g_2 \in \mathcal{O}_W(U')$ satisfy $g_1 \circ f = g_2 \circ f$, then $g_1|_{f(U)} = g_2|_{f(U)}$, so $g_1 = g_2$).

1.2.2 Rational functions and rational maps

Consider $V \subset W$ an open subset of an affine subvariety, and suppose V is irreducible, i.e. every nonempty open subset $U \subset V$ is dense in V , and intersection of any two nonempty open subsets is nonempty and dense.

Definition 1.2.30. Let V be an irreducible open subset of an affine subvariety and W an open subset of the variety. A *rational function* $f : V \dashrightarrow k$ (with respect to a rational map $f : V \dashrightarrow W$) is an equivalence class of pairs (U, f) where U is nonempty and open in V and f is a regular function on U , with equivalence defined as: $(U, f) \sim (U', f')$ if $f|_{U \cap U'} = f'|_{U \cap U'}$.

The equivalence relation is clearly reflexive and symmetric. Now suppose $(U, f) \sim (U', f')$ and $(U', f') \sim (U'', f'')$, i.e. $f|_{U \cap U'} = f'|_{U \cap U'}$ and $f'|_{U' \cap U''} = f''|_{U' \cap U''}$, then $f|_{U \cap U' \cap U''} = f''|_{U \cap U' \cap U''}$, but $U \cap U' \cap U''$ is dense in $U \cap U''$, so the relation is transitive as well as expected.

If $(U, f) \sim (U', f')$, we can glue f and f' to get a regular map $f'' : U \cup U' \rightarrow W$, and $(U \cup U', f'') \sim (U, f) \sim (U', f')$. This means we can find a maximal representative defined on the union of the open subsets. This leads to:

Definition 1.2.31. The *domain* of a rational function ϕ , denoted by $\text{dom } \phi$, is the maximal open subset on which ϕ is defined, i.e. the set of points $x \in V$ such that $\exists U \ni x$ an open neighbourhood with $f : U \rightarrow W$ representing ϕ . These points are called *regular points*.

One can define addition (and subtraction) on rational functions:

$$(U, f) + (U', f') := (U \cap U', f|_{U \cap U'} + f'|_{U \cap U'})$$

and multiplication:

$$(U, f)(U', f') := (U \cap U', f|_{U \cap U'} f'|_{U \cap U'})$$

with multiplicative inverse

$$(U, f)^{-1} = \left(U \setminus f^{-1}(0), \frac{1}{f} \right),$$

so rational functions on U form a field, denoted by $k(U)$.

Week 3, lecture 2, 23rd January

The lecture was not recorded; see Proposition 6.1 to Proposition 6.2 in the given lecture notes.

Week 3, lecture 3, 24th January

1.2.3 Hypersurfaces and birational equivalence

Recall that we now have the algebra–geometry dictionary:

- V is an irreducible affine subvariety $\iff k[V]$ is a domain
- Interpret $k(V)$ as the field of fractions of $k[V]$, and just like 1.2.10, we have a bijection between dominant rational maps $V \dashrightarrow W$ and k -field homomorphisms $k(W) \rightarrow k(V)$ by sending $\varphi : V \dashrightarrow W$ to $\varphi^* : k(W) \rightarrow k(V) : g \mapsto g \circ \varphi$. In particular, V and W are birational iff $k(V)$ and $k(W)$ are k -isomorphic.

We now use this dictionary to interpret some algebraic results in terms of geometry. First we have a lemma from algebra:

Lemma 1.2.32. Let k be an algebraically closed field and K be a finitely generated extension (NB not finite extension) of k . Then $\exists t_1, \dots, t_d, u \in K$ such that

1. $K = k(t_1, \dots, t_d, u)$.
2. There is no nonzero $f \in k[x_1, \dots, x_d]$ such that $f(t_1, \dots, t_d) = 0$. (In this case we say t_1, \dots, t_d are *algebraically independent* over k .)
3. u is algebraic over $k(t_1, \dots, t_d)$.

Proof. This follows from the primitive element theorem. □

Proposition 1.2.33. Let V be an irreducible affine subvariety, then V is birational to a hypersurface in some affine space \mathbb{A}_k^{d+1} .

Proof. Since V is irreducible, $k[V]$ is a domain, hence a finitely generated k -algebra, hence $K = k[V]$ is a finitely generated field extension of k . Apply the lemma and write $K = k(t_1, \dots, t_d, u)$. Let $p(U) \in k(t_1, \dots, t_d)[U]$ be the minimal polynomial of u over $k(t_1, \dots, t_d)$. Write

$$p(U) = \sum_{k=0}^n \frac{g_k(t_1, \dots, t_d)}{h_k(t_1, \dots, t_d)} U^k$$

where g and h have no common factor, and let h be the lowest denominator of the h_k 's, so

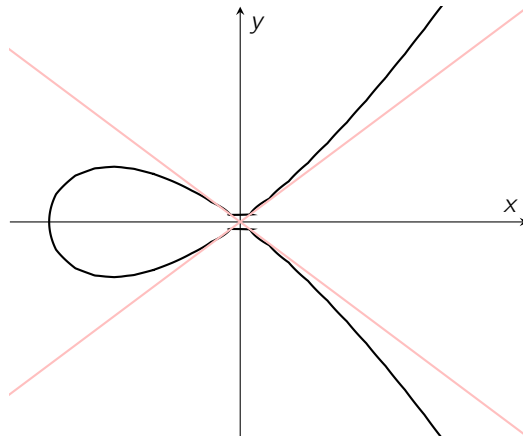
$$p(U) = \frac{g(t_1, \dots, t_d, U)}{h(t_1, \dots, t_d)}$$

for some g . But now $p(u) = 0$, so $g(t_1, \dots, t_d, U) = 0$. Note that g is irreducible by construction. Then let $H = V(g) \subset \mathbb{A}_k^{d+1}$ be a hypersurface. Since g is irreducible, so $(g) \subset k[T_1, \dots, T_d, U]$ is radical, so $I(H) = (g)$, and so $k[H] = k[T_1, \dots, T_d, U]/(g)$.

Now consider the map $k[H] \rightarrow K : T_1, \dots, T_d, U \mapsto t_1, \dots, t_d, u$, then it's surjective since K is proved to be generated by these $d+1$ images, and it has kernel (g) , hence $k[H] \cong K = k[V]$, hence V is birational to H . □

Example 1.2.34. Consider $W = V(y^2 - x^3) \subset \mathbb{A}_k^2$ with $\varphi : W \dashrightarrow \mathbb{A}_k^1 : (x, y) \mapsto \frac{y}{x}$ and $\psi : \mathbb{A}_k^1 \dashrightarrow W : t \mapsto (t^2, t^3)$. Clearly $\text{dom } \psi = \mathbb{A}_k^1$ and $\text{dom } \varphi = W \setminus \{(0, 0)\}$. Then W is birational to \mathbb{A}_k^1 .

A slightly more tricky one is $V = V(y^2 - x^3 - x^2) \subset \mathbb{A}_k^2$, which has a node at the origin.



$\varphi : V \dashrightarrow \mathbb{A}_k^1 : (x, y) \mapsto \frac{y}{x}$ is rational. Now parameterising the points by $y = tx$, we have $t^2x^2 - x^3 - x^2 = x^2(t^2 - x - 1) = 0$, so either $x = y = 0$ or $x = t^2 - 1$ and $y = t^3 - t$, so we have the rational map $\psi : \mathbb{A}_k^1 : t \mapsto (t^2 - 1, t^3 - t)$, and

$$\psi \circ \varphi : V \dashrightarrow V : (x, y) \mapsto \left(\frac{y^2}{x^2} - 1, \frac{y^3}{x^3} - \frac{y}{x} \right) = \left(\frac{x^3 + x^2}{x^2} - 1, y \left(\frac{x^3 + x^2}{x^2} - 1 \right) \right) = (x, y),$$

with

$$\varphi \circ \psi : \mathbb{A}_k^1 \dashrightarrow \mathbb{A}_k^1 : t \mapsto \frac{t^3 - t}{t^2 - 1} = t,$$

so V is birational with \mathbb{A}_k^1 as well.

Exercise 1.2.35. 1. Let $\varphi : V \rightarrow W$ be a regular map between affine subvarieties. Prove that $\varphi^* : k[W] \rightarrow k[V]$ is injective iff $\text{im } \varphi$ is dense in W .

Solution. If $f \in k[W]$ has $f \circ \varphi = 0$, then $f|_{\text{im } \varphi} = 0$, but now if $\text{im } \varphi$ is dense in W , then $f = 0$ everywhere in W , hence φ^* is injective.

Now if $\text{im } \varphi \subset W$ is not dense, then by definition there is a nonempty open $U \subset W$ with $\text{im } \varphi \cap U = \emptyset$, so in particular $W \setminus U$ is closed in W , but precisely by definition of Zariski topology, there is a $f \in k[W]$ that vanishes on $W \setminus U$ and not identically 0, but $f|_{\text{im } \varphi} = 0$, i.e. $f \circ \varphi = 0$, hence φ^* is not injective.

2. Determine all isomorphisms $\mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$.

Solution. All isomorphisms $\mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ are given by polynomials of degree 1. Indeed, the inverse of $x \mapsto \lambda x + a$ is $x \mapsto \lambda^{-1}x - a\lambda^{-1}$.

Now suppose $\varphi : \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1 : x \mapsto P(x)$ and $\psi^{-1} : \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1 : x \mapsto Q(x)$ satisfy $\varphi \circ \psi = \text{id}$. Then $PQ = x$, so $\deg P = \deg Q = 1$.

Week 4, lecture 1, 27th January

2 Projective

If we look at solutions of polynomial equations over \mathbb{R} , we are missing points in the algebraically closed field \mathbb{C} . But over \mathbb{C} we still “miss” points, leading us to projective varieties. For example, two lines in general position will intersect at one point, but two parallel lines don't intersect in the affine sense; we add a “point at infinity” to make it more consistent.

Definition 2.0.1. The n dimensional *projective space* \mathbb{P}^n is the quotient of $k^{n+1} \setminus \{(0, \dots, 0)\}$ by the equivalence relation

$$(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n) \quad \text{for } \lambda \in k^\times.$$

Denote the points of \mathbb{P}^n by $[x_0 : \dots : x_n]$, called *homogeneous coordinates*.

One can embed \mathbb{A}^n into \mathbb{P}^n by $(x_1, \dots, x_n) \mapsto [1 : x_1 : \dots : x_n]$. The complementary is the set $\mathbb{P}^n \supset \{[0 : x_1 : \dots : x_n]\} \cong \mathbb{P}^{n-1}$, and hence $\mathbb{P}^n \cong \mathbb{A}^n \cup \mathbb{P}^{n-1}$, i.e. a projective space is an affine space together with points at infinity. For example, $\mathbb{P}^1 = \mathbb{A}^1 \cup \{[0 : 1]\}$.

2.1 Topology on \mathbb{P}^n

Definition 2.1.1. The *Zariski topology* on \mathbb{P}^n is the quotient topology induced by $\pi : \mathbb{A}^{n+1} \setminus \{(0, \dots, 0)\} \rightarrow \mathbb{P}^n$.

Remark 2.1.2. By this definition, $V \subset \mathbb{P}^n$ is closed iff $\pi^{-1}(V) \subset \mathbb{A}^{n+1} \setminus \{(0, \dots, 0)\}$ is closed, i.e. $\overline{\pi^{-1}(V)} \subset \mathbb{A}^{n+1}$ is closed, where

$$\overline{\pi^{-1}(V)} = \begin{cases} \pi^{-1}(V) \cup \{(0, \dots, 0)\} & \text{if } V \neq \emptyset \\ \emptyset & \text{if } V = \emptyset \end{cases}.$$

Now of course, by the usual definition, $\overline{\pi^{-1}(V)} \subset \mathbb{A}^{n+1}$ is closed iff $\overline{\pi^{-1}(V)} = V(I)$ for some ideal $I \subset k[x_0, \dots, x_n]$. But note that since we have the scaling equivalence relation in our projective space, if $(x_0, \dots, x_n) \in \overline{\pi^{-1}(V)}$, then $f(x_0, \dots, x_n) = 0$ for $f \in I$, and for any $\lambda \in k$ we have $(\lambda x_0, \dots, \lambda x_n) \in \overline{\pi^{-1}(V)}$, i.e. $f(\lambda x_0, \dots, \lambda x_n) \in I$. We give this type of ideal I a name:

Definition 2.1.3. An ideal $I \subset k[x_0, \dots, x_n]$ is *homogeneous* if $f \in I, \lambda \in k \implies f(\lambda x_0, \dots, \lambda x_n) \in I$.

In particular, if f is homogeneous, then (f) is homogeneous. More generally,

Proposition 2.1.4. Let $I \subset k[x_0, \dots, x_n]$ be an ideal. The following are equivalent.

1. I is homogeneous.
2. $\forall f \in I$, each homogeneous part $f^{(d)} \in I$ (the sum of all terms of degree d in f).
3. I is generated by homogeneous polynomials.

Proof. $2 \implies 3$: Since $k[x_0, \dots, x_n]$ is Noetherian, suppose f_i 's generated I , then I is also generated by homogeneous parts of these generators, hence generated by homogeneous polynomials.

$3 \implies 1$: Write $f = \sum_i g_i f_i \in I$ where f_i 's are homogeneous of degree d_i , then

$$f(\lambda x_0, \dots, \lambda x_n) = \sum_i g_i \lambda^{d_i} f_i(x_0, \dots, x_n) \in I.$$

$1 \implies 2$: take $f \in I$ and write $f = f^{(0)} + \dots + f^{(d)}$. Since k is infinite, let $c_0, \dots, c_d \in k^\times$ be distinct elements, i.e. $c_i - c_j \neq 0 \forall 0 \leq i, j \leq d$. In particular, the $(d+1) \times (d+1)$ matrix $(c_i^j)_{0 \leq i, j \leq d}$ is invertible since it has Vandermonde determinant $\prod_{0 \leq i < j \leq d} (c_i - c_j)$. Now

$$f(c_i x_0, \dots, c_i x_n) = \sum_j c_i^j f^{(j)}(x_0, \dots, x_n) \in I,$$

so applying the inverse of $(c_i^j)_{0 \leq i, j \leq d}$ gives us that each $f^{(j)} \in I$. □

In particular, by Hilbert basis theorem, a homogeneous ideal has a finite basis of homogeneous polynomials.

We now claim that there is indeed a bijection

$$\{\text{homogeneous ideals of } k[x_0, \dots, x_n]\} \setminus \{(x_0, \dots, x_n)\} \leftrightarrow \{\text{closed subsets of } \mathbb{P}^n\}$$

via

$$V \mapsto I(\overline{\pi^{-1}(V)}) = \{f \in k[x_0, \dots, x_n] : f(x_0, \dots, x_n) = 0 \forall [x_0 : \dots : x_n] \in V\} =: \tilde{I}(V)$$

and

$$I \mapsto \pi(V(I) \setminus \{0, \dots, 0\}) = \{[x_0 : \dots : x_n] \in \mathbb{P}^n : f(x_0, \dots, x_n) = 0 \forall f \in I\} =: \tilde{V}(I).$$

One can check that $\tilde{V}(\tilde{I}(V)) = V$. What about $\tilde{I}(\tilde{V}(I))$?

Proposition 2.1.5 (Projective Nullstellensatz). Let $I \subset k[x_0, \dots, x_n]$ be homogeneous and different from the irrelevant ideal (x_0, \dots, x_n) . Then $\tilde{I}(\tilde{V}(I)) = \sqrt{I}$.

Proof. If $I = (x_0, \dots, x_n)$ then $V(I) = \{(0, \dots, 0)\}$, so $\tilde{V}(I) = \emptyset = \tilde{V}(k[x_0, \dots, x_n])$. Removing this we can directly apply the classical Nullstellensatz. □

Definition 2.1.6. A *projective subvariety* of \mathbb{P}^n is a closed subset $V = \tilde{V}(I)$ for a homogeneous ideal I , i.e.

$$V = \{[x_0 : \dots : x_n] \in \mathbb{P}^n : f_1(x_0, \dots, x_n) = \dots = f_r(x_0, \dots, x_n) = 0\}$$

for some homogeneous polynomials f_1, \dots, f_r .

Example 2.1.7. Let $V = \tilde{V}(wx - y^2) \subset \mathbb{P}^2 = \{[w : x : y] \in \mathbb{P}^2 : wx - y^2 = 0\}$. We make sense of V by considering $\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$:

$$V \cap \mathbb{A}^2 = \{[1 : x : y] : x - y^2 = 0\} = V(x - y^2),$$

and

$$V \cap \mathbb{P}^1 = \{[0 : x : y] : -y^2 = 0\} = \{[1 : 0]\},$$

i.e. a parabola together with a point at infinity in the direction $(1, 0)$, i.e. the x -axis. Informally, the two arms of the parabola meet at infinity.

But we can also consider the embedding $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$ in another way: $(w, x) \mapsto [w : x : 1]$, then

$$V \cap \mathbb{A}^2 = \{[w : x : 1] : wx - 1 = 0\} = V(wx - 1)$$

and

$$V \cap \mathbb{P}^1 = \{[w : x : 0] : wx = 0\} = \{[1 : 0], [0 : 1]\},$$

i.e. a hyperbola together with the two asymptotes.

2.1.1 Homogenisation

We now know how to interpret projective subvarieties with affine ones, but how do we go from affine to projective? By adding a variable, we can homogenise a polynomial $f \in k[x_1, \dots, x_n]$ to $\bar{f} \in k[x_0, \dots, x_n]$:

$$f = f^{(0)} + \dots + f^{(d)} \rightsquigarrow \bar{f} = x_0^d f^{(0)} + x_0^{d-1} f^{(1)} + \dots + x_0 f^{(d-1)} + f^{(d)}.$$

Using the above example, $x - y^2$ homogenises to $wx - y^2$ and $wx - 1$ to $wx - y^2$, as expected.

Example 2.1.8 (Twisted cubic). Let $C = \{(t, t^2, t^3) \in \mathbb{A}^3 : t \in k\} = V(y - x^2, z - xy)$. Then C defines a projective subvariety

$$C' = \{[w : x : y : z] \in \mathbb{P}^3 : wy - x^2 = wz - xy = 0\}.$$

By our construction, naturally $C' \cap \mathbb{A}^3 = C$. What's more interesting is

$$C' \cap \mathbb{P}^2 = \{[0 : x : y : z] \in \mathbb{P}^3 : -x^2 = -xy = 0\} = \{[0 : y : z] \in \mathbb{P}^2\} = \mathbb{P}^1,$$

so in some sense we have a "curve at infinity", which is not quite what we expect since C itself is a curve. In fact, the smallest projective subvariety that C embeds into is not C' , but

$$C'' = \{[w : x : y : z] \in \mathbb{P}^3 : wy - x^2 = wz - xy = zx - y^2 = 0\},$$

then again

$$C'' \cap \mathbb{A}^3 = \{[1 : x : y : z] \in \mathbb{P}^3 : y - x^2 = z - xy = zx - y^2 = 0\} = C$$

(since $y - x^2 = z - xy = 0 \implies zx - y^2 = 0$) but

$$C'' \cap \mathbb{P}^2 = \{[0 : x : y : z] \in \mathbb{P}^3 : -x^2 = -xy = zx - y^2 = 0\} = \{[0 : 0 : 1]\},$$

now a single point. In general, for $V = V(I) \subset \mathbb{A}^n$, define $\bar{V} \subset \mathbb{P}^n$ to be the projective subvariety defined by the ideal generated by homogenisations of each $f \in I$; i.e. it's generally not enough to just homogenise the generators of I .

Week 4, lecture 2, 30th January

2.2 Regular maps between projective varieties

Let $V \subset \mathbb{P}^n, W \subset \mathbb{P}^m$ be projective subvarieties. We want to say that a map $f : V \rightarrow W$ is regular if we can write it as polynomials, i.e.

$$f[x_0 : \dots : x_n] = [f_0(x_0, \dots, x_n) : \dots : f_m(x_0, \dots, x_n)]$$

for some polynomials f_0, \dots, f_m . But for this to be well-defined it has to be independent of choice of coordinates, i.e.

$$[f_0(\lambda x_0, \dots, \lambda x_n) : \dots : f_m(\lambda x_0, \dots, \lambda x_n)] = [f_0(x_0, \dots, x_n) : \dots : f_m(x_0, \dots, x_n)],$$

so f_0, \dots, f_m must be homogeneous of the same degree; and since $[0 : \dots : 0]$ is not defined, at least one f_i has $f_i(x_0, \dots, x_n) \neq 0$. The problem is these two are very strong conditions and most of the time it's hard to find maps which can be written in this form. But we can still study local behaviour like before. Let's first look at an example.

Example 2.2.1. Let $V = \tilde{V}(wy - x^2) \subset \mathbb{P}^2$. Recall that $V' = V \cap \{[w : x : y] : w \neq 0\}$ is $V(y - x^2)$, a parabola, which is isomorphic with \mathbb{A}^1 with $\psi : \mathbb{A}^1 \rightarrow V' : t \mapsto (t, t^2)$ and $\varphi : V' \rightarrow \mathbb{A}^1 : (x, y) \mapsto x$.

Can we then extend this to a bijection between V and \mathbb{P}^1 ? We want $\psi' : \mathbb{P}^1 \rightarrow V : [1 : t] \mapsto [1 : t : t^2]$, hence homogenise this and write $\psi' : [s : t] \mapsto [s^2 : st : t^2]$, which indeed does not vanish for $s \neq 0$ or $t \neq 0$, and one has that point of infinity $[0 : 1]$ mapped to point of infinity $[0, 0, 1]$.

Now if we try to extend φ to $[1 : x : y] \mapsto [1 : x]$ and homogenise to $[w : x : y] \mapsto [w : x]$, we have a problem: if $w = 0$ then $x^2 = wy = 0$ so $x = 0$, and $[0 : 0 : 1] \mapsto [0 : 0]$, an undefined point. But since $x^2 = wy$, one can rewrite $[w : x] = [wx : x^2] = [wx : wy] = [x : y]$, and then in particular $[0 : 0 : 1] \mapsto [0 : 1]$ now as expected. This of course requires $w, x \neq 0$, so define φ' by

$$[w : x : y] \mapsto \begin{cases} [w : x] & \text{if } w \neq 0 \\ [x : y] & \text{if } y \neq 0 \end{cases}$$

which is an inverse of ψ' .

Definition 2.2.2. A *quasiprojective variety* is an open subset in a projective variety.

Example 2.2.3. Since the whole space is always both open and closed, any projective variety itself is quasiprojective. Any affine variety is quasiprojective as well: it's the intersection of \mathbb{A}^n (which is open in \mathbb{P}^n) with a projective variety.

Definition 2.2.4. Given two quasiprojective varieties $V \subset \mathbb{P}^n, W \subset \mathbb{P}^m$, a *regular map* $\varphi : V \rightarrow W$ is a function such that $\forall x \in V, \exists$ an open neighbourhood $V \supset U \ni x$ and $f_0, \dots, f_m \in k[x_0, \dots, x_n]$ homogeneous of the same degree such that:

1. For each $y \in U$, at least one $f_i(y) \neq 0$
2. For each $y = [y_0 : \dots : y_n] \in U$, one can write $\varphi(y) = [f_0(y_0, \dots, y_n) : \dots : f_m(y_0, \dots, y_n)]$

(i.e. the properties we deemed necessary to satisfy).

If we want to practically build such a map, consider open subsets U_α covering V and families of homogeneous polynomials of the same degree (which depends on α). For instance, in 2.2.1, we have $\{[w : x : y] : w \neq 0\}$ and $\{[w : x : y] : y \neq 0\}$ covering \mathbb{P}^2 , and two families of homogeneous polynomials: w, x and x, y . By our definition, we need to check 4 things to see this is a regular map:

1. each covering set is indeed open (check, $\{w = 0\}$ and $\{y = 0\}$ are closed)
2. an expression never gives $[0 : \dots : 0]$ (check, we specifically write in this form to avoid this)
3. two expressions agree wherever they are both defined (check; when $w, y \neq 0$, we have $x \neq 0$ and so $[w : x] = [wy : xy] = [x^2 : xy] = [x : y]$)
4. the image is contained in W (check, this is trivial since in the above case W is the whole \mathbb{P}^1)

Lemma 2.2.5. Let $\varphi, \psi : V \rightarrow W$ be regular maps between quasiprojective varieties. If there is a dense subset $A \subset V$ such that $\varphi|_A = \psi|_A$ then $\varphi = \psi$. (cf. 1.2.8)

Proof. Let $Z = \{x \in V : \varphi(x) = \psi(x)\}$. Then $Z \supset A$ where $A \subset V$ is dense. We want to show $Z = V$, so it suffices to show Z is closed in V . Take an open cover $\{U_\alpha\}$ of V and write

$$\varphi|_{U(\alpha)} = [f_{\alpha,0} : \dots : f_{\alpha,m}], \quad \psi|_{U(\alpha)} = [g_{\alpha,0} : \dots : g_{\alpha,m}],$$

then it suffices to show

$$Z \cap U_\alpha = \{x \in U_\alpha : [f_{\alpha,0} : \dots : f_{\alpha,m}] = [g_{\alpha,0} : \dots : g_{\alpha,m}]\}$$

is closed to show Z is closed. But then this is precisely the zero set of polynomials

$$f_{\alpha,i}(x)g_{\alpha,j}(x) - f_{\alpha,j}(x)g_{\alpha,i}(x) \quad i, j \in \{0, m\}$$

by the definition of equivalence in projective space, and note that these polynomials are homogeneous. □

Lemma 2.2.6. A regular map between quasiprojective varieties is continuous. (cf. 1.2.7)

Proof. Let $V \subset \mathbb{P}^n, W \subset \mathbb{P}^m$ be quasiprojective, $\varphi : V \rightarrow W$ be regular and $Z \subset W$ be closed, so write

$$Z = \{y \in W : g_0(y) = \dots = g_r(y) = 0\}$$

where g_i 's are homogeneous. Cover V with $\{U_\alpha\}$ and take $f_{\alpha,0}, \dots, f_{\alpha,m}$ representing φ . Then

$$\begin{aligned} \varphi^{-1}(Z) &= \{x \in V : [f_{\alpha,0}(x) : \dots : f_{\alpha,m}(x)] \in Z\} \\ &= \{x \in V : g_0[f_{\alpha,0}(x) : \dots : f_{\alpha,m}(x)] = \dots = g_r[f_{\alpha,0}(x) : \dots : f_{\alpha,m}(x)] = 0\} \end{aligned}$$

is defined by homogeneous polynomials, hence closed. □

Lemma 2.2.7. Let $V \subset \mathbb{P}^n, W \subset \mathbb{P}^m, Z \subset \mathbb{P}^r$ be quasiprojective and $\varphi : V \rightarrow W, \psi : W \rightarrow Z$ be regular. Then $\psi \circ \varphi : V \rightarrow Z$ is regular.

Remark 2.2.8. Consider the embedding of $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$ by $(x_1, \dots, x_n) \mapsto [1 : x_1 : \dots : x_n]$. Then a regular map $\mathbb{P}^n \rightarrow \mathbb{P}^m$ is locally $[x_0 : \dots : x_n] \mapsto [f_0(x_0, \dots, x_n) : \dots : f_m(x_0, \dots, x_n)]$, i.e.

$$[1 : x_1 : \dots : x_n] \mapsto \left[1 : \frac{f_1(1, x_1, \dots, x_n)}{f_0(1, x_1, \dots, x_n)} : \dots : \frac{f_m(1, x_1, \dots, x_n)}{f_0(1, x_1, \dots, x_n)} \right],$$

which can be mapped back to a rational function $\mathbb{A}^n \rightarrow \mathbb{A}^m$ (one can reverse this too).

First two problems are covered before in lecture.

Exercise 2.2.9. 3. Let $C = V(x^2 + y^2 - 1) \subset \mathbb{A}^2$ and $\varphi : V \dashrightarrow \mathbb{A}^1$ be the rational function given by $(x, y) \mapsto \frac{1-y}{x}$. Find $\text{dom } \varphi$.

Solution. A priori the function is undefined on $x = 0$, i.e. on points $(0, -1)$ and $(0, 1)$. But note that over C one has $x \cdot x = x^2 = 1 - y^2 = (1 + y)(1 - y)$ so $\frac{1-y}{x} = \frac{x}{1+y}$, hence $(0, 1) \in \text{dom } \varphi$, but we still have a problem with $(0, -1)$. Suppose $(0, -1) \in \text{dom } \varphi$. Then $\exists f, g$ with $g(0, -1) \neq 0$ and $\frac{1-y}{x} = \frac{f}{g}$ in $k(C)$, i.e. $g \cdot (1 - y) = fx$ in $k[C]$, then at $(0, -1)$ one has $2g = 0$, so if $\text{char } k \neq 2$ then $g = 0$, a contradiction. If $\text{char } k = 2$ then $x^2 + y^2 = (x + y)^2$ and $1 = -1$ so $C = V(x + y - 1)$ and $\varphi = 1$.

4. Determine the irreducible components of $V = V(y^2 - xz, x^2y - z) \subset \mathbb{A}^3$.

Solution. Write $f = y^2 - xz, g = x^2y - z$. Then $f - xg = y^2 - xz - x^3y + xz = y^2 - x^3y = y(y - x^3)$, and so $V = V(y(y - x^3), x^2y - z) = V(y, x^2y - z) \cup V(y - x^3, x^2y - z) = V(y, z) \cup V(y - x^3, z - x^5)$. Both components are isomorphic to \mathbb{A}^1 ; the first is x -axis, the second by the parameterisation $t \mapsto (t, t^3, t^5)$ and projection $(x, y, z) \mapsto x$.

5. Let $V = V(xy) \subset \mathbb{A}^2$, $W = V(y(y - x^2)) \subset \mathbb{A}^2$.

- Determine the irreducible components of V and W .
- Which points lie in more than one irreducible component of W ?
- Prove that $f : W \rightarrow k$ via $(x, 0) \mapsto x^2$ and $(x, y) \mapsto 0$ for $y \neq 0$ is regular on W .
- Prove that $g : W \rightarrow k$ via $(x, 0) \mapsto x$ and $(x, y) \mapsto 0$ for $y \neq 0$ is not regular on W .
- Write down polynomials in $k[x, y]$ which define a surjective regular map $V \rightarrow W$.
- Is W isomorphic to V ?

Solution.

- $V = V(x) \cup V(y)$ and $W = V(y) \cup V(y - x^2)$. One is two axes and the other is x -axis with the parabola $y = x^2$, so both are two lines with an intersection at $(0, 0)$, hence intuitively they should be isomorphic?
- $(0, 0)$.
- W is affine, so f is regular if $\exists F \in k[x, y] : F(x, y) = f(x, y) \forall (x, y) \in W$. In particular, $y \neq 0$ means $(x, y) \in V(y - x^2)$, so we want $F(x, 0) = x^2$ and $F(x, x^2) = 0$. Clearly $F(x, y) = x^2 - y$ works.
- By the same analysis, we want $F \in k[x, y]$ with $F(x, 0) = x$, $F(x, x^2) = 0$. Then write $F(x, y) = x + yG(x, y)$, so $F(x, x^2) = x + x^2G(x, y) = 0$, which is impossible.
- By (a), we can try to fix $V(y)$ and send $V(x)$ to $V(y - x^2)$, i.e. $(x, 0) \mapsto (x, 0)$ and $(0, y) \mapsto (y, y^2)$. This can be done by simply adding them together: define $f : V \rightarrow W$ via $(x, y) \mapsto (x + y, y^2)$.
- No. Suppose per contra $\psi : W \rightarrow V$ is an isomorphism. Since $V(x), V(y)$ are symmetric components of V and ψ maps irreducible components to irreducible components, WLOG one has ψ gives isomorphisms $V(y) \cong V(y)$ and $V(y - x^2) \cong V(x)$, and $\psi(0, 0) = (0, 0)$. Write $\psi(x, y) = (g(x, y), h(x, y))$. Since $V(y) \cong \mathbb{A}^1$, by a previous exercise (2), we have $g(x, y) = \lambda x$ for some $\lambda \in k^\times$, and for $y \neq 0$, $(x, y) \in V(y - x^2)$, so $\psi(x, y) \in V(x)$ and $g(x, y) = 0$. This contradicts (d).

6. Let $C = V(y^2 - x^3) \subset \mathbb{A}^2$ and $\varphi \in k(C)$ be given by $\frac{y}{x}$. Show that φ is not regular at $(0, 0)$.

Solution. Suppose $\exists f, g \in k[C]$ where $g(0, 0) \neq 0$ such that $\frac{f}{g} = \frac{y}{x}$ over C , i.e. $fx = yg$. Parameterise C by $t \mapsto (t^2, t^3)$. Then $f(t^2, t^3)t^2 = t^3g(t^2, t^3)$, i.e. $f(t^2, t^3) = tg(t^2, t^3)$. Hence $t^2 \mid f(t^2, t^3)$, so $t \mid g(t^2, t^3)$, thus $g(0, 0) = 0$, a contradiction.

Week 5, lecture 1, 3rd February

Definition 2.2.10. Given V, W quasiprojective and V irreducible, a *rational map* $\varphi : V \dashrightarrow W$ is an equivalence class defined by: for open subsets $U, U' \subset V$ and regular maps f, f' on them, $(U, f) \sim (U', f')$ if $f|_{U \cap U'} = f'|_{U \cap U'}$.

Definition 2.2.11. Equivalently, let S be the set of tuples $(f_0, \dots, f_n) \in k[x_0, \dots, x_m]^{n+1}$ such that

- f_0, \dots, f_n are homogeneous of the same degree

2. $(f_0, \dots, f_n) \neq (0, \dots, 0)$
3. For an nonempty subset $U \subset V$ we have $[f_0(x) : \dots : f_n(x)] \in W$ for all $x \in U$.

Define the equivalence relation on S by: $(f_0, \dots, f_n) \sim (g_0, \dots, g_n)$ if $f_i g_j = f_j g_i \forall i, j$. Then a rational map $V \dashrightarrow W$ is an equivalence class in S .

Remark 2.2.12. Given $\varphi : V \dashrightarrow W$, $\text{dom } \varphi$ depends on W . For example, $\varphi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ by $[s : t] \mapsto [s^2 : st : t^2]$ has $\text{dom } \varphi = \mathbb{P}^1$, i.e. φ is regular everywhere. But now consider the map $\mathbb{P}^1 \dashrightarrow W$ given by the same formula where $W = \{[w : x : y] : w \neq 0\}$. Then $[0 : 1] \mapsto [0 : 0 : 1] \notin W$ so $\text{dom } \varphi \subsetneq \mathbb{P}^1$.

Example 2.2.13. Let $C = V(y - x^3) \subset \mathbb{A}^2$ and $\bar{C} = \tilde{V}(w^2y - x^3) = C \cup \{[0 : 0 : 1]\} \subset \mathbb{P}^2$. Consider the regular map $\varphi : C \rightarrow \mathbb{A}^1$ via $(x, y) \mapsto x$ and $\bar{\varphi} : \bar{C} \rightarrow \mathbb{P}^1$ via $[w : x : y] \mapsto [w : x]$. But this expression is not regular at $[0 : 0 : 1]$. We will show that $\text{dom } \bar{\varphi} = C \setminus \{[0 : 0 : 1]\}$.

Definition 2.2.14. Let $\varphi : V \dashrightarrow W$ be rational and write $\text{im } \varphi = \{\varphi(x) : x \in \text{dom } \varphi\}$. Then φ is *dominant* if $\text{im } \varphi$ is dense in W .

For V, W irreducible and $\varphi : V \dashrightarrow W$ dominant and $\psi : W \dashrightarrow T$ rational, one can define $\psi \circ \varphi : V \dashrightarrow T$. In particular, if $T = V$ and ψ is dominant and $\psi \circ \varphi = \text{id}_V$, $\varphi \circ \psi = \text{id}_W$, then V, W are *birational*, and φ is called a *birational equivalence*.

Lemma 2.2.15. Two irreducible quasiprojective varieties V, W are birational iff there are nonempty open subsets $U \subset V, U' \subset W$ which are isomorphic.

Proof. $A_1 = \text{dom } \varphi \subset V$ and $B_1 = \text{dom } \psi \subset W$ are nonempty and open. Now φ gives a regular $\varphi : A_1 \rightarrow W$, so continuous, so $A = \varphi^{-1}(B_1) \subset A_1 \subset V$ is open. Similarly, $\psi : B_1 \rightarrow V$ is continuous so $B = \psi^{-1}(A_1) \subset B_1 \subset W$. Hence $\varphi|_{A_1} : A \rightarrow B$ and $\psi|_{B_1} : B \rightarrow A$ are inverse isomorphisms. \square

In particular, \mathbb{A}^n is birational to \mathbb{P}^n .

Definition 2.2.16. A *rational function* on V is a rational map $\varphi : V \dashrightarrow \mathbb{A}^1$. The rational functions on V form a field $k(V)$.

2.3 Linear spaces in \mathbb{P}^n

Definition 2.3.1. A *hyperplane* in \mathbb{P}^n is a closed subset defined by a single homogeneous linear equation:

$$H = \{[x_0 : \dots : x_n] \in \mathbb{P}^n : h_0 x_0 + \dots + h_n x_n = 0\}$$

where $(h_0, \dots, h_n) \neq (0, \dots, 0)$.

A *linear subspace* of \mathbb{P}^n is a closed subset defined by any set of homogeneous linear equations.

Example 2.3.2. Hypersurfaces, \mathbb{P}^n itself, $\emptyset = \{[x_0 : \dots : x_n] : x_0 = \dots = x_n = 0\}$,
 $\{x\} = \{[y_0 : \dots : y_n] \in \mathbb{P}^n : [y_0 : \dots : y_n] = [x_0 : \dots : x_n]\} = \{[y_0 : \dots : y_n] \in \mathbb{P}^n : y_i x_j - y_j x_i = 0 \forall i, j\}$.

Definition 2.3.3. Given $\Lambda \subset \mathbb{P}^n$, define $C(\Lambda) = \{(x_0, \dots, x_n) \in k^{n+1} : [x_0 : \dots : x_n] \in \Lambda\}$. If Λ is a linear subspace of \mathbb{P}^n , then the cone $C(\Lambda)$ is a linear subspace of k^{n+1} . The dimension of $C(\Lambda)$ is already defined, so define $\dim \Lambda = \dim C(\Lambda) - 1$. In particular, hypersurfaces have dimension $n - 1$ and points have dimension 0.

Definition 2.3.4. A *line* in \mathbb{P}^n is a linear subspace of dimension 1, e.g. \mathbb{P}^1 .

Lemma 2.3.5. For any two distinct points $x, y \in \mathbb{P}^n$, there is a unique line passing by x and y .

Proof. Write $x = [x_0 : \dots : x_n]$ and $y = [y_0 : \dots : y_n]$. Then Λ contains $x, y \iff C(\Lambda)$ contains (x_0, \dots, x_n) and (y_0, \dots, y_n) . But $x \neq y$ so (x_0, \dots, x_n) and (y_0, \dots, y_n) are not collinear, so there is a unique 2-dimensional linear subspace containing them, hence Λ has dimension 1.

Explicitly,

$$L_{xy} = \{[sx_0 + ty_0 : \dots : sx_n + ty_n] \in \mathbb{P}^n : [s : t] \in \mathbb{P}^1\}.$$

\square

Let $H \subset \mathbb{P}^n$ be a hyperplane and up to change of coordinates write $H = \{[x_0 : \dots : x_n] \in \mathbb{P}^n : x_n = 0\}$. Let $p \in \mathbb{P}^n \setminus H$ and write $p = [p_0 : \dots : p_n]$ (so $p_n \neq 0$). Let $x \in \mathbb{P}^n \setminus \{p\}$, and consider L_{xp} , in particular $L_{xp} \cap H$. Intuitively, a plane intersects with a line that's not in the plane at only one point. Indeed, write

$$\begin{aligned} L_{xp} \cap H &= \{[sx_0 + tp_0 : \dots : sx_n + tp_n] \in \mathbb{P}^n : sx_n + tp_n = 0\} \\ &= \{[p_n x_0 - p_0 p_n : \dots : p_n x_{n-1} - p_{n-1} x_n : 0]\}. \end{aligned}$$

Now consider the map $\varphi : \mathbb{P}^n \dashrightarrow H \cong \mathbb{P}^{n-1}$ via $[x_0 : \dots : x_n] \mapsto [p_n x_0 - p_0 p_n : \dots : p_n x_{n-1} - p_{n-1} x_n : 0]$. It is regular on $\mathbb{P}^n \setminus \{p\}$, but not at p . This can be considered as a projection of x onto H from p by sending x to the unique intersection point of L_{xp} and H .

Lemma 2.3.6. The projection φ is not regular at p .

Proof. Consider $y \in H$. For any $x \in L_{yp} \setminus \{p\}$ one has $\varphi(x) = y$. If φ is regular over all \mathbb{P}^n then $\varphi|_{L_{yp}}$ is regular, but $L_{yp} \setminus \{p\}$ is dense so $\varphi|_{L_{yp}} = y$, in particular $\varphi(p) = y$. But this argument works the same for any $y' \in H$, so $\varphi(p)$ is undefined. \square

Week 5, lecture 2, 6th February

2.4 Products of quasiprojective varieties

With affine spaces it's easy to define products since naturally $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$. But the projective counterpart $\mathbb{P}^n \times \mathbb{P}^m = \mathbb{P}^{n+m}$ is not true; in particular consider

$$\mathbb{P}^1 \times \mathbb{P}^1 = (\mathbb{A}^1 \cup \{\text{pai}\}) \times (\mathbb{A}^1 \cup \{\text{pai}\}) = \mathbb{A}^2 \cup \mathbb{A}^1 \cup \mathbb{A}^1 \cup \{\text{pai}\} \neq \mathbb{A}^2 \cup \mathbb{A}^1 \cup \{\text{pai}\} = \mathbb{A}^2 \cup \mathbb{P}^1 = \mathbb{P}^2$$

(where pai means "point at infinity").

Now explicitly, $\mathbb{P}^n \times \mathbb{P}^m = \{[x_0 : \dots : x_n], [y_0 : \dots : y_m]\}$, and a closed subset of which would be

$$\{[x_0 : \dots : x_n], [y_0 : \dots : y_m] : f_1(x_0, \dots, x_n, y_0, \dots, y_m) = \dots = f_r(x_0, \dots, x_n, y_0, \dots, y_m) = 0\}$$

where $f_i \in k[x_0, \dots, x_n, y_0, \dots, y_m]$ are homogeneous of degrees d_i, d'_i , meaning one can write

$$f_i = \sum_{\alpha, \beta} \lambda_{\alpha, \beta} x_0^{\alpha_1} \dots x_n^{\alpha_n} y_0^{\beta_0} \dots y_m^{\beta_m}$$

with $\sum_{k=0}^n \alpha_k = d_i$ and $\sum_{k=0}^m \beta_k = d'_i$. In this case we say f_i is *bihomogeneous*.

The Segre embedding. We would want to embed $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^N$ for some N so that the set above can be considered as a closed subset of \mathbb{P}^N . Consider the map

$$\begin{aligned} \sigma_{m,n} : \mathbb{P}^n \times \mathbb{P}^m &\hookrightarrow \mathbb{P}^N \\ [x_0 : \dots : x_n], [y_0 : \dots : y_m] &\mapsto [(x_i y_j)_{0 \leq i \leq n, 0 \leq j \leq m}] \end{aligned}$$

where $N = (n+1)(m+1) - 1 = mn + m + n$. Denote the image by $\Sigma_{m,n}$.

Example 2.4.1. Consider $\sigma_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$ via $[x_0, x_1], [y_0, y_1] \mapsto [x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1]$. Consider the image as a matrix $\begin{pmatrix} x_0 y_0 & x_0 y_1 \\ x_1 y_0 & x_1 y_1 \end{pmatrix}$, which has rank 1. In fact, the image is precisely 2×2 rank 1 matrices and we can write it as

$$\Sigma_{1,1} = \{[z_{00} : z_{01} : z_{10} : z_{11} : z_{00} z_{11} - z_{01} z_{10} = 0]\} \subset \mathbb{P}^3$$

In particular this is defined by a homogeneous polynomial, hence a hypersurface.

More generally,

$$\begin{aligned} \Sigma_{n,m} &= \{[z_{ij}]_{0 \leq i \leq n, 0 \leq j \leq m} \in \mathbb{P}^N : (z_{ij})_{i,j} \text{ has rank } 1\} \\ &= \{[z_{ij}]_{0 \leq i \leq n, 0 \leq j \leq m} \in \mathbb{P}^N : z_{ij} z_{kl} - z_{kj} z_{il} = 0 \forall 0 \leq i, k \leq n, 0 \leq j, l \leq m\}, \end{aligned}$$

hence a closed subset.

Lemma 2.4.2. $\sigma_{m,n}$ is a bijection between $\mathbb{P}^n \times \mathbb{P}^m \leftrightarrow \Sigma_{n,m}$.

Proof. Since we defined $\Sigma_{m,n} = \text{im } \sigma_{m,n}$, it suffices to show $\sigma_{m,n}$ is injective. We do this by constructing an inverse $\Sigma_{n,m} \rightarrow \mathbb{P}^n \times \mathbb{P}^m$. Let $(z_{ij}) \in \Sigma_{n,m}$. Since $(z_{ij}) \neq 0$, for some $J = 1, \dots, m$, the J th column of (z_{ij}) contains a nonzero entry. Consider the projection $\Sigma_{n,m} \rightarrow \mathbb{P}^n : (z_{ij}) \mapsto [z_{0J} : \dots : z_{nJ}]$. But then this is independent of J since (z_{ij}) has rank 1, so every column is a multiple of the J th column. We can do this similarly with rows and get a projection $\Sigma_{n,m} \rightarrow \mathbb{P}^m : (z_{ij}) \mapsto [z_{I0} : \dots : z_{Im}]$. \square

Lemma 2.4.3. Let $V \subset \mathbb{P}^n \times \mathbb{P}^m$. $\sigma_{m,n}(V) \subset \Sigma_{n,m}$ is closed iff V is of the form

$$\{[x_0 : \dots : x_n], [y_0 : \dots : y_m] : f_1(x_0, \dots, x_n, y_0, \dots, y_m) = \dots = f_s(x_0, \dots, x_n, y_0, \dots, y_m) = 0\}$$

for some bihomogeneous polynomials $f_1, \dots, f_s \in k[x_0, \dots, x_n, y_0, \dots, y_m]$.

Proof. Suppose $\sigma_{m,n}(V)$ is closed and defined by homogeneous polynomials $g_r \in k[z_{ij}]$. Then

$$\begin{aligned} V &= \{[x_0 : \dots : x_n], [y_0 : \dots : y_m] : [x_i y_j] \in \sigma_{n,m}(V)\} \\ &= \{[x_0 : \dots : x_n], [y_0 : \dots : y_m] : f_1(x_i y_j) = \dots = f_s(x_i y_j) = 0\} \end{aligned}$$

Conversely, suppose V is defined as above by bihomogeneous polynomials f_r of degrees (d_r, d'_r) . WLOG suppose $d'_r \geq d_r$. Then $f_r = 0 \iff x_0^{d'_r - d_r} f = \dots = x_n^{d'_r - d_r} f = 0$, which are now homogeneous polynomials of degree d'_r . \square

Now everything is well-defined: products of closed subsets (i.e. projective varieties) are closed, and products of quasiprojective sets (intersection of an open subset and a closed subset) is also quasiprojective, hence the product actually means something.

Definition 2.4.4. Let $V \subset \mathbb{P}^n, W \subset \mathbb{P}^m$ be quasiprojective and $\varphi : V \rightarrow W$ regular map. The *graph* of φ is defined to be $\Gamma = \{(x, y) \in V \times W : y = \varphi(x)\}$.

Note that this is the preimage of $\iota \circ \varphi, \iota$ (where $\iota : W \hookrightarrow \mathbb{P}^m$ is the inclusion map) of $\Delta = \{x, y \in \mathbb{P}^m : x = y\} \subset \mathbb{P}^m \times \mathbb{P}^m$, and one can write Δ as

$$\{[x_0 : \dots : x_m], [y_0 : \dots : y_m] : x_i y_j - x_j y_i = 0 \forall i, j\},$$

evidently a closed set.

Week 5, lecture 3, 7th February

2.5 Completeness

In general, quasiprojective varieties are very not Hausdorff, and on the other hand, by the algebra–geometry dictionary we have Noetherianity (any descending chain of closed subsets stabilises), which implies quasicompactness. But then the notion of compactness here is not useful, since intersection of two compact sets is not necessarily compact (unlike in Euclidean topology).

Note that with \mathbb{A}^1 and $V = V(xy - 1) \subset \mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^2$, the second projection $p_2 : \mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1 : (x, y) \mapsto y$ maps V to $\{y \in \mathbb{A}^1 : y \neq 0\} = \mathbb{A}^1 \setminus \{0\}$, which is not closed. In this case we say \mathbb{A}^1 is not complete:

Definition 2.5.1. A quasiprojective variety V is *complete* if for any quasiprojective variety W , the second projection $p_2 : V \times W \rightarrow W$ is closed, i.e. maps closed sets to closed sets.

We give a formal consequence of this to capture this notion of completeness.

Theorem 2.5.2. A quasiprojective variety is complete iff it is projective.

Definition 2.5.3. A regular map $\varphi : V \rightarrow W$ is *proper* if for any quasiprojective variety Z , the map $\varphi \times \text{id} : V \times Z \rightarrow W \times Z$ is closed. In particular, V is complete if $V \rightarrow \{\text{pt}\}$ is proper.

Lemma 2.5.4. The composition of two proper maps is proper.

Proof. This follows from that composition of two closed maps is closed. \square

Lemma 2.5.5. A closed embedding $i : V \rightarrow W$ is proper.

Proof. i is a homeomorphism from V to $i(V)$, so any closed subset of V is also closed in W , i.e. i is a closed map, hence $i \times \text{id} : V \times Z \rightarrow W \times Z$ is a closed map as well. \square

Corollary 2.5.6. A closed subset of a complete variety is complete.

Proof. Let $V \subset W$ be closed and W complete. Consider the diagram

$$\begin{array}{ccc} V & \xrightarrow{i} & W \\ & \searrow k & \downarrow j \\ & & \{\text{pt}\} \end{array}$$

closed and proper, and j is proper since W is complete by assumption. Hence k is proper by 2.5.4, V is complete. \square

Therefore, to prove 2.5.2, it suffices to show \mathbb{P}^n is complete.

Lemma 2.5.7. Let $f : V \rightarrow W$ be proper. Then for any $y \in W$, we have $f^{-1}(y)$ is complete.

Proof. Let Z be a quasiprojective variety and $Y = f^{-1}(y) \times Z$. Since $\{y\}$ is closed, $f^{-1}(y)$ is closed in V , hence Y is closed in $V \times Z$, and since f is proper, $f(Y)$ is closed in $W \times Z$. \square

Lemma 2.5.8. Let $f : V \rightarrow W$ be proper. Then for any quasiprojective Z , the map $f \times \text{id} : V \times Z \rightarrow W \times Z$ is proper.

Proof. Let Y be a quasiprojective variety, and consider $f \times \text{id} \times \text{id} : (V \times Z) \times Y \rightarrow (W \times Z) \times Y$ where $(V \times Z) \times Y = V \times (Z \times Y)$ and $(W \times Z) \times Y = W \times (Z \times Y)$. By definition of proper, this map is closed, hence again by definition $f \times \text{id} : V \times Z \rightarrow W \times Z$ is proper. \square

Corollary 2.5.9. A product of two complete varieties is complete.

Proof. Let V, W be complete. Then $V \rightarrow \{\text{pt}\}$ and $W \rightarrow \{\text{pt}\}$ are proper, so by above $V \times W \rightarrow W$ is proper, hence $V \times W \rightarrow W \rightarrow \{\text{pt}\}$ is proper, so $V \times W$ is complete. \square

Proposition 2.5.10. If V is a complete variety and $f : V \rightarrow W$ is a surjective regular map, then W is complete.

Proof. Let Z be a quasiprojective variety and $Y \subset W \times Z$ closed. Let $Y' = (f \times \text{id})^{-1}(Y) \subset V \times Z$. Since f is regular, Y' is closed.

$$\begin{array}{ccccc} V \times Z & \xrightarrow{f \times \text{id}} & W \times Z & \xrightarrow{p_2} & Z \\ \downarrow & & \downarrow & & \\ Y' & & Y & & \end{array}$$

Since V is complete, the projection of Y' to Z is closed, but f is surjective, so this is a projection of Y to S , hence p_2 is closed, so W is complete. \square

Proposition 2.5.11. If V is complete, then any regular map $f : V \rightarrow W$ is proper, i.e. image of V under any regular map is closed.

Proof. Consider the graph $\Gamma_f = \{(x, y) : y = f(x)\} \subset V \times W$ which is closed. Then the map $V \rightarrow V \times W$ via $x \mapsto (x, f(x))$ is a closed embedding, hence by 2.5.5 is proper. But V is complete, so $V \rightarrow \{\text{pt}\}$ is proper, and $V \times W \rightarrow W$ is proper, hence $f : V \rightarrow V \times W \rightarrow W$ is proper. \square

Corollary 2.5.12. Every regular function on a connected complete variety is constant.

Proof. Consider regular $\varphi : V \rightarrow \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ with V connected and complete. Then $\varphi(V)$ is closed in \mathbb{A}^1 , but it's not \mathbb{A}^1 since it's not closed in \mathbb{P}^1 , so $\varphi(V)$ is a finite set of points. But V is connected, so $\varphi(V)$ is a single point. \square

Corollary 2.5.13. Every regular function from a connected complete variety to an affine variety is constant.

Proof. Consider regular $\varphi : V \rightarrow W \subset \mathbb{A}^n$ with V connected and completed. But each projection $x_i \circ \varphi : V \rightarrow \mathbb{A}^1$ is constant by above, so φ is constant. \square

Lemma 2.5.14. If $V \subset \mathbb{P}^n$ is an infinite complete variety and $H \subset \mathbb{P}^n$ is a hyperplane, then $V \cap H \neq \emptyset$.

Proof. Suppose per contra $V \cap H = \emptyset$, i.e. $V \subset \mathbb{P}^n \setminus H \cong \mathbb{A}^n$. Then since V is complete, apply the above using the trivial embedding $V \hookrightarrow \mathbb{A}^n$ to see that each connected component of V is a point, but there is a finite number of connected components, contradicting V is infinite. \square

Lemma 2.5.15. Let V be a quasiprojective variety. For any point $x \in V$, there is an open neighbourhood $U \subset V$ of x which is affine.

Proof. By definition, $V = U_0 \cup V_0$ where U_0 is open and V_0 is closed. Note that x lies in some \mathbb{A}^n ; write $x \in V \cap \mathbb{A}^n = (\mathbb{A}^n \cap U_0) \cap (V_0 \cap \mathbb{A}^n)$. By 1.2.18, $\mathbb{A}^n \cap U_0$ is covered by distinguished open subsets $D_V(f)$, hence $x \in D_V(f) \subset V$ for some f . \square

Week 6, lecture 1, 10th February

Proof of 2.5.2. Suppose V is complete. As V is quasiprojective, there is a regular immersion $i : V \hookrightarrow \mathbb{P}^n$, but by completeness $i(V) \cong V$ is closed in \mathbb{P}^n , so V is projective.

Conversely let $V \subset \mathbb{P}^m$ be projective and W be quasiprojective. We need to show $p_2 : V \times W \rightarrow W$ is closed. Let $Z \subset V \times W$ be closed. Cover W by affine open subsets U_α , then $p_2(Z)$ is closed in $W \iff p_2(Z) \cap U_\alpha$ is closed in U_α , but $p_2(Z) \cap U_\alpha = p_2(Z \cap (V \times U_\alpha))$ so we can assume W is affine; more precisely let $W \subset \mathbb{A}^n$ be closed. But now it suffices to show: for any $Z \subset \mathbb{P}^m \times \mathbb{A}^n$ closed, $p_2(Z) \subset \mathbb{A}^n$ is closed. Since Z is closed, it is defined by $f_1, \dots, f_r \in k[x_0, \dots, x_m, y_1, \dots, y_n]$ where f_i 's are homogeneous of degree d_i with respect to x and no condition with respect to y . Now for $y = (y_1, \dots, y_n) \in \mathbb{A}^n$, define

$$Z_y = \{x \in \mathbb{P}^m : (x, y) \in Z\} = \{[x_0 : \dots : x_m] \in \mathbb{P}^m : f_i(x_0, \dots, x_m, y_1, \dots, y_n) = 0 \forall i\}.$$

Then $p_2(Z) = \{y \in \mathbb{A}^n : Z_y \neq \emptyset\} \subset \mathbb{A}^n$. By 2.1.5, $Z_y \neq \emptyset \iff \sqrt{I_y} \neq k[x_0, \dots, x_m]$, where $I_y = (f_i(x_0, \dots, x_m, y_1, \dots, y_n)) \subset k[x_0, \dots, x_m]$, i.e. $S_d \not\subset I_y \forall d \geq 0$ where

$$S_d = \{\text{homogeneous polynomials of degree } d\} \subset k[x_0, \dots, x_m].$$

Hence one can write $p_2(Z) = \bigcap_{d \geq 0} W_d$ where $W_d = \{y \in \mathbb{A}^n : S_d \not\subset I_y\}$. So it suffices to show for $d \geq \max(d_1, \dots, d_r)$, $W_d \subset \mathbb{A}^n$ is closed. Let $g \in S_d$. Then $g \in I_y \iff \exists h_i \in k[x_0, \dots, x_m]$ such that $g = \sum_i h_i f_i$. Hence h_i is homogeneous of degree $d - d_i$. Then there is a linear map

$$\alpha_{d,y} : \bigoplus_{i=1}^r S_{d-d_i} \rightarrow S_d : (h_1, \dots, h_r) \mapsto \sum_{i=1}^r h_i f_i,$$

and one can write $S_d \cap I_y = \text{im } \alpha_{d,y}$, hence

$$\begin{aligned} W_d &= \{y \in \mathbb{A}^n : \alpha_{d,y} \text{ is not surjective}\} = \{y \in \mathbb{A}^n : \text{rk } \alpha_{d,y} < \dim S_d\} \\ &= \{y \in \mathbb{A}^n : \text{all } \dim S_d \times \dim S_d \text{ submatrices of } \alpha_{d,y} \text{ have det } 0\}, \end{aligned}$$

hence defined by polynomial equations, i.e. closed. \square

2.6 Dimension

2.6.1 Geometric definition

We want to talk about dimension of irreducible quasiprojective varieties. What do we expect?

1. $\dim \mathbb{A}^n = n$;
2. If $U \subset V$ is open then $\dim U = \dim V$;
- 1.+2. $\dim \mathbb{P}^n = \dim \mathbb{A}^n = n$ since $\mathbb{A}^n \subset \mathbb{P}^n$ is open;
3. If V, W are birational, by 2.2.15 they have nonempty isomorphic open subsets, hence by 2 one has $\dim V = \dim W$.
4. If $f : V \rightarrow W$ is finite-to-one and dominant, then $\dim V = \dim W$.

Note that 2 and 3 are equivalent again by 2.2.15, since if $U \subset V$ is open then both U, V contain the open subset U , which is clearly isomorphic to itself, hence U, V are birational.

It turns out that these characterise the dimension uniquely!

Lemma 2.6.1 (Geometric approach). Let V be an irreducible projective variety. Then there exists a surjective finite-to-one map $V \rightarrow \mathbb{P}^d$ for some d .

Proof. Consider V as a closed subset of \mathbb{P}^N . If $V = \mathbb{P}^N$ then we are done. Otherwise, there is some $p \in \mathbb{P}^N \setminus V$, so consider the projection from p to a hyperplane $H \cong \mathbb{P}^{N-1}$ $\pi : V \rightarrow \mathbb{P}^{N-1}$. By completeness, $\pi(V) \subset \mathbb{P}^{N-1}$ is closed. Let $y \in \pi(V)$, then $\pi^{-1}(y) = L_{yp} \cap V$ which is closed in $L_{yp} \cong \mathbb{P}^1$ and doesn't contain p , so it's a finite set of points. Define $V_1 = \pi(V)$ and iterate this (replace V by V_1 and get V_2 and so on) until $V_r = \mathbb{P}^{n-r}$, and we have $V \rightarrow V_1 \rightarrow V_2 \rightarrow \dots \rightarrow V_r = \mathbb{P}^{n-r}$ where each map is finite-to-one. \square

2.6.2 Algebraic definition

Theorem 2.6.2 (Preparation for algebraic approach). Let V and W be irreducible varieties over k and $f : V \rightarrow W$ a dominant, finite-to-one rational map. Then $k(W) \rightarrow k(V)$ is an algebraic extension of fields.

We begin with the lemma for 1-d case:

Lemma 2.6.3. Let W be an irreducible affine variety over k and $V \subset W \times \mathbb{A}^1$ an irreducible closed subvariety. Then $k(W) \rightarrow k(V)$ is an algebraic extension of fields.

Proof. Consider a coordinate T on \mathbb{A}^1 . The coordinate ring of $W \times \mathbb{A}^1$ is $k[W][T]$. Write $I = I(V) \subset k[W][T]$. As $V \rightarrow W$ is dominant, $I \cap k[W] = 0$. If $V \subsetneq \mathbb{A}^1 \times W$ then by the Nullstellensatz, $I \supsetneq \{0\}$, in particular I contains a nonconstant polynomial $P(T) \in k[W][T]$, so $k[V] = k[W][T]/I$ is a quotient of $k[W][T]/P(T)$, but $k[V]$ is generated over $k[W]$ by T , and T is a root of $P(T)$, so $k(W) \rightarrow k(V)$ is an algebraic extension. \square

Proof of 2.6.2. By replacing V, W by open subsets, we can assume they are affine. Consider V as a closed subset of \mathbb{A}^n and the graph $\Gamma_f \subset \mathbb{A}^n \times W$. Then $V \subseteq \Gamma_f$. Write $V_0 = \Gamma_f$ and define V_1 to be the closure of image of V_0 under projection $\mathbb{A}^n \times W \rightarrow \mathbb{A}^{n-1} \times W$. Iterate this and we have $V = V_0 \rightarrow V_1 \rightarrow \dots \rightarrow V_n = \mathbb{A}^0 \times W = W$, and by the lemma, each $k(V_{i-1}) \rightarrow k(V)$ is algebraic, so $k(V) \rightarrow k(W)$ is algebraic. \square

Week 6, lecture 2, 13th February

Definition 2.6.4. Let $k \rightarrow K$ be a field extension. The *transcendence degree* of K over k is the cardinal of a maximal k -algebraically independent set in K . Such a set is called a transcendence basis. By commutative algebra, this number is independent of choice of basis.

For example, $k(x_1, \dots, x_n)$ has transcendence degree n over k .

Now if $V \subset \mathbb{P}^n$, then $k(V) = k(V \cap \mathbb{A}^n)$ is a quotient of $k(x_1, \dots, x_n)$, so for an irreducible quasiprojective variety V , $k(V)$ has finite transcendence degree over k . We hence an algebraic definition of dimension of a irreducible quasiprojective variety:

Definition 2.6.5. Given an irreducible quasiprojective variety V , the *dimension* of V , denoted by $\dim V$, is the transcendence degree of $k(V)$ over k .

Let's check if it satisfies our expectations listed last time:

1. $k(\mathbb{A}^n) = k(x_1, \dots, x_n)$ does have degree n over k as mentioned above;
2. $\dim V$ depends only on $k(V)$, which is birational invariant;
3. If $f : V \rightarrow W$ is finite-to-one and dominant, by 2.6.2 $k(V)$ is algebraic over $k(W)$, hence they have the same transcendence degree.

Lemma 2.6.6. If V, W are irreducible quasiprojective varieties and $f : V \rightarrow W$ is dominant, then $\dim V \geq \dim W$.

Proof. f induces a field extension $k(W) \rightarrow k(V)$, so the transcendence degree of $k(V)$ over k is larger or equal to that of $k(W)$ over k . \square

Lemma 2.6.7. If V, W are irreducible quasiprojective varieties, then $\dim V \times W = \dim V + \dim W$.

Proof. Replace V and W with affine open subsets and then their projective closures doesn't change the dimension, so assume V and W are projective. Then we have surjective, finite-to-one maps $f : V \rightarrow \mathbb{P}^d$, $g : W \rightarrow \mathbb{P}^e$ where $d = \dim V$, $e = \dim W$. Then $f \times g : V \times W \rightarrow \mathbb{P}^d \times \mathbb{P}^e$ is surjective, finite-to-one as well, so $\dim(V \times W) = \dim(\mathbb{P}^d \times \mathbb{P}^e) = \dim(\mathbb{A}^d \times \mathbb{A}^e) = d + e$ since $\mathbb{A}^n \subset \mathbb{P}^n$ is open. \square

For projective nonempty $V \subset \mathbb{P}^n$, recall the projective cone of V is the closed subset

$$\tilde{V} = \{0\} \cup \{(x_0, \dots, x_n) \in \mathbb{A}^{n+1} : [x_0 : \dots : x_n] \in V\}.$$

Corollary 2.6.8. Let $V \subset \mathbb{P}^n$ be irreducible and projective and $\tilde{V} \subset \mathbb{A}^{n+1}$ the projective cone of V . Then $\dim \tilde{V} = \dim V + 1$.

Proof. Choose some i such that V is not contained in the hyperplane $x_i = 0$. WLOG $i = 0$. Then $\dim V = \dim V \cap \mathbb{A}^n$ and write $V \cap \mathbb{A}^n = U$. The cone \tilde{U} of U is the set $\{(t, tx_1, \dots, tx_n) : t \neq 0, (1 : x_1 : \dots : x_n) \in U\} \subset \tilde{V}$, which is open, so $\dim \tilde{V} = \dim \tilde{U}$. Hence it now suffices to show $\dim \tilde{U} = \dim U + 1$. But we have the isomorphism $(\mathbb{A}^1 \setminus \{0\}) \times U \rightarrow \tilde{U} \setminus \{(0, \dots, 0)\}$ by $(t, [1 : x_1 : \dots : x_n]) \mapsto (t, tx_1, \dots, tx_n)$. So

$$\dim \tilde{U} = \dim \tilde{U} \setminus \{(0, \dots, 0)\} = \dim(\mathbb{A}^1 \setminus \{0\}) \times U = \dim \mathbb{A}^1 + \dim U = \dim U + 1$$

as desired. \square

2.6.3 Dimension and intersection

Recall two facts about intersection of vector spaces from linear algebra:

Proposition 2.6.9. Let $E \subset k^n$ be a sub-vector space and $H \subset k^n$ a hyperplane not containing E , then $\dim E \cap H = \dim E - 1$.

Proposition 2.6.10. Let $E, F \subset k^n$ be two sub-vector spaces of dimension d, e such that $d + e \geq n$, then $\dim E \cap F \geq d + e - n$.

Theorem 2.6.11. Let $V \subset \mathbb{P}^n$ be an irreducible quasiprojective variety of dimension ≥ 1 and $H \subset \mathbb{P}^n$ a hyperplane not containing V , then every irreducible component of $V \cap H$ has dimension $\dim V - 1$. Moreover, if V is projective then $V \cap H \neq \emptyset$.

Proof. Let \bar{V} be the closure of V in \mathbb{P}^n , then V is open in \bar{V} and $V \cap H$ is open in $\bar{V} \cap H$, so $\dim V = \dim \bar{V}$ and $\dim V \cap H = \dim \bar{V} \cap H$, so we can continue the proof working with a projective V .

We prove the theorem by induction on n . If $n = 1$ then V has to be the whole \mathbb{P}^1 since it has dimension ≥ 1 , so $V \cap H = H$ (which is a point) is clearly not empty and has dimension 0.

Now suppose the result is proven for $n - 1$. If $V = \mathbb{P}^n$ then we're done by a similar argument. If $V \subsetneq \mathbb{P}^n$, we first show $H \not\subset V$. Indeed, if $H \subset V$, then $I(V) \subset I(H)$, so $I(V)$ is a homogeneous prime ideal contained in the principal ideal generated by the polynomial L defining H . Let P be an irreducible homogeneous polynomial in $I(V)$. Then $L \mid P$, but P is irreducible so $L = P$, hence $I(V) = I(H)$ but then $V = H$, contradicting that H does not contain V .

Fix a point $p \in H \setminus V$, and a hyperplane $\mathbb{P}^{n-1} \neq H$ not containing p . Consider the projection $\pi : \mathbb{P}^n \rightarrow \mathbb{P}^{n-1}$ from p which is regular on V . Then $\pi(V) = V'$ is closed in \mathbb{P}^{n-1} by completeness of projective varieties, and $\pi : V \rightarrow V'$ is surjective and finite-to-one, so $\dim V = \dim V'$. On the other hand, $H' = \pi(H)$ is a hyperplane in \mathbb{P}^{n-1} , and $\pi : V \cap H \rightarrow V' \cap H'$ is surjective and finite-to-one, so it maps each irreducible component of $V \cap H$ to an irreducible component of $V' \cap H'$ of the same dimension. By inductive hypothesis on $V' \subset \mathbb{P}^{n-1}$, $V' \cap H' \neq \emptyset$ and each of its irreducible component has dimension $\dim V' - 1 = \dim V - 1$, so the same holds for $V \cap H$. \square

Proposition 2.6.12. Let $V \subsetneq \mathbb{P}^n$ be an irreducible closed subset and $H \subset \mathbb{P}^n$ a hyperplane. If $H \subset V$ then $H = V$.

Proof. $H \subset V$ implies $\tilde{I}(V) \subset \tilde{I}(H) = (L)$ where L is some polynomial of degree 1. Then since V is irreducible, $\tilde{I}(V)$ is prime, hence generated by some nonconstant irreducible $P \in \tilde{I}(V)$, but then $L \mid P$ and since P is irreducible, $L \sim P$. \square

Week 6, lecture 3, 14th February: problem class (sheet 3)

Exercise 2.6.13. 1. (a) Let $\varphi : \mathbb{P}^n \rightarrow \mathbb{P}^m$ be regular. Prove that there exists homogeneous polynomial $f_0, \dots, f_m \in k[x_0, \dots, x_n]$ such that $[f_0(x_0, \dots, x_n) : \dots : f_m(x_0, \dots, x_n)]$ defined φ on all \mathbb{P}^n .

(b) Prove that if $\varphi : \mathbb{P}^n \rightarrow \mathbb{P}^1$ is nonconstant and regular, then $\text{im } \varphi \not\subset \mathbb{A}^1$.

(c) Prove that every rational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ is regular.

Solution.

(a) Proof is similar to the affine case (which is quite technical so omitted).

- (b) This is a consequence of completeness, but we can use (a). Suppose $\text{im } \varphi \subset \mathbb{P}^1$. Then we have f, g homogeneous of the same degree with $\varphi([x_0 : \dots : x_n]) = [f(x_0, \dots, x_n), g(x_0, \dots, x_n)]$ and f does not vanish on $\mathbb{A}^{n+1} \setminus \{(0, \dots, 0)\}$, so f is constant, but then g is constant too, hence φ is constant.
- (c) It suffices to show that the induced map φ on each \mathbb{A}^1 covering \mathbb{P}^1 is regular. Consider a local expression given by $f_0, \dots, f_n \in k[x]$, i.e. $\varphi(x) = [f_0(x) : \dots : f_n(x)]$. Divide them by their gcd and assume they are coprime. Then $\forall x \in \mathbb{A}^1, (f_0(x), \dots, f_n(x)) \neq (0, \dots, 0)$, so this gives a regular expression defined on \mathbb{A}^1 .

Remark 2.6.14. There are indeed nonregular rational maps from, for example, \mathbb{P}^2 , e.g. $\psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 : [x : y : z] \mapsto [yz : zx : xy]$ is not regular at $[1 : 0 : 0], [0 : 1 : 0]$ and $[0 : 0 : 1]$.

2. (a) Prove that if $\varphi : \mathbb{A}^2 \dashrightarrow k$ is a rational function, then $\exists f, g \in k[x, y] : \varphi = \frac{f}{g}$ and $g \neq 0$ on all $\text{dom } \varphi$.
- (b) Deduce that if $\varphi : \mathbb{A}^2 \dashrightarrow k$ is regular on $\mathbb{A}^2 \setminus \{(0, 0)\}$ then it is regular at $(0, 0)$ too.
- (c) Consider $V = \mathbb{A}^2 \setminus \{(0, 0)\}$. Prove that the ring of regular functions on V is $k[x, y]$.
- (d) Prove that V is not isomorphic to any affine variety.

Solution.

- (a) Consider a representative $\varphi = \frac{f}{g}$ and assume f, g are coprime. If $\frac{f'}{g'}$ is another regular expression then $\frac{f}{g} = \frac{f'}{g'}$, i.e. $fg' = f'g$, in particular $g \mid fg'$, but f, g are coprime so $g \mid g'$, hence $g'(x) \neq 0 \implies g(x) \neq 0$.
- (b) The assumption means we can find f, g such that $\varphi = \frac{f}{g}$ with $g(x) \neq 0 \forall x \in \mathbb{A}^2 \setminus \{(0, 0)\}$. Suppose per contra $V(g) = \{(0, 0)\}$, then by Nullstellensatz $I(V(g)) = (x, y) = \sqrt{(g)}$, but (g) is principal and (x, y) is not. Hence $V(g) = \emptyset$ and φ is regular everywhere.
- (c) Consider the restriction $k[\mathbb{A}^2] \rightarrow \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \setminus \{(0, 0)\})$. But $\mathbb{A}^2 \setminus \{(0, 0)\}$ is dense, so by 1.2.8 the map is injective. The previous question implies the map is surjective.
- (d) If V is affine, the ideal (x, y) of $k[x, y]$ would be associated to a nonempty closed subset, but this is not the case.
3. Let $\overline{C} = \{[w : x : y] \in \mathbb{P}^2 : w^2y = x^3\}$ and the rational map $\varphi : \overline{C} \dashrightarrow \mathbb{P}^1$ via $[w : x : y] \mapsto [w : x]$. Show that $\text{dom } \varphi = \overline{C} \setminus \{[0 : 0 : 1]\}$.

Solution. Let $C = V(y - x^3) \subset \mathbb{A}^2$ and parameterise C by $\mathbb{A}^1 \rightarrow C : t \mapsto (t, t^3)$, which extends to $[1 : t] \mapsto [1 : t : t^3]$, so write φ^{-1} by $[s : t] \mapsto [s^3 : s^2t : t^3]$. Now suppose φ is represented by f, g near $[0 : 0 : 1]$, then $xf = wg$, so for $(s, t) \neq (0, 0)$ one has $s^2tf(s^3, s^2t, t^3) = s^3g(s^3, s^2t, t^3)$, i.e. $tf(s^3, s^2t, t^3) = sg(s^3, s^2t, t^3)$, in particular $s \mid f(s^3, s^2t, t^3)$, so $s^2 \mid f(s^3, s^2t, t^3)$, hence $s \mid g(s^3, s^2t, t^3)$. Thus $f(0, 0, 1) = g(0, 0, 1) = 0$, so $[w : x : y] \mapsto [f(w, x, y) : g(w, x, y)]$ is not regular at $[0 : 0 : 1]$.

Week 7, lecture 1, 20th February: problem class (sheet 3 continued)

4. Let

$$f(x_0, \dots, x_n) = \sum_{i,j=0}^n a_{ij}x_ix_j \in k[x_0, \dots, x_n]$$

where $(a_{ij}) \neq (0)$. Note that this is homogeneous of degree 2. $V = V(f)$ is then called *quadric*.

- (a) Prove that V is irreducible iff f is a power of an irreducible polynomial.
- (b) Let $p = [0 : \dots : 0 : 1] \in \mathbb{P}^n$. What condition on a_{ij} is equivalent to $p \in V$?
From now on, assume f is irreducible and $p \in V$.
- (c) Let H be the hyperplane $x_n = 0$ (so $p \notin H$). For $x \in H$, denote by L_x the line through x and p . Prove that either $L_x \subset V$ or $L_x \cap V$ contains 1 or 2 points.
- (d) Denote $U = \{x \in H : |L_x \cap V| = 2\} \subset H$. Prove U is open by finding equations defining $H \setminus U$.
- (e) What condition on a_{ij} is equivalent to $U \neq \emptyset$?
From now on, assume $U \neq \emptyset$.
- (f) Write $L_x \cap V = \{p, \psi(x)\}$. Find the coordinates of $\psi(x)$ and conclude ψ is a rational map $H \dashrightarrow V$.
- (g) Consider the projection from p to H and denote it by $\pi : V \dashrightarrow H$, which is regular on $V \setminus \{p\}$. Prove that V is birational to H .
- (h) Determine $\text{dom } \psi$. What is $\psi(x)$ for $x \in \text{dom } \psi \setminus U$?

- (i) Apply this to the quadric in \mathbb{P}^3 defined by $xy - wz$. Write down ψ and $\text{dom } \psi$.
(j) Determine $\text{dom } \pi$.

Solution.

- (a) This is seen in sheet 1. Moreover, since $\deg f = 2$, either f is irreducible or $f = l^2$ where $\deg l = 1$, but then V is a hyperplane. This is a degenerate case (if f is not irreducible then V is the union of two hyperplanes).
(b) $a_{nn} = 0$.
(c) Write $x = [x_0 : \cdots : x_n]$ and parameterise $L_x = \{p\} \cup \{[x_0 : \cdots : x_{n-1} : t] : t \in k\}$. Then

$$[x_0 : \cdots : x_{n-1} : t] \in V \iff f(x_0, \dots, x_{n-1}, t) = 0 \iff t \sum_{i=0}^{n-1} (a_{in} + a_{ni})x_i + \sum_{i,j=0}^{n-1} a_{ij}x_i x_j = 0.$$

Denote the last polynomial by $g \in k[t]$. Then $g = 0 \implies L_x \subset V$, $\deg g = 0 \implies L_x \cap V = \{p\}$ since g has no solution, and $\deg g = 1 \implies L_x \cap V = \{p, [x_0 : \cdots : x_{n-1} : t]\}$ where t is a solution to g , i.e.

$$t = -\frac{\sum_{i,j=0}^{n-1} a_{ij}x_i x_j}{\sum_{i=0}^{n-1} (a_{in} + a_{ni})x_i}.$$

- (d) $\sum_{i=0}^{n-1} (a_{in} + a_{ni})x_i$.
(e) $U = \emptyset \iff \sum_{i=0}^{n-1} (a_{in} + a_{ni})x_i = 0 \forall x \iff a_{in} + a_{ni} = 0 \forall i$, hence

$$U \neq \emptyset \iff (a_{0n} + a_{n0}, \dots, a_{n-1,n} + a_{n,n-1}) \neq (0, \dots, 0).$$

(f)

$$\begin{aligned} \psi(x) &= \left[x_0 : \cdots : x_{n-1} : -\frac{\sum_{i,j=0}^{n-1} a_{ij}x_i x_j}{\sum_{i=0}^{n-1} (a_{in} + a_{ni})x_i} \right] \\ &= \left[\left(\sum_{i=0}^{n-1} (a_{in} + a_{ni})x_i \right) x_0 : \cdots : \left(\sum_{i=0}^{n-1} (a_{in} + a_{ni})x_i \right) x_{n-1} : -\sum_{i,j=0}^{n-1} a_{ij}x_i x_j \right]. \end{aligned}$$

- (g) For $y \in V \setminus \{p\}$, $L_{yp} \cap H = \{\pi(y)\}$, i.e. $y, p, \pi(y)$ are collinear, so $y \in L_{p\pi(y)} \cap V$, hence $y = \psi\pi(y)$.
For $x \in U \subset H$, $L_{xp} \cap V = \{p, \psi(x)\}$, i.e. $x, p, \psi(x)$ are collinear, so $x \in L_{p\psi(x)} \cap H$, hence $x = \pi\psi(x)$.
Therefore ψ, π are birational inverses.
(h) Using (f), ψ is regular on $H \setminus W$ where

$$W = \left\{ [x_0 : \cdots : x_{n-1} : 0] \in H : \sum_{i=0}^{n-1} (a_{in} + a_{ni})x_i = \sum_{i,j=0}^{n-1} a_{ij}x_i x_j = 0 \right\}.$$

On $(H \setminus W) \setminus U$, by (d), $\psi(x) = [0 : \cdots : 0 : -\sum_{i,j=0}^{n-1} a_{ij}x_i x_j] = [0 : \cdots : 0 : 1] = p$.

- (i) By (h), $\text{dom } \psi = H \setminus W$ where $W = \{[w : x : y : 0] \in H : -w = xy = 0\}$, and

$$\psi : [w : x : y : 0] \mapsto [-w : w : -wx : -wy : -xy] = [w^2 : wx : wy : xy].$$

Week 7, lecture 2, 20th February: problem class (sheet 3 continued)

- (j) We know π is regular on $V \setminus \{p\}$. The question is if π is regular at p . The answer is different for $n = 2$ and $n \geq 3$.

For $n = 2$, by (d), $H \setminus U$ is defined a single nontrivial linear equation, hence just a point in \mathbb{P}^1 . Write $H \setminus U = \{y\}$. Now for $x \in U$, $L_{xp} \cap V = \{p, \psi(x)\}$, and for y , $L_{yp} \cap V = \{p\}$, so we can extend π by sending p to y .

Algebraically, suppose V is defined by $f(x, y, z) = ax^2 + by^2 + gz^2 + dxy + exz + fyz$. Since $p \in V$, by (b), $g = 0$. By a change of coordinates, one can assume $a = 0$ and $e \neq 0$. Embed $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$ as $\{[x : y : 1]\}$, and write $V \cap \mathbb{A}^2 = \{[x : y : 1] : by^2 + dxy + ex + fy = 0\}$. On $(V \cap \mathbb{A}^2) \setminus \{p\}$, $\pi([x : y : 1]) = [x : y]$, but on V , $[x : y] = [ex : ey] = [-by^2 - dxy - fy : ey] = [-by - dx - f : e] \neq [0 : 0]$, so π is regular at $p = [0 : 0 : 1]$ as well.

Note that by (h), ψ is regular on $H \setminus W$ where $W = \{[x : y : 0] \in H : ex + fy = by^2 + dxy = 0\}$. Now if $[x : y : 0] \in W$ and $y = 0$ then $ex = 0$, then since $e \neq 0$ we have $x = 0$, but $[0 : 0 : 0]$ is not defined.

If $y \neq 0$ then $ex + fy = by + dx = 0$, but for any $(b, d, e, f) \neq (0, 0, 0, 0)$ this either implies $x = y = 0$ which again implies $W = \emptyset$, or $be = df$, but then $by^2 + dxy + exz + fyz = (by + dx)(y + \frac{f}{b}z)$ is not irreducible. Hence ψ is regular on the whole H and ψ and π are inverse isomorphisms, i.e. any nondegenerate quadric in \mathbb{P}^2 is isomorphic to \mathbb{P}^1 .

For $n \geq 3$, W has at least two points. Let $x_1 \neq x_2 \in W$, but then for $y \in L_{x_p} \setminus \{p\} : \pi(y) = x_1$, by continuity $\pi(y) = x_2$, a contradiction.

5. A quadric in \mathbb{P}^2 is called a *conic*. Let $f(x, y, z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz \in k[x, y, z]$ where $(a, b, c, d, e, f) \neq (0, 0, 0, 0, 0, 0)$. Let $V = \tilde{V}(f) \subset \mathbb{P}^2$.
- Prove that two quadratics f, g defined the same conic iff $f = \lambda g$ for some $\lambda \in k^*$.
 - For $p \in \mathbb{P}^2$, prove that the set of conics containing p is a hyperplane in \mathbb{P}^5 .
 - Prove that given 5 points in \mathbb{P}^2 , there is at least 1 conic passing through them.
 - Find conditions on 5 points in \mathbb{P}^2 equivalent to
 - There is an irreducible (i.e. nondegenerate) conic passing through them.
 - There is a unique conic passing through them.

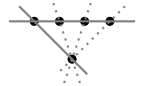
Solution.

- (a) Suppose $\tilde{V}(f) = \tilde{V}(g)$. If $f = f_1 f_2$ where $f_1 \neq f_2$ are linear polynomials, then $\tilde{V}(f) = \tilde{V}(f_1) \cup \tilde{V}(f_2)$; if $f = l^2$ then $\tilde{V}(f) = \tilde{V}(l)$; and if f is irreducible then $\tilde{V}(f)$ is irreducible.
- In the first case $g = g_1 g_2$ with $\tilde{V}(g_1) = \tilde{V}(f_1)$ and $\tilde{V}(g_2) = \tilde{V}(f_2)$, so $(f_1) = (g_1)$ and $(f_2) = (g_2)$, hence $g_1 = \lambda_1 f_1$ and $g_2 = \lambda_2 f_2$ for some $\lambda_1, \lambda_2 \in k^*$, thus $g = g_1 g_2 = \lambda_1 \lambda_2 f_1 f_2 = \lambda_1 \lambda_2 f$.
- In the other cases $\tilde{V}(f) = \tilde{V}(g)$ is irreducible, so by Nullstellensatz $\sqrt{(f)} = \sqrt{(g)}$. But then if f is irreducible and $g = l'^2$ for some linear l' then $\sqrt{(f)} = (f) = (l') = \sqrt{(g)}$ which is impossible, so g is irreducible too and $(f) = (g)$ which implies $f = \lambda g$ for some $\lambda \in k^\times$; and if $f = l^2$ and g is irreducible then $\sqrt{(f)} = (l) = (g) = \sqrt{(g)}$ which is again impossible, so $g = l'^2$ for some linear l' and $(l) = (l')$ which implies $l' = \lambda l$ for some $\lambda \in k^\times$ and so $g = \lambda^2 f$.

Week 7, lecture 3, 21st February: last bit of sheet 3, and dimension continued

Note that by this, two conics are the same if the polynomials define them differ by a scalar, so we can identify the set of conics with \mathbb{P}^5 via $f(x, y, z) \mapsto [a : b : c : d : e : f]$.

- Write $p = [x : y : z]$, then the set in interest is $\{[a : b : c : d : e : f] \in \mathbb{P}^5 : ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0\}$, which is defined by a single homogeneous polynomial in $k[a, b, c, d, e, f]$ of degree 1, hence a hyperplane.
- By above, to find a conic passing through 5 given points is equivalent to finding intersection of 5 hyperplanes in \mathbb{P}^5 , which is nonempty.
- If 3 of the 5 are collinear, then since any 2 points are collinear, the conic is a union of lines, hence degenerate. Conversely, if the conic is degenerate, there must be at least 3 points collinear by pigeonhole. Therefore the condition is no 3 points are collinear.
 - If 4 of the 5 are collinear, then the remaining one is collinear with any one of the 4, so it's not unique. Conversely, if no 4 are collinear, then choose 3 not collinear and by change of coordinates write them as $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$. Write the other 2 points on the conic as $[x_1 : y_1 : z_1], [x_2 : y_2 : z_2]$.



Then $f(1, 0, 0) = f(0, 1, 0) = f(0, 0, 1) = 0$ implies $a = b = c = 0$, so it remains to show $\begin{cases} dx_1 y_1 + ex_1 z_1 + f y_1 z_1 = 0 \\ dx_2 y_2 + ex_2 z_2 + f y_2 z_2 = 0 \end{cases}$ has only one solution (d, e, f) up to scaling (we've seen that (d, e, f) and $(\lambda d, \lambda e, \lambda f)$ define the same conic by (a)). By calculating relevant determinants, this is equivalent to $[x_1 : y_1 : z_1], [x_2 : y_2 : z_2]$ are not the same point, which is assumed.

Corollary 2.6.15. Let $V, W \in \mathbb{A}^n$ be irreducible and quasiprojective of dimensions d, e . Then either $V \cap W = \emptyset$ or every irreducible component of $V \cap W$ has dimension $\geq d + e - n$.

Proof. Embed $\mathbb{A}^n \hookrightarrow \mathbb{A}^n \times \mathbb{A}^n$ via $x \mapsto (x, x)$. The image Δ is $V(x_1 - y_1, \dots, x_n - y_n) = \bigcap_{i=1}^n H_i$ where $H_i = V(x_i - y_i)$ is a hyperplane; in particular Δ is a closed subset. Then $V \cap W = \{x \in \mathbb{A}^n : x \in V, x \in W\}$ can be identified with

$$(V \times W) \cap \Delta = \{(x, y) \in \mathbb{A}^n \times \mathbb{A}^n : x \in V, y \in W, x = y\} = (V \times W) \cap \bigcap_{i=1}^n H_i,$$

where $\dim V \times W = \dim V + \dim W = d + e$ by 2.6.7. The rest follows by induction and 2.6.11. \square

Theorem 2.6.16. Let $V, W \subset \mathbb{P}^n$ be irreducible and quasiprojective of dimensions d, e with $d + e \geq n$. Then either $V \cap W = \emptyset$ or every irreducible component of $V \cap W$ has dimension $\geq d + e - n$. If V, W are projective, $V \cap W \neq \emptyset$.

Proof. By replacing V, W by projective closure, we can assume they are projective. Consider $\tilde{V}, \tilde{W} \subset \mathbb{A}^{n+1}$ the projective cones of V, W . Then $\dim \tilde{V} = d + 1, \dim \tilde{W} = e + 1$ by 2.6.8, so each irreducible component of $\tilde{V} \cap \tilde{W}$ has dimension $\geq d + 1 + e + 1 - (n + 1) = d + e - n + 1 \geq 1$, in particular not empty. Since $\tilde{V} \cap \tilde{W}$ is the cone of $V \cap W$, it follows $V \cap W \neq \emptyset$, and dimension follows from 2.6.15 and again 2.6.8. \square

Week 8, lecture 1, 24th February: correction of coursework 1, dimension continued

1. You can omit $\{(0, 0)\}$ since it's contained in the line $\{x = y\}$.
2. Note that we also need to consider the case $\text{char} = 2$: for $V(f, g)$, if $\text{char } k = 2$ then $\{(t, -t, -t^3)\}$ and $\{t, t, t^3\}$ are the same cubic, and similarly for $V(f, h)$, $i^2 = -1 = 1$ so $i = 1$ and the 4 parabolas are in fact the same one, and finally $V(f, g, h)$ is just 2 points.
3. Writing $\text{im } \phi = V(y^p - x^q)$ is not entirely correct. One should take $d = \gcd(p, q)$, write $p = dp', q = dq'$ and $\text{im } \phi = V(y^{p'} - x^{q'})$. For the second part, again we need to consider other characteristics: $\zeta^p = \zeta^q = 1$ can also happen if $\gcd(p, q)$ is some power of $\text{char } k$.
4. This is very similar to some of the exercises we've done in lecture before.
5. Explicitly, we have the projection map $\pi : V \dashrightarrow \mathbb{P}^2 : [w : x : y : z] \mapsto [w : x : y]$ with inverse $\psi : \mathbb{P}^2 \dashrightarrow V : [w : x : y] \mapsto [wy : xy : y^2 : wx]$.

We want to generalise 2.6.11 by **Veronese embedding**. Let $d, n \in \mathbb{Z}^+$ and consider monomials of degree d in $k[x_0, \dots, x_n]$. In total there are $\binom{n+d}{d}$ of such monomials. Let $N = \binom{n+d}{d} - 1$. Define the Veronese embedding

$$\nu_{n,d} : \mathbb{P}^n \rightarrow \mathbb{P}^N : [x_0 : \dots : x_n] \mapsto [\text{all monomials of degree } d \text{ in } x_0, \dots, x_n]$$

this is regular since if $(x_0, \dots, x_n) \neq (0, \dots, 0)$ then suppose $x_i \neq 0$ so $x_i^d \neq 0$.

For example,

$$\nu_{2,2} : \mathbb{P}^2 \rightarrow \mathbb{P}^5 : [x_0 : x_1 : x_2] \mapsto [x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_1x_2 : x_0x_2].$$

We see that the Veronese embedding is an isomorphism on its image which is closed in \mathbb{P}^N (in particular determined by homogeneous polynomials of degree d). Also, if we have $\tilde{V}(f) \subset \mathbb{P}^n$ where f is homogeneous of degree d , then write $f = \sum \alpha_\mu x_0^{\mu_0} \dots x_n^{\mu_n}$, so that $\nu_{n,d}(\tilde{V}(f)) = \nu_{n,d}(\mathbb{P}^n) \cap \tilde{V}(\sum \alpha_\mu u x_\mu)$ where $\sum \alpha_\mu u x_\mu$ is now a linear polynomial, hence $\tilde{V}(\sum \alpha_\mu u x_\mu)$ is some hyperplane in \mathbb{P}^N .

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We therefore have a generalised 2.6.11 which follows from 2.6.11 itself:

Theorem 2.6.17. Let $V \subset \mathbb{P}^n$ be irreducible and quasiprojective and H a hyperplane of \mathbb{P}^n such that $V \not\subset H$. Then either $V \cap H = \emptyset$ or every irreducible component of $V \cap H$ has dimension $\dim V - 1$. If V is projective then $V \cap H \neq \emptyset$.

2.6.4 Topological definition

Lemma 2.6.18. Let V be irreducible and quasiprojective and W be an irreducible and closed subset of V . Then $\dim W \leq \dim V$ with equality iff $V = W$.

Proof. For the first part:

- Geometric proof: restrict to affine open subset of V , i.e. assume V is affine, and as usual replace V, W by their projective closures, i.e. assume V, W are projective. Then there is a finite-to-one, surjective map $V \rightarrow \mathbb{P}^n$ where $n = \dim V$. By iterating projections to hyperplane, there is a finite-to-one, surjective map $W \rightarrow \mathbb{P}^{n-r}$ for some $r \geq 0$. Hence $\dim W = n - r \leq n = \dim V$.
- Algebraic proof: still assume V is affine, then there is a surjective ring homomorphism $k[V] \rightarrow k[W]$ and in particular a surjective field homomorphism $k(V) \rightarrow k(W)$. Hence $k(W) \subset k(V)$ is a field extension, so transcendence degree of $k(W)$ is greater than or equal to that of $k(V)$, which implies the desired by the algebraic definition of dimension.

For the second part, write $V \subset \mathbb{P}^n$ is quasiprojective and $W \subsetneq V$. Then there is a homogeneous polynomial $f \in k[x_0, \dots, x_n]$ such that f vanishes on W but not on V . Denote by H the hyperplane defined by f . Then $V \not\subset H$ and $W \subset V \cap H$, so by 2.6.17 each irreducible component of $V \cap H$ has $\dim V - 1$, so $\dim W \leq \dim V - 1 < \dim V$. \square

Theorem 2.6.19. Let V be irreducible and quasiprojective. The dimension of V is the maximal integer d such that there is a chain of irreducible closed subsets $V = V_d \supsetneq V_{d-1} \supsetneq \dots \supsetneq V_0 \supsetneq \emptyset$.

Proof. Take an irreducible component V_d of V , find a hyperplane H such that $V_d \not\subset H$, apply 2.6.17 and take V_{d-1} to be $V_d \cap H$, and iterate this until $\dim V = 0$ and label this V_0 . At each step, $\dim V_i = \dim V_{i+1} - 1$, hence $d = \dim V$. It also follows from previous lemma that there's no such sequence with $d > \dim V$. \square

The theorem itself is not strong enough to be useful, i.e. even if we successfully write down such a sequence and we cannot insert other irreducible closed subsets into it, we still don't have a way to know its length is maximal. But we don't have to worry about this since we have Krull dimension (you know this from Atiyah's *Commutative algebra*), and as long as the chain is *maximal* in the sense that we cannot insert more, then it's *maximal* in the sense that there's no other longer chain.

2.6.5 Fibre theorem

Let $f : V \rightarrow W$ be a regular map of quasiprojective varieties and $p \in W$ a point. Call the closed subset $f^{-1}(p) \subset V$ the *fibre* of f at p . Recall that $\dim V \times W = \dim V + \dim W$, and for projection $p_2 : V \times W \rightarrow W$ and $p \in W$ we have $f^{-1}(p) = V \times \{p\}$, so its dimension is $\dim V = \dim V \times W - \dim W$. We want to do this in more generality.

(!) We assume that f is dominant for dimension consistency (if we have $\{0\} \hookrightarrow \mathbb{A}^n$ then we would have that fibre of any point is dimension 0).

Also, consider $f : \mathbb{A}^3 \rightarrow \mathbb{A}^2 : (x, y, z) \mapsto (xy, xz)$. Over $p = (s, t) \neq (0, 0)$ one has $f^{-1}(p) = \{(x, y, z) : xy = s, xz = t\}$. By symmetry assume $s \neq 0$. Then $x, y \neq 0$ and $y = sx^{-1}, z = tx^{-1}$, so $f^{-1}(p) = \{(x, sx^{-1}, tx^{-1}) : x \in k^\times\} \cong \mathbb{A}^1 \setminus \{0\}$ which has dimension $1 = \dim \mathbb{A}^3 - \dim \mathbb{A}^2$. But $f^{-1}((0, 0)) = \{(x, y, z) : xy = xz = 0\} = V(x) \cup V(y, z)$ is not irreducible and its components have dimension 2 and 1.

Theorem 2.6.20 (Fibre dimension). Let V, W be irreducible and quasiprojective and $f : V \rightarrow W$ dominant and regular. Then

1. $\forall w \in W$, each irreducible component of $f^{-1}(w)$ has dimension at least $\dim V - \dim W$.
2. There is a nonempty open subset $U \subset W$ such that each irreducible component of $f^{-1}(w)$ has dimension exactly $\dim V - \dim W \forall w \in U$.

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Proof. 1. Cover W with affine open subsets U_i and $f^{-1}(U_i)$ with U'_{ij} . Then $f_{ij} : U'_{ij} \rightarrow U_i$ is dominant, so we can assume V, W to be affine. Now f corresponds to $f^* : k[W] \rightarrow k[V]$, which by 1.2.29 is injective. Let $f_1, \dots, f_n \in k[W]$ be a transcendence basis of $k(W)$ and consider $g_1, \dots, g_m \in k[V]$ such that $f_1, \dots, f_n, g_1, \dots, g_m$ is a transcendence basis of $k(V)$. Then $\dim W = n$ and $\dim V = n + m$.

Recall Nullstellensatz and 1.2.3 and see that a point $u \in W$ corresponds to a maximal ideal $\mathfrak{m}_u \subsetneq k[W] \hookrightarrow k[V]$ and $k[f^{-1}(u)] = k[V]/\mathfrak{m}_u$. Now $g_1, \dots, g_m \in k[V]$ are algebraically independent over $k[W]$, hence gives m algebraically independent elements of a prime ideal $I \subset k[V]/\mathfrak{m}_u = k[f^{-1}(u)]$ over $k[W]/\mathfrak{m}_u = k$, so every irreducible component of $f^{-1}(u)$ has degree at least $m = \dim V - \dim W$.

2. Consider $h_1, \dots, h_r \in k[V]$ such that $f_1, \dots, f_n, g_1, \dots, g_m, h_1, \dots, h_r$ generate $k[V]$. By construction h_i 's are algebraic in $f_1, \dots, f_n, g_1, \dots, g_m$, i.e. for each h_i we can take $F_i \in k[f_1, \dots, f_n, g_1, \dots, g_m][x]$ such that $F_i(h_i) = 0$. In particular, write $F_i(h_i) = c_d h_i^d + c_{d-1} h_i^{d-1} + \dots + c_0$ where $c_j \in k[f_1, \dots, f_n, g_1, \dots, g_m]$ and $c_d \neq 0$. Consider $c_d \in k[g_1, \dots, g_m][f_1, \dots, f_n]$ and let $U_i = \{u \in W : c_d(f_1(u), \dots, f_n(u)) \neq 0\}$ and $U = \bigcap_{i=1}^r U_i$. Then $u \in U$ means h_i 's are algebraic over g_1, \dots, g_m in $k[f^{-1}(u)]$, so combining with part 1 we have that g_1, \dots, g_m is a transcendence basis of $k(V')$ where V' is an irreducible component of $f^{-1}(u)$, i.e. every irreducible component of $f^{-1}(u)$ has dimension m . It just remains to see that U is open since it's defined by the non-vanishing of a polynomial. \square

The theorem is useful when we know $\dim V$ or $\dim W$ but not the other, and we work out $\dim f^{-1}(w)$ for some $w \in W$ to get an inequality. In particular if some $\dim f^{-1}(w) = 0$ then $\dim V - \dim W \leq 0$, i.e. $\dim V \leq \dim W$, but we know $\dim W \leq \dim V$ by the proof of part 1 of the theorem, so $\dim V = \dim W$.

Example 2.6.21. Consider $\varphi : \mathbb{A}^2 \rightarrow \mathbb{A}^2 : (x, y) \mapsto (x, xy)$ and the line $L_x = \{(x, y) : y \in k\}$ where x is fixed. If $x \neq 0$ then $\varphi : L_x \rightarrow L_x$ is an isomorphism; if $x = 0$ then φ maps the whole line L_0 to 0. This verifies the theorem: on the open set $\{(x, y) \in L_x : x \neq 0\}$ we then have $\dim \varphi^{-1}(x, y) = 0 = \dim \mathbb{A}^2 - \dim \mathbb{A}^2$ (a single point), and $\dim \varphi^{-1}((0, 0)) = \dim L_0 = 1 \geq 0$.

Another usage is a criteria for irreducibility. Note that if $f : V \rightarrow W$ is a dominant and regular map between quasiprojective varieties and V is irreducible, then W is irreducible; indeed, if $W = W_1 \cup W_2$ then $V = f^{-1}(W_1) \cup f^{-1}(W_2)$, so WLOG $V = f^{-1}(W_1)$ and hence $W = W_1$. In general the converse (if W is irreducible then V is) is false but we can use the fibre theorem to impose some condition to have a partial converse.

Proposition 2.6.22. Let $f : V \rightarrow W$ be surjective and suppose V is projective, W is irreducible, and fibres of V are all irreducible of same dimension d . Then V is irreducible.

Proof. Let V_1, \dots, V_n be irreducible components of V . By 2.5.2 V is complete, so $f(V_i)$ is closed, and since f is surjective, $W = \bigcup_{i=1}^n f(V_i)$. Since W is irreducible, $W = f(V_i)$ for some i . Fix this i and we have a surjective restriction $f_i : V_i \rightarrow W$. Let $U_i = V_i \setminus \bigcup_{j \neq i} V_j$. This is a nonempty and open subset of V_i , and also dense in V_i since V_i is irreducible. Then $f_i(U_i)$ is dense in W . Indeed, take any nonempty open subset $U \subset W$, then $f_i^{-1}(U \cap f_i(U_i)) \supset f_i^{-1}(U) \cap U_i$ is nonempty since U_i is dense, so $U \cap f_i(U_i) \neq \emptyset$. Hence f_i is dominant.

Now let $w \in W$ and write $f^{-1}(w) = \bigcup_j f^{-1}(w) \cap V_j$. But we assumed fibres are irreducible, so $f^{-1}(w) \subset V_j$ for some j , and if $w \in f_i(U_i)$ then clearly $f^{-1}(w) \subset V_i$, so $f^{-1}(w) = f_i^{-1}(w)$ and by the fibre dimension theorem, $\dim V_i = \dim W + \dim f_i^{-1}(w) = \dim W + \dim f^{-1}(w) = \dim W + d$.

But by assumption, since f is surjective (so in particular dominant), we also have $\dim V = \dim W + d$, hence $\dim V = \dim V_i$, which forces $V = V_i$. \square

(!) We do need that V is projective: if $V \subset \mathbb{A}^2$ is the union of the hyperbola $V(xy - 1)$ and the origin, which is clearly reducible, then V projects surjectively on \mathbb{A}^1 with each fibre a single point (hence in particular irreducible of same dimension).

Week 9, lecture 1, 3rd March: problem class (sheet 5)

Exercise 2.6.23. 1. Let $V \subset \mathbb{P}^n$ be a projective variety with all components of dimension $n - 1$. Prove V is a hypersurface.

Solution. Write $V = V_1 \cup \dots \cup V_n$ where V_i 's are irreducible components of V and by assumption $\dim V_i = n - 1$. Consider a nontrivial homogeneous polynomial g_i which vanishes on V_i . As V_i is irreducible, each irreducible factor of g_i vanishes on V_i . Let f_i be a irreducible factor of g_i . Then $V_i \subset \tilde{V}(f_i) \subset \mathbb{P}^n$ where $V(f_i)$ is a irreducible hypersurface of dimension $n - 1$. But same is true for V_i , so $V_i = \tilde{V}(f_i)$ and $V = \tilde{V}(f_1 \dots f_n)$, a hypersurface.

This shows that $V \subset \mathbb{P}^n$ has irreducible components all of dimension $n - 1$ iff V is a hypersurface (the converse direction is trivial).

2. Let $H \subset \mathbb{P}^n$ be a hyperplane and $V \subset H$ a irreducible projective variety. Let $x \in \mathbb{P}^n \setminus H$ and $C = \bigcup_{p \in V} L_{xp}$ (the cone over V with vertex x). Prove that C is projective, irreducible, and $\dim C = \dim V + 1$.

Solution. Choose coordinates such that

$$H = \{[0 : x_1 : \dots : x_n] : (x_1, \dots, x_n) \in k^n \setminus \{0\}\} \quad \text{and} \quad x = [1 : 0 : \dots : 0].$$

Then $C = \{[s : tx_1 : \dots : tx_n] : [0 : x_1 : \dots : x_n] \in V, s, t \in k^\times\}$. Let $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ be homogeneous of the same degree such that $V = \{[0 : x_1 : \dots : x_n] : f_i(x_1, \dots, x_n) = 0 \forall i = 1, \dots, r\}$. But then $C = \tilde{V}(g_1, \dots, g_r)$ where $g_i(x_0, \dots, x_n) = f_i(x_1, \dots, x_n)$, in particular C is projective. Since V is irreducible, $\sqrt{f_i}$'s are prime, so $\sqrt{g_i}$'s are prime, so C is irreducible. Now note that $C \setminus \{x\} \cong V \times \mathbb{A}^1$, then $\dim C = \dim C \setminus \{x\} = \dim V + \dim \mathbb{A}^1 = \dim V + 1$.

3. Let $V_{n,d}$ be the vector space of homogeneous polynomials of degree d in $k[x_0, \dots, x_n]$ and $P_{n,d}$ be the projective space to $V_{n,d}$.

(a) Show that for each $e : 0 \leq e \leq d$, multiplication of polynomials $V_{n,e} \times V_{n,d-e} \rightarrow V_{n,d}$ induces a regular map $\mu_{n,d,e} : P_{n,e} \times P_{n,d-e} \rightarrow P_{n,d}$.

- (b) By applying completeness to $\mu_{n,d,e}$, show that $P_{n,d,\text{irr}} = \{[f] \in P_{n,d} : f \text{ is irreducible}\}$ is an open subset of $P_{n,d}$.
- (c) Let $n \geq 2$. Prove that $P_{n,d,\text{irr}} \neq \emptyset$.
- (d) Prove that $P_{n,d,\text{rad}} = \{[f] \in P_{n,d} : f \text{ generates a radical ideal}\}$ is an open subset of $P_{n,d}$.
- (e) Give an example of $f, g \in V_{n,d}$ with $[f], [g] \in P_{n,d} \setminus P_{n,d,\text{rad}}$ such that f is not a scalar multiple of g but f, g defines the same hypersurface in \mathbb{P}^n .

Solution.

- (a) Note that $([f], [g]) \mapsto [fg]$ is bilinear, so we only need to check multiplication of polynomials which are homogeneous of degree 1 in one variable, which is clear.
- (b) Note that f is reducible iff it's in the image of $\mu_{n,d,e}$ for $1 \leq e \leq d-1$, which is closed.
- (c) Note that $x_0^2 - x_1x_2 \in P_{2,2,\text{irr}}$.
Generally, note that $\dim V_{n,d} = \binom{n+d}{d}$, so $\dim P_{n,d} = \binom{n+d}{d} - 1$, and
$$\dim P_{n,e} \times P_{n,d-e} = \binom{n+e}{e} + \binom{n+d-e}{d-e} - 2 < \binom{n+d}{d} - 1,$$
i.e. $\dim P_{n,d,\text{irr}} > 1$.
- (d) f generates a radical ideal iff each of its irreducible factor appears only once, i.e. one cannot write $f = gh^r$ for $h \geq 2$. Now consider $a, b, r : ar + b = d$ with $r \geq 2$, then image of $\mu_{n,a,b,r} : P_{n,a} \times P_{n,b} \rightarrow P_{n,d} : ([f], [g]) \mapsto [f^r g]$ is closed, and $P_{n,d,\text{rad}}$ is precisely the complement of its union over a, b, r .
- (e) $x_0^{d-1}x_1, x_0x_1^{d-1}$.

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4. Let $V \subset \mathbb{P}^n$ be an irreducible, projective variety of dimension d .

- (a) Let $\Sigma = \{(p, q, r) \in V \times V \times \mathbb{P}^n : p \neq q \text{ and } r \in L_{pq}\}$ and $\bar{\Sigma}$ be its Zariski closure. Suppose $\bar{\Sigma}$ is irreducible, and $\bar{\Sigma} \cap \{(p, q, r) \in V \times V \times \mathbb{P}^n : p \neq q\} = \Sigma$. Prove that projection $\pi_{1,2} : \bar{\Sigma} \rightarrow V \times V$ is surjective.
- (b) Show that $\dim \bar{\Sigma} \leq 2d + 1$ by applying the fibre dimension theorem to $\pi_{1,2} : \bar{\Sigma} \rightarrow V \times V$.
- (c) Let

$$S = \bigcup_{\substack{(p,q) \in V \times V \\ p \neq q}} L_{pq}$$

and denote by π_3 the projection $\bar{\Sigma} \rightarrow \mathbb{P}^n$ to the third factor. Prove that $\pi_3(\bar{\Sigma}) = S$ and S is contained in some closed subset of \mathbb{P}^n of dimension at most $2d + 1$.

- (d) Deduce that if $2d + 1 < n$ then there is a point $r \in \mathbb{P}^n$ such that every line through r intersects V in at most one point.

Solution.

- (a) As V is projective, $V \times V \times \mathbb{P}^n$ is projective, and since $\bar{\Sigma} \subset V \times V \times \mathbb{P}^n$ is closed, it is projective and in particular complete, so $\pi_{1,2}(\bar{\Sigma})$ is closed in $V \times V$. But the image of $\pi_{1,2} : \Sigma \rightarrow V \times V$ is the dense open subset $\{(p, q) \in V \times V : p \neq q\}$, and clearly $\pi_{1,2}(\bar{\Sigma}) \supset \pi_{1,2}(\Sigma)$, so $\pi_{1,2}(\bar{\Sigma}) = V \times V$.
- (b) We know $\dim(V \times V) = 2 \dim V = 2d$. To have the desired inequality using fibre dimension theorem, it suffices to show each irreducible component of a fibre has dimension at least 1. Let $(p, q) \in V \times V$ with $p \neq q$. Then $\pi_{1,2}^{-1}((p, q)) = \{r \in \mathbb{P}^n : (p, q, r) \in \bar{\Sigma}\} = \{r \in \mathbb{P}^n : r \in L_{pq}\} = L_{pq}$ which has dimension 1.
- (c) $r \in \pi_3(\bar{\Sigma}) \iff \exists p \neq q \in V : r \in L_{pq} \iff r \in S$. Now $S \subset \pi_3(\bar{\Sigma})$, where $\pi_3(\bar{\Sigma})$ is closed (similar to in (a) that $\pi_{1,2}(\bar{\Sigma})$ is closed). Now $\bar{\Sigma}$ is irreducible and by (b) has dimension at most $2d + 1$, so its image has dimension at most $2d + 1$ too.
- (d) By (c), the assumption means S is contained in a closed subset of \mathbb{P}^n of dimension $< n$, so $S \subsetneq \mathbb{P}^n$, hence $\exists r \in \mathbb{P}^n \setminus S$, i.e. r is not in any line through two different points of V , i.e. any line through r intersects V in at most one point, exactly as desired.

2.7 Parameter spaces

Let B be a quasiprojective variety. A *family of projective algebraic varieties* over B is a Zariski-closed subset $\mathcal{V} \subset B \times \mathbb{P}^n$. For each $b \in B$, write $\mathcal{V}_b = \{x \in \mathbb{P}^n : (b, x) \in \mathcal{V}\}$ and call this a *fibre* of \mathcal{V} . B is then the *base* or *parameter space* of the family \mathcal{V} .

Example 2.7.1. In fact we've already seen some example. A hypersurface of degree d in \mathbb{P}^n is defined by a homogeneous polynomial of degree d in n variables. Denote by $V_{n,d}$ the vector space of homogeneous polynomials of degree d in $k[x_0, \dots, x_n]$ and let $P_{n,d} = V_{n,d} \setminus \{0\} / k^\times$. Let $\mathcal{H} = \{([f], x) \in P_{n,d} \times \mathbb{P}^n : f(x) = 0\}$. This is closed, so it's a family of projective algebraic varieties over $P_{n,d}$. Then a fibre is

$$\mathcal{H}_{[f]} = \{x \in \mathbb{P}^n : ([f], x) \in \mathcal{H}\} = \{x \in \mathbb{P}^n : f(x) = 0\} = \text{the hypersurface defined by } f$$

(two different homogeneous polynomials differ by scaling define the same hypersurface).

Fix $x \in \mathbb{P}^n$ and write coordinates $x = [0 : \dots : 0 : 1]$. Consider the set of hypersurfaces containing x : $\{[f] \in P_{n,d} : x \in \mathcal{H}_{[f]}\}$, which can be written as $\{[f] \in P_{n,d} : f_{i_0 \dots i_n} = 0\}$ where $f_{i_0 \dots i_n}$ is the coefficients of $f \in V_{n,d}$ (with respect to the basis $x_0^{i_0} \dots x_n^{i_n}$), hence the set itself is a hypersurface in $P_{n,d}$, in particular closed. One therefore sometimes says that the condition that "the hypersurface f defines contains a given point x " is a *closed condition* on f . Similarly, as we've seen in exercise 3 above, irreducibility and generating a radical ideal are *open conditions*. In general it's more in our interest to study closed conditions since we can use the dimension theory we just developed on closed sets, while recall that if $U \subset V$ is open then $\dim U = \dim V$.

2.7.1 Intersection of hypersurfaces

Consider the intersection of $n+1$ hypersurfaces in \mathbb{P}^n . From our previous result on dimensions of intersections (2.6.17), generally if they are different hypersurfaces this intersection would be empty. For simplicity, suppose the $n+1$ hypersurfaces all have degree d . Consider the parameter space $(P_{n,d})^{n+1}$ and its subset

$$S = \left\{ ([f_0], \dots, [f_n]) \in (P_{n,d})^{n+1} : \bigcap_{i=0}^n \mathcal{H}_{[f_i]} \neq \emptyset \right\}.$$

Let

$$\Sigma = \{([f_0], \dots, [f_n], x) \in (P_{n,d})^{n+1} \times \mathbb{P}^n : x \in \mathcal{H}_{[f_i]} \forall i\}$$

and consider the projection $\pi_1 : \Sigma \rightarrow (P_{n,d})^{n+1}$. Note that

$$\pi_1^{-1}([f_0], \dots, [f_n]) \cap \Sigma \neq \emptyset \iff ([f_0], \dots, [f_n]) \in S,$$

i.e. $S = \pi_1(\Sigma)$, in particular $\pi_1 : \Sigma \rightarrow S$ is surjective. Now together with the second projection $\Sigma \rightarrow \mathbb{P}^n$, we can use fibre dimension theorem to compute $\dim S$ to "measure" how far away the $n+1$ hypersurfaces are from general.

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Let $x \in \mathbb{P}^n$, then its fibre is

$$\pi_2^{-1}(x) = \{([f_0], \dots, [f_n], x) : x \in \mathcal{H}_{[f_i]} \forall i\} = \{([f_0], \dots, [f_n], x) : f_i(x) = 0 \forall i\} \cong (\{[f] \in P_{n,d} : f(x) = 0\})^{n+1}$$

since the $n+1$ polynomials are independent. As x is fixed, f is a homogeneous polynomial of degree d in its coefficients, so $\pi_2^{-1}(x)$ is a product of $n+1$ hypersurfaces in $P_{n,d}$, hence is irreducible of degree

$$(n+1)(\dim P_{n,d} - 1) = (n+1) \left(\binom{n+d}{d} - 2 \right).$$

As x is arbitrary, this means π_2 is surjective, and clearly \mathbb{P}^n is irreducible so by 2.6.22 Σ is irreducible. Hence by fibre dimension theorem (since all fibres have same dimension we can directly use the strong version),

$$\dim \pi_2^{-1}(x) = (n+1) \left(\binom{n+d}{d} - 2 \right) = \dim \Sigma - \dim \mathbb{P}^n = \dim \Sigma - n,$$

i.e.

$$\dim \Sigma = (n+1) \left(\binom{n+d}{d} - 2 \right) + n = (n+1) \binom{n+d}{d} - n - 2$$

Now $S = \pi_1(\Sigma)$ and Σ is irreducible imply S is irreducible, so we can apply the fibre dimension theorem to π_1 . Note that for a fixed $([f_0], \dots, [f_n]) \in S$, its fibre is

$$\pi_1^{-1}([f_0], \dots, [f_n]) = \left\{ ([f_0], \dots, [f_n], x) : x \in \bigcap_{i=0}^n \mathcal{H}_{[f_i]} \right\} \cong \bigcap_{i=0}^n \mathcal{H}_{[f_i]} \subset \mathbb{P}^n$$

So choose $f_0 = x_1, f_1 = x_1, f_2 = x_2, \dots, f_n = x_n$, then $\bigcap_{i=0}^n \mathcal{H}_{[f_i]}$ is $\{[1 : 0 : \dots : 0]\}$, a point, hence has dimension 0. By the theorem we have $0 \geq \dim \Sigma - \dim S$, but of course $\dim \Sigma \geq \dim S$ since π_2 is surjective, so $\dim S = \dim \Sigma = (n+1) \binom{n+d}{d} - n - 2 = \dim(P_{n,d})^{n+1} - 1$, in particular S is a hypersurface in $(P_{n,d})^{n+1}$, so there is a polynomial in coefficients of f_0, \dots, f_n which vanishes iff $\bigcap_{i=0}^n \mathcal{H}_{[f_i]} \neq \emptyset$. This is completely nontrivial without dimension theory!

2.8 Tangent space and smoothness (mastery material)

2.8.1 Tangent spaces with coordinates

Let $V \subset \mathbb{A}^n$ be an affine variety with $x \in V$ and $f \in k[x_1, \dots, x_n]$. Define

$$df_x : k^n \rightarrow k : (a_1, \dots, a_n) \mapsto \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i} \Big|_x.$$

(!) Differentials are weird in characteristic p , for example $\frac{\partial x^p}{\partial x} = px^{p-1} = 0$, in particular that differential of f vanishes doesn't imply f is constant.

Definition 2.8.1. The *tangent space* of an affine variety V at point $x \in V$ is $T_x V = \bigcap_{f \in I(V)} \ker df_x \subset k^n$.

(!) Polynomials that generate $I(V)$ are not necessarily those that define V . Take for example $V = V(x^2) \subset \mathbb{A}^1$, then $V = \{0\}$ and $dx_x^2 = \frac{\partial x^2}{\partial x} = 2x$, so $\ker dx_0^2 = \{x \in k : 2 \cdot 0 = 0\} = k$, but

$$T_0 V = \ker dx_0 = \left\{ x \in k : dx = \frac{\partial x}{\partial x} = 1 = 0 \right\} = \emptyset$$

Example 2.8.2. Consider the line $L = V(x) \subset \mathbb{A}^2$, then

$$dx_x(a_1, a_2) = a_1 \frac{\partial x}{\partial x} + a_2 \frac{\partial x}{\partial y} = a_1,$$

hence

$$T_x L = \ker dx_x = \{(a_1, a_2) \in k^2 : a_1 = 0\} = \langle (0, 1) \rangle.$$

Example 2.8.3. Let $g \in k[x]$ and consider the graph $\Gamma = \{(x, g(x)) : x \in k\} = V(y - g(x))$. Now $y - g(x)$ is monic of degree 1, hence irreducible, so $I(\Gamma) = (y - g(x))$. Now

$$d(y - g(x))_x = a_1 \frac{\partial (y - g(x))}{\partial x} + a_2 \frac{\partial (y - g(x))}{\partial y} = -a_1 g'(x) + a_2,$$

so

$$T_x \Gamma = \ker d(y - g(x))_x = \{(a_1, a_2) \in k^2 : a_2 = a_1 g'(x)\} = \langle (1, g'(x)) \rangle.$$

Example 2.8.4 (Cuspidal cubic). Let $V = V(y^2 - x^3) \subset \mathbb{A}^2$. Clearly $y^2 - x^3$ is irreducible so $I(V) = (y^2 - x^3)$. We have

$$d(y^2 - x^3)_x(a_1, a_2) = a_1 \frac{\partial (y^2 - x^3)}{\partial x} + a_2 \frac{\partial (y^2 - x^3)}{\partial y} = -3a_1 x^2 + 2a_2 y,$$

so

$$T_x V = \ker d(y^2 - x^3)_x = \{(a_1, a_2) \in k^2 : 2a_2 y = 3a_1 x^2\},$$

in particular for the cusp at the origin, $T_{(0,0)} V = k^2$ and for any other point it's a line, which is a nice way of saying that the tangent line at $(0, 0)$ is undefined.

Lemma 2.8.5. Let $V \subset \mathbb{A}^m, W \subset \mathbb{A}^n$ be affine varieties and $\phi : V \rightarrow W$ regular. For any $x \in V$, ϕ induces a linear map $d\phi_x : T_x V \rightarrow T_{\phi(x)} W$.

Proof. Since ϕ is regular, write $\phi(x) = (f_1(x), \dots, f_n(x))$ where $f_1, \dots, f_n \in k[x_1, \dots, x_m]$. Consider the matrix $\left(\frac{\partial f_i}{\partial x_j} \right)_{ij}$ which gives a linear map $d\phi : k^m \rightarrow k^n$. Now for any $g \in k[y_1, \dots, y_n]$, chain rule says $\frac{\partial g(f_1, \dots, f_n)}{\partial x_j} = \frac{\partial g}{\partial y_i} \frac{\partial f_i}{\partial x_j}$, hence $d\phi$ maps $\ker dg_x$ to $\ker dg_{\phi(x)}$, in particular maps $T_x V$ to $T_{\phi(x)} W$. This also does not depend on the choice of f_i 's. \square

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In particular, if $\phi : V \rightarrow W$ is an isomorphism, then $d\phi_x : T_x V \rightarrow T_{\phi(x)} W$ is an isomorphism too with inverse $d\phi_{\phi(x)}^{-1} : T_{\phi(x)} W \rightarrow T_x V$, so we can define the tangent space for any quasiprojective variety.

2.8.2 Singular points

Intuitively, at "smooth points", $\dim V = \dim T_x V$. But we would have a problem if V is reducible, so we want to define smoothness as a local property.

Definition 2.8.6. Let V be a quasiprojective variety. The *local dimension* of V at x , denoted by $\dim_x V$, is the maximum of dimensions of irreducible components of V containing x .

Definition 2.8.7. Let V be a quasiprojective variety and $x \in V$. We say x is a *singular point* of V if $\dim T_x V \neq \dim_x V$. Otherwise x is a *smooth point* of V .

If we write $V = \bigcup_i V_i$ where V_i 's are irreducible components and have $x \in V_i \setminus \bigcup_{j \neq i} V_j$ for some i , then by definition $\dim_x V = \dim_x V_i$, so V is singular at x iff V_i is singular at x . It's possible to prove that if $x \in V_i \cap V_j$ with $i \neq j$ then V is singular at x , but this requires Nakayama's lemma which won't be covered in this course. We focus on the proof of the following theorem.

Theorem 2.8.8. Let V be an irreducible quasiprojective variety. Then the set of singular points of V , denoted by $\text{Sing } V$, is a proper closed subset of V .

Suppose V is an irreducible hypersurface $V = V(f) \subset \mathbb{A}^n$ where $f \in k[x_1, \dots, x_n]$ generates $I(V)$ and is irreducible. Then $\dim_x V = \dim V = n - 1 \forall x \in V$. Now for any $x \in V$, $T_x V = \ker df_x \subset k^n$ has two possibilities of dimensions: if df_x is constantly 0 then $T_x V = k^n$ so $\dim T_x V = n$, or if it's not then $T_x V$ is a hypersurface in k^n with $\dim T_x V = n - 1$. Hence by definition, in the latter case x is smooth and in the previous x is singular. Hence

$$\text{Sing } V = V \left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) \subset \mathbb{A}^n,$$

in particular closed. Recall that $\frac{\partial f}{\partial x_i} = 0 \forall i \not\Rightarrow f$ is constant in characteristic p . But we can prove that this problem only happen if the exponent is a multiple of p .

Lemma 2.8.9. Let k be a field with $\text{char } k = p > 0$ and $f \in k[x]$. Then

$$\frac{\partial f}{\partial x} = 0 \iff f = \sum_{i=0}^d a_{ip} x^{ip}.$$

Proof. The \Leftarrow direction is trivial. Now write $f = \sum a_i x^i$ in its general form. Then

$$\frac{\partial f}{\partial x} = \sum i a_i x^{i-1} = 0 \iff i a_i = 0 \forall i \iff a_i = 0 \forall i : p \nmid i \iff f = \sum_{i=0}^d a_{ip} x^{ip}.$$

\square

Proposition 2.8.10. If $V \subset \mathbb{A}^n$ is a nonempty hypersurface, then $\text{Sing } V \subsetneq V$.

Proof. Write $V = V(f)$ where f generates $I(V)$. Suppose per contra $\text{Sing } V = V$. Then

$$V \left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) = V(f),$$

so $f \mid \frac{\partial f}{\partial x_i} \forall i$, but $\deg \frac{\partial f}{\partial x_i} < \deg f$, so $\frac{\partial f}{\partial x_i} = 0 \forall i$. If $\text{char } k = 0$ then f is constant, contradicting that V is a hypersurface. If $\text{char } k = p > 0$, then

$$f = \sum a_{p i_1 \dots p i_n} x_1^{p i_1} \dots x_n^{p i_n} = \sum (b_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n})^p = \left(\sum b_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n} \right)^p,$$

in particular f is not irreducible, contradicting that f generates $I(V)$. \square

Hence we've proved 2.8.8 for the special case of hypersurfaces.

Lemma 2.8.11. Let V be a quasiprojective variety. For any $d \in \mathbb{Z}$, the set $\Sigma_d(V) = \{x \in V : \dim T_x V > d\}$ is closed.

Proof. A subset is closed iff its intersection with an open cover is closed, so assume $V \subset \mathbb{A}^n$ is affine. Write $I(V) = (f_1, \dots, f_m)$. Then for $x \in V$, $T_x V = \bigcap_{i=1}^m \ker df_{f_i}$. Interpret this as the kernel of the $m \times n$ matrix $M_x = \left(\frac{\partial f_i}{\partial x_j} \right)_{ij}$. By the rank-nullity theorem, $\dim T_x V = \dim k^n - \text{rk } M_x$, so

$$\Sigma_d(V) = \{x \in V : n - d > \text{rk } M_x\} = \{x \in V : \text{every } (n-d) \times (n-d) \text{ submatrix of } M_x \text{ has det } 0\},$$

hence $\Sigma_d(V)$ is defined by polynomials, i.e. closed. \square

Lemma 2.8.12. Let V be an irreducible quasiprojective variety. The set of smooth points of V , denoted by V_{ns} , is dense in V .

Proof. Let $U \subset V$ be open and nonempty. We prove $U \cap V_{\text{ns}} \neq \emptyset$. Since V is irreducible, U is irreducible, so by 1.2.33, U is birational to a hypersurface H , hence some open subset U' of U is isomorphic to an open subset of H . Since $\text{Sing } H$ is closed and proper subset of H (2.8.10), $H_{\text{ns}} = H \setminus \text{Sing } H$ is a nonempty open subset of H , but since H is irreducible, H_{ns} is dense in H . So $H_{\text{ns}} \cap U' \neq \emptyset$, i.e. $U' \cap V_{\text{ns}} \neq \emptyset$, and in particular $U \cap V_{\text{ns}} \neq \emptyset$. \square

Lemma 2.8.13. Let V be an irreducible quasiprojective variety. For every $x \in V$, $\dim T_x V \geq \dim_x V$.

Proof. By definition, since V is already irreducible, $\dim_x V = \dim V =: d$. To prove the lemma, it's equivalent to show $\Sigma_{d-1}(V) = V$. By 2.8.11, $\Sigma_{d-1}(V)$ is closed, and since $V_{\text{ns}} = \{x \in V : \dim T_x V = d\} \subset \Sigma_{d-1}(V)$, it's also dense. The desired then follows immediately. \square

Now it remains to put everything together.

Proof of 2.8.8. By definition and 2.8.13, $\text{Sing } V = \{x \in V : \dim T_x V > \dim_x V = \dim V =: d\} = \Sigma_d$ which is closed by 2.8.11 and proper by 2.8.12. \square

Exercise 2.8.14 (sheet 6). 1. Suppose $\text{char } k \neq 2, 3$. For $a \in k$, denote by V_a the surface

$$V(x^3 + y^3 + z^3 - 3a(x^2 + y^2 + z^2) - a^2).$$

Suppose that this polynomial generates $I(V_a)$. Describe $\text{Sing } V_a$.

Solution. Denote the polynomial defining V_a by f_a . Then

$$\begin{aligned} \text{Sing } V_a &= V \left(f_a, \frac{\partial f_a}{\partial x}, \frac{\partial f_a}{\partial y}, \frac{\partial f_a}{\partial z} \right) = V(f_a, 3x^2 - 6ax, 3y^2 - 6ay, 3z^2 - 6az) \\ &= V_a \cap \{(x, y, z) : x = 0 \text{ or } 2a, y = 0 \text{ or } 2a, z = 0 \text{ or } 2a\} \end{aligned}$$

Week 10, lecture 2, 13th March: problem class (sheet 6 continued)

Now if $a = 0$ then $\text{Sing } V_a = \{(0, 0, 0)\}$. If $a \neq 0$, consider a point (x, y, z) where n of its coordinates are equal to $2a$ and $3 - n$ coordinates equal to 0. Then

$$f_a(x, y, z) = n((2a)^3 - 3a(2a)^2) - a^2 = -a^2(1 + 4na) = 0 \iff a = \frac{-1}{4n}.$$

2. Let $V \subset \mathbb{A}^n$, $W \subset \mathbb{A}^m$ be affine and consider $v \in V$, $w \in W$. Prove that $V \times W$ is smooth at (v, w) iff V is smooth at v and W is smooth at w .

Solution. Consider the irreducible components V_i of V and W_j of W . Then

$$V \times W = \left(\bigcup_i V_i \right) \times \left(\bigcup_j W_j \right) = \bigcup_{i,j} V_i \times W_j$$

is the irreducible decomposition of $V \times W$. Then

$$\begin{aligned} \dim_{(v,w)} V \times W &= \max_{i,j:(v,w) \in V_i \times W_j} \dim V_i \times W_j = \max_{i,j:v \in V_i, w \in W_j} (\dim V_i + \dim W_j) \\ &= \max_{i:v \in V_i} \dim V_i + \max_{j:w \in W_j} \dim W_j = \dim_v V + \dim_w W. \end{aligned}$$

Now it suffices to prove $\dim T_{(v,w)}V \times W = \dim T_vV + \dim T_wW$, since then by definition (v, w) is a smooth point of $V \times W$ iff $\dim T_{(v,w)}V \times W = \dim T_vV + \dim T_wW = 0$. Consider f_1, \dots, f_r generating $I(V) \subset k[x_1, \dots, x_n]$ and g_1, \dots, g_s generating $I(W) \subset k[y_1, \dots, y_m]$. Then $\bar{f}_1, \dots, \bar{f}_r, \bar{g}_1, \dots, \bar{g}_s$ where $\bar{f}_i(x_1, \dots, x_n, y_1, \dots, y_m) = f_i(x_1, \dots, x_n)$ and $\bar{g}_i(x_1, \dots, x_n, y_1, \dots, y_m) = g_i(y_1, \dots, y_m)$ generate $I(V \times W)$. Then $d(\bar{f}_i)_{(v,w)} : k^n \times k^m \rightarrow k : (a, b) \mapsto d(f_i)_v(a)$, so $\ker d(\bar{f}_i)_{(v,w)} = \ker d(f_i)_v \times k^m$. Similarly $\ker d(\bar{g}_i)_{(v,w)} = k^n \times \ker d(g_i)_w$. Hence

$$\begin{aligned} T_{(v,w)}V \times W &= \bigcap_{i=1}^r \ker d(\bar{f}_i)_{(v,w)} \cap \bigcap_{j=1}^s \ker d(\bar{g}_j)_{(v,w)} \\ &= \bigcap_{i=1}^r (\ker d(f_i)_v \times k^m) \cap \bigcap_{j=1}^s (k^n \times \ker d(g_j)_w) \\ &= (T_vV \times k^m) \cap (k^n \times T_wW) = T_vV \times T_wW, \end{aligned}$$

as desired.

3. Let $V \subset \mathbb{A}^n$ be an affine variety with irreducible components V_1, V_2 . Consider $x \in V_1 \cap V_2$. Prove that $T_xV_1 + T_xV_2 \subset T_xV$.

Solution. We have $V_1 \subset V \implies I(V) \subset I(V_1)$. Let $a \in T_xV_1$ and write $T_xV_1 = \ker df_x$. Then $f \in I(V) \subset I(V_1)$, so $a \in T_xV$, i.e. $T_xV_1 \subset T_xV$. Similarly $T_xV_2 \subset T_xV$, thus $T_xV_1 + T_xV_2 \subset T_xV$.

Consider $V(y(y - x^2)) = V(y) \cup V(y - x^2) =: V_1 \cup V_2$ (union of x-axis and the parabola $y = x^2$) and let $x = (0, 0)$. Then $T_xV_1 = T_xV_2 = x$ -axis, so $T_xV_1 + T_xV_2 = x$ -axis. But

$$T_xV = \ker d(y(y - x^2))_x = \{(a_1, a_2) \in k^2 : a_1(-2xy) + a_2(2y - x^2) = 0 \text{ where } x = y = 0\} = k^2,$$

so $T_xV_1 + T_xV_2 \subsetneq T_xV$.

4. Let f be a nonzero polynomial in $k[x_1, \dots, x_n]$ with no repeated factors, and $V \subset \mathbb{A}^n$ be the hypersurface defined by f . In particular f generates $I(V)$.

(a) Let $x \in V$ and $u \in k^n$. Prove that $f(x + Tu) \in k[T]$ has a repeated root at $T = 0$ iff $u \in T_xV$.

Solution. We have

$$\begin{aligned} f(x + Tu) &= f(x) + T \sum_{i=1}^n u_i \frac{\partial f}{\partial x_i} \Big|_x + T^2 g(T) \in k[x] \\ &= f(x) + df_x(u)T + g(T)T^2 \in k[T] \end{aligned}$$

and since $x \in V$, $f(x) = 0$, so the order of root $T = 0$ of $f(x + Tu)$ is 2 iff $df_x(u) = 0$, i.e. $u \in \ker df_x = T_xV$.

Week 10, lecture 3, 14th March

Week 11, lecture 1, 17th March

The last two lectures went through the 2024 paper.