

# MA3G6 Commutative algebra :: Lecture notes

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What is this module about?

- Continuation of MA249,
- Back engine for algebraic geometry and (algebraic) number theory,
- Connection to other areas (combinatorics, applied maths, ...),
- Fun in its own right.

## Recall

**Definition 0.0.1.** A *ring*  $(R, +, \times)$  is a set  $R$  with binary operations  $+: R \times R \rightarrow R$ ,  $\times: R \times R \rightarrow R$  such that

1.  $(R, +)$  is an abelian group (identity denoted  $0_R$  or given clear context simply 0),
2.  $\times$  is associative and distributes over  $+$ ,
3.  $\exists 1_R \in R: 1_R \cdot a = a \cdot 1_R = a \forall a \in R$ .

Within context of this module, we always add a 4th axiom:

4.  $ab = ba \forall a, b \in \mathbb{R}$  commutativity

**Example 0.0.2.** •  $\mathbb{Z}$

- Polynomial ring
- $S = \mathbb{C}[x_1, \dots, x_n]$ ,  $f \in S$ ,  $f = \sum_{u \in \mathbb{N}^n} c_u x^u$ ,  $c_u \in \mathbb{C}$ ,  $x^u = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$  (this is called multiindex notation) and only finitely many  $c_u \neq 0$ . e.g.  $x_1 x_3 + 7x_2 \in \mathbb{C}[x_1, x_2, x_3]$  is written as  $x^{(1,0,1)} + 7x^{(0,2,0)}$ . One can also replace  $\mathbb{C}$  with any field.

**Definition 0.0.3.** A *ring homomorphism* is a function  $\varphi: R \rightarrow S$  where  $R, S$  rings that respects addition and multiplication:  $\varphi(a + b) = \varphi(a) + \varphi(b)$ ,  $\varphi(ab) = \varphi(a)\varphi(b)$  and  $\varphi(1_R) = 1_S$ .

The definition implies that homomorphisms preserve 0.

**Definition 0.0.4.** The *kernel* of a homomorphism  $\varphi$  is  $\ker(\varphi) = \{a \in R: \varphi(a) = 0_S\}$ .

**Definition 0.0.5.** A nonempty  $I \subseteq R$  is an *ideal* if  $a, b \in I \Rightarrow a + b \in I$  and  $a \in I, r \in R \Rightarrow ra \in I$ .

It immediately follows from the definition that kernel of  $\varphi: R \rightarrow S$  is an ideal of  $R$ .

**Example 0.0.6.**  $\varphi: \mathbb{Z} \rightarrow \mathbb{Z}/5\mathbb{Z}$  by  $\varphi(n) = n \bmod 5$ .

**Definition 0.0.7.** We say  $I$  is *generated* by  $f_1, \dots, f_s \in R$  if

$$I = \left\{ \sum_{i=1}^s h_i f_i : h_i \in R \right\} =: \langle f_1, \dots, f_s \rangle$$

More generally,  $I$  is generated by  $G \subseteq R$  if

$$I = \left\{ \sum_{i=1}^s h_i f_i : h_i \in R, f_i \in G, s \geq 0 \right\}.$$

This is closed under addition and multiplication by an element of  $R$ , hence an ideal.

*Week 1, lecture 2 starts here*

## 1 Gröbner basis

**Example 1.0.1** (Motivating questions). 1. Is  $14 \in \langle 6, 26 \rangle \subseteq \mathbb{Z}$ ? Yes, since  $14 = -2 \times 6 + 26$ .

Do note that  $\mathbb{Z}$  is a PID, and  $\langle 6, 26 \rangle = \langle 2 \rangle$  where  $2 = \gcd(6, 26)$ .

2. Is  $x + 7 \in \langle x^2 - 4x + 3, x^2 + x - 2 \rangle \subseteq \mathbb{Z}[x]$ ? No, since  $x^2 - 4x + 3 = (x - 1)(x - 3)$  and  $x^2 + x - 2 = (x - 1)(x + 2)$ , and  $x - 1 \nmid x + 7$ .

3. Is  $x + 3y - 2z \in \langle x + y - z, y - z \rangle$ ? No, since any linear combination of the two generators have same coefficients for  $y$  and  $z$ . In linear algebra jargon,  $(1, 3, -2)$  is not in rowspace of  $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}$ .

We do have enough specific knowledge to solve these, but not their general forms.

**Example 1.0.2.** Is  $xy^2 - x \in \langle xy + 1, y^2 - 1 \rangle$ ?

If we were not careful, we would try to divide  $xy^2 - x$  by  $xy + 1$  which leads to  $xy^2 - x = y(xy + 1) + (-x - y)$ , a dead end. But note that  $xy^2 - x = x(y^2 - 1)$ , which means it is in the ideal.

We now want to know how we can be ‘careful’.

**Definition 1.0.3.** A *term order* (or monomial order) is a total order on monomials  $x^u$  in  $S = K[x_1, \dots, x_n]$  (where  $K$  is a field) such that

1.  $1 \prec x^u \forall u \neq 0$
2.  $x^u \prec x^v \Rightarrow x^{u+w} \prec x^{v+w} \forall u, v, w \in \mathbb{N}^n$

**Example 1.0.4.** 1. Lexicographic term order:  $x^u \prec x^v$  if the first nonzero element of  $v - u$  is positive.

e.g.  $x_2^2 \prec x_2^{10} \prec x_1 x_3 \prec x_1^2$ . We can write them in multiindex notation:

$$x^{(0,2,0)}, x^{(0,10,0)}, x^{(1,0,1)}, x^{(2,0,0)},$$

and the result is clear. This is analogous to how we order words in a dictionary.

2. Degree lexicographic order:  $x^u \prec x^v$  if  $\deg(x^u) < \deg(x^v) = v_1 + \dots + v_n$ , or if they are equal,  $x^u \prec_{\text{lex}} x^v$ . e.g.  $x_2^2 \prec x_1x_3 \prec x_1^2 \prec x_2^{10}$ .
3. (Degree) reverse lexicographic order (revlex):  $x^u \prec x^v$  if  $\deg(x^u) < \deg(x^v) = v_1 + \dots + v_n$ , or if they are equal, the last nonzero entry of  $v - u$  is negative. e.g.  $x_1x_3 \prec x_2^2 \prec x_1^2 \prec x_2^{10}$ .

**Definition 1.0.5.** Fix a term order  $\prec$  on  $K[x_1, \dots, x_n]$ . The *initial term*  $\text{in}_\prec(f)$  of a polynomial  $f = \sum c_u x^u$  is  $c_v x^v$  if  $x^v = \max_\prec \{x^u : c_u \neq 0\}$ .

**Example 1.0.6.** Let  $f = 3x^2 - 8xz^9 + 9y^{10}$ . Then

- If  $\prec = \text{lex}$ ,  $\text{in}_\prec(f) = 3x^2$
- If  $\prec = \text{deglex}$ ,  $\text{in}_\prec(f) = -8xz^9$
- If  $\prec = \text{revlex}$ ,  $\text{in}_\prec(f) = 9y^{10}$

**Definition 1.0.7.** Let  $I \subseteq S$  be an ideal. The *initial ideal* of  $I$  is  $\text{in}_\prec(I) := \langle \text{in}_\prec(f) : f \in I \rangle$ .

**Remark.** If  $I = \langle f_1, \dots, f_s \rangle$  then  $\langle \text{in}_\prec(f_1), \dots, \text{in}_\prec(f_s) \rangle \subseteq \text{in}_\prec(I)$ , but not necessarily equal.

**Example 1.0.8.**  $I = \langle x+y+z, x+2y+3z \rangle$ . Then  $\text{in}_\prec(f_1) = \text{in}_\prec(f_2) = x$ , so  $\langle \text{in}_\prec(f_1), \text{in}_\prec(f_2) \rangle = \langle x \rangle$ , but  $y+2z \in I$ ,  $\text{in}_\prec(y+2z) = y \notin \langle x \rangle$ .

**Definition 1.0.9.** A set  $\{g_1, \dots, g_s\} \subseteq I$  is a *Gröbner basis* for  $I$  if  $\text{in}_\prec(I) = \langle \text{in}_\prec(g_1), \dots, \text{in}_\prec(g_s) \rangle$ .

With this language, we can express Example 1.0.8 by saying ‘ $\{x+y+z, x+2y+3z\}$  is not a Gröbner basis of the ideal’. We will see that every ideal in  $S$  has a Gröbner basis, and long division using a Gröbner basis solves the ideal membership problem ( $f \in I$  iff the remainder on dividing by the Gröbner basis is 0).

*Week 2, lecture 1 starts here*

## 1.1 Division algorithm

Let  $S = K[x_1, \dots, x_n]$ .

- Input:  $f_1, \dots, f_s, f \in S$  and  $\prec$  the term order
- Output: an expression  $f = \sum_{i=1}^s h_i f_i + r$ , where
  1.  $h_i, r \in S$ ,  $r = \sum c_u x^u$
  2. If  $c_u \neq 0$ , then  $x^u$  is not divisible by any  $\text{in}_\prec(f_i)$
  3. If  $\text{in}_\prec(f) = c_u x^u$ ,  $\text{in}_\prec(h_i f_i) = c_{v_i} x^{v_i}$  then  $x^u \succeq x^{v_i} \forall i$
- The algorithm:
  1. Initialize:  $h_1, \dots, h_s = 0$ ,  $r = 0$ ,  $p = f$ ,  $f = p + \sum h_i f_i + r$ .
  2. Loop: At each stage, if  $\text{in}_\prec(p)$  is divisible by some  $\text{in}_\prec(f_i)$ , subtract  $\frac{\text{in}_\prec(p)}{\text{in}_\prec(f_i)} f_i$  from  $p$  and add  $\frac{\text{in}_\prec(p)}{\text{in}_\prec(f_i)}$  to  $h_i$ .  
If  $\text{in}_\prec(p)$  is not divisible by any  $\text{in}_\prec(f_i)$ , subtract it from  $p$  and add it to  $r$ .
  3. Termination: stop when  $p = 0$  and output  $h_1, \dots, h_s, r$ .

**Example 1.1.1.**  $f = \underline{x} + 2y + 3z$ ,  $f_1 = \underline{x} + y + z$ ,  $f_2 = \underline{5y} + 3z$ , term order is  $\prec_{\text{lex}}$  and  $x \succ y \succ z$ .

1. Initialize:  $h_1 = h_2 = r = 0$ ,  $p = x + 2y + 3z$
2. 1st loop: The underlined are initial terms, and  $\text{in}_{\prec}(p) = x$  is divisible by  $\text{in}_{\prec}(f_1) = x$ , so

$$p = p - \frac{\text{in}_{\prec}(p)}{\text{in}_{\prec}(f_1)} f_1 = x + 2y + 3z - (x + y + z) = y + 2z$$

$$\text{and } h_1 = 0 + \frac{\text{in}_{\prec}(p)}{\text{in}_{\prec}(f_1)} = 1.$$

3. 2nd loop:  $\text{in}_{\prec}(p) = y$  is divisible by  $\text{in}_{\prec}(f_2) = 5y$ , so

$$p = p - \frac{\text{in}_{\prec}(p)}{\text{in}_{\prec}(f_2)} f_2 = y + 2z - \frac{1}{5}(5y + 3z) = \frac{7}{5}z$$

$$\text{and } h_2 = 0 + \frac{\text{in}_{\prec}(p)}{\text{in}_{\prec}(f_2)} = \frac{1}{5}.$$

4. Termination:  $\text{in}_{\prec}(p) = \frac{7}{5}z$  is not divisible by any  $\text{in}_{\prec}(f_i)$ , so

$$p - \text{in}_{\prec}(p) = 0, \quad r = \text{in}_{\prec}(p) = \frac{7}{5}z$$

and we have the expression

$$x + 2y + 3z = 1(x + y + z) + \frac{1}{5}(5y + 3z) + \frac{7}{5}z.$$

**Example 1.1.2.** Divide  $f = x^2$  by  $f_1 = x + y + z$  and  $f_2 = y - z$  with  $\prec_{\text{lex}}$  and  $x \succ y \succ z$ .

1.  $h_1 = h_2 = r = 0$ ,  $p = f = x^2$
2.  $p = p - \frac{\text{in}_{\prec}(p)}{\text{in}_{\prec}(f_1)} f_1 = x^2 - \frac{x^2}{x}(x + y + z) = -xy - xz$ ,  $h_1 = 0 + x = x$
3.  $p = p - \frac{\text{in}_{\prec}(p)}{\text{in}_{\prec}(f_1)} f_1 = -xy - xz - (-y)(x + y + z) = -xz + y^2 + yz$ ,  $h_1 = h_1 - y = x - y$
4.  $p = p - \frac{\text{in}_{\prec}(p)}{\text{in}_{\prec}(f_1)} f_1 = -xz + y^2 + yz + z(x + y + z) = y^2 + 2yz + z^2$ ,  $h_1 = h_1 - z = x - y - z$
5.  $p = p - \frac{\text{in}_{\prec}(p)}{\text{in}_{\prec}(f_2)} f_2 = y^2 + 2yz + z^2 - y(y - z) = 3yz + z^2$ ,  $h_2 = 0 + y = y$
6.  $p = p - \frac{\text{in}_{\prec}(p)}{\text{in}_{\prec}(f_2)} f_2 = 3yz + z^2 - 3z(y - z) = 4z^2$ ,  $h_2 = h_2 + 3z = y + 3z$
7.  $4z^2$  not divisible by any  $\text{in}_{\prec}(f_i)$ , so terminate.  $p = p - \text{in}_{\prec}(p)$ ,  $r = \text{in}_{\prec}(p)$ , output  $h_1 = x - y - z$ ,  $h_2 = y + 3z$ ,  $r = 4z^2$ , and check:

$$x^2 = (x - y - z)(x + y + z) + (y + 3z)(y - z) + 4z^2.$$

The coming punchline is that if  $f_i$ 's are a Gröbner basis then remainder  $r$  is unique.

**Lemma 1.1.3.** Let  $I = \langle x^u : u \in A \rangle$  for some  $A \subseteq \mathbb{N}^n$ , then

1.  $x^v \in I$  iff  $x^u \mid x^v$  for some  $u \in A$
2. if  $f = \sum c_v x^v \in I$ , then each  $x^v$  is divisible by  $x^u$  for some  $u \in A$

**Proposition 1.1.4.** If  $\{g_1, \dots, g_s\}$  is a Gröbner basis for  $I$  with respect to  $\prec$ , then  $f \in I$  iff the division algorithm dividing  $f$  by  $g_1, \dots, g_s$  gives remainder 0.

*Proof.*  $\Rightarrow$  Division algorithm writes  $f = \sum h_i g_i + r$ , so if  $r = 0$  we have  $f \in I$ .

$\Leftarrow$  We prove the contrapositive: suppose  $r \neq 0$ . If  $f \in I$  then  $r \in I$ , so  $\text{in}_\prec(r) \in \text{in}_\prec(I)$ . But by construction,  $\text{in}_\prec(r)$  is not divisible by  $\text{in}_\prec(g_i)$  for any  $i$ . This contradicts that  $\text{in}_\prec(I) = \langle \text{in}_\prec(g_1), \dots, \text{in}_\prec(g_s) \rangle$ .

□

Week 2, lecture 2 starts here (Chunyi Li)

## 2 Noetherian ring

**Definition 2.0.1.** A ring  $R$  is *Noetherian* if every ideal of  $R$  is finitely generated.

**Example 2.0.2.** 1.  $\mathbb{R}$  and  $\mathbb{C}$  are fields, so they only have two ideals  $\langle 0 \rangle, \langle 1 \rangle$ , so Noetherian.

2.  $\mathbb{Z}$  and  $\mathbb{C}[x]$  are principal ideal domains, this implies they are Noetherian.

3.  $\mathbb{C}[x, y]$  and  $\mathbb{Z}[x]$ ?

4.  $R := \{f : \mathbb{R} \rightarrow \mathbb{R} : f \text{ continuous}\}$ , probably not?

5.  $\mathbb{C}[x_1, \dots, x_n, \dots] = \bigcup_{n=1}^{\infty} \mathbb{C}[x_1, \dots, x_n]$ , a polynomial ring which has infinite variables but finite nonzero terms.

**Definition 2.0.3.** A ring  $R$  satisfies *ascending chain condition* (ACC) if every chain of ideals  $I_1 \subseteq I_2 \subseteq \dots \subseteq I_n \subseteq \dots$  eventually stabilizes, i.e.  $\exists n \in \mathbb{N} : I_m = I_n \forall m \geq n$ , i.e.  $\nexists$  strictly ascending chain of ideals  $I_1 \subsetneq I_2 \subsetneq \dots \subsetneq I_n \subsetneq \dots$ .

**Proposition 2.0.4.**  $R$  is Noetherian iff  $R$  satisfies ACC.

*Proof.*  $\Rightarrow$  Let  $I_1 \subseteq I_2 \subseteq \dots \subseteq I_n \subseteq \dots \triangleleft R$  and consider  $J = \bigcup_{k=1}^{\infty} I_k$ . Note  $\forall r, s \in J$ ,  $r \in I_j$ ,  $s \in I_t$ . WLOG assume  $j \leq t$ , then  $r, s \in I_t$  and  $r \pm s \in I_t \subset J$ , and more generally  $J \triangleleft R$ . Since  $J$  is finitely generated, we write  $J = \langle f_1, \dots, f_m \rangle$ . By definition  $f_i \in I_{n_i}$ , so  $\exists N : f_i \in I_N \forall i$ , implying  $J \subseteq I_N$ . But  $J$  is already the union of all ideals, so the chain must stabilize at  $I_N$ .

$\Leftarrow$  Let  $I \triangleleft R$  and suppose  $I$  is not finitely generated. We know  $\exists f_1 \neq 0 \in I$  and  $I \neq \langle f_1 \rangle$ , also  $\exists f_2 \in I \setminus \langle f_1 \rangle$  and  $I \neq \langle f_1, f_2 \rangle$ . We can keep doing this and in general

$$\exists f_{n+1} \in I \setminus \langle f_1, \dots, f_n \rangle \Rightarrow I \neq \langle f_1, \dots, f_{n+1} \rangle \quad \forall n \in \mathbb{N}$$

This gives us a strictly ascending chain  $\langle f_1 \rangle \subsetneq \langle f_1, f_2 \rangle \subsetneq \dots \subsetneq \langle f_1, \dots, f_n \rangle \subsetneq \dots$  which is a contradiction.

□

**Example 2.0.5.** 1. We now know the 4th of Example 2.0.2 is not Noetherian, since

$$\langle \sin x \rangle \subsetneq \left\langle \sin \frac{x}{2} \right\rangle \subsetneq \left\langle \sin \frac{x}{4} \right\rangle \subsetneq \dots \subsetneq \left\langle \sin \frac{x}{2^n} \right\rangle \subsetneq \dots$$

is a strictly ascending chain of ideals.

2. Also,

$$\langle x_1 \rangle \subsetneq \langle x_1, x_2 \rangle \subsetneq \cdots \subsetneq \langle x_1, \dots, x_n \rangle \subsetneq \cdots$$

so the 5th is also not Noetherian.

**Theorem 2.0.6** (1st isomorphism theorem). Let  $R, S$  be rings. If  $\varphi : R \rightarrow S$  is a ring homomorphism then  $\text{im } \varphi \cong R / \ker \varphi$ . If  $\varphi$  is surjective then  $\text{im } \varphi = S$  so we have  $S \cong R / \ker \varphi$ .

$\forall I \triangleleft R$ ,  $R/I$  is a ring, and there is a natural surjective homomorphism  $\varphi : R \rightarrow R/I$  defined by  $r \mapsto r + I$ . Note that  $I = \ker \varphi$ , so this is an isomorphism.

**Theorem 2.0.7** (4th isomorphism theorem). For the same  $\varphi$  as above, there is a 1-1 correspondence

$$\varphi^{-1} : \{J \triangleleft R/I\} \rightarrow \{\tilde{J} \triangleleft R : J \supseteq I \triangleleft R\}.$$

**Proposition 2.0.8.** If  $R$  is Noetherian then  $R/I$  is Noetherian  $\forall I \triangleleft R$ .

*Week 2, lecture 3 starts here*

*Proof.* Suppose  $\exists J_1 \subsetneq \cdots \subsetneq J_n \subsetneq \cdots \triangleleft R/I$ . Then by 4th isomorphism theorem,

$$\exists \varphi^{-1}(J_1) \subsetneq \cdots \subsetneq \varphi^{-1}(J_n) \subsetneq \cdots \triangleleft R,$$

a contradiction. □

**Theorem 2.0.9** (Hilbert basis theorem). If  $R$  is Noetherian then  $R[x]$  is Noetherian.

*Proof (nonexamenable).* Let  $I \triangleleft R[x]$ . Suppose  $I$  is not finitely generated.  $\exists f_1 \in I$  with the minimal degree such that  $I \neq \langle f_1 \rangle$ . Now choose  $f_2 \in I \setminus \langle f_1 \rangle$  with the minimal degree so that  $I \neq \langle f_1, f_2 \rangle$ . We proceed inductively and have

$$\exists f_{n+1} \in I \setminus \langle f_1, \dots, f_n \rangle \text{ with minimal degree so that } I \neq \langle f_1, \dots, f_{n+1} \rangle.$$

For every  $f_i$  we can write  $f_i = r_i x^{n_i} + \text{lower degree terms}$  and  $n_1 \leq n_2 \leq \cdots \leq n_m \leq \cdots$ . We now claim that

$$\langle r_1 \rangle \subsetneq \langle r_1, r_2 \rangle \subsetneq \cdots \subsetneq \langle r_1, \dots, r_m \rangle \subsetneq \cdots$$

is a strictly ascending chain of ideals in  $R$ , which gives a contradiction. To see this, suppose  $r_{m+1} \in \langle r_1, \dots, r_m \rangle$ , i.e.

$$r_{m+1} = s_1 r_1 + \cdots + s_m r_m \quad \text{for some } s_1, \dots, s_m \in R,$$

Now consider

$$\tilde{f}_{m+1}(x) := f_{m+1}(x) - s_1 x^{n_{m+1}-n_1} f_1(x) - s_2 x^{n_{m+1}-n_2} f_2(x) - \cdots - s_m x^{n_{m+1}-n_m} f_m(x),$$

whose leading terms cancel and  $\deg \tilde{f}_{m+1} < \deg f_{m+1}$ . But  $\tilde{f}_{m+1}$  still satisfies that it's not in  $\langle f_1, \dots, f_m \rangle$ , contradicting the minimality of  $\deg f_{m+1}$ . □

**Corollary 2.0.10.** If  $R$  is Noetherian then  $R[x_1, \dots, x_n]$  is Noetherian.

*Proof.* One knows  $R[x]$  is Noetherian. Now assume  $R[x_1, \dots, x_m]$  is Noetherian. Then

$$R[x_1, \dots, x_{m+1}] = (R[x_1, \dots, x_m])[x_{m+1}]$$

is Noetherian, so by induction one has what's desired. □

**Example 2.0.11.** 1.  $\mathbb{Z}$  is a PID, so Noetherian, so  $\mathbb{Z}[x]$  is Noetherian.

2.  $\mathbb{Z}[\sqrt{5}] \cong \mathbb{Z}[x]/\langle x^2 - 5 \rangle$  is Noetherian.

3.  $\mathbb{Z}[\sqrt{5}, \sqrt[4]{7}] \cong \mathbb{Z}[x, y]/\langle x^2 - 5, x^4 - 7 \rangle$  is Noetherian.

4. We have already seen that all fields are Noetherian, and any ring is a subring of its field of fractions. So it's not true that a subring of a Noetherian ring is Noetherian.

**Definition 2.0.12.** An ideal  $I \triangleleft R$  is *prime* if

1.  $I \neq R$

2.  $\forall fg \in I, f \text{ or } g \in I$

**Example 2.0.13.** In  $\mathbb{Z}$ ,  $\langle p \rangle$  where  $p$  prime is a prime ideal by Euclid's lemma. Also  $\langle 0 \rangle$  is prime, but  $\langle 1 \rangle$  is not since it's the whole ring.

*Week 3, lecture 1 starts here*

## 2.1 Every ideal $I$ in $\mathbb{C}[x_1, \dots, x_n]$ has a finite Gröbner basis

*Proof of Lemma 1.1.3.* Note that 1 is a special case of 2, so it suffices to prove the latter.

If  $f \in I$  write  $f = \sum c_v x^v = \sum_{u \in A} h_u x^u$  with only finitely many  $h_u \neq 0$ . We expand the RHS as a sum of monomials, each monomial is divisible by some  $x^u$  with  $u \in A$ . Hence the same is true for  $x^v$  with  $c_v \neq 0$  since these are terms remaining after cancellation.  $\square$

**Theorem 2.1.1** (Dickson's lemma). Let  $I = \langle x^u : u \in A \rangle \subseteq S = K[x_1, \dots, x_n]$  for some  $A \subseteq \mathbb{N}^n$ . Then  $\exists a_1, \dots, a_s \in A$  with  $I = \langle x^{a_1}, \dots, x^{a_s} \rangle$ .

Before diving into the proof let's think about two special cases.

$n = 1$  Consider  $I = \langle x_1^3, x_1^7, x_1^{70000}, x_1^{1234}, \dots \rangle$ . One can see that  $x_1^3$  is sufficient to generate the whole  $I$ .

$n = 2$  Consider  $u, v \in \mathbb{N}^2$  as points on a lattice grid. Then  $x^u$  is divisible by  $x^v$  if it's top right of it, so we can get rid of unnecessary ones in a similar fashion.

Now let's turn these intuitions into a general proof.

*Proof by induction.* Straightforwardly, when  $n = 1$ ,  $I = \langle x_1^{\alpha_1} \rangle$  for  $\alpha = \min\{j : x_j^I\}$ . Now assume  $n > 1$  and the theorem is true for  $n - 1$ .

Write the variables in  $S$  as  $x_1, \dots, x_{n-1}, y$  and let  $I$  be an ideal in  $S$ . Let  $J = \langle x^u : x^u y^c \in I \text{ for some } c \geq 0 \rangle \subseteq K[x_1, \dots, x_{n-1}]$ . By inductive hypothesis,  $J$  is finitely generated, so write  $J = \langle x^{a_{m_1}}, \dots, x^{a_{m_r}} \rangle$  for  $x^{a_{m_i}} y^{m_i} \in I$ .

Let  $m = \max\{m_i\}$ . For  $0 \leq l \leq m - 1$ , let  $J_l = \langle x^u : x^u y^l \in I \rangle \subseteq K[x_1, \dots, x_{n-1}]$ . Again  $J_l$  is finitely generated and write  $J_l = \langle x^{a_{j_1}}, \dots, x^{a_{j_{r_l}}} \rangle$ . We claim that  $I$  is generated by  $\{x^{a_{m_i}} y^{m_i} : 1 \leq i \leq r\} \cup \{x^{a_{j_i}} y^j : 1 \leq j \leq m - 1, 1 \leq i \leq r_j\}$ . Indeed, if  $x^u y^j \in I$  then either

1.  $j < m$ , so  $x^u \in J_j$ , so  $x^{j_i} \mid x^u$  for some  $i$ , and so  $x^{a_{j_i}} y^j \mid x^u y^j$ .

2.  $j \geq m$ , so  $x^u \in J$ , so  $x^{a_{m_i}} \mid x^u$  for some  $i$ , and so since  $m_i \leq m$ ,  $x^{a_{m_i}} y^{m_i} \mid x^u y^j$ .



So every monomial in  $I$  is a multiple of one of the claimed generators.

If any of these generators is not in our original set  $A$ , we can replace it by a monomial with exponent in  $A$ , and by Lemma 1.1.3 if they generate all monomials then they generate the whole  $I$ .  $\square$

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**Corollary 2.1.2.** Every ideal in  $S = K[x_1, \dots, x_n]$  has a finite Gröbner basis with respect to a term order.

*Proof.* The initial ideal in  $\text{in}_\prec(I) = \langle \text{in}_\prec(f) : f \in I \rangle$  is a monomial ideal (using that coefficients can be omitted since we are in a field). By Dickson's lemma, there are  $g_1, \dots, g_s \in I$  with  $\langle \text{in}_\prec(g_1), \dots, \text{in}_\prec(g_s) \rangle = \text{in}_\prec(I)$ . Thus  $\{g_1, \dots, g_s\}$  is a Gröbner basis for  $I$  by definition.  $\square$

**Proposition 2.1.3.** If  $\{g_1, \dots, g_s\}$  is a Gröbner basis for  $I$  with respect to  $\prec$ , then  $I = \langle g_1, \dots, g_s \rangle$ .

*Proof.* By division algorithm, any  $f \in I$  can be written as  $f = \sum h_i g_i$  with remainder 0 since  $f \in I$ . It follows that  $f \in \langle g_1, \dots, g_s \rangle$ , which gives the desired since  $f$  is arbitrary.  $\square$

**Corollary 2.1.4** (Special case of Hilbert basis theorem). Every ideal in  $S = K[x_1, \dots, x_n]$  is finitely generated.

*Proof.* Immediate from previous two results.  $\square$

**Exercise 2.1.5.** Claim:  $y = \left\{ \underline{x_2^2} - x_1x_3, \underline{x_2x_3} - x_1x_4, \underline{x_3^2} - x_2x_4 \right\}$  is a Gröbner basis with respect to revlex. Find the remainder on dividing  $x_2^2x_3^2$  by  $y$ .

$$\begin{aligned} f_1 : x_2^2x_3^2 &\xrightarrow{f_1} x_1x_3 \xrightarrow{f_3} x_1x_2x_3x_4 \xrightarrow{f_2} x_1^2x_4^2 \\ f_2 : x_2^2x_3^2 &\xrightarrow{f_2} x_1x_2x_3x_4 \xrightarrow{f_2} x_1^2x_4^2 \\ f_3 : x_2^2x_3^2 &\xrightarrow{f_3} x_2^3x_4 \xrightarrow{f_1} x_1x_2x_3x_4 \xrightarrow{f_2} x_1^2x_4^2 \end{aligned}$$

The remainders are the same: this shouldn't surprise us. But we haven't proved it, so why did this work?

### 3 General commutative ring

**Definition 3.0.1.** An ideal  $I \subseteq R$  is *prime* if it's proper and  $f, g \in I \Rightarrow f$  or  $g \in I$ .

**Notation.**  $\text{Spec}(R) := \{\text{prime ideals in } R\}$ .

**Example 3.0.2.**  $R = \mathbb{Z}/6\mathbb{Z}$ ,  $\text{Spec}(R) = \{\langle 2 \rangle, \langle 3 \rangle\}$ . Note that although 5 is prime but  $\langle 5 \rangle$  is not a prime ideal since  $5^2 = 1$  in  $\mathbb{Z}/6\mathbb{Z}$  so it's not proper.

**Lemma 3.0.3.** An ideal  $P \subseteq R$  is prime iff  $R/P$  is a domain.

*Proof.*  $P$  is prime iff

$$fg \in P \Rightarrow f \text{ or } g \in P. \quad (*)$$

$R/P$  is a domain iff  $fg + P = 0 + P \Rightarrow f + P$  or  $g + P = 0 + P$ , which is equivalent to  $(*)$ .  $\square$

**Definition 3.0.4.** An ideal  $I \subseteq R$  is *maximal* if it's proper and there is no ideal  $J : I \subsetneq J \subsetneq R$ .

Do maximal ideals always exist? Yes, if we assume axiom of choice.

Recall: a *partially ordered* set is a set  $S$  with transitive, reflexive binary relation  $\leq$  (e.g.  $\leq$  on  $\mathbb{R}$  or power set (inclusion)). Given a subset  $U \subseteq S$ , an *upper bound* for  $U$  is  $s \in S$  with  $u \leq s \forall u \in U$ . An element  $m \in S$  is *maximal* if  $\nexists s \in S$  with  $s > m$ .

**Axiom 3.0.5** (Zorn's lemma). Let  $S$  be a nonempty partially ordered set with the property that any totally ordered subset  $U \subseteq S$  (a 'chain') has an upper bound. Then  $S$  has a maximal element.

This is equivalent to:

1. The axiom of choice: every product  $\prod_{a \in A} S_a$  of nonempty sets is nonempty.
2. Well-ordering principle: every set can be well-ordered.

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**Proposition 3.0.6.** Let  $R$  be a ring and let  $I$  be a proper ideal of  $R$ . Then there is a maximal ideal  $M$  containing  $I$ .

*Proof.* Let  $\mathcal{I}$  be the set of proper ideals in  $R$  containing  $I$ , ordered by inclusion ( $J_1 \leq J_2$  if  $J_1 \subseteq J_2$ ). Note that if  $\{J_\alpha : \alpha \in A\}$  is a totally ordered (any two are comparable) subset of  $\mathcal{I}$  then  $J = \bigcup_{\alpha \in A} J_\alpha$  is an ideal. [This uses the total order, e.g. in  $K[x, y]$ ,  $\langle x \rangle \cup \langle y \rangle$  is not an ideal since  $x + y$  is not in there.] Since  $J_\alpha \subseteq J \forall \alpha$  and  $I \subseteq J$ , one has  $J \in \mathcal{I}$ . Hence  $J$  is an upper bound for  $\{J_\alpha\}$ . Thus by Zorn's lemma,  $\mathcal{I}$  has a maximal element.  $\square$

**Lemma 3.0.7.**  $I \subseteq R$  is maximal iff  $R/I$  is a field.

*Proof.* Exercise (see Algebra II notes).  $\square$

**Corollary 3.0.8.** Maximal ideals are prime.

*Proof.* If  $I$  is maximal then  $R/I$  is a field, and in particular a domain.  $\square$

### 3.1 Localisation

**Definition 3.1.1.** A ring  $R$  is *local* if it has a unique maximal ideal  $M$ .

**Example 3.1.2.** Every field is local.  $\mathbb{Z}$  is not local since  $\langle 2 \rangle, \langle 3 \rangle$  are both maximal.

Consider

$$\mathbb{Z}_{\langle 2 \rangle} := \left\{ \frac{a}{b} \in \mathbb{Q} : a, b \in \mathbb{Z}, 2 \nmid b \right\}.$$

This is a subring of  $\mathbb{Q}$ . Note that proper ideals are those generated by even integers, but  $\langle 6 \rangle = \langle 2 \rangle$  since  $\frac{1}{3} \in \mathbb{Z}_{\langle 2 \rangle}$ . So in fact they are all generated by powers of 2, and  $\langle 2 \rangle$  is maximal, so  $\mathbb{Z}_{\langle 2 \rangle}$  is local.

$\mathbb{C}[x]$  is not local, since we can build (at least two) quotient rings which is a field by first isomorphism theorem, e.g.  $\varphi_1 : x \rightarrow 1$  and  $\varphi_2 : x \rightarrow i$ .

Now consider

$$\mathbb{C}[x]_{\langle x \rangle} := \left\{ \frac{f}{g} : f, g \in \mathbb{C}[x], x \nmid g \right\}.$$

This is analogous to  $\mathbb{Z}_{\langle x \rangle}$  and its proper ideals are of the form  $\langle x^j \rangle$  with  $\langle x \rangle$  being maximal.

**Definition 3.1.3.** A set  $U \subseteq R$  is *multiplicatively closed* if  $1 \in U$  and  $f, g \in U \Rightarrow fg \in U$ .

**Example 3.1.4.** In any  $R$  with  $f \in R$ ,  $U = \{1, f, f^2, \dots\}$  is multiplicatively closed.

Suppose  $P \subseteq R$  is prime. Then  $1 \notin P$ , i.e.  $1 \in R \setminus P$ , and  $fg \in P \Rightarrow f \in P$  or  $g \in P$ , so  $f, g \in R \setminus P \Rightarrow fg \in R \setminus P$ . By definition this means  $R/P$  is multiplicatively closed.

$U = \{r \in R : \exists s \in R : rs = 1\} = \{\text{units of } R\}$  is multiplicatively closed. In particular, if  $R$  is a domain then  $U = R \setminus \{0\}$  is.

**Definition 3.1.5.** Let  $R$  be a ring and let  $U \subseteq R$  be multiplicatively closed. Then

$$R[U^{-1}] := \left\{ \frac{r}{u} : r \in R, u \in U \right\}$$

modulo the equivalence relation  $\sim$

$$\frac{r}{u} \sim \frac{r'}{u'} \quad \text{if} \quad \exists \tilde{u} \in U : \tilde{u}(ru' - r'u) = 0.$$

**Example 3.1.6.**  $R = \mathbb{Z}$ ,  $U = \mathbb{Z} \setminus \{0\}$ . Then  $R[U^{-1}] = \mathbb{Q}$ . We don't have to worry about the  $\tilde{u}$  condition since  $\mathbb{Z}$  is a domain.

$R = \mathbb{Z}$ ,  $U = \mathbb{Z} \setminus \langle 2 \rangle$ . Then  $R[U^{-1}] = \mathbb{Z}_{(2)}$ .

$R = \mathbb{C}[x]$ ,  $U = \mathbb{C}[x] \setminus \langle x \rangle$ . Then  $R[U^{-1}] = \mathbb{C}[x]_{\langle x \rangle}$ .

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**Lemma 3.1.7.** 1. The  $\sim$  in Definition 3.1.5 is indeed an equivalence relation.

2.  $R[U^{-1}]$  is a ring with addition and multiplication defined

$$\frac{r}{u} + \frac{r'}{u'} := \frac{ru' + r'u}{uu'}, \quad \left(\frac{r}{u}\right) \left(\frac{r'}{u'}\right) := \frac{rr'}{uu'}$$

3. The map  $\varphi : R \rightarrow R[U^{-1}]$  given by  $r \mapsto \frac{r}{1}$  is a ring homomorphism.

*Proof.* 1. It's reflexive since  $1(ru - ru) = 0$ . It's symmetric since  $\tilde{u}(ru' - r'u) = 0 \Rightarrow -1\tilde{u}(r'u - ru') = 0$  and  $-1\tilde{u} \in U$  by multiplicative closedness.

Now suppose

$$\frac{r}{u} \sim \frac{r'}{u'}, \quad \frac{r'}{u'} \sim \frac{r''}{u''},$$

then  $\exists \tilde{u} \in U : \tilde{u}(ru' - r'u) = 0$  and  $\exists \tilde{u}' \in U : \tilde{u}'(r'u'' - r''u') = 0$ . So

$$\tilde{u}'u''(\tilde{u}(ru' - r'u)) + \tilde{u}u(\tilde{u}'(r'u'' - r''u')) = 0.$$

which is equal to

$$\tilde{u}\tilde{u}'(ru'u'' - r'uu'' + r'uu'' - r''uu') = \tilde{u}\tilde{u}'u'(ru'' - r''u)$$

where  $\tilde{u}\tilde{u}'u' \in U$ . Therefore it's transitive.

2. (Exercise) One needs to check:

- The two operations are well-defined, i.e. they don't depend on choice of representatives

- Ring axioms, in particular  $\frac{0}{1}$  is additive identity and  $\frac{1}{1}$  is multiplicative identity
3. One has  $\varphi(r+r') = \frac{r+r'}{1} = \frac{r}{1} + \frac{r'}{1} = \varphi(r) + \varphi(r')$  and  $\varphi(rr') = \frac{rr'}{1} = \left(\frac{r}{1}\right) \left(\frac{r'}{1}\right) = \varphi(r)\varphi(r')$ .  $\square$

**Remark.** 1. If  $U$  contains 0 then it's very boring:  $R[U^{-1}] = 0$  iff  $0 \in U$ . Indeed, for  $R[U^{-1}] = 0$  one needs  $\exists u \in U : u \cdot 1 = 0$ , and the only such  $u$  is 0, and if  $0 \in U$  then  $0(r \cdot 1 - 0 \cdot 1) = 0r = 0 \forall r$  hence  $\frac{r}{1} \sim \frac{0}{1} \forall r$ .

2.  $\varphi$  is not always injective, e.g.  $R = \mathbb{Z}/6\mathbb{Z}$ ,  $U = \{1, 3, 5\}$ . Then  $\varphi(2) = \frac{2}{1}$  but  $\frac{2}{1} \sim \frac{0}{1}$  since  $3(2 \times 1 - 0 \times 1) = 0$ . Furthermore,  $\ker \varphi = \{r \in R : \frac{r}{1} \sim \frac{0}{1}\} = \{r \in R : \exists u \in U : ur = 0\}$ .

**Notation** (Important special case). In the case of  $U = R \setminus P$  where  $P$  is prime, we write  $R_P$  for  $R[(R \setminus P)^{-1}]$ . An example would be, again,  $\mathbb{Z}_{(2)}$ .

Why is this important?

**Proposition 3.1.8.** The set  $P_P := \{\frac{r}{u} \in R_P : r \in P\}$  is an ideal of  $R_P$  and is the unique maximal ideal.

*Proof.* If  $\frac{r}{u} \notin P_P$  then  $r \notin P$ , so  $\frac{u}{r} \in R_P$  and hence  $\frac{r}{u}$  is a unit. Now suppose there is a maximal ideal  $I$  and in particular  $\exists \frac{r}{u} \in I \setminus P_P$ . But then  $I$  would be the whole ring  $R_P$  since it contains a unit. This argument also justifies that  $P_P$  is maximal itself.  $\square$

**Corollary 3.1.9** (A fortunate byproduct of the proof).  $I \subseteq R$  is the unique maximal ideal iff every  $r \notin I$  is a unit.

Week 4, lecture 2 starts here

### 3.1.1 Effect of localisation on ideals

We want to investigate the relationship between  $\text{Spec}(R)$  and  $\text{Spec}(R[U^{-1}])$ .

We have the ring homomorphism  $\varphi$ , but  $I \mapsto \varphi(I) := \{\varphi(r) : r \in I\}$  is not good enough, since if  $R = \mathbb{Z}$ ,  $U = \mathbb{Z} \setminus \{0\}$  then  $R[U^{-1}] = \mathbb{Q}$  is a field, so it has only two ideals, and  $\varphi(I) = \{\frac{n}{1} : n \text{ even}\}$  obviously is not one of them. Rather we need  $I \mapsto \varphi(I)R[U^{-1}] := \langle \varphi(r) : r \in I \rangle$ . In the above case,  $\varphi(I)R[U^{-1}] = \mathbb{Q}$ .

For the other way, we can simply consider  $J \mapsto \varphi^{-1}(J)$  as a map without the 'generated by'.

**Lemma 3.1.10.** There is a bijection between ideals  $J \subseteq R[U^{-1}]$  and ideals  $I \subseteq R$  with property

$$ru \in I \text{ for some } u \in U \Rightarrow r \in I. \quad (\star)$$

**Example 3.1.11.** In the above example,  $\langle 6 \rangle$  is not such ideal  $I \subseteq \mathbb{Z}$  since  $6 = 6 \times 1 \in \langle 6 \rangle$ ,  $6 \in U$  but  $1 \notin \langle 6 \rangle$ . Note that this argument works for any  $\langle n \rangle$  where  $n > 1$ . In fact, the only two ideals that satisfy this are  $\langle 0 \rangle, \langle 1 \rangle$  which indeed have a natural bijection to ideals in  $\mathbb{Q}$ .

*Proof.* To show  $J \mapsto \varphi^{-1}(J)$  is injective, we show  $\varphi(\varphi^{-1}(J))R[U^{-1}] = J$ .

$\subseteq$  is clear:  $\varphi^{-1}(J) = \{r : \frac{r}{1} \in J\}$ , so  $\varphi(\varphi^{-1}(J)) = \{\frac{r}{1} : \frac{r}{1} \in J\}$ , and if you take the ideal generated by a subset of  $J$  of course you get something contained in  $J$ .

To see  $\supseteq$ , note that

$$\frac{r}{u} \in J \Rightarrow \frac{u}{1} \frac{r}{u} = \frac{r}{1} \in J,$$

so  $r \in \varphi^{-1}(J)$  and  $\frac{r}{1} \in \varphi(\varphi^{-1}(J))R[U^{-1}]$  and furthermore for any

$$u \in U, \quad \frac{1}{u} \frac{r}{1} = \frac{r}{u} \in \varphi(\varphi^{-1}(J))R[U^{-1}].$$

To show  $J \mapsto \varphi^{-1}(J)$  is surjective, fix  $I \subseteq R$  satisfying  $\star$  and let  $J = \varphi(I)R[U^{-1}]$ . The proof is then complete if we show  $I = \varphi^{-1}(J)$ .

$\frac{r}{1} \in \varphi(I)R[U^{-1}]$  means

$$\begin{aligned} \frac{r}{1} &= \sum \frac{h_i}{u_i} \frac{r_i}{1} \text{ where } r_i \in I, h_i \in R, u_i \in U \\ &= \frac{\tilde{r}}{u} \text{ for some } \tilde{r} \in I, u \in U. \end{aligned}$$

By definition, this implies  $\exists \tilde{u} \in U : \tilde{u}(ur - \tilde{r}) = 0$ , i.e.  $(\tilde{u}u)r = \tilde{u}\tilde{r} \in I$  since  $\tilde{r} \in I$ . By assumption,  $r \in I$ . This shows  $\varphi^{-1}(J) \subseteq I$ , and since  $\frac{r}{1} \in J \forall r \in I$ ,  $I \subseteq \varphi^{-1}(J)$ .  $\square$

**Exercise 3.1.12** (\*). What ideals  $I \subseteq \mathbb{Z}$  satisfy  $\star$  when  $U = \{\text{odd numbers}\}$  and when  $U = \{1, 2, 4, 8, \dots\}$ ?

For  $U = \{\text{odd numbers}\}$ , recall Example 3.1.2.  $U = \mathbb{Z} \setminus \langle 2 \rangle$ , so ideals  $I \subseteq \mathbb{Z}$  satisfy  $\star$  corresponds to ideals of  $\mathbb{Z}_{\langle 2 \rangle}$ , which are generated by powers of 2 (and also the 0 ideal).

**Corollary 3.1.13.**  $J \mapsto \varphi^{-1}(J)$  maps  $\text{Spec}(R[U^{-1}])$  to  $\{P \in \text{Spec}(R) : P \cap U = \emptyset\}$ .

*Proof.* In Homework 2 it will be proved that for any ring homomorphism  $\varphi : R \rightarrow S$ , if  $P \subseteq S$  is prime then  $\varphi^{-1}(P) \subseteq R$  is prime. Now if a prime  $P \subseteq R$  satisfies  $P \cap U = \emptyset$  and if  $ru \in P$  for some  $u \in U$ , then  $r \in P$  since  $P$  is prime and  $u \notin P$ , so it's indeed the image. Conversely, if  $\star$  holds then  $P \cap U = \emptyset$  since if  $u \in P \cap U$ ,  $u = u \cdot 1 \in P$  but  $1 \notin P$ , a contradiction.  $\square$

Week 4, lecture 3 starts here

## 4 Module

**Definition 4.0.1.** Let  $R$  be a ring. An  $R$ -module is an abelian group  $M$  with multiplication  $R \times M \rightarrow M$  (sometimes called  $R$ -action) satisfying

1.  $r(m + n) = rm + rn$
2.  $(r + r')m = rm + r'm$
3.  $(rr')m = r(r'm)$
4.  $1_R m = m$

$\forall r, r' \in R, m, n \in M$ .

**Example 4.0.2.** If  $R = K$  is a field then  $M$  is a  $K$ -vector space. In fact, the definition should remind you of that of vector spaces.

If  $R = \mathbb{Z}$  then  $R$ -modules are abelian groups with  $R \times M \rightarrow M$  given by  $n \times g := \underbrace{g + \dots + g}_{n \text{ times}}$ .

One is forced to define multiplication like this by definition.

If  $R$  is an arbitrary ring and  $I$  is an ideal in  $R$ , then  $R$  itself is a  $R$ -module with multiplication the same as ring multiplication in  $R$ , and  $I$ ,  $R/I$  are also  $R$ -modules.

**Remark.** Much of commutative algebra is generalising linear algebra to modules, and every theorem you see about modules, ask what it says for vector spaces/abelian groups.

**Definition 4.0.3.** A subset  $N \subseteq M$  is a *submodule* if

1.  $m, n \in N \Rightarrow m + n \in N$  and
2.  $m \in N, r \in R \Rightarrow rm \in N$ .

**Example 4.0.4.** A submodule of the  $R$ -module  $R$  is precisely an ideal.

Like any other algebraic objects, it's important to understand functions between modules. We want a definition that can be generalised to group homomorphisms since modules are abelian groups, and can be specified to linear maps since vector spaces are modules.

**Definition 4.0.5.** A function  $\varphi : M \rightarrow N$  where  $M, N$  are  $R$ -modules is an  *$R$ -module homomorphism* if

1.  $\varphi$  is a group homomorphism and
2.  $\varphi(rm) = r\varphi(m)$ .

**Example 4.0.6.** As expected, if  $R$  is a field then an  $R$ -module homomorphism is a linear map, and if  $R = \mathbb{Z}$  then it's a group homomorphism. Also  $R \rightarrow R/I$  and  $I \rightarrow R$  for  $I$  an ideal given by  $r \mapsto r$  are  $R$ -module homomorphisms.

**Definition 4.0.7.** The *kernel* of an  $R$ -module homomorphism  $\varphi : M \rightarrow N$  is

$$\ker \varphi := \{m \in M : \varphi(m) = 0_N\},$$

and the *image* of  $\varphi$  is

$$\operatorname{im} \varphi = \{\varphi(m) : m \in M\}.$$

**Exercise 4.0.8.** Show that these are both submodules of  $M$  and  $N$  respectively.

**Definition 4.0.9.** If  $N$  is a submodule of an  $R$ -module  $M$ , then it is also a subgroup of the abelian group  $M$ , so we can construct quotient group  $M/N$ . This is an  $R$ -module with  $r(m+N) = rm + N$  and called a *quotient module*.

**Theorem 4.0.10** (Isomorphism theorems). 1. If  $\varphi : M \rightarrow N$  is an  $R$ -module homomorphism then  $M/\ker \varphi \cong \operatorname{im} \varphi$ . (The morally equivalence of this in linear algebra is the rank-nullity theorem.)

2. If  $L \subseteq M \subseteq N$  with  $L$  a submodule of  $M$  and  $M$  a submodule of  $N$ , then  $N/M \cong (N/L)/(M/L)$ .

3. If  $L, M$  are submodules of  $N$  then  $(L + M)/L \cong M/(M \cap L)$  where  $L + M := \{l + m : l \in L, m \in M\}$ . (This is a generalisation of the proposition about dimensions of subspaces in linear algebra.)

*Week 5, lecture 1 starts here*

## 4.1 Free module

Recall that every finite dimensional  $K$ -vector space is isomorphic to  $K^n$ .

**Definition 4.1.1.** The  $R$ -module  $R^n$  is defined to be

$$\{(r_1, \dots, r_n) : r_i \in R\}$$

with  $R$ -action

$$r(r_1, \dots, r_n) = (rr_1, \dots, rr_n), \quad (r_1, \dots, r_n) + (r'_1, \dots, r'_n) = (r_1 + r'_1, \dots, r_n + r'_n).$$

**Remark.** 1. More generally, for any index set  $A$  (might be uncountable),  $M_1 := \{(r_\alpha : \alpha \in A) : r_\alpha \in R\}$  (functions  $A \rightarrow R$ ) and  $M_2 := \{(r_\alpha : \alpha \in A) : r_\alpha \in R, \text{ only finitely many } r_\alpha \neq 0\}$  are  $R$ -modules.

2. Note that every element of  $R^n$  can be written as an  $R$ -linear combination of the standard basis  $e_i$  as expected.

**Definition 4.1.2.** Let  $M$  be an  $R$ -module and  $\mathcal{G} = \{m_\beta : \beta \in B\} \subseteq M$ . Then  $\mathcal{G}$  generates  $M$  (as an  $R$ -module) if every element  $m \in M$  can be written as

$$m = \sum_{i=1}^s r_i m_{\beta_i} \text{ for some } \beta_1, \dots, \beta_s \in B, \quad r_1, \dots, r_s \in R.$$

Note that  $B$  might be infinite but the sum must be finite.

**Example 4.1.3.** 1. If  $R$  is a field then the verb generate is the same as ‘span’ as in linear algebra.

2. If  $R = \mathbb{Z}$  then  $\mathcal{G}$  generates  $M$  iff it generates  $M$  as an abelian group.

3. If  $M = I \subseteq R$  is an ideal, then  $\mathcal{G}$  generates  $M$  as an  $R$ -module iff  $g$  generates  $I$  as an ideal.

**Exercise 4.1.4.** 1. Give an example of an  $R$ -module  $M$  with  $g \subseteq M$  that generates  $M$  as an  $R$ -module but not as an abelian group.

Consider  $M = R = \mathbb{R}$ . Then  $\mathcal{G} = \{1\}$  generates  $M$  as an  $R$ -module, but the abelian group it generates is  $\mathbb{Z} \subsetneq \mathbb{R}$ .

2. Generators for  $M_1, M_2$  in above remark?

**Definition 4.1.5.** A set  $\mathcal{G} \subseteq M$  is a *basis* for  $M$  if  $\mathcal{G}$  generates  $M$  as an  $R$ -module and every element of  $M$  can be written uniquely as an  $R$ -linear combination of finitely many elements of  $\mathcal{G}$ . Equivalently, if  $\sum_{i=1}^s r_i g_i = 0_M$  for  $g_i \in \mathcal{G}$ ,  $r_i \in R$  then  $r_i = 0 \forall i = 0$ .

**Example 4.1.6.** 1. If  $R$  is a field then a basis for  $M$  as an  $R$ -module is a basis for  $M$  as a  $R$ -vector space.

2. A basis for  $R^2$  is  $\{(1, 0), (0, 1)\}$ .

3.  $\{e_\alpha\}$  is a basis for  $M_2$ . It's not a basis for  $M_1$  since the sum could be infinite.  $\nexists$  There are modules with no basis (that's kind of the point of this section), e.g.  $R = \mathbb{C}[x, y]$ ,  $M = \langle x, y \rangle$ . Suppose  $M$  has a basis  $\mathcal{G}$ . First of all  $|\mathcal{G}| > 1$  since  $M$  is not a principal ideal. Now pick  $f, g \in \mathcal{G}$ . Then  $fg \in M$ , but  $fg = gf$ , so not uniquely expressed, a contradiction.

**Definition 4.1.7.** A  $R$ -module is *free* if it has a basis.

**Example 4.1.8.**  $R^n$  and  $M_2$  are free.  $\langle x, y \rangle \subseteq \mathbb{C}[x, y]$  is not a free  $\mathbb{C}[x, y]$ -module, but it's a free  $\mathbb{C}$ -module since  $\mathbb{C}$  is a field so it's a vector space, which always has a basis.

$\mathbb{Q}$  is not a free  $\mathbb{Z}$ -module. First a basis would have to be infinite, but  $bc\left(\frac{a}{b}\right) - ad\left(\frac{c}{d}\right) = 0$ , a nontrivial linear dependence relation.

## 4.2 Cayley–Hamilton theorem

**Remark.** Matrices make sense over an arbitrary ring and give a  $R$ -module homomorphism  $\varphi : R^n \rightarrow R^n$ . Determinants still make sense (as a indicator of whether  $\varphi$  is invertible).

**Definition 4.2.1.** Let  $M$  be an  $R$ -module. The set of all  $R$ -module homomorphisms  $\varphi : M \rightarrow M$  forms a noncommutative ring with identity and  $(\varphi + \psi)(m) = \varphi(m) + \psi(m)$  and  $(\varphi\psi)(m) = \varphi(\psi(m))$ , denoted  $\text{End}(M)$ .

**Example 4.2.2.** If  $R$  is a field and  $M = R^n$  then  $\text{End}(M) = n \times n$  matrices.

**Notation.**  $\{\varphi_s : s \in R\}$  where  $\varphi_s(m) := sm$ .

**Definition 4.2.3.** Given an  $n \times n$  matrix  $A$ , the subring  $R[A]$  of  $\text{End}(R^n)$  is the smallest subring of  $\text{End}(R^n)$  containing  $A$  and all  $\{\varphi_s : s \in R\}$ . Explicitly,

$$R[A] = \left\{ \sum_{i=0} a_i A^i : a_i \in R \right\}$$

and set  $A^0 = I$ .

**Remark.** Note that  $R[A]$  is commutative, and (suggestive by notation) we have a ring homomorphism

$$\begin{aligned} \psi : R[x] &\rightarrow R[A] \\ x &\mapsto A. \end{aligned}$$

This is of course not an isomorphism by Cayley–Hamilton theorem.

Also,  $R^n$  is an  $R[A]$  module with action given by

$$\left( \sum_{i=0} a_i A^i \right) \underline{v} = \sum_{i=0} a_i (A^i \underline{v})$$

where  $\underline{v} = (r_1, \dots, r_n) \in R^n$ .

**Definition 4.2.4.** The *characteristic polynomial* of  $A \in \text{End}(R^n)$  is  $\det(xI - A) \in R[x]$ .

**Example 4.2.5.** 1.  $A = \begin{pmatrix} x & x^2 \\ x^3 & x^4 \end{pmatrix}$  and  $R = \mathbb{C}[x]$ . Then

$$\begin{aligned} \det \begin{pmatrix} t-x & x^2 \\ x^3 & t-x^4 \end{pmatrix} &= (t-x)(t-x^4) - x^5 \\ &= t^2 - (x+x^4)t + x^5 - x^5 \\ &= t^2 - (x+x^4)t. \end{aligned}$$



2.  $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$  and  $R = \mathbb{Z}/6\mathbb{Z}$ . Then

$$\det \begin{pmatrix} x-1 & 2 \\ 3 & x-4 \end{pmatrix} = (x-1)(x-4) - 6 = x^2 - 5x - 2 = x^2 + x + 4.$$

**Remark.** Recall the adjoint of an  $n \times n$  matrix  $B$  is the matrix  $C$  with

$$C_{ij} = (-1)^{i+j} \det(B \setminus i\text{th column and } j\text{th row}),$$

which makes sense over any ring. We claim  $BC = CB = \det B I_n$  (which implies if  $\det B$  is a unit then  $B$  is invertible). Indeed,

$$\begin{aligned} (BC)_{ij} &= \sum_{k=1}^n B_{ik} C_{kj} \\ &= \sum_{k=1}^n (-1)^{k+j} B_{ik} \det(B \setminus k\text{th column and } j\text{th row}) \\ &= \det(B \text{ with } j\text{th row replaced by } i\text{th row}) \\ &= \begin{cases} 0 & \text{if } i \neq j \\ \det B & \text{if } i = j \end{cases} \\ &= (\det B I_n)_{ij}. \end{aligned}$$

**Theorem 4.2.6** (Cayley–Hamilton). Let  $R$  be a ring and let  $A$  be an  $n \times n$  matrix with entries in  $R$ . Set  $p_A(x) = \det(xI - A) \in R[x]$ , then  $p_A(A) = 0$ , i.e.  $p_A \in \ker \psi$  where  $\psi$  is as in remark after Definition 4.2.3.

**Definition 4.2.7.** An  $R$ -module  $M$  is *finitely generated* if it has a finite generating set.

**Theorem 4.2.8.** Let  $M$  be a finitely generated  $R$ -module with  $n$  generators and  $\varphi : M \rightarrow M$  an  $R$ -module homomorphism. Suppose  $I$  is an ideal of  $R$  with  $\varphi(M) \subseteq IM := \langle rm : r \in I, m \in M \rangle$ . Then  $\varphi$  satisfies a relation of the form

$$\varphi^n + a_1 \varphi^{n-1} + a_{n-1} \varphi + a_n = 0 \in \text{End}(M) \text{ where } a_i \in I.$$

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*Proof of Cayley–Hamilton theorem.* Write  $e_1, \dots, e_n$  for the standard basis for  $R^n$  and one has  $Ae_k = \sum_{j=1}^n a_{jk} e_j$  (the  $k$ th column of  $A$ ). Write  $\delta_{jk}$  for the Kronecker delta. Then

$$\sum_{j=1}^n (A\delta_{jk} - a_{jk}I) e_j = \sum_{j=1}^n A\delta_{jk} e_j - \sum_{j=1}^n a_{jk} I e_j = Ae_k - Ae_k = 0.$$

Let  $B = (B_{jk})$  be the  $n \times n$  matrix with entries in  $R[A]$  with  $B_{jk} = A\delta_{jk} - a_{jk}I$  and  $C$  the

adjoint of  $B$ . Then

$$\begin{aligned} 0 &= \sum_{k=1}^n C_{kj} \left( \sum_{i=1}^n A\delta_{ik} - a_{ik}I \right) e_i = \sum_{i=1}^n \left( \sum_{k=1}^n C_{kj} (A\delta_{ik} - a_{ik}I) e_i \right) \\ &= \sum_{i=1}^n \sum_{k=1}^n B_{ik} C_{kj} e_i = \sum_{i=1}^n (BC)_{ij} e_i = (\det B) e_j = \begin{pmatrix} 0 \\ \vdots \\ \det B \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{the } j\text{th position,} \end{aligned}$$

so  $\det B = 0$ .

Now  $p_A(x) \in R[x]$ . Then

$$\psi(p_A(x)) = p_A(A) = \det \left( \begin{pmatrix} A & & 0 \\ & A & \\ 0 & & A \end{pmatrix} - A \right) = \det B = 0.$$

□

**Exercise 4.2.9.** Let

$$D = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\ \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} & \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \end{pmatrix}.$$

Then

$$\begin{aligned} \det D &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} - \begin{pmatrix} 3 & 8 \\ 9 & 16 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -2 \\ -2 & -28 \end{pmatrix}. \end{aligned}$$

*Proof of Theorem 4.2.8.* Let  $\{m_1, \dots, m_n\}$  be a generating set for  $M$ . Since  $\varphi(M) \in IM$ , one can write

$$\varphi(m_i) = \sum_{j=1}^n a_{ji} m_j \text{ with } a_{ji} \in I.$$

So  $\sum_{j=1}^n (\delta_{ji}\varphi - a_{ji})m_j = 0$  where  $\delta_{ji}\varphi - a_{ji}$  can be analogously be viewed as an element of  $R[\varphi]$ , which is a commutative ring. Write  $B$  for the  $n \times n$  matrices with entries in  $R[\varphi]$  and  $B_{ij} = \delta_{ji}\varphi - a_{ij}$ . Let  $C$  be the adjoint of  $B$ . Then

$$0 = \sum_{i=1}^n C_{ki} \left( \sum_{j=1}^n B_{ij} m_j \right) = \sum_{j=1}^n \left( \sum_{i=1}^n C_{ki} B_{ij} \right) m_j = \sum_{j=1}^n (CB)_{kj} m_j = (\det B) m_k,$$

so  $(\det B)m = 0 \forall m \in M$ , hence  $\det B = 0 \in \text{End}(M)$ . Expanding  $\det B$  as a polynomial in  $\varphi$  gives us the desired. □

*Week 6, lecture 2 starts here*

### 4.2.1 Corollaries of C–H theorem: Nakayama’s lemma(s)

**Corollary 4.2.10.** If  $M$  is a finitely generated  $R$ -module and  $I$  an ideal of  $R$  with  $IM = M$ , then  $\exists r \in R : r - 1 \in I, rM = 0$ .

*Proof.* Let  $\varphi = \text{id}$ . Then  $\varphi(M) \subset M \subset IM$ . So by Theorem 4.2.8, one has

$$\varphi^n + \sum_{i=1}^{n-1} a_i \varphi^{n-1} + a_n = \left(1 + \sum_{i=1}^n a_i\right) \varphi = 0 \text{ where } a_i \in I^i.$$

Set  $r := 1 + \sum_{i=1}^n a_i$ . Then  $r - 1 \in I$  and  $rm = 0 \forall m \in M$ . □

**Corollary 4.2.11.** Let  $R$  be a local ring with maximal ideal  $\mathfrak{m}$  and  $M$  a finitely generated  $R$ -module. If  $\mathfrak{m}M = M$ , then  $M = 0$ .

*Proof.* By corollary above,  $\exists r \in R : r - 1 \in \mathfrak{m}$  and  $rm = 0 \forall m \in M$ . But then  $r \notin \mathfrak{m}$  since otherwise  $r - (r - 1) = 1 \in \mathfrak{m}$  hence not maximal. So  $r$  is a unit. But then

$$m = 1m = (r^{-1}r)m = r^{-1}(rm) = r^{-1}0 = 0 \quad \forall m \in M$$

and thus  $M = 0$ . □

**Corollary 4.2.12.** Let  $R$  be a local ring with maximal ideal  $\mathfrak{m}$ . If  $M$  is a finitely generated  $R$ -module and  $m_1, \dots, m_n \in M$  are elements whose images span the (residue field)  $k = R/\mathfrak{m}$ -vector space  $\overline{M} = M/\mathfrak{m}M$ , then  $m_i$  generate  $M$ .

**Remark** (Sanity check before the proof). Verify that this is well-defined, i.e.  $r + \mathfrak{m} = r' + \mathfrak{m}$ ,  $m + \mathfrak{m}M = m' + \mathfrak{m}M \Rightarrow rm + \mathfrak{m}M = r'm' + \mathfrak{m}M$ . Indeed, if  $r - r' \in \mathfrak{m}$  and  $m - m' \in \mathfrak{m}M$ , then  $rm - r'm' = (r - r')m + (m - m')r' \in \mathfrak{m}M$ .

*Proof.* Let  $N$  be the submodule of  $M$  generated by  $m_i$ . Since  $m_i + \mathfrak{m}M$  span  $M/\mathfrak{m}M$ , each element of  $M$  can be written as

$$m = \sum_{i=1}^n r_i m_i + rm' \text{ where } r_i \in R, r \in \mathfrak{m}, m' \in M.$$

Thus  $m = n + rm'$  for  $n \in N$ , i.e.  $M/N = \mathfrak{m}M/N$ . So by corollary above  $M/N = 0$ , so  $M = N$  and one has the desired. □

**Remark** ( $\frac{1}{2}$ ). These all needed  $M$  to be finitely generated. If (recall Example 3.1.2)  $R = \mathbb{Z}_{\langle 2 \rangle}$  and  $M = \mathbb{Q}$ , then  $\langle 2 \rangle \mathbb{Q} = \mathbb{Q}$ , but clearly  $\mathbb{Q} \neq 0$ . This is not a counterexample since  $\mathbb{Q}$  is not finitely generated.

Also, the last two needed  $R$  to be local. Consider  $\mathbb{Z}$  as a  $\mathbb{Z}$ -module.  $2\mathbb{Z}$  is maximal and the image of 5 (which is 1) generates  $\mathbb{Z}/2\mathbb{Z}$ , but 5 clearly does not generate  $\mathbb{Z}$ .

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### 4.3 Localisation of modules

**Definition 4.3.1.** Let  $M$  be an  $R$ -module and  $U \subseteq R$  a multiplicatively closed set. Then

$$M[U^{-1}] := \left\{ \frac{m}{u} : m \in M, u \in U \right\}$$

modulo the equivalence relation

$$\frac{m}{u} \sim \frac{m'}{u'} \text{ if } \exists \tilde{u} \in U : \tilde{u}(u'm - um') = 0.$$

This is an  $R[U^{-1}]$ -module.

Readers should verify that the  $R[U^{-1}]$ -action  $\frac{r}{u} \cdot \frac{m}{u'} = \frac{rm}{uu'}$  is well-defined and  $M[U^{-1}]$  obeys the module axioms.

**Lemma 4.3.2.** Let  $R$  be a ring and  $M$  an  $R$ -module.

1. If  $m \in M$ ,  $m = 0 \Leftrightarrow \frac{m}{1} = \frac{0}{1}$  in every localisation  $M_{\mathfrak{m}} = M[U^{-1}]$  at a maximal ideal  $\mathfrak{m}$  where  $U = R \setminus \mathfrak{m}$ .
2.  $M = 0 \Leftrightarrow M_{\mathfrak{m}} = 0$  for every maximal ideal  $\mathfrak{m}$  of  $R$ .

**Definition 4.3.3.** The *annihilator* of an element  $m \in M$  is  $\text{ann}(m) := \{r \in R : rm = 0_M\}$ .

**Example 4.3.4.**  $R = \mathbb{Z}$ ,  $M = \mathbb{Z}/6\mathbb{Z}$ . Then  $\text{ann}(1) = \langle 6 \rangle$ ,  $\text{ann}(2) = \langle 3 \rangle$ . In general,  $\text{ann}(m)$  would be an ideal.

*Proof.* 1.  $\frac{m}{1} = \frac{0}{1} \Leftrightarrow \exists u \notin \mathfrak{m} : um = 0 \Rightarrow \text{ann}(m) \not\subseteq \mathfrak{m}$ . If  $m \neq 0$  then  $\text{ann}(m) \neq R$  since  $1m = m \neq 0$ . So by definition  $\exists$  a maximal ideal  $\mathfrak{m}$  with  $\text{ann}(m) \subseteq \mathfrak{m}$ , so  $\frac{m}{1} \neq \frac{0}{1}$  in  $M_{\mathfrak{m}}$ . The other direction is clear.

2.  $M = 0 \Leftrightarrow m = 0 \forall m \in M \Leftrightarrow \forall m \in M, \frac{m}{1} = \frac{0}{1} \in M_{\mathfrak{m}} \forall \mathfrak{m} \Leftrightarrow M_{\mathfrak{m}} = 0 \forall \mathfrak{m}$ .

□

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**Definition 4.3.5** (/Lemma). Let  $\varphi : M \rightarrow N$  be an  $R$ -module homomorphism and  $P$  a prime ideal of  $R$ . Define  $\varphi_P : M_P \rightarrow N_P$  (where again  $M_P$  denotes  $M[U^{-1}]$  where  $U = R \setminus P$ ) by  $\frac{m}{u} \mapsto \frac{\varphi(m)}{u}$ .

Check this is well defined: if  $\frac{m}{u} = \frac{m'}{u'}$  then  $\exists \tilde{u} \in U : \tilde{u}(u'm - um') = 0$ , so  $0_N = \varphi(0_M) = \varphi(\tilde{u}(u'm - um')) = \tilde{u}(u'\varphi(m) - u\varphi(m'))$ , i.e.  $\frac{\varphi(m)}{u} = \frac{\varphi(m')}{u'}$ . Readers should also check it's a homomorphism.

**Corollary 4.3.6.** If  $\varphi : M \rightarrow N$  is an  $R$ -module homomorphism, then  $\varphi$  is injective (or surjective, or bijective) iff for every maximal  $\mathfrak{m}$  of  $R$ ,  $\varphi_{\mathfrak{m}} : M_{\mathfrak{m}} \rightarrow N_{\mathfrak{m}}$  is injective (or surjective, or bijective).

*Proof.* One has

$$\begin{aligned}\ker \varphi_{\mathfrak{m}} &= \left\{ \frac{m}{u} : \frac{\varphi(m)}{u} = \frac{0}{1} \right\} = \left\{ \frac{m}{u} : \exists \tilde{u} \notin \mathfrak{m} : \tilde{u}(\varphi(m)) = 0 \right\} \\ &= \left\{ \frac{m}{u} : \exists \tilde{u} \notin \mathfrak{m} : \varphi(\tilde{u}m) = 0 \right\} \subseteq (\ker \varphi)_{\mathfrak{m}}\end{aligned}$$

since  $\frac{m}{u} = \frac{\tilde{u}m}{\tilde{u}u}$ . Now if  $\frac{m}{u} \in (\ker \varphi)_{\mathfrak{m}}$  then  $\varphi_m\left(\frac{m}{u}\right) = \frac{\varphi(m)}{u} = 0$ , so  $\frac{m}{u} \in \ker \varphi_m$ . Hence

$$\begin{aligned}\varphi \text{ injective} &\Leftrightarrow \ker \varphi = 0 \Leftrightarrow (\ker \varphi)_{\mathfrak{m}} = 0 \quad \forall \mathfrak{m} \Leftrightarrow \ker \varphi_{\mathfrak{m}} = 0 \quad \forall \mathfrak{m} \\ &\Leftrightarrow \varphi_{\mathfrak{m}} \text{ injective} \quad \forall \mathfrak{m}.\end{aligned}$$

Now consider cokernel  $N/\text{im } \varphi$ , which is 0 iff  $\varphi$  is surjective. We claim  $(N/\text{im } \varphi)_{\mathfrak{m}} \cong N_{\mathfrak{m}}/\text{im } \varphi_{\mathfrak{m}}$  by the map  $\frac{n + \text{im } \varphi}{u} \mapsto \frac{n}{u} + \text{im } \varphi$ . Rest of proof is left as an exercise.  $\square$

## 5 Integral closure

**Definition 5.0.1.** Let  $R$  be a ring. A ring  $S$  is an  $R$ -algebra if  $\exists$  a homomorphism  $\varphi : R \rightarrow S$ . This implies  $S$  is a  $R$ -module with  $R$ -action given by  $r \cdot s := \varphi(r)s$ .

**Example 5.0.2.**  $\mathbb{C}[x_1, x_2]$  is a  $\mathbb{C}$ -algebra,  $\mathbb{Q}(\sqrt{3})$  is a  $\mathbb{Q}$ -algebra,  $\mathbb{Z}/6\mathbb{Z}$  is a  $\mathbb{Z}$  algebra.

**Definition 5.0.3.**  $S$  is *finite* over  $R$  if  $S$  is a finitely generated  $R$ -module.

**Definition 5.0.4.**  $s \in S$  is *integral* over  $R$  if there is a monic polynomial  $f(y) = y^n + a_1y^{n-1} + \dots + a_n \in R[y]$  with  $f(s) = 0$ . If every element of  $S$  is integral over  $R$ , we say  $S$  is *integral* over  $R$ .

**Example 5.0.5** (Why ‘integral’?).  $\mathbb{Q}$  is a  $\mathbb{Z}$ -algebra, and  $q \in \mathbb{Q}$  is integral iff  $q \in \mathbb{Z}$ .

$\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$  is a  $\mathbb{Z}$ -algebra, and  $\varphi = \frac{1+\sqrt{5}}{2}$  is integral since  $\varphi^2 - \varphi - 1 = 0$ .

$\mathbb{Z}\left[\frac{1}{3}\right] = \mathbb{Z}_{\langle 3 \rangle}$  is a  $\mathbb{Z}$ -algebra. Let’s prove  $\frac{1}{3}$  is not integral. Suppose

$$f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^n + \sum_{i=0}^{n-1} a_{n-i} \left(\frac{1}{3}\right)^i = 0.$$

Then

$$1 + \sum_{i=0}^{n-1} a_{n-i} 3^{n-i} = 0,$$

which is impossible since LHS = 1 mod 3 and RHS = 0 mod 3.

**Notation.** For a  $R$ -algebra  $S$  with  $R \subseteq S$  and  $s_1, \dots, s_m$  elements of  $S$ , write  $R[s_1, \dots, s_m]$  for the smallest subring of  $S$  containing  $R$  and  $s_1, \dots, s_m$ .

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**Lemma 5.0.6.** We can explicitly write

$$R[s_1, \dots, s_m] = \left\{ \sum_{u \in \mathbb{N}^n}^k a_u s^u : a_u \in R \text{ and only finitely many } a_u \neq 0 \right\}.$$

**Definition 5.0.7.**  $S' \subseteq S$  is an  $R$ -subalgebra if  $S'$  is a subring and  $\varphi : R \rightarrow S$  has image in  $S'$ .

**Proposition 5.0.8.** Let  $S$  be an  $R$ -algebra with  $R \subseteq S$  and fix  $s \in S$ . The following are equivalent:

1.  $s$  is integral over  $R$ .
2. Subring  $R[s] \subseteq S$  is finite over  $R$ .
3. There is an  $R$ -subalgebra  $R' \subseteq S : R[s] \subseteq R'$  and  $R'$  is finite over  $R$ .

*Proof.*

- $1 \Rightarrow 2$ : If  $s$  satisfies a relation  $s^n + a_1 s^{n-1} + \cdots + a_n = 0$  with  $a_i \in R$ , then  $R[s]$  is generated by  $1, s, \dots, s^{n-1}$  as an  $R$ -module. Indeed, if  $f \in R[s]$  is a polynomial in  $s$  of degree  $\geq n$ , then we can use the relation to lower the degree.
- $2 \Rightarrow 3$ : Take  $R' = R[s]$ .
- $3 \Rightarrow 1$ : Consider the  $R$ -module homomorphism  $\varphi : R' \rightarrow R'$  given by  $r \mapsto sr$ . Since  $R'$  is a finitely generated  $R$ -module,  $\varphi$  satisfies a relation  $\varphi^n + a_1 \varphi^{n-1} + \cdots + a_n = 0$  where  $a_i \in R$  by Theorem 4.2.8 and taking  $I = R$ . Applying this to  $1_R$  gives  $s^n + a_1 s^{n-1} + \cdots + a_n = 0$ , so  $s$  is integral.

□

**Theorem 5.0.9.** Let  $S$  be a  $R$ -algebra with  $R \subseteq S$ .

1. If  $R \subseteq S \subseteq S'$ ,  $S'$  is finite over  $S$  and  $S$  is finite over  $R$ , then  $S'$  is finite over  $R$ .
2. If  $s_1, \dots, s_m \in S$  are integral, then  $R[s_1, \dots, s_m]$  is finite over  $R$  and in particular integral over  $R$ .
3. If  $R \subseteq S \subseteq S'$ ,  $S'$  is integral over  $S$  and  $S$  is integral over  $R$ , then  $S'$  is integral over  $R$ .
4. The subset

$$\tilde{R} := \{s \in S : s \text{ integral over } R\} \subseteq S,$$

is a subring of  $S$  with  $\tilde{\tilde{R}} = \tilde{R}$ , i.e. if  $s \in S$  is integral over  $\tilde{R}$ , it is integral over  $R$ .

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*Proof.* 1. If  $s_1, \dots, s_m$  generate  $S$  as an  $R$ -module and  $s'_1, \dots, s'_n$  generate  $S'$  as an  $S$ -module, then  $\{s_i s'_j : 1 \leq i \leq m, 1 \leq j \leq n\}$  generates  $S'$  as an  $R$ -module. Indeed, if  $s \in S'$  then  $s = \sum_{i=1}^n a_i s'_i$  where  $a_i \in S$ , and  $a_i = \sum_{j=1}^m r_{ij} s_j$  where  $r_{ij} \in R$ , so

$$s = \sum_{i=1}^n \left( \sum_{j=1}^m r_{ij} s_j \right) s'_i = \sum_{i=1}^n \sum_{j=1}^m r_{ij} (s'_i s_j).$$

2. We prove by induction on  $m$  and Proposition 5.0.8 provides the base case. Suppose the claim is true for  $m' < m$ . One has  $R[s_1, \dots, s_m] = R[s_1, \dots, s_{m-1}][s_m]$ ,  $R[s_1, \dots, s_{m-1}]$  is finite over  $R$  and  $s_m$  is integral over  $R[s_1, \dots, s_{m-1}]$ , so again by Proposition 5.0.8,  $R[s_1, \dots, s_m]$  is finite over  $R[s_1, \dots, s_{m-1}]$ . Hence by part 1,  $R[s_1, \dots, s_m]$  is finite over  $R$ . To see it is integral, one uses part 3 of Proposition 5.0.8.

3. Let  $s' \in S'$ . One has  $s'$  satisfies a relation  $s'^n + b_1 s'^{n-1} + \dots + b_n = 0$  where  $b_i \in S$ , so each  $b_i$  is integral over  $R$ . By part 2,  $R[b_1, \dots, b_n]$  is finite over  $R$ , so by Proposition 5.0.8,  $R[b_1, \dots, b_n, s']$  is finite over  $R[b_1, \dots, b_n]$  and so  $R[b_1, \dots, b_n, s']$  is finite over  $R$ . Therefore again by part 3 of Proposition 5.0.8,  $s'$  is integral over  $R$ .
4. To see  $\tilde{R}$  is a ring, consider  $s_1, s_2 \in \tilde{R}$ . Then by part 2,  $R[s_1, s_2]$  is integral over  $R$ , so  $-s_1, s_1 + s_2, s_1 s_2 \in R[s_1, s_2]$  are integral over  $R$ , i.e. in  $\tilde{R}$ . The fact that  $\tilde{\tilde{R}} = \tilde{R}$  follows from part 3 with  $S = \tilde{R}$ ,  $S' = \tilde{R}[s]$ .

□

**Definition 5.0.10.**  $\tilde{R}$  is called *integral closure* of  $R$  in  $S$ .

If  $\tilde{R} = R$  then  $R$  is *integrally closed* in  $S$ .

If  $R$  is a domain and is integrally closed in its field of fractions  $R_{(0)}$ , then  $R$  is *integrally closed* (or *normal*).

**Example 5.0.11.** For  $R = \mathbb{Z}$  and  $S = \mathbb{Q}(\sqrt{d})$  where  $d$  is a squarefree integer,  $\tilde{R} = \mathbb{Z}[d]$  if  $d \equiv 2, 3 \pmod{4}$  and  $\mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right]$  if  $d \equiv 1 \pmod{4}$ . When one sees this, one should recall definition of *ring of integers* in algebraic number theory.

The integral closure of  $\mathbb{C}[t^2, t^3] \subseteq \mathbb{C}[t]$  is all of  $\mathbb{C}[t]$  since  $t$  is integral ( $x^2 - t^2 = 0$ ).

## 6 Variety

**Definition 6.0.1.** Let  $I \subseteq K[x_1, \dots, x_n]$  be an ideal where  $K$  is a field. The *variety* of  $I$  is  $V(I) := \{(a_1, \dots, a_n) \in K^n : f(a_1, \dots, a_n) = 0 \forall f \in I\}$ .

**Example 6.0.2.**  $V(\langle x^2 + y^2 - 1 \rangle) \subseteq \mathbb{R}^2$  is a circle,  $V(\langle y^2 - x^3 + x - 1 \rangle) \subseteq \mathbb{C}^2$  is an elliptic curve,  $V(\langle xy, xz \rangle) \subseteq \mathbb{C}^2$  is  $y$ - $z$ -plane and  $x$ -axis.

By Fermat's last theorem,  $V(\langle x^4 + y^4 - 1 \rangle) \subseteq \mathbb{Q}^2$  is  $\{(0, \pm 1), (\pm 1, 0)\}$ .

$V(\langle xz - y^2, xw - yz, yw - z^2 \rangle)$  is ...

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Note that  $(a, b, c, d) \in X \Rightarrow (\lambda a, \lambda b, \lambda c, \lambda d) \in X \forall \lambda \in K$ . So we look at 2 cases:

1°  $w = 0$ , then solutions are  $\{(a, 0, 0, 0) : a \in K\}$ .

2°  $w = 1$ , then solutions are  $\{(wz^3, wz^2, wz, w) : w, z \in K\}$ , which is equivalent to  $\{(a^3, a^2b, ab^2, b^3) : a, b \in K\}$ .

**Remark.** Note that by the way we define  $V(I)$ ,  $I = \langle f_1, \dots, f_r \rangle$  is finitely generated by Hilbert. In particular,

$$V(I) = \{(a_1, \dots, a_n) \in K^n : f_1(a_1, \dots, a_n) = \dots = f_r(a_1, \dots, a_n)\}.$$

**Definition 6.0.3.** A field  $K$  is *algebraically closed* if  $\forall f \in K[x], \exists a \in K : f(a) = 0$ .

Within context of this module, we assume  $K$  to be algebraically closed when talking about varieties.

**Definition 6.0.4.** Let  $X \subseteq K^n$  be a set. The *ideal* of  $X$  is

$$I(X) = \{f \in K[x_1, \dots, x_n] : f(a_1, \dots, a_n) = 0 \forall (a_1, \dots, a_n) \in X\}.$$

**Remark.** One has  $X \subseteq V(I(X))$  and  $I \subseteq I(V(I))$ . Both can be proper, e.g.

1.  $X = \mathbb{Z} \subseteq \mathbb{C}$ . Then  $V(I(X)) = V(\langle 0 \rangle) = \mathbb{C}$ .
2.  $I = \langle x^2 \rangle \subseteq \mathbb{C}[x]$ . Then  $I(V(I)) = I(\{0\}) = \langle x \rangle$ .

**Theorem 6.0.5** (Nullstellensatz). Let  $K$  be algebraically closed and  $I \subseteq K[x_1, \dots, x_n]$ .

1.  $V(I) = \emptyset \Leftrightarrow I = \langle 1 \rangle = K[x_1, \dots, x_n]$ . weak form
2.  $I(V(I)) = \sqrt{I} := \{f \in K[x_1, \dots, x_n] : f^m \in I \text{ for some } m \geq 1\}$ . strong form

**Remark.** 1. 2 implies 1, but we will prove it first anyway since it's in fact harder.

2.  $K$  must be algebraically closed, e.g.  $V(\langle x^2 + 1 \rangle) \subseteq \mathbb{R}$  is empty.
3.  $\sqrt{I}$ , called the *radical* of  $I$ , is an ideal. Indeed, if  $f, g \in \sqrt{I}$ , then  $f^m, g^r \in I$  so

$$(f + g)^{m+r} = \sum_{k=0}^{m+r} \binom{m+r}{k} f^k g^{m+r-k} \in I,$$

so  $f + g \in \sqrt{I}$ . It's also clear that  $\sqrt{I}$  is multiplicatively closed.

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**Lemma 6.0.6.**  $I(X)$  is an ideal.

*Proof.* If  $f(a), g(a) = 0 \forall a \in X$ , then  $(f + g)(a) = f(a) + g(a) = 0 + 0 = 0 \forall a \in X$ , and  $(hf)(a) = h(a)f(a) = h(a)0 = 0$ . □

**Proposition 6.0.7.** If  $R \subseteq S$  are domains with  $S$  integral over  $R$ , then  $R$  is a field iff  $S$  is a field.

*Proof.*  $\Rightarrow$ : Fix  $s \in S$  with  $s \neq 0$ . Then one can write

$$s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n = 0 \quad \text{where } a_i \in R$$

Since  $S$  is domain,  $a_n \neq 0$ , and since  $R$  is a field,

$$1 = -a_n^{-1} s (s^{n-1} + a_1 s^{n-2} + \dots + a_{n-2} s + a_{n-1}),$$

i.e.  $s$  has an inverse, so  $S$  is a field.

$\Leftarrow$ : Fix  $r \in R$  with  $r \neq 0$ . Then since  $S$  is a field,  $r^{-1} \in S$ , so one can write

$$(r^{-1})^n + a_1 (r^{-1})^{n-1} + \dots + a_{n-1} r^{-1} + a_n = 0 \quad \text{where } a_i \in R.$$

Multiplying by  $r^{n-1}$  gives

$$r^{-1} + a_1 + \dots + a_{n-1} r^{n-2} + a_n r^{n-1} = 0,$$

so  $r^{-1} \in R$ . □



**Definition 6.0.8.** An  $R$ -algebra  $S$  is *finitely generated* if  $\exists s_1, \dots, s_n \in S : S$  the smallest  $R$ -subalgebra containing  $s_1, \dots, s_n$ .

**Remark.** Finitely generated as an  $R$ -algebra is weaker than finite over  $R$  (i.e. finitely generated as an  $R$ -module), e.g.  $K[x_1, \dots, x_n]$  as a  $K$ -algebra/module.

**Theorem 6.0.9.** Let  $K$  be a field and  $L$  a finitely generated  $K$ -algebra which is also a field. Then  $L$  is finite over  $K$  (in particular integral over  $K$ ).

*Proof.* Write

$$L = K[x_1, \dots, x_n]/I \cong K[\overline{x}_1, \dots, \overline{x}_n] \quad \text{where } \overline{x}_i = x_i + I \text{ is the image of } x_i \text{ in } L.$$

We will prove (by induction) that each  $\overline{x}_i$  is integral over  $K$ , so that by Theorem 5.0.9.2 we would have the desired.

When  $n = 1$ ,  $L = K[\overline{x}_1]$ . Since  $\overline{x}_1^{-1} \in L$ , it's a polynomial  $p$  in  $\overline{x}_1$ , so  $\ast(\overline{x}_1) = \overline{x}_1 p(\overline{x}_1) - 1 = 0$ . Since  $K$  is a field, we can make  $\ast$  monic and  $\overline{x}_1$  is hence integral over  $K$ , so  $L$  is finite over  $K$ .

Now suppose the statement is true for  $n - 1$ . Since  $L$  is a field, it contains the field  $K(\overline{x}_1)$ , so we can really think of  $L$  as  $K(\overline{x}_1)[\overline{x}_2, \dots, \overline{x}_{n-1}]$ . By induction hypothesis,  $L$  is finite over  $K(\overline{x}_1)$ , so each  $\overline{x}_j$  where  $2 \leq j \leq n$  is integral over  $K(\overline{x}_1)$ , i.e. one has a monic polynomial  $p_j \in K(\overline{x}_1)[x] : p_j(\overline{x}_j) = 0$ . Let  $f \in K[\overline{x}_1]$  be a common denominator of coefficients of all  $p_j$ . Then one can see  $p_j$  as monic with coefficients in  $K[\overline{x}_1][f^{-1}]$ , and  $L$  is integral over  $K[\overline{x}_1][f^{-1}]$ . By Proposition 6.0.7,  $K[\overline{x}_1][f^{-1}] \subseteq K(\overline{x}_1)$  is a field. By minimality of  $K(\overline{x}_1)$ ,  $K[\overline{x}_1][f^{-1}] \cong K(\overline{x}_1)$ .

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Consider the ring homomorphism  $\varphi : K[t] \rightarrow K[\overline{x}_1] : t \mapsto \overline{x}_1$ . This is surjective by construction. If  $\varphi$  is injective (hence an isomorphism), then  $K(t) \cong K(\overline{x}_1)$ . Let  $p(t) = \varphi^{-1}(f)$  and choose  $g \in K[t]$  with  $p \nmid g$ . Then

$$\frac{1}{q} \notin K[t][p^{-1}] \cong K[\overline{x}_1][f^{-1}] \cong K(\overline{x}_1) \cong K(t),$$

a contradiction, so  $\varphi$  is not injective, i.e.  $\exists p \in K[t] : p(\overline{x}_1) = 0$ . Again, since  $K$  is a field, one can make  $p$  monic and see that  $\overline{x}_1$  is integral over  $K$ , so  $K[\overline{x}_1]$  is finite over  $K$ . Now by 6.0.7,  $K[\overline{x}_1]$  is a field and so  $K[\overline{x}_1] \cong K(\overline{x}_1)$ , hence  $K(\overline{x}_1)$  is finite over  $K$ . We conclude that  $L$  is finite over  $K$ .  $\square$

**Corollary 6.0.10.** The maximal ideals of  $K[x_1, \dots, x_n]$  where  $K$  is an algebraically closed field are of the form  $\langle x_1 - a_1, \dots, x_n - a_n \rangle$  where  $a_i \in K$ .

**Remark.** 1. Again we do need the assumption of  $K$  being algebraically closed. Consider  $\langle x^2 + 1 \rangle \subseteq \mathbb{R}[x]$ , which is maximal since any ideal strictly containing it would be generated by something that strictly divides  $x^2 + 1$ , which is however irreducible over  $\mathbb{R}$ .

2. If  $K$  is algebraically closed then there are no fields  $L \supsetneq K$  with  $L$  finite over  $K$ , since if there is such  $L$  then  $l \in L \setminus K$  would satisfy a monic polynomial with coefficients in  $K$  and minimal degree, but then by definition  $l \in K$ .

*Proof.* Let  $\mathfrak{m}$  be a maximal ideal. Then  $L = K[x_1, \dots, x_n]/\mathfrak{m}$ , a finitely generated  $K$ -algebra, is a field, so by theorem above  $L$  is finite over  $K$ . Since  $K$  is algebraically closed, by remark

above  $L \cong K$ , i.e.  $\exists \varphi : K[x_1, \dots, x_n]/\mathfrak{m} \rightarrow K$  with  $\varphi(a) = a \ \forall a \in K$ . Set  $a_i = \varphi(x_i)$ , then  $\varphi(x_i - a_i) = \varphi(x_i) - a_i = 0$ , i.e.  $x_i - a_i \in \ker \varphi = \mathfrak{m}$ , so  $\mathfrak{m} \supseteq \langle x_1 - a_1, \dots, x_n - a_n \rangle$ . We claim

$$f \in \mathfrak{m} \setminus \langle x_1 - a_1, \dots, x_n - a_n \rangle \Rightarrow f(a_1, \dots, a_n) \in \mathfrak{m},$$

proof of which is left as an exercise. This completes the proof.  $\square$

We are now ready to prove the **Nullstellensatz**.

*Proof of 6.0.5.* 1. Suppose  $I \subsetneq K[x_1, \dots, x_n]$ . Then  $I \subseteq \mathfrak{m}$  for some maximal ideal  $\mathfrak{m} = \langle x_1 - a_1, \dots, x_n - a_n \rangle$ . One has  $\bar{I} \subseteq J \Rightarrow V(J) \subseteq V(I)$ , so since  $(a_1, \dots, a_n) \in V(\mathfrak{m})$ ,  $V(I) \neq \emptyset$ .

2. (by Rabinowitch trick) Note that if  $f^m \in I$  then  $f^m(a) = 0 \ \forall a \in V(I)$ , so  $(f(a))^m = 0$  and so  $f(a) = 0 \ \forall a \in V(I)$ , so  $f \in I(V(I))$ . We have just proved that  $\sqrt{I} \subseteq I(V(I))$ .

Now fix  $f \in I(V(I))$  and set  $J = \langle f : f \in I \rangle + \langle fy - 1 \rangle \subseteq K[x_1, \dots, x_n, y]$ . One has  $V(J) = \emptyset$  since  $f$  and  $fy - 1$  cannot be zero at the same time. By weak Nullstellensatz,  $1 \in J$ , i.e.

$$1 = \sum_{i=1}^r h_i(x, y) f_i(x) + h(x, y)(fy - 1) \in K[x_1, \dots, x_n, y] \subseteq K(x_1, \dots, x_n, y).$$

Now if one just lets  $y = \frac{1}{f}$  then

$$1 = \sum_{i=1}^r h_i \left( x, \frac{1}{f} \right) f_i(x)$$

and for some  $m$  one can clear the denominator and have,

$$f^m = \sum_{i=1}^r \underbrace{h_i \left( x, \frac{1}{f} \right) f^m}_{\text{polynomial in } x} f_i(x) \in I,$$

i.e.  $I(V(I)) \subseteq \sqrt{I}$ .  $\square$

*Week 9, lecture 1 starts here*

*Week 9, lecture 2 starts here*

*Week 9, lecture 3 starts here*