MA3E1 Groups and representations :: Lecture notes

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Contents

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1 Reminders

Definition 1.0.1. A *group* is ...

Example 1.0.2.
- $\mathbb{Z}$ with addition
- $\mathbb{C}^\times$ with multiplication
- A subgroup of above: $\{g \in \mathbb{C} : g^n = 1\}$, the nth roots of unity ζ_n^i with $\zeta_n = e^{\frac{2\pi i}{n}}$. ζ_n^j is primitive if $\mathrm{ord}(\zeta_n^j) = n$
- General linear group $GL_d(K)$
- A subgroup of above: special linear group $SL_d(K)$

Given G and $g \in G$, one can define the *cyclic* group generated by g, denoted $\langle g \rangle$, an abelian subgroup of G, of order $\mathrm{ord}(g)$.

Recall *symmetric* group S_n and cycle notation; verify that $|S_n| = n!$; recall elements of S_n can be written as either even or odd number of transpositions (cycles of length 2) but not both, and *alternating* group A_n, a subgroup of S_n.

1.1 Group action

Definition 1.1.1. Let G be a group and X a set. A *left action* of G on X is a map $G \times X \to X : (g, x) \mapsto g * x$ which satisfies

1. $1_G * x = x \ \forall x \in X$
2. $(gh) * x = g * (h * x) \ \forall g, h \in G, x \in X$

Example 1.1.2.
- $X = \{1, \ldots, n\},\ G = S_n,\ \pi * i := \pi(i)$
- $X = \mathbb{R}^n,\ G = GL_n(\mathbb{R}),\ A * v := Av$

Definition 1.1.3. For $x, y \in X$, write $x \sim y$ if $\exists g \in G : g * x = y$. This is an equivalence relation and an equivalence class of $\sim$ is an *orbit*.

Example 1.1.4. $\mathrm{orb}_{GL_n(\mathbb{R})}((1, 0, \ldots, 0)) = \mathbb{R}_n \backslash \{0\}$ and $\mathrm{orb}_{GL_n(\mathbb{R})}(0) = \{0\}$, so there are exactly two orbits of 1.1.2.2.

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Definition 1.1.5. G acts *transitively* on X if there is only one orbit.

e.g. 1.1.2.1.

Definition 1.1.6. Define the *stabiliser* $\mathrm{stab}_G(x) := \{g \in G : g * x = x\}$. This is a subgroup of G, sometimes called *symmetry group*.

Theorem 1.1.7 (Orbit–Stabiliser)**.** For a finite G acting on X and $x \in X$,

$$|G| = |\mathrm{orb}_G(x)| \cdot |\mathrm{stab}_G(x)|.$$

Theorem 1.1.8. G acts on itself by conjugation ($G \times G \to G : g \cdot h = ghg^{-1}$). In this case, orbit is *conjugacy class* and stabiliser is *centraliser*. An obvious corollary then follows from O–S.

Example 1.1.9. If $G = S_n$, then the conjugacy classes correspond to cycle types (ordered list of lengths of cycles), since

$$\pi(a_1\ a_2\ \cdots\ a_k)\pi^{-1} = (\pi(a_1)\ \pi(a_2)\ \cdots\ \pi(a_k)).$$

1.2 Normal subgroup

Definition 1.2.1. A subgroup is *normal* if ...

Lemma 1.2.2. Let H be a subgroup of G. The following are equivalent.

1. H is normal in G
2. $gHg^{-1} = H \ \forall g \in G$ (definition)
3. $gH = Hg \ \forall g \in G$

Example 1.2.3. $SL_d(K) \trianglelefteq GL_d(K)$ by determinant product.

1.3 Homomorphism

Definition 1.3.1. A *group homomorphism* is ...
The *kernel* and *image* of a homomorphism are ...

Example 1.3.2. Consider $\phi : S_n \to GL_n(K)$ given by $\phi(e_i) = e_{\pi(i)}$, e.g.

$$\pi = (1\ 2\ 3), \quad \phi(\pi) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Verify this is a group homomorphism and $\operatorname{im}(\phi) = \{1, -1\}$. Since $GL_n(K) \to K^\times$ by taking determinant is a also a homomorphism, one has

$$\begin{tikzcd} S_n \arrow[r, "\phi"] \arrow[dr, "\mathrm{sgn}"'] & GL_n(K) \arrow[d, "\det"] \\ & K^\times \end{tikzcd}$$

where sign is a homomorphism and $\mathrm{sgn}(\pi) \in \{1, -1\}$. In fact, $\mathrm{sgn}(\pi) = 1$ if π is even and -1 if odd.

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Theorem 1.3.3 (1st isomorphism theorem)**.** If $\phi : G \to H$ is a homomorphism of groups, then

1. $\ker \phi \trianglelefteq G$
2. $\operatorname{im} \phi \leq H$
3. $\hat{\phi} : G/\ker\phi \to \operatorname{im}\phi : g\ker\phi \mapsto \phi(g)$ is a well defined isomorphism.

1.4 Dihedral group

Definition 1.4.1. $D_{2n} := \left\langle r, s \mid r^n = 1, s^2 = 1, srs^{-1} = r^{-1} \right\rangle$ is called the *dihedral group*.
It has two cyclic subgroups $\langle r \rangle \cong C_n, \langle s \rangle \cong C_2$.

1.5 Linear map

Definition 1.5.1. Let V, W be vector spaces over K. A map $T : V \to W$ is *linear* if

1. $T(\alpha v) = \alpha T(v) \ \forall \alpha \in K, v \in V$
2. $T(v + w) = T(v) + T(w) \ \forall v, w \in V$

Example 1.5.2. $A \in M_{m\times n}(K)$ gives a linear map $T_A : K^n \to K^m,\ T_A(v) = Av$.

Theorem 1.5.3 (Rank–nullity)**.** If V is finite dimensional and $T : V \to W$ a linear map, then

$$\dim V = \dim\ker T + \dim\operatorname{im} T.$$

Corollary 1.5.4. If V is finite dimensional and $T : V \to V$ a linear map, then the following are equivalent.

1. T is injective
2. T is surjective
3. T is an isomorphism

Notation. $GL(V) := \{T : V \to V \text{ isomorphism}\}$. This is a group.
If $V = K^n$ then $GL(V) \cong GL_n(K)$.

2 Group presentation

In general, a group can be given uniquely (*presented*) by $\langle S \mid R\rangle$ where S is a set of symbols and R relations. If $\exists S, R$ that are finite then G is *finitely presented*.

Example 2.0.1. $C_n = \langle x \mid x^n = 1\rangle$.
$C_\infty = \langle x \mid \ \rangle = \{1, x, x^{-1}, x^2, x^{-2}, \dots\} \cong (\mathbb{Z}, +)$.

Theorem 2.0.2. Let $G = \langle s_1, \dots, s_n \mid R\rangle$ and H a group with $h_1, \dots, h_n \in H$. Then $\exists$ a homomorphism $\phi : G \to H$ with $\phi(s_i) = h_i \ \forall i$ iff every relation $r \in R$ holds where all s_i are replaced by h_i.

Example 2.0.3. Consider C_n and $\widehat{C_n}$, the set of group homomorphisms $C_n \to GL_1(\mathbb{C}) = \mathbb{C}^\times$, called the 1-dimensional complex representations of C_n. A candidate of $\phi(x)$ is a root of unity $\zeta = e^{\frac{2\pi i}{n}}$. If we write $\phi_j(x) := \zeta^j$ then

$$\widehat{C_n} = \{\phi_0, \dots, \phi_{n-1}\}.$$

Example 2.0.4. Consider the 1-dimensional complex representations of D_{2n}. Note that $\phi(r)^n = 1$, $\phi(s)^2 = 1$ and $\phi(s)\phi(r)\phi(s)^{-1} = \phi(r)^{-1}$, i.e. $\phi(r)^2 = 1$. If n is even then we can have $\phi(r) = \pm 1$, $\phi(s) = \pm 1$, 4 representations. If n is odd then we can only have $\phi(r) = 1$ and $\phi(s) = \pm 1$, 2 representations.

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3 Representation

3.1 Matrix representation

Definition 3.1.1. Let G be a group. A degree d *matrix representation* of G over a field K is a group homomorphism $\rho : G \to GL_d(K)$.

Example 3.1.2. Last time, we classified the degree 1 representations of C_n and D_{2n} over $\mathbb{C}$.
Consider a degree 2 representation of D_{2n} over $\mathbb{R}$, i.e. a group homomorphism $D_{2n} \to GL_2(\mathbb{R})$. Intuitively, we want to map to the corresponding rotation/reflection matrix, i.e.

$$\phi(r) = R_{2\pi/n} = \begin{pmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix} \qquad \phi(s) = S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Example 3.1.3 (Trivial degree d matrix representation of G over K)**.** For all $g \in G$, define $\rho(g) := I_d \in GL_d(K)$, the identity matrix.

Example 3.1.4. Fix $A \in GL_d(K)$ and define $\rho : C_\infty \to GL_d(K)$ to be $\rho(x) = A$ (so that $\rho(x^i) = A^i$).

Example 3.1.5. Let $\theta \in \mathbb{R}$ and $R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Is there a degree 2 real representation of C_n with $\rho(x) = R_\theta$? By 2.0.2, it's sufficient and necessary that $R_\theta^n = R_{n\theta} = I_2$, i.e. $n\theta \in 2\pi\mathbb{Z}$, i.e.

$$\theta \in \{2\pi k/n : k \in \{0, \dots, n-1\}\}.$$

Example 3.1.6. $\operatorname{sgn} : S_n \to \mathbb{C}^\times$ is a degree 1 complex representation of S_n.

Lemma 3.1.7. Let $\rho : G \to GL_d(K)$ be a matrix representation and $P \in GL_d(K)$. Then $\rho' : G \to GL_d(K) : g \mapsto P\rho(g)P^{-1}$ is also a matrix representation.

Proof. One has $\rho'(gh) = P\rho(gh)P^{-1} = P\rho(g)\rho(h)P^{-1} = P\rho(g)P^{-1}P\rho(h)P^{-1} = \rho'(g)\rho'(h)$. ☐

Definition 3.1.8. Two degree d matrix representations $\rho_1, \rho_2 : G \to GL_d(K)$ are *isomorphic* or *equivalent* if $\exists P \in GL_d(K) : \rho_2(g) = P\rho_1(g)P^{-1}\ \forall g \in G$, denoted $\rho_1 \sim \rho_2$.

Lemma 3.1.9. Two degree 1 representations $\theta_1, \theta_2 : G \to GL_1(K) = K^\times$ are isomorphic iff they are equal.

Proof. If θ_1, θ_2 are isomorphic then $\exists : P \in K^\times : \theta_2(g) = P\theta_1(g)P^{-1} = \theta_1(g)$ since $P, \theta_1(g), P^{-1} \in K^\times$, a subset of a field.
If they are equal then they are isomorphic by definition. □

Example 3.1.10. By lemma above, none of the two representations of Example 2.0.3 are isomorphic.

Definition 3.1.11. A representation $\rho : G \to GL_d(K)$ is *faithful* if ρ is injective.

3.2 Complex representations of C_n

Lemma 3.2.1. Let $A \in GL_d(\mathbb{C})$ and suppose $A^n = I_d$ for some n. Then $\exists Q \in GL_d(\mathbb{C}) : Q^{-1}AQ$ is diagonal with roots of unity $\theta_1, \ldots, \theta_d$ on the diagonal.

Proof. It suffices to prove A is diagonalisable and all eigenvalues are roots of unity. Let $f(x) = x^n - 1$, so that $f(A) = 0$. Then $\mu_A(x)$ divides $f(x)$, so all its roots are distinct and are roots of unity. □

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Theorem 3.2.2. Let $C_n = \langle x \mid x^n = 1 \rangle$ and $\rho : C_n \to GL_d(\mathbb{C})$ a matrix representation. Then $\exists$ nth roots of unity $\theta_1, \ldots, \theta_d$ and a representation $\rho' : C_n \to GL_d(\mathbb{C})$ with $\rho \sim \rho'$ and

$$\rho'(x^k) = \begin{pmatrix} \theta_1^k & & 0 \\ & \ddots & \\ 0 & & \theta_d^k. \end{pmatrix}$$

Proof. Let $A = \rho(x)$. Since $x^n = 1,\ A^n = \rho(x^n) = I_d$. By lemma above, we can define $\rho'(x^k) = Q^{-1}\rho(x^k)Q$. By definition, $\rho' \sim \rho$. Now

$$\rho'(x^k) = Q^{-1}\rho(x^k)Q = Q^{-1}A^kQ = (Q^{-1}AQ)^k,$$

a power of a diagonal matrix, so it indeed has its desired form. □

Example 3.2.3. Suppose $n \geq 3$ and $\rho : C_n \to GL_2(\mathbb{R}) \subseteq GL_2(\mathbb{C}) : x \mapsto R_{2\pi/n}$. Then $R_{2\pi/n}$ has complex eigenvalues ζ and ζ^{n-1} where ζ is the nth root of unity. So $\exists Q \in GL_2(\mathbb{C}) : Q^{-1}R_{2\pi/n}Q = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{n-1} \end{pmatrix}$, and we can define $\rho' : C_n \to GL_2(\mathbb{C})$ to be

$$x^k \mapsto Q^{-1}\rho(x^k)Q = (Q^{-1}R_{2\pi/n}Q)^k = \begin{pmatrix} \zeta^k & 0 \\ 0 & \zeta^{(n-1)k} \end{pmatrix}.$$

Note that by notation used in Example 2.0.3, we can write $\rho'(g)$ as $\begin{pmatrix} \phi_1(g) & 0 \\ 0 & \phi_{n-1}(g). \end{pmatrix}$ More generally, this is called *decomposing* the representation and denoted $\rho' = \phi_1 \oplus \phi_{n-1}$.

Theorem 3.2.4. Every element of $Q_8 = \left\langle a, b \mid a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} \right\rangle$ can be written as a^ib^j where $0 \leq i \leq 3,\ 0 \leq j \leq 1$. Moreover, $|Q_8| = 8$.

Proof. One has $a^{-1} = a^3$ and $b^{-1} = b^3$ since $b^4 = (b^2)^2 = (a^2)^2 = a^4 = 1$, so we get rid of the inverses. Then we use $ba = a^7b$ to move all b to the right, and use $a^4 = 1$ to reduce power of a to under 3.
To prove the $4 \times 2 = 8$ elements are distinct, define the group homomorphism $\phi : Q_8 \to GL_2(\mathbb{C}) : \phi(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \phi(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $|\langle \phi(a) \rangle| = 4 \mid |\operatorname{im}\phi|$, and since $\phi(b) \notin \langle \phi(a) \rangle,\ |\operatorname{im}\phi| > 4$, and since $|\operatorname{im}\phi| \leq 8$, one concludes $|\operatorname{im}\phi| = 8$. None of these matrices are similar, so $|Q_8| = 8$. □

4 Character: first encounter

Definition 4.0.1. Let $\rho : G \to GL_d(K)$ be a representation. The *character* of ρ is $\chi_\rho : G \to \mathbb{C} : g \mapsto \operatorname{tr}(\rho(g))$. Note that this is not a homomorphism.

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Example 4.0.2. $\rho : G \to \mathbb{C}^\times$ is a 1-dim representation. Then $\chi_\rho(g) = \rho(g)$. In this case, character is a group homomorphism since it's the same as the representation itself.

Example 4.0.3. $\rho : D_{2n} \to GL_2(\mathbb{C}) : r \mapsto R_{2\pi/n}, s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (as in Example 3.1.2).

Compute the values of the character:

$$\chi_\rho(r^k) = \operatorname{tr} R_{2\pi k/n} = \operatorname{tr} \begin{pmatrix} \cos \frac{2\pi k}{n} & -\sin \frac{2\pi k}{n} \\ \sin \frac{2\pi k}{n} & \cos \frac{2\pi k}{n} \end{pmatrix} = 2 \cos \frac{2\pi k}{n},$$

and

$$\chi_\rho(sr^k) = \operatorname{tr} \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos \frac{2\pi k}{n} & -\sin \frac{2\pi k}{n} \\ \sin \frac{2\pi k}{n} & \cos \frac{2\pi k}{n} \end{pmatrix} \right) = \operatorname{tr} \begin{pmatrix} \cos \frac{2\pi k}{n} & -\sin \frac{2\pi k}{n} \\ -\sin \frac{2\pi k}{n} & -\cos \frac{2\pi k}{n} \end{pmatrix} = 0.$$

4.1 Isomorphic representations have same character

Recall that the character polynomial expands

$$c_A(x) = \det(xI_d - A) = x^d - \operatorname{tr}(A) x^{d-1} + \cdots + (-1)^d \det(A).$$

Lemma 4.1.1. Similar matrices have same character polynomial. In particular, they have same trace.

Proof. Let $B = Q^{-1}AQ$. Then

$$\begin{aligned} c_B(x) &= \det(xI_d - B) = \det\left(Q^{-1}xI_dQ - Q^{-1}AQ\right) = \det\left(Q^{-1}(xI_d - A)Q\right) \\ &= \det\left(Q^{-1}\right) \det(xI_d - A) \det(Q) = \det(xI_d - A) \\ &= c_A(x). \end{aligned}$$

□

Lemma 4.1.2. Isomorphic representations have same character.

Proof. Let $\rho_1 \sim \rho_2$, i.e. $\forall g,\ \rho_1(g) \sim \rho_2(g)$. By previous lemma, $\chi_{\rho_1}(g) = \operatorname{tr}(\rho_1(g)) = \operatorname{tr}(\rho_2(g)) = \chi_{\rho_2}(g)$. □

We will see later the converse also holds.

4.2 Matrix of finite order

Lemma 4.2.1. Let $A \in GL_d(\mathbb{C})$ with $A^n = I_d$ for some $n \in \mathbb{N}$. Then

1. $|\operatorname{tr}(A)| \le d$
2. $|\operatorname{tr}(A)| = d$ iff $A = \theta I_d$ for an nth root of unity θ
3. $\operatorname{tr}(A) = d$ iff $A = I_d$
4. $\operatorname{tr}\left(A^{-1}\right) = \overline{\operatorname{tr}(A)}$

Proof.

1. Recall Lemma 3.2.1 which says $A \sim \begin{pmatrix} \theta_1 & & \\ & \ddots & \\ & & \theta_d \end{pmatrix}$, so by Lemma 4.1.1 one has $\operatorname{tr}(A) = \theta_1 + \cdots + \theta_d \le d$. Triangle inequality gives
$$|\operatorname{tr}(A)| \le |\theta_1| + \cdots + |\theta_d| = d.$$
2. The triangle inequality has equality iff $\theta_1 = \cdots = \theta_d = \theta$, so $A = Q^{-1} \begin{pmatrix} \theta & & \\ & \ddots & \\ & & \theta \end{pmatrix} Q = Q^{-1}\theta Q = \theta I_d$.

3. The 'if' is clear. If $\operatorname{tr}(A) = d$ then 2 tells us $\theta d = d$ so $\theta = 1$ and $A = 1I_d = I_d$.
4. Note that if A has finite order then so does A^{-1}, so

$$A^{-1} \sim Q^{-1}A^{-1}Q = \left(QAQ^{-1}\right)^{-1} = \begin{pmatrix} \theta_1 & & \\ & \ddots & \\ & & \theta_d \end{pmatrix}^{-1} = \begin{pmatrix} \theta_1^{-1} & & \\ & \ddots & \\ & & \theta_d^{-1} \end{pmatrix},$$

hence $\operatorname{tr}\left(A^{-1}\right) = \theta_1^{-1} + \cdots + \theta_d^{-1} = \overline{\theta_1} + \cdots + \overline{\theta_d} = \overline{\theta_1 + \cdots + \theta_d} = \operatorname{tr}(A)$.

□

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4.3 First properties of character

Proposition 4.3.1. Let G be a finite group and $\rho : G \to GL_d(\mathbb{C})$ a representation with character $\chi = \chi_\rho$. Then

1. $|\chi(g)| \leq d \ \forall g \in G$
2. $\chi(g) = d$ iff $\rho(g) = I_d$. In particular, $\chi(e) = d$.
3. $\chi\left(g^{-1}\right) = \overline{\chi(g)} \ \forall g \in G$
4. $\chi\left(h^{-1}gh\right) = \chi(g) \ \forall g, h \in G$, i.e. χ is constant on a conjugacy class (hence called *class function*)

Proof. Since G is finite, every $g \in G$ has finite order, so its representation matrix also has finite order, hence 1–3 follow from 4.2.1. For part 4, note that since ρ is a homomorphism,

$$\chi\left(h^{-1}gh\right) = \operatorname{tr}\left(\rho\left(h^{-1}gh\right)\right) = \operatorname{tr}\left(\rho(h)^{-1}\rho(g)\rho(h)\right) = \operatorname{tr}(\rho(g)) = \chi(g).$$

by 4.1.1. □

5 Linear representation and KG-module

Definition 5.0.1. Let G be a group. A *linear representation* of G is a pair (V, ρ) where V is a vector space and $\rho : G \to GL(V)$ is a group homomorphism. $\dim V$ is the *degree* or *dimension* of (V, ρ). We also say '$\rho : G \to GL(V)$ is a linear representation.'

Example 5.0.2. Trivial representation $\rho : G \to GL(V) : g \mapsto I_V$.

Example 5.0.3. $C_2 = \left\langle x \mid x^2 = 1\right\rangle, \ \rho : C_2 \to GL(V) : 1 \mapsto I_V, x \mapsto -I_V$.

Example 5.0.4. $C_n = \langle x \mid x^n = 1\rangle, \ \rho : C_n \to GL(V) : x^i \mapsto \zeta_n^i I_V$ where V is over $\mathbb{C}$.

5.1 Correspondence between matrix representations and linear representations

Let $\rho : G \to GL_d(K)$ be a matrix representation. For all $g \in G$, define $\theta_g : K^d \to K^d : v \mapsto \rho(g)v$. Clearly $\theta_g \in GL(K^d) \ \forall g \in G$. Now consider the map $\theta : G \to GL(K^d) : g \mapsto \theta_g$. We claim this is a group homomorphism, and therefore is a linear representation. Indeed, $\theta(gh)(v) = \theta_{gh}(v) = \rho(gh)v = \rho(g)\rho(h)v = (\theta_g\theta_h)(v)$.

Now let (V, θ) be a linear representation with $\dim V = d < \infty$ and $(v_1, \ldots, v_d)$ a K-basis of V. For all $g \in G$, $\theta(g) : V \to V$ has an associated matrix. Denote it $\rho(g) \in GL_d(K)$. (Verify that $\rho : G \to GL_d(K)$ is a group homomorphism.) If we take a different basis $w_1, \ldots, w_d$, we get ρ' and there exists $P \in GL_d(K)$ (depending only on $v_1, \ldots, v_d, \ w_1, \ldots, w_d$) with $\rho'(g) = P\rho(g)P^{-1} \ \forall g \in G$, hence $\rho \sim \rho'$.

5.2 The regular representation

Let $|G| = n$ and V the linear span of the n many linearly independent vectors v_g, indexed by the group elements. Then $\dim V = n$. For $h \in G$, let $\operatorname{reg}_h \in \operatorname{Hom}(V, V)$ be defined via $\operatorname{reg}_h(v_g) := v_{hg}$. In particular, $\operatorname{reg}_h(\alpha_1 v_{g_1} + \cdots + \alpha_n v_{g_n}) = \alpha_1 v_{hg_1} + \cdots + \alpha_n v_{hg_n}$.

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Example 5.2.1. $C_3 = \langle x \mid x^3 = 1 \rangle$, $V = \operatorname{linspan}\{v_1, v_x, v_{x^2}\}$. Then $\operatorname{reg}_x(v_1) = v_x$, $\operatorname{reg}_x(v_x) = \operatorname{reg}_{x^2}$, $\operatorname{reg}_x(v_{x^2}) = v_1$, and the matrix of reg_x with respect to bases (v_1, v_x, v_{x^2}) is

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that $\rho : C_3 \to GL_3(\mathbb{C}) : x \mapsto M$ is a group homomorphism.

Lemma 5.2.2. $\operatorname{reg}_h \in GL(V)$ $\forall h \in G$.

Proof. One has to show bijectivity. Using Corollary 1.5.4, showing surjectivity suffices. Let $g \in G$. Then

$$\operatorname{reg}_h(v_{h^{-1}g}) = v_{hh^{-1}g} = v_g,$$

hence $\operatorname{im}\operatorname{reg}_h$ contains every basis vector v_g. □

This gives a map $\operatorname{reg} : G \to GL(V)$.

Lemma 5.2.3. $\operatorname{reg} : G \to GL(V) : h \mapsto \operatorname{reg}_h$ is a linear representation.

Proof. Let $h_1, h_2, g \in G$. Then

$$\begin{aligned}(\operatorname{reg}(h_1)\operatorname{reg}(h_2))(v_g) &= \operatorname{reg}(h_1)(\operatorname{reg}(h_2)(v_g)) = \operatorname{reg}_{h_1}(\operatorname{reg}_{h_2}(v_g)) \\ &= \operatorname{reg}_{h_1}(v_{h_2 g}) = v_{h_1 h_2 g} = \operatorname{reg}_{h_1 h_2}(v_g) \\ &= \operatorname{reg}(h_1 h_2)(v_g),\end{aligned}$$

so $\operatorname{reg}(h_1)\operatorname{reg}(h_2) = \operatorname{reg}(h_1 h_2)$. □

5.3 KG-module

Definition 5.3.1. A *linear action* of a group G on a vector space V over field K is a map $\gamma : G \times V \to V : (g, v) \mapsto \gamma(g, v)$ such that $\forall u, v \in V,\ a \in K,\ g, h \in G$:

$$\left.\begin{array}{l} 1.\ \gamma(e, v) = v \\ 2.\ \gamma(hg, v) = \gamma(h, \gamma(g, v)) \end{array}\right\} \text{a group action of } G \text{ on } V$$

$$\left.\begin{array}{l} 3.\ \gamma(g, u + v) = \gamma(g, u) + \gamma(g, v) \\ 4.\ \gamma(g, av) = a\gamma(g, v) \end{array}\right\} v \mapsto \gamma(g, v) \text{ is a linear map } \forall g \in G$$

Definition 5.3.2. A *KG-module* is a vector space V over K equipped with a linear action γ of G on V.

Example 5.3.3. $C_n = \langle x \mid x^n = 1 \rangle$ and V is any $\mathbb{C}$-vector space. Let x act by multiplication with ζ_n, i.e. $\gamma(x, v) = \zeta_n v$. This is sufficient to define the action, since, for example, $\gamma(x^2, v) = \gamma(x, \gamma(x, v)) = \gamma(x, \zeta_n v) = \zeta_n^2 v$ by definition, and in general $\gamma(x^i, v) = \zeta_n^i v$.

Notation. $gv := \gamma(g, v) = \rho(g)(v)$.

Proposition 5.3.4. Specifying a KG-module structure on a K-vector space V is the same as specifying a linear representation $G \to GL(V)$.

Proof. Let $\gamma : G \times V \to V$ be a KG-module. Define $\rho_g : V \to V : v \mapsto \gamma(g, v)$. By parts 3 and 4 of definition, ρ_g is a linear map. By part 1, $\rho_e(v) = \gamma(e, v) = v$, so $\rho_e = I_V \in GL(V)$. Also, $(\rho_g \rho_h)(v) = \rho_g(\rho_h(v)) = \gamma(g, \gamma(h, v)) = \gamma(gh, v) = \rho_{gh}(v)$, so $\rho_{gh} = \rho_g \rho_h$. In particular, $\rho_g \rho_{g^{-1}} = \rho_e = I_V$, so $\rho_g \in GL(V)$. Therefore $\rho : G \to GL(V) : g \mapsto \rho_g$ is a group homomorphism.

For the converse, we start with a linear representation $\rho : G \to GL(V)$ and define $\gamma : G \times V \to V : (g, v) \mapsto \rho(g)(v)$. Check this gives a linear action: 1 and 2 hold since ρ is a group homomorphism, and 3 and 4 hold since each $\rho(g)$ is a linear map. □

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Example 5.3.5. $C_2 = \left\langle x \mid x^2 = 1 \right\rangle$, $V = \mathbb{C}^2$. Let x act on V via multiplication by $A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then γ is determined: $\gamma(x, v) = Av \ \forall v \in V$. Also, $\rho : C_2 \to GL(V)$ is determined: $\rho(e)(v) = v$ (identity), $\rho(x)(v) = Av \ \forall v \in V$. Note that not every arbitrary A works; verify the γ and ρ satisfy the definition axioms.

Example 5.3.6. $\rho : Q_8 \to GL_2(\mathbb{C})$, $\rho(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\rho(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This makes $\mathbb{C}^2$ a $\mathbb{C}Q_8$-module via $\gamma(g, v) = \rho(g)(v)$. In other language, a and b act on $\mathbb{C}^2$ by multiplication with $A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

6 Submodule and morphism

6.1 Submodule and reducibility

Definition 6.1.1. Let G be a group, K a field and V a KG-module. $W \subseteq V$ is a *KG-submodule* of V if

1. $W \subseteq V$ is a K-subspace
2. $gw \in W \ \forall w \in W, g \in G$

Example 6.1.2. $C_2 = \left\langle x \mid x^2 = 1 \right\rangle$, $V = \mathbb{C}^2$. Let x act on V via multiplication by $A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The submodules are $\{0\}$, $\mathbb{C}^2$ (the trivial ones), $\mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbb{C}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Lemma 6.1.3. A KG-submodule is a KG-module. In the language of presentations, if $\rho : G \to GL(V)$ is a linear representation and $W \subseteq V$ is a KG-submodule, then $\rho' : G \to GL(W)$ is also a linear representation, called a *subrepresentation*.

Definition 6.1.4. A KG-submodule of V is *proper* if $W \neq V$, *nontrivial* if $W \neq \{0\}$.
A nontrivial KG-module V is *reducible* if V has a nontrivial proper submodule. Otherwise, it is *irreducible* or *simple*.

Example 6.1.5. $C_n = \langle x \mid x^n = 1 \rangle$, $\rho : C_n \to GL_2(\mathbb{R})$, $\rho(x) = R_{2\pi/n}$. We claim ρ is irreducible if $n \geq 3$. It suffices to show any 1-d subspace $\mathbb{R}u$ where $u \neq 0$ of $\mathbb{R}^2$ are not KG-submodules. Indeed; let $\alpha u \in \mathbb{R}u$, then $x\alpha u = \alpha x u = \alpha R_{2\pi/n} u \notin \mathbb{R}u$.

Example 6.1.6. $C_\infty = \langle x \mid \ \rangle$, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Consider the $\mathbb{C}C_\infty$-module $V = \mathbb{C}^2$ with x acting by multiplication with A. One can see $\mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a 1-d subrepresentation, and we claim there are no other 1-d subrepresentations (i.e. no other nontrivial proper subrepresentations). Indeed, suppose $\mathbb{C}v$ where $v \neq 0$ is one, i.e. $Av = \lambda v$ for some $\lambda \in \mathbb{C}$, but A only has one eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If A were $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ then there would be two nontrivial proper subrepresentations, $\mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbb{C}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Example 6.1.7. If a group is generated by $g_1, \ldots, g_n$ and V is a KG-module, then V has a 1-dim KG-submodule iff $\rho(g_1), \ldots, \rho(g_n)$ have a common eigenvector. Indeed; the $\Leftarrow$ is trivial, and the $\Rightarrow$ follows from that if $Ku \subseteq V$ is a submodule, implying $g_i \alpha u \in Ku \ \forall i$, then u is an eigenvector of $\rho(g_i)$ by definition.

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Example 6.1.8 (6.1.5 but over $\mathbb{C}$)**.** $C_n = \langle x \mid x^n = 1 \rangle$, $\rho : C_n \to GL_2(\mathbb{C})$, $\rho(x) = R_{2\pi/n}$ with $n \geq 3$. Now $R_{2\pi/n}$ has eigenvectors $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ i \end{pmatrix}$ with eigenvalues ζ and ζ^{-1}, so there are 4 submodules: $\{0\}$, $\mathbb{C}\begin{pmatrix} 1 \\ -i \end{pmatrix}$, $\mathbb{C}\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\mathbb{C}^2$.

Example 6.1.9 (3.1.2 but over $\mathbb{C}$)**.** $D_{2n} = \left\langle r, s \mid r^n = s^2 = 1,\ srs^{-1} = r^{-1} \right\rangle$, $V = \mathbb{C}^2$ with the same action and $n \geq 3$. There's no common eigenvectors of $R_{2\pi/n}$ and S, so V has not proper nontrivial subrepresentations, hence irreducible.

6.2 Reducible representation in terms of matrices

Let V be a d-dimensional KG-module with submodule $U \subseteq V$. Choose a basis $v_1, \dots, v_r$ of U and extend it to a basis $v_1, \dots, v_r, v_{r+1}, \dots, v_d$ of V. Let $\theta : G \to GL_d(K)$ be the matrix representation with respect to this basis. Write

$$\theta(g) = (a_{ij}(g))_{1 \le i \le d,\ 1 \le j \le d} \qquad \text{with } \theta(g)(v_j) = a_{1j}(g)v_1 + \cdots + a_{dj}(g)v_d,$$

but note that $\theta(g)(v_i)$ for $i = 1, \dots, r$ are expressed by solely $v_1, \dots, v_r$, so the bottom left $d-r$ by $d-r$ is 0, i.e.

$$\theta(g) = \left(\begin{array}{cccc|ccc} a_{11}(g) & a_{12}(g) & \cdots & a_{1r}(g) & a_{1r+1}(g) & \cdots & a_{1d}(g) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{r1}(g) & a_{r2}(g) & \cdots & a_{rr}(g) & \vdots & & \vdots \\ \hline 0 & 0 & \cdots & 0 & a_{r+1r+1}(g) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_{dr+1}(g) & \cdots & a_{dd}(g) \end{array}\right) = \left(\begin{array}{c|c} \phi(g) & \psi(g) \\ \hline 0 & \eta(g) \end{array}\right).$$

We also know θ is a homomorphism, hence

$$\theta(gh) = \begin{pmatrix} \phi(gh) & \psi(gh) \\ 0 & \eta(g) \end{pmatrix} = \begin{pmatrix} \phi(g) & \psi(g) \\ 0 & \eta(g) \end{pmatrix} \begin{pmatrix} \phi(h) & \psi(h) \\ 0 & \eta(h) \end{pmatrix} = \theta(g)\theta(h)$$
$$\begin{pmatrix} \phi(g)\phi(h) & \psi(g)\psi(h) \\ 0 & \eta(g)\eta(h) \end{pmatrix} = \begin{pmatrix} \phi(g)\phi(h) & \phi(g)\psi(h) + \psi(g)\eta(h) \\ 0 & \eta(g)\eta(h) \end{pmatrix},$$

so $\underbrace{\phi : G \to GL_r(K)}_{U}, \underbrace{\eta : G \to GL_{d-r}(K)}_{V/U}$ are homomorphisms, hence matrix representations.

6.3 Permutation representation

Definition 6.3.1. Given a group action $\gamma : G \times X \to X$ where $X = \{x_1, \dots, x_d\}$, define K-vector space of formal linear combination of $v_{x_1}, \dots, v_{x_d}$, and linear action $g \cdot v_{x_i} := v_{gx_i}$. This gives an element of $GL_d(K)$ determined by g, i.e. a representation $g(\alpha_1 v_{x_1} + \cdots + \alpha_d v_{x_d}) = \alpha_1 v_{gx_1} + \cdots + \alpha_d v_{gx_d}$ called the *permutation representation* or *permutation module* to γ.

Example 6.3.2. G can act on itself by left multiplication $(g, h) \mapsto gh$ (which gives the regular representation; see 5.2), $(g, h) \mapsto hg^{-1}$ or $(g, h) \mapsto ghg^{-1}$.

Example 6.3.3. S_n acts on $\{1, \dots, n\}$ via $\pi i = \pi(i)$. Let $V = \text{linspan}\{v_1, \dots, v_n\}$ with $\pi v_i = v_{\pi(i)}$. Then $v_1 + \cdots + v_n$ is a 1-dimensional subrepresentation of V.

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6.4 Morphism

Definition 6.4.1. Let V, W be KG-modules. A K-linear map $f : V \to W$ is a *G-morphism* (or an *equivariant map*, or simply *morphism* of KG-modules) if $gf(v) = f(gv)\ \forall v \in V, g \in G$.

Notation. $\text{Hom}_G(V, W) = \{f : V \to W : f \text{ is a } G\text{-morphism}\}$. This is a vector space.

Definition 6.4.2. A *G-isomorphism* is a bijective G-morphism.

Lemma 6.4.3. If $f : V \to W$ is a G-morphism, then $\ker f$ and $\operatorname{im} f$ are subrepresentations of V and W respectively.

Proof. Since f is linear, $\ker f$ and $\operatorname{im} f$ are linear subspaces of V and W respectively. It remains to show that

1. $gv \in \ker f\ \forall g \in G, v \in \ker f$. Indeed, $f(gv) = gf(v) = g0 = 0$ by definition, and
2. $gw \in \operatorname{im} f\ \forall g \in G, w \in \operatorname{im} f$. Indeed, let $v \in V : f(v) = w$, then $gw = gf(v) = f(gv)$.

∎

Example 6.4.4. Let $X = \{1, 2, 3\}$, $G = S_3$, V the permutation module $\{a_1e_1 + a_2e_2 + a_3e_3 : a_1, a_2, a_3 \in \mathbb{C}\}$ and $W = \mathbb{C}$ the trivial $\mathbb{C}S_3$-module, i.e. $gw = w\ \forall w \in W, g \in S_3$. Fix $0 \neq w \in W$ and define $f : V \to W : a_1e_1 + a_2e_2 + a_3e_3 \mapsto (a_1 + a_2 + a_3)w$. Verify f is a G-morphism: f is clearly a linear map, and one has

$$\begin{aligned} gf(a_1e_1 + a_2e_2 + a_3e_3) &= g(a_1 + a_2 + a_3)w = (a_1 + a_2 + a_3)w \\ &= \left(a_{g^{-1}(1)} + a_{g^{-1}(2)} + a_{g^{-1}(3)}\right) w = f(g(a_1e_1 + a_2e_2 + a_3e_3)). \end{aligned}$$

6.5 Schur's lemma

Theorem 6.5.1 (Schur's lemma I)**.** Let G be a group, K a field and $f : U \to V$ a G-morphism of irreducible KG-modules. Then either $f = 0$ or f is an isomorphism.

Proof. One has $f = 0$ iff $\ker f = U$ and $\operatorname{im} f = \{0\}$. Now suppose $f \neq 0$, then $\ker f \subsetneq U$ and $\{0\} \subsetneq \operatorname{im} f \subseteq V$, but by Lemma 6.4.3 and the assumption that U, V are irreducible, $\ker f = \{0\}$ and $\operatorname{im} f = V$, i.e. f is injective and surjective, i.e. f is an isomorphism. ☐

Theorem 6.5.2 (Schur's lemma over $\mathbb{C}$)**.** Let G be a group, V a finite dimensional irreducible $\mathbb{C}G$-module and $f : V \to V$ a G-morphism. Then $f = \lambda I_V$ for some $\lambda \in \mathbb{C}$. In particular, $\dim \operatorname{Hom}_G(V, V) = 1$.

Proof. Let λ be an eigenvalue of f with eigenvector u. Let $f' : V \to V : v \mapsto f(v) - \lambda v$. We claim f' is a G-morphism. Indeed; it's clearly a linear map, and

$$f'(gv) = f(gv) - \lambda gv = gf(v) - g\lambda v = g(f(v) - \lambda v) = gf'(v).$$

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By Schur's lemma I, since $f'(u) = 0$ and $u \neq 0$, one has $f' = 0$, i.e. $f(v) = \lambda v\ \forall v \in V$, so equivalently $f' = \lambda I_V$ which is what's desired. ☐

Example 6.5.3 (Schur's lemma over $\mathbb{R}$)**.** $C_3 = \langle x \mid x^3 = 1 \rangle$, V the regular C_3-representation with basis v_e, v_x, v_{x^2}, $W = \operatorname{linspan}_{\mathbb{R}}\{v_e - v_x, v_x - v_{x^2}\}$ a subrepresentation. The matrix for this action of x on W is then $\rho(x) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, which has no real eigenvalues, hence no 1-dim subrepresentation, so irreducible.

To calculate the $\mathbb{R}$-vector space of C_3-morphisms $W \to W$, note that one needs by definition

$$\begin{pmatrix} -c & -d \\ a-c & b-d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} b & -a-b \\ d & -c-d \end{pmatrix},$$

i.e. $c = -b, d = a + b$ and the matrix is $\begin{pmatrix} a & b \\ -b & a+b \end{pmatrix}$ which has two degrees of freedom a and b, so $\dim_{C_3}(W, W) = 2$.

7 Maschke's theorem

7.1 Projection

Definition 7.1.1. A map f is called *idempotent* if $f \circ f = f$. A such linear map $V \to U$ is a *projection* if $f(u) = u\ \forall u \in U$.

Example 7.1.2. $V = \mathbb{R}^2$, $U = \mathbb{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \subseteq V$, $f : V \to U : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a \\ 0 \end{pmatrix}$ is a projection. Note that $V = U \oplus \ker f$.

Lemma 7.1.3. Let V be a finitely dimensional vector space and $U \subseteq V$ a linear subspace. Then $\exists$ a projection $f : V \to U$.

Proof. Let $v_1, \dots, v_r$ be a basis for U and $v_1, \dots, v_r, v_{r+1}, \dots, v_d$ a basis for V. Define $f : V \to U$ by

$$\alpha_1v_1 + \dots + \alpha_dv_d \mapsto \alpha_1v_1 + \dots + \alpha_rv_r,$$

which is a projection. ☐

Theorem 7.1.4. Let $f : V \to U$ be a projection. Then $V = U \oplus \ker f$.

Proof. 1. To show $V = U + \ker f$, let $v \in V$ and write $v = f(v) + v - f(v)$. Clearly $f(v) \in U$ and it remains to show $f(v - f(v)) = 0$, but $f(v - f(v)) = f(v) - f(f(v)) = f(v) - f(v) = 0$ by idempotence.

2. To show $U \cap \ker f = \{0\}$, let $u \in U \cap \ker f$, then $f(u) = u$ and $f(u) = 0$, so $u = 0$.

$\square$

7.2 Semisimplicity and complementary modules

Definition 7.2.1. A KG-module V is *semisimple* if $\forall KG$-submodules U, $\exists$ a KG-submodule $W \subseteq V$ such that $V = U \oplus W$, where U and W are *complementary.*

Example 7.2.2. If V is irreducible then the only submodules are $\{0\}$ and V, which are complementary, hence every irreducible representation is semisimple.

Example 7.2.3. Recall Example 6.1.6 where we have three submodules $\{0\}, \mathbb{C}\begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbb{C}^2$. Hence the representation is not semisimple since $\mathbb{C}\begin{pmatrix}1\\0\end{pmatrix}$ has no complementary submodule. If we again replace A by a diagonal matrix then it would be semisimple ($\mathbb{C}\begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbb{C}\begin{pmatrix}0\\1\end{pmatrix}$ are complementary).

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7.3 Maschke's theorem

Lemma 7.3.1 (Averaging)**.** Let G be a finite group, K a field with $|G| \cdot 1_K \neq 0_K$ (i.e. $\operatorname{char} K \nmid |G|$) and U, V be KG-modules with $f : U \to V$ a linear map. Define

$$f' : V \to U : v \mapsto \frac{1}{|G|} \sum_{g \in G} g\left(f\left(g^{-1}v\right)\right),$$

then f' is a G-morphism.
(cf. HW5, Exe 3)

Proof. Let $h \in G$, then

$$\begin{aligned} f'(hv) &= \frac{1}{|G|} \sum_{g \in G} g\left(f\left(g^{-1}hv\right)\right) = h\frac{1}{|G|} \sum_{g \in G} h^{-1}gf\left(\left(h^{-1}g\right)^{-1} v\right) \\ &= h\frac{1}{|G|} \sum_{h^{-1}g \in G} h^{-1}gf\left(\left(h^{-1}g\right)^{-1} v\right) = h(f'(v)). \end{aligned}$$

$\square$

Theorem 7.3.2 (Maschke's)**.** Let G be a finite group and K a field with $|G| \cdot 1_K \neq 0_K$. Then every finite dimensional KG-module is semisimple.

Proof. Let $U \subseteq V$ be a KG-submodule. We want to show $\exists W \subseteq V$ a KG-submodule such that $V = U \oplus W$. Let $f : V \to U$ be a projection and $f' \in \operatorname{Hom}_G(V, U)$ as in lemma above. We claim f' is idempotent and $\operatorname{im} f' = U$. Indeed; since $f'(v) \in U\ \forall v \in V$, it suffices to show $f'(u) = u\ \forall u \in U$:

$$\begin{aligned} f'(u) &= \frac{1}{|G|} \sum_{g \in G} g\left(f\left(g^{-1}u\right)\right) \\ &= \frac{1}{|G|} \sum_{g \in G} g\left(g^{-1}u\right) \qquad \text{since } g^{-1}u \in u \text{ and } f \text{ is a projection} \\ &= \frac{1}{|G|} \sum_{g \in G} u \\ &= \frac{1}{|G|}|G|u = u. \end{aligned}$$

Hence, by Theorem 7.1.4, $V = U \oplus \ker f'$ where $\ker f'$ is indeed a KG-submodule by 6.4.3. $\square$

Corollary 7.3.3. Let G be a group, K a field with $|G| \cdot 1_K \neq 0_K$ and V a finite dimensional KG-module. Then $\exists$ irreducible submodules $U_1, \ldots, U_j$ such that $V = U_1 \oplus U_2 \oplus \cdots \oplus U_j$.

Proof. Induction on $\dim V$. If $\dim V = 1$ then V is irreducible hence we are done. Now let $\dim V > 1$. If V is irreducible then we are again done, so suppose V is reducible and let $U \subseteq V$ be a nontrivial proper subrepresentation with complementary W, whose existence is guaranteed by Maschke's theorem. Note that $\dim U, \dim W < \dim V$, so by inductive hypothesis $U = U_1 \oplus \cdots \oplus U_r, W = U_{r+1} \oplus \cdots \oplus U_k$ where U_i irreducible, hence $V = U \oplus W = U_1 \oplus \cdots \oplus U_k$. $\square$

Remark (On cyclic groups)**.** We actually have seen Maschke's theorem and its corollary for specifically cyclic groups C_n already, and as corollaries, all irreducible representations of C_n are 1-dimensional, and there are exactly n many non-isomorphic irreducible representations of C_n.

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7.4 Orthogonality relations of characters

Notation. $\mathbb{C}^G := \{f : G \to \mathbb{C}\}$. Note that $\mathbb{C}^G \cong \mathbb{C}G$ as a vector space and $\dim \mathbb{C}^G = |G|$.

Lemma 7.4.1. Let $V = U_1 \oplus \cdots \oplus U_k$ be a decomposition of a KG-module V, then $\chi_V = \chi_{U_1} + \cdots + \chi_{U_k}$.

Remark. Note that Maschke's theorem does not give us uniqueness of the decomposition, but the equation stated will independently hold.

Proof. Choose a basis of V by choosing a basis for each U_i, then matrices $\rho(g)$ are block diagonal with respect to this basis (cf. Section 6.2):

$$\rho_V(g) = \left(\begin{array}{c|c|c} \rho_{U_1}(g) & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & \rho_{U_k}(g) \end{array}\right),$$

and by definition of character (trace of the matrix) one has what's desired. $\square$

From now on we fix the field $\mathbb{C}$ and group G to be finite. Write $V \in \text{Mod-}G$ to say 'V is a finite dimensional $\mathbb{C}G$-module'.

Lemma 7.4.2. Let $V \in \text{Mod-}G$ be irreducible and $f \in \text{Hom}(V, V)$. Define

$$\widetilde{f} \in \text{Hom}_G(V, V) \quad \text{by} \quad v \mapsto \frac{1}{|G|} \sum_{g \in G} g\left(f\left(g^{-1}v\right)\right).$$

Then

$$\widetilde{f} = \frac{\operatorname{tr}(f)}{\dim V} I_V$$

Proof. Schur's lemma over $\mathbb{C}$ (6.5.2) tells us indeed $\widetilde{f} = \lambda I_V$ for some $\lambda \in \mathbb{C}$. Now one has

$$\begin{aligned} \lambda \dim V = \operatorname{tr}(\lambda I_V) &= \operatorname{tr}\left(\frac{1}{|G|} \sum_{g \in G} \rho(g) \circ f \circ \rho\left(g^{-1}\right)\right) \\ &= \frac{1}{|G|} \sum_{g \in G} \operatorname{tr}\left(\rho(g) \circ f \circ \rho(g)^{-1}\right) \\ &= \operatorname{tr}(f). \qquad \text{by 4.1.1} \end{aligned}$$

$\square$

Definition 7.4.3. For $\varphi, \psi \in \mathbb{C}^G$, define the *inner product*

$$\langle \varphi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \varphi(g)\overline{\psi(g)}.$$

Note that this is a Hermitian inner product on $\mathbb{C}^G$, i.e. $\forall \varphi, \psi, \xi \in \mathbb{C}^G,\ \alpha \in \mathbb{C}$,

1. $\langle \varphi, \psi \rangle = \overline{\langle \psi, \varphi \rangle}$

2. $\langle \alpha\varphi + \xi, \psi \rangle = \alpha \langle \varphi, \psi \rangle + \langle \xi, \psi \rangle$
3. $\langle \psi, \alpha\varphi + \xi \rangle = \overline{\alpha} \langle \varphi, \psi \rangle + \langle \xi, \psi \rangle$
4. $\langle \psi, \psi \rangle \geq 0$

Theorem 7.4.4 (Orthogonality relations)**.** Let $U, V \in$ Mod-G be irreducible. Then

$$\langle \chi_U, \chi_V \rangle = \begin{cases} 1 & \text{if } U \sim V \\ 0 & \text{otherwise} \end{cases}$$

Proof. One has

$$\begin{aligned}
\langle \chi_U, \chi_V \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_u(g) \chi_V(g^{-1}) \qquad \text{by 4.3.1} \\
&= \frac{1}{|G|} \sum_{g \in G} \left(\sum_i \rho_U(g)_{i,i} \right) \left(\sum_j \rho_V\left(g^{-1}\right)_{j,j} \right) \qquad \text{by definition} \\
&= \sum_{i,j} \left(\frac{1}{|G|} \sum_{g \in G} \rho_U(g)_{i,i} \rho_V\left(g^{-1}\right)_{j,j} \right) \\
&= \sum_{i,j} \left(\frac{1}{|G|} e_i^T \rho_U(g) e_i e_j^T \rho_V\left(g^{-1}\right) e_j \right) \\
&= \sum_{i,j} \left(e_i^T \left(\frac{1}{|G|} \sum_{g \in G} \rho_U(g) E_{i,j} \rho_V\left(g^{-1}\right) \right) e_j \right) \\
&= \sum_{i,j} \left(e_i^T \underbrace{\widetilde{E_{i,j}}}_{\in \mathrm{Hom}_G(V,U)} e_j \right) \qquad \text{by definition in 7.4.2}
\end{aligned}$$

By Schur's lemma (6.5.1), if $U \not\sim V$ then $\widetilde{E_{i,j}} = 0$. If $U \sim V$ then $\chi_U = \chi_V$, so it suffices to treat the case $U = V$. $\widetilde{E_{i,i}}$ is then diagonal by 6.5.2, hence

$$\sum_i e_i^T \widetilde{E_{i,i}} e_i = \mathrm{tr}\left(\widetilde{E_{i,i}}\right) = \dim V \frac{\mathrm{tr}(E_{i,i})}{\dim V} = 1$$

by Lemma 7.4.2. ☐

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Corollary 7.4.5. The number of pairwise nonisomorphic irreducible finite-dimensional $\mathbb{C}G$-modules is at most the number of conjugacy classes in G.

Proof. By 7.4.4, the characters of pairwise nonisomorphic irreducible finite-dimensional $\mathbb{C}G$-modules form an orthonormal system in the vector space $V = \{\chi \in \mathbb{C}^G : \chi \text{ class function}\}$, which implies the number of them cannot exceed $\dim V$ (you cannot have four vectors pairwise perpendicular in a 3-d space), which is the number of conjugacy class in G. ☐

Corollary 7.4.6. For $U, V \in$ Mod-G, one has $U \sim V$ iff $\chi_U = \chi_V$.

Proof. It suffices to show the $\Leftarrow$ by Lemma 4.1.2. Let $W_1, \ldots, W_r \in$ Mod-G be a complete list of pairwise nonisomorphic irreducibles. Now, by Maschke's theorem (7.3.2) one can write $U \sim \bigoplus_{i=1}^r W_i^{\oplus n_i}$ and $V \sim \bigoplus_{i=1}^r W_i^{\oplus m_i}$ where $n_i, m_i \in \mathbb{N}$. By 7.4.1 and assumption,

$$\chi_U = \sum_{i=1}^r n_i \chi_{W_i} = \sum_{i=1}^r m_i \chi_{W_i} = \chi_V.$$

Now by 7.4.4, χ_{W_i} are linearly independent, so the coefficients are uniquely determined and $n_i = m_i$ $\forall i$, and $U \sim V$ immediately follows. ☐

Definition 7.4.7. Let $U \in \text{Mod-}G$ be irreducible and $W \in \text{Mod-}G$. Define the *multiplicity* of U in W as

$$\text{mult}_U(W) := \langle \chi_U, \chi_W \rangle .$$

Proposition 7.4.8. Let $U \in \text{Mod-}G$ be irreducible and $W \in \text{Mod-}G$. For any decomposition $W = \bigoplus_{i=1}^k U_i$, one has

$$\text{mult}_U(W) = |\{i \in \{1, \dots, k\} : U \sim U_i\}|.$$

Proof. Let $W_1, \dots, W_r \in \text{Mod-}G$ be a complete list of pairwise nonisomorphic irreducibles. One then has

$$\chi_W = \sum_{i=1}^k \chi_{U_i} = \sum_{j=1}^r n_j \chi_{W_j} \quad \text{where } n_j = |\{i \in \{1, \dots, k\} : U_i \sim W_j\}|.$$

By 7.4.4, one sees

$$\begin{aligned} \text{mult}_U(W) = \left\langle \chi_U, \sum_{j=1}^r n_j \chi_{W_j} \right\rangle &= \sum_{j=1}^r n_j \langle \chi_U, \chi_{W_j} \rangle \\ &= 0 + \dots + n_{j_0} \langle \chi_U, \chi_{j_0} \rangle + \dots + 0 = n_{j_0} \end{aligned}$$

where $j_0 \in \mathbb{N} : U \sim W_{j_0}$. □

Lemma 7.4.9. $U \in \text{Mod-}G$ is irreducible iff $\langle \chi_U, \chi_U \rangle = 1$.

Proof. It suffices to show the $\Leftarrow$ by Theorem 7.4.4. Let $W_1, \dots, W_k \in \text{Mod-}G$ be a complete list of pairwise nonisomorphic irreducibles. Use Maschke's (7.3.2) to write

$$U \sim \bigoplus_{j=1}^k W_j^{\oplus n_j} \text{ and hence } \chi_U = \sum_{j=1}^k n_j \chi_{W_j}.$$

where $n_j \in \mathbb{N}$, then by 7.4.4 and assumption,

$$\langle \chi_U, \chi_U \rangle = \sum_{i,j=1}^k n_i n_j \langle \chi_{W_i}, \chi_{W_j} \rangle = \sum_{i=1}^k (n_i)^2 = 1,$$

which means one $n_i = 1$ and all other $n_i = 0$, so $U \sim W_i$ for some i, i.e. U is irreducible. □

7.5 Decomposition of regular representation

Lemma 7.5.1. Let $W_1, \dots, W_k \in \text{Mod-}G$ be a complete list of pairwise nonisomorphic irreducibles. Then

$$\sum_{i=1}^k (\dim W_i)^2 = |G|.$$

Proof. Let $\mathbb{C}G$ denote the regular representation. First note $\dim(\mathbb{C}G) = |G|$, and since reg_g, a permutation of basis vectors, has no fixed points as long as $g \neq e$ and hence only zeros along the diagonal, one has

$$\begin{aligned} \text{mult}_{W_i}(\mathbb{C}G) = \langle \chi_{\mathbb{C}G}, \chi_{W_i} \rangle &= \frac{1}{|G|} \sum_{g \in G} \underbrace{\overline{\chi_{\mathbb{C}G}(g)}}_{=0 \text{ if } g \neq e} \chi_{W_i}(g) \\ &= \frac{1}{|G|} \overline{\chi_{\mathbb{C}G}(e)} \chi_{W_i}(e) = \frac{1}{|G|} \dim W_i = \dim W_i. \end{aligned}$$

Now since

$$\mathbb{C}G \sim \bigoplus_{i=1}^k W_i^{\oplus \text{mult}_{W_i}(\mathbb{C}G)} = \bigoplus_{i=1}^k W_i^{\dim W_i},$$

one has $|G| = \dim \mathbb{C}G = \sum_{i=1}^k (\dim W_i)^2$. □

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Definition 7.5.2. A character χ is *irreducible* if χ is the character of an irreducible representation $V \in \text{Mod-}G$.

Example 7.5.3. $G = C_3 = \langle x \mid x^3 = 1 \rangle$. Recall the 3 irreducible characters: let $\zeta \in \mathbb{C}$ a primitive 3rd root of unity. Note since G is abelian it has $|G| = 3$ conjugacy classes. Consider the character table

	$\{1\}$	$\{x\}$	$\{x^2\}$
χ_0	1	1	1
χ_1	1	ζ	ζ^2
χ_2	1	ζ^2	$\zeta^4 = \zeta$

One verifies that

$$\begin{aligned}
\langle \chi_0, \chi_1 \rangle &= \frac{1}{3}\left(1 \cdot 1 + 1 \cdot \overline{\zeta} + 1 \cdot \overline{\zeta^2}\right) = \frac{1}{3}\left(1 + \zeta^2 + \zeta\right) = 0, \\
\langle \chi_1, \chi_1 \rangle &= \frac{1}{3}\left(1 \cdot 1 + \zeta \cdot \overline{\zeta} + \zeta^2 \cdot \overline{\zeta^2}\right) = \frac{1}{3}\left(1 + \zeta^3 + \zeta^3\right) = 1, \\
\langle \chi_2, \chi_1 \rangle &= \frac{1}{3}\left(1 \cdot 1 + \zeta^2 \cdot \overline{\zeta} + \zeta \cdot \overline{\zeta^2}\right) = \frac{1}{3}\left(1 + \zeta^4 + \zeta^2\right) = 0.
\end{aligned}$$

Example 7.5.4. $G = C_n$ and ζ is a primitive nth root of unity. Generalising from example above, one sees the character table is now an $n \times n$ matrix whose (i, j)th entry (counting from zero) is ζ^{ij}, $0 \leq i, j < n$. (Known as the Vandermonde matrix.)

Example 7.5.5. $G = S_3$, $S = \{1, 2, 3\}$ and V is the corresponding permutation representation (note $\dim V = 3$). We've seen in Example 1.3.2 the 1-d representation sign with character

$$\chi_{\text{sign}}(e) = 1, \qquad \chi_{\text{sign}}((12)) = -1, \qquad \chi_{\text{sign}}((123)) = 1.$$

Now let $U := \mathbb{C}(e_1 + e_2 + e_3)$ and consider V/U with basis $(e_1 + U, e_2 + U)$ (and $e_3 = -e_1 - e_2$), then

$$\rho_{V/U}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho_{V/U}((12)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \rho_{V/U}((123)) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$$\text{tr} = 2 \qquad\qquad \text{tr} = 0 \qquad\qquad \text{tr} = -1$$

We can use 7.4.9 to check that V/U is irreducible:

$$\langle \chi_{V/U}, \chi_{V/U} \rangle = \frac{1}{6}\left(2^2 + 3 \times 0^2 + 2 \times (-1)^2\right) = \frac{1}{6} \times 6 = 1.$$

Verify 7.5.1: $2^2 + 1^2 + 1^2 = 6$.

7.5.1 The Wedderburn isomorphism

Definition 7.5.6. A $\mathbb{C}$-algebra A is a $\mathbb{C}$-vector space and a ring such that the scalar multiplication and ring multiplication are compatible, i.e. $\exists$ an injective ring homomorphism $\iota : \mathbb{C} \to A$ with

$$\alpha \cdot_{\mathbb{C}} a = \iota(\alpha) \cdot_A a \qquad \forall \alpha \in \mathbb{C}, a \in A.$$

Example 7.5.7. Let $\text{End}(V) := \text{Hom}(V, V)$, which is a $\mathbb{C}$-algebra via $\iota(\alpha) = \alpha I_V$. Note $GL(V) \subsetneq \text{End}(V)$. Also $\mathbb{C}G$ is a $\mathbb{C}$-algebra via the product

$$\left(\sum_{g \in G} \alpha_g g\right)\left(\sum_{h \in G} \beta_h h\right) = \sum_{g' \in G, gh = g'} (\alpha_g \beta_h) g',$$

the 'linear continuation' of action of G on regular representation $\mathbb{C}G$.

Theorem 7.5.8 (Wedderburn's)**.** Let $W_1, \ldots, W_k \in$ Mod-G be a complete list of pairwise nonisomorphic irreducibles and

$$\begin{aligned}
f : \mathbb{C}G &\to \text{End}(W_1) \times \cdots \times \text{End}(W_k) \\
g &\mapsto \left(\rho_{W_1}(g), \ldots, \rho_{W_k}(g)\right).
\end{aligned}$$

Then f is an isomorphism of $\mathbb{C}$-algebras.

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Remark. Let V be a $\mathbb{C}$-algebra and a G-representation whose group action is compatible with the ring multiplication $\cdot_V$ as follows:

$$(gh)1_V = (g1_V) \cdot_V (h1_V).$$

A G-homomorphism from $\mathbb{C}G$ with $f(1_{\mathbb{C}G}) = 1_V$ is always a ring homomorphism, hence a $\mathbb{C}$-algebra homomorphism, since

$$\begin{aligned} f\left(\left(\sum_{g\in G}\alpha_g g\right)\left(\sum_{h\in H}\beta_h h\right)\right) &= f\left(\sum_{g,h\in G}(\alpha_g\beta_h)gh\right) = \sum_{g,h\in G}\alpha_g\beta_h f(gh) \\ &= \sum_{g,h\in G}\alpha_g\beta_h f(g)f(h) = \left(\sum_{g\in G}\alpha_g f(g)\right)\left(\sum_{h\in G}\beta_h f(h)\right) \\ &= f\left(\sum_{g\in G}\alpha_g g\right) f\left(\sum_{h\in G}\beta_h h\right). \end{aligned}$$

Proof of 7.5.8. f is a linear map and a G-morphism, hence a $\mathbb{C}$-algebra morphism. By 7.5.1, the dimensions are equal so by 1.5.3 it suffices to show either injectivity or surjectivity. Consider $a = \sum_{g\in G}\alpha_g g \in \ker f$. Then

$$\forall i \in \{1,\dots,k\},\ \sum_{g\in G}\alpha_g \rho_{W_i}(g) =: \rho_{W_i}(a) = 0,$$

i.e. $\forall w \in W_i,\ \rho_{W_i}(a)(w) = 0$. By construction of W_i's and Maschke's theorem (7.3.2), one has $\forall V \in$ Mod-G, $\rho_V(a) = 0$. In particular for $V = \mathbb{C}G$, $\forall b \in \mathbb{C}G,\ a \cdot_{\mathbb{C}G} b = 0$, hence $a = a \cdot_{\mathbb{C}G} 1_G = 0$. ☐

Definition 7.5.9. The *centre* of a $\mathbb{C}$-algebra A is the linear subspace $Z(A) \subseteq A$ defined as

$$Z(A) = \{a \in A : ab = ba\ \forall b \in A\}.$$

Notation. $\mathrm{Cl}_G :=$ {conjugacy classes in G}.

Proposition 7.5.10. $\dim Z(\mathbb{C}G) = |\mathrm{Cl}_G|$.

Proof. First note that $\forall b \in \mathbb{C}G,\ ab = ba \iff \forall h \in G,\ ah = ha \iff \forall h \in G,\ hah^{-1} = a$. Write $a = \sum_{g\in G}\alpha_g g$. One has $hah^{-1} = a$ iff

$$\sum_{g\in G}\alpha_g g = \sum_{g\in G}\alpha_g hgh^{-1} = \sum_{g'\in G}\alpha_{h^{-1}g'h}g' \iff \forall g \in G,\ \alpha_g = \alpha_{h^{-1}gh},$$

so $a \in Z(G) \iff \alpha : G \to \mathbb{C}$ is constant on conjugacy classes. The vector space of such α hence has dimension $|\mathrm{Cl}_G|$. ☐

Corollary 7.5.11. The number of pairwise nonisomorphic irreducible representations of G equals $|\mathrm{Cl}_G|$.

Proof. By 7.5.8 one has

$$\dim Z(\mathbb{C}G) = |\mathrm{Cl}_G| = \dim Z(\mathrm{End}(W_1) \times \cdots \times \mathrm{End}(W_1)).$$

Note that $Z(\mathrm{End}(W)) = \mathbb{C}I_w$ (the only matrices that commute with any other matrix are the ones that are diagonal with same entries on the diagonal), which is 1-dimensional. More generally,

$$Z(\mathrm{End}(W_1) \times \cdots \times \mathrm{End}(W_1)) = Z(\mathrm{End}(W_1)) \times \cdots \times Z(\mathrm{End}(W_k))$$

which is k-dimensional. ☐

Notation. $\mathbb{C}^{\mathrm{Cl}_G} = \{f : \mathrm{Cl}_G \to \mathbb{C}\}$, which we identify with the set of class functions $\mathbb{C}^G$.

Corollary 7.5.12. The characters of irreducible representations of G form a basis of vector space $\mathbb{C}^{\mathrm{Cl}_G}$.

Proof. By 7.4.4, the irreducibles characters are linearly independent, and by 7.5.11 the number of such characters equals $\dim \mathbb{C}^{\mathrm{Cl}_G} = |\mathrm{Cl}_G|$. ☐

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7.5.2 Character tables

Definition 7.5.13. The *character table* of G is the square matrix whose columns are indexed by conjugacy classes $\mathrm{Cl}_G(g_i)$ and rows are index by W_j with entries $\chi_{W_j}(g_i)$.

Example 7.5.14. The character table of S_3 (the subscripts indicate sizes of conjugacy classes):

S_3	id_1	$(12)_3$	$(123)_2$
triv	1	1	1
sign	1	-1	1
$\langle e_1, e_2, e_3\rangle / \mathbb{C}(e_1+e_2+e_3)$	2	0	-1

Theorem 7.4.4 tells us if one multiplies each column g in the table by $\sqrt{\frac{|\mathrm{Cl}_G(g)|}{|G|}}$ one obtains a matrix A with orthogonal rows of norm 1 (in the sense of standard Hermitian inner product $\langle v, w\rangle := \sum_{i=1}^n v_i\overline{w_i}$ for $v, w \in \mathbb{C}^n$), i.e. orthonormal rows:

$G = S_3$	1	x	x^2
triv	$\frac{1}{\sqrt{6}}$	$\frac{3}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$
sign	$\frac{1}{\sqrt{6}}$	$-\frac{3}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$
U/V	$\frac{2}{\sqrt{6}}$	0	$-\frac{2}{\sqrt{6}}$

Proposition 7.5.15. A matrix A with orthonormal rows also has orthonormal columns.

Proof. For a matrix A with orthonormal rows, let $A^\dagger$ denote its conjugate transpose. One has

$$(AA^\dagger)_{i,j} = \sum_{l=1}^k A_{i,l}A^\dagger_{l,j} = \sum_{l=1}^k A_{i,l}\overline{A_{j,l}} = \langle A_{\text{row }i}, A_{\text{row }j}\rangle = \delta_{i,j},$$

so $A^\dagger = A^{-1}$. But conversely,

$$\delta_{i,j} = (A^{-1}A)_{i,j} = (A^\dagger A)_{i,j} = \left\langle A^\dagger_{\text{row }i}, A^\dagger_{\text{row }j}\right\rangle = \langle \overline{A_{\text{col }i}}, \overline{A_{\text{col }j}}\rangle = \langle A_{\text{col }i}, A_{\text{col }j}\rangle\,.$$

□

Definition 7.5.16. Matrices A with $A^\dagger = A^{-1}$ are *unitary*.

Corollary 7.5.17 (Orthogonal columns)**.**

$$\forall g \in G,\ \sum_\chi \chi(g)\overline{\chi(g)} = \frac{|G|}{|\mathrm{Cl}_G(g)|}$$

where the sum is over all irreducible characters χ. If g_1 and g_2 are not conjugates then

$$\sum_\chi \chi(g_1)\overline{\chi(g_2)} = 0.$$

Proof. Rescaling every column of the character table T by $\sqrt{\frac{|\mathrm{Cl}_G(g)|}{|G|}}$ gives a matrix A with orthonormal rows by 7.4.4, hence orthonormal columns by 7.5.15. □

7.6 The isotypic decomposition

Theorem 7.6.1. Let $W_1, \ldots, W_k$ be a complete list of pairwise nonisomorphic irreducibles of G. For a fixed $i \in \{1, \ldots, k\}$, let

$$a_i := \frac{\dim W_i}{|G|}\sum_{g\in G}\overline{\chi_{W_i}(g)}g \in \mathbb{C}G.$$

and let $V \in \text{Mod-}G$. Consider the decomposition into irreducibles

$$V = \bigoplus_{l=1}^k \underbrace{\bigoplus_{j=1}^{\mathrm{mult}_{W_l}(V)} U_{l,j}}_{V_l} \qquad \text{with each } U_{l,j} \sim W_l.$$

Then $\rho_V(a_i) \in \mathrm{End}(V)$ is the projection onto V_i. In particular, the space V_i is independent of the finer decomposition of V into the $U_{l,j}$.

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Proof. Fix $i \in \{1, \dots, k\}$ and let $U \in$ Mod-G be irreducible such that $U \sim W_j$. Consider $\rho_U(a_i) \in \mathrm{End}(U)$. We claim $a_i \in Z(\mathbb{C}G)$. Indeed, for $h \in G$,

$$\begin{aligned} ha_i &= h\frac{\dim W_i}{|G|}\sum_{g\in G}\overline{\chi_{W_i}(g)}g = \frac{\dim W_i}{|G|}\sum_{g\in G}\overline{\chi_{W_i}(g)}hg = \frac{\dim W_i}{|G|}\sum_{g\in G}\overline{\chi_{W_i}(h^{-1}gh)}hh^{-1}gh \\ &= \frac{\dim W_i}{|G|}\sum_{g\in G}\overline{\chi_{W_i}(g)}gh = a_i h, \end{aligned}$$

and therefore $\rho_U(h)\rho_U(a_i) = \rho_U(ha_i) = \rho_U(a_ih) = \rho_U(a_i)\rho_U(h)$, i.e. $\rho_U(a_i) \in \mathrm{End}_G(U)$. By 6.5.2, $\rho_U(a_i) = \lambda_{i,j} I_U$ for some $\lambda_{i,j} \in \mathbb{C}$. Note

$$\lambda_{i,j}\dim U = \mathrm{tr}(\rho_U(a_i)) = \frac{\dim W_i}{|G|}\sum_{g\in G}\overline{\chi_{W_i}(g)}\underbrace{\mathrm{tr}(\rho_U(g))}_{\chi_{W_j}(g)} = \dim W_i\left\langle \chi_{W_j}, \chi_{W_i}\right\rangle = \dim W_i \delta_{i,j},$$

and note that if $i = j$ then $\dim U = \dim W_j = \dim W_i$, so $\lambda_{i,j} = \delta_{i,j}$.
Hence, if we take a basis of V that respects the decomposition

$$V = \bigoplus_{l=1}^{k}\bigoplus_{j=1}^{\mathrm{mult}_{W_l}(V)} U_{l,j},$$

then $\rho_V(a_i)$ is a block diagonal matrix, one block for each $U_{l,j}$ and it is the zero matrix for all $i \neq l$ and is identity for all $U_{i,j}$. This is the projection to $\bigoplus_j U_{i,j} = V_i$. ☐

Example 7.6.2. For W_0 being the trivial representation, one has

$$a_0 = \frac{1}{|G|}\sum_{g\in G} g \in \mathbb{C}G,$$

the projection to the invariant space.

Example 7.6.3. Let $G = C_2 = \left\langle x \mid x^2 = 1\right\rangle$, $V = \mathbb{C}^{2\times 2}$ with the action $xA = A^T$. Then

$$a_{\mathrm{triv}} = \frac{1}{2}(1+x), \qquad a_{\mathrm{sign}} = \frac{1}{2}(1-x)$$

so in particular if A is symmetric then $a_{\mathrm{triv}}A = \frac{1}{2}\left(A + A^T\right) = A$ (i.e. the 3-dimensional space of symmetric matrices is invariant under a_{triv}) and $a_{\mathrm{sign}}A = \frac{1}{2}\left(A - A^T\right) = 0$. But if B is any matrix then $a_{\mathrm{triv}}B$ will be symmetric, so a_{triv} is idempotent, hence a projection. Similar for a_{sign}, it's a projection to the 1-dimensional space of skew-symmetric matrices (matrices of the form $\begin{pmatrix} 0 & a \\ -a & 0\end{pmatrix}$).

Definition 7.6.4. Theorem 7.6.1 gives a decomposition $V = \bigoplus_{i=1}^k V_i$. We call V_i an *isotypic component*, which are unique up to reordering of the summands. A representation that contains only on nonzero isotypic component is *isotypic*.

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8 Induced representation

Definition 8.0.1. Let $H \leq G$ be a subgroup and let $V \in$ Mod-G. Then H acts linearly on V and we denote the corresponding $\mathbb{C}H$-module by $V\downarrow_H^G \in$ Mod-H, called the *restriction* of V.
We write $\chi_V\downarrow_H^G := \chi_{V\downarrow_H^G}$.

Note that if $V \in \text{Mod-}G$ is irreducible then $V \downarrow_H^G$ might not be irreducible. For example, if $\dim V = 2$ and $H = \{e\}$ is the trivial group.

In the following, let $H \leq G$ and fix a set of coset representatives $t_1, \dots, t_l : G = t_1H \sqcup t_2H \sqcup \dots \sqcup t_lH$. The set $\{t_1, \dots, t_l\}$ is called a *transversal.*

Definition 8.0.2 (The coset module)**.** Let $\mathcal{H} = \{t_1H, \dots, t_lH\}$. The group G acts on $\mathcal{H}$ via $g(t_iH) := (gt_i)H$. Let $\mathbb{C}\mathcal{H} \in \text{Mod-}G$ denote the corresponding permutation representation, called the coset module.

Example 8.0.3. Let $G = S_3,\ H = \{\text{id}, (23)\}$ and $\mathcal{H} = \{H, (12)H, (13)H\}$. Then

$$\mathbb{C}\mathcal{H} = \{\alpha_1 H + \alpha_2(12)H + \alpha_3(13)H : \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}\}.$$

We determine $\rho_{\mathbb{C}\mathcal{H}}((12)) \in GL_3(\mathbb{C})$ with respect to the basis $\mathcal{H}$:

$$\begin{aligned} (12)H &= (12)H \\ (12)(12)H &= H \\ (12)(13)H = (132)H = (132)(23)H &= (13)H \end{aligned}$$

since $(23) \in H$, so the matrix is

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Definition 8.0.4. If $\rho : H \to GL_n(\mathbb{C})$ is a H-representation, define $\rho \uparrow_H^G : G \to \text{End}(\mathbb{C}^{nl})$ via

$$\rho \uparrow_H^G (g) := \begin{pmatrix} \rho(t_1^{-1}gt_1) & \cdots & \rho(t_1^{-1}gt_l) \\ \vdots & \ddots & \vdots \\ \rho(t_l^{-1}gt_1) & \cdots & \rho(t_l^{-1}gt_l) \end{pmatrix}$$

where $\rho(g) = 0$ if $g \notin H$.

$\rho \uparrow_H^G$ is called the *induced representation* of ρ.

Proposition 8.0.5. Let $1 : H \to GL_1(\mathbb{C})$ denote the trivial representation of H. Then $1 \uparrow_H^G \in \text{Mod-}G$ and one has $1 \uparrow_H^G \sim \mathbb{C}\mathcal{H}$.

Proof. Let $\rho := \rho_{1\uparrow_H^G}$ and $\psi := \rho_{\mathbb{C}\mathcal{H}}$. We claim that $\forall g \in G,\ \rho(g) = \psi(g)$. Note $\forall g \in G$, both $\rho(g)$ and $\psi(g)$ contain only 0s and 1s. Now $\forall g \in G$:

$$\rho(g)_{i,j} = 1 \iff t_i^{-1}gt_j \in H \iff g(t_jH) = t_iH \iff \psi(g)_{i,j} = 1.$$

$\square$

Theorem 8.0.6. $\rho \uparrow_H^G : G \to GL_{nl}(\mathbb{C})$ is a matrix representation.

Proof. We prove that $\rho \uparrow_H^G (g)$ is a block matrix whose coarse structure is a permutation matrix, i.e. in every row and column of blocks there is exactly one nonzero block. Now for the jth column, the blocks are $\rho(t_1^{-1}gt_j), \rho(t_2^{-1}gt_j), \dots, \rho(t_l^{-1}gt_j)$. But $t_i^{-1}gt_j \in H \iff gt_j \in t_iH$ which is true for exactly one i since the t_iH's form a disjoint union of G. Analogously for rows. It's also easy to check $\rho \uparrow_H^G (e) = I_{nl}$ since $t_i^{-1}t_j \in H \iff t_j \in t_iH \iff i = j$. It remains to prove $\forall g, h \in G$,

$$\rho \uparrow_H^G (gh) = \rho \uparrow_H^G (g) \rho \uparrow_H^G (h).$$

Consider the (i, j)th block on both sides, it suffices to prove

$$\sum_{k=1}^{l} \rho(\underbrace{t_i^{-1}gt_k}_{a_k})\rho(\underbrace{t_k^{-1}ht_j}_{b_k}) = \rho(\underbrace{t_i^{-1}ght_j}_{c}). \tag{$*$}$$

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Note $\forall k,\ a_kb_k = t_i^{-1}gt_kt_k^{-1}ht_j = t_i^{-1}ght_j = c$.

If $\rho(c) = 0$ then $c \notin H$ so either $a_k \notin H$ or $b_k \notin H\ \forall k$, i.e. $\rho(a_k) = 0$ or $\rho(b_k) = 0\ \forall k$, thus $\sum_k \rho(a_k)\rho(b_k) = 0$, which proves $*$.

If $\rho(c) \neq 0$ then let m be the unique index with $a_m \in H$ (see previous block structure argument), then $b_m = a_m^{-1}c \in H$ and $\sum_k \rho(a_k)\rho(b_k) = \rho(a_m)\rho(b_m) = \rho(a_mb_m) = \rho(c)$ since ρ is representation of H. $\square$

Theorem 8.0.7. A priori the construction process of $\rho\uparrow_H^G$ depends on the set of coset representations. Consider $\rho\uparrow_H^{G,t}$ and $\rho\uparrow_H^{G,s}$ constructed from $\rho : H \to GL(V)$ using two sets of coset representations $t = (t_1, \dots, t_l)$ and $s = (s_1, \dots, s_l)$ respectively:

$$G = t_1H \sqcup \cdots \sqcup t_lH = s_1H \sqcup \cdots \sqcup s_lH,$$

then $\rho\uparrow_H^{G,t} \sim \rho\uparrow_H^{G,s}$.

Proof. By 7.4.6 it suffices to show $\chi\uparrow_H^{G,t} = \chi\uparrow_H^{G,s}$. One has

$$\chi\uparrow_H^{G,t} = \sum_{i=1}^{l} \operatorname{tr}(\rho(t_i^{-1}gt_i)) = \sum_{i=1}^{l} \chi(t_i^{-1}gt_i) \tag{8.0.7.1}$$

and similarly

$$\chi\uparrow_H^{G,s} = \sum_{i=1}^{l} \chi(s_i^{-1}gs_i). \tag{8.0.7.2}$$

Now note that $t_iH = s_iH\ \forall i$ (after relabelling), which implies $\forall i,\ \exists h_i \in H : t_i = s_ih_i$, so

$$t_i^{-1}gt_i = h_i^{-1}s_i^{-1}gs_ih_i,$$

which means

- $t_i^{-1}gt_i \in H$ iff $s_i^{-1}gs_i \in H$
- when both in H, they are conjugate

Hence $\chi(t_i^{-1}gt_i) = \chi(s_i^{-1}gs_i)$. □

Lemma 8.0.8. Let $\rho \in$ Mod-H with character χ. Then

$$\chi\uparrow_H^G(g) = \frac{1}{|H|}\sum_{x\in G}\chi(x^{-1}gx)$$

where $\chi(g) = 0$ if $g \notin H$.

Proof. Cf. proof of 8.0.7. Observe

$$\chi(t_i^{-1}gt_i) = \frac{1}{|H|}\sum_{h\in H}(h^{-1}t_i^{-1}gt_ih)$$

which, plugged into 8.0.7.1, gives

$$\chi\uparrow_H^G(g) = \frac{1}{|H|}\sum_{i\in\{1,\dots,l\},h\in H}\chi(h^{-1}t_i^{-1}gt_ih)$$

but by going through all the i's (all the cosets) and $h \in H$ (all elements in the subgroup), t_ih gives us precisely all elements of G, hence

$$\chi\uparrow_H^G(g) = \frac{1}{|H|}\sum_{x\in G}\chi(x^{-1}gx).$$

□

Theorem 8.0.9 (Frobenius reciprocity)**.** Let $H \leq G$ and let ψ, χ be characters of H and G respectively. Then

$$\left\langle \psi\uparrow_H^G, \chi\right\rangle = \left\langle \psi, \chi\downarrow_H^G\right\rangle.$$

Proof.

$$
\begin{aligned}
\langle \psi \uparrow_H^G, \chi \rangle &= \frac{1}{|G|} \sum_{g \in G} \psi \uparrow_H^G (g) \chi(g^{-1}) \\
&= \frac{1}{|G| \cdot |H|} \sum_{x \in G} \sum_{g \in G} \psi(x^{-1} g x) \chi(g^{-1}) \qquad \text{by 8.0.8} \\
&= \frac{1}{|G| \cdot |H|} \sum_{x \in G} \sum_{y \in G} \psi(y) \chi(x y^{-1} x^{-1}) \qquad \text{writing } y = x^{-1} g x \\
&= \frac{1}{|G| \cdot |H|} \sum_{x \in G} \sum_{y \in G} \psi(y) \chi(y^{-1}) \qquad \text{by 4.3.1.4} \\
&= \frac{1}{|G| \cdot |H|} |G| \sum_{y \in G} \psi(y) \chi(y^{-1}) = \frac{1}{|H|} \sum_{y \in G} \psi(y) \chi(y^{-1}) \qquad \text{independence of } x \\
&= \frac{1}{|H|} \sum_{y \in H} \psi(y) \chi(y^{-1}) \qquad \text{since } \psi(y) = 0 \text{ if } y \notin H \\
&= \langle \psi, \chi \downarrow_H^G \rangle .
\end{aligned}
$$

□

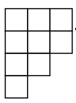
9 An in-depth example: the symmetric group S_n

9.1 Young subgroup, tableau, tabloid

Definition 9.1.1. A *partition* λ of n is a list $(\lambda_1, \dots, \lambda_l) \in \mathbb{N}^l$ with $\lambda_1 \geq \cdots \geq \lambda_l > 0$ with $\sum_{i=1}^{l} \lambda_i = n$. One writes $\lambda \vdash n$. The number $l(\lambda) = l$ is the *length* of λ and $\lambda_i = 0$ for $i > l(\lambda)$.

Week 8, lecture 1 starts here

We have seen that # conjugacy classes in S_n = # partitions of n.

Definition 9.1.2. For each partition λ we can draw its *Ferrers (or Young) diagram*, for example for $\lambda = (3, 3, 2, 1)$ (or $(3^2, 2, 1)$) the diagram is .

Notation. For a set A write $S_A := \{\pi : A \to A \text{ bijective}\}$. In particular $S_n = S_{\{1,\dots,n\}}$.

Definition 9.1.3. Let $\lambda \vdash n$. The *Young subgroup* $S_\lambda \leq S_n$ is

$$S_\lambda = S_{\{1,2,\dots,\lambda_1\}} \times S_{\{\lambda_1+1,\dots,\lambda_1+\lambda_2\}} \times \cdots \times S_{\{n-\lambda_l+1,\dots,n\}}.$$

Example 9.1.4.

$$S_{\{3,3,2,1\}} = S_{\{1,2,3\}} \times S_{\{4,5,6\}} \times S_{\{7,8\}} \times S_{\{9\}}.$$

In general,

$$S_\lambda \cong S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_l}.$$

Now consider $1 \uparrow_{S_\lambda}^{S_n}$. If $\pi_1, \dots, \pi_k$ is a transversal, then S_n acts linearly on

$$V^\lambda = \operatorname{linspan}\{\pi_1 S_\lambda, \dots, \pi_k S_\lambda\}$$

and one has $V^\lambda \sim 1 \uparrow_{S_\lambda}^{S_n}$. See 8.0.5.

Definition 9.1.5. Let $\lambda \vdash n$. A *Young tableau* (or just *tableau*) t of shape λ is an array obtained by writing numbers $1, 2, \dots, n$ into the boxes of the Young diagram of λ, each number exactly once.

The shape $\operatorname{sh}(t)$ of a Young tableau is the partition associated to its Young diagram, e.g.

$$\operatorname{sh}\left(\begin{array}{|c|c|c|} \hline 2 & 1 & 4 \\ \hline 5 & 3 \\ \cline{1-2} \end{array} \right) = (3, 2).$$

A Young tableau of shape λ is also called a λ-tableau. For $\lambda \vdash n$ there are $n!$ λ-tableaux.

Let $t_{i,j}$ denote the entry of t at position i, j.

Definition 9.1.6. Two λ-tableaux are *row-equivalent*, denoted $t_1 \sim t_2$, if the corresponding rows contain the same elements. An equivalence class of this is a *tabloid* of shape λ or λ-tabloid, denoted $\{t_1\}$ (so $t_1 \sim t_2 \implies \{t_1\} = \{t_2\}$). We use lines between rows to denote tabloids:

$$\begin{array}{c}\hline 2\ 1\ 4 \\ \hline 5\ 3 \\ \hline\end{array} = \begin{array}{c}\hline 4\ 2\ 1 \\ \hline 5\ 3 \\ \hline\end{array} = \begin{array}{c}\hline 1\ 2\ 4 \\ \hline 3\ 5 \\ \hline\end{array} = \cdots$$

$\pi \in S_n$ acts on a Young tableau t via $(\pi t)_{i,j} = \pi(t_{i,j})$, which induces an action on tabloids also: $\pi\{t\} = \{\pi t\}$.

Definition 9.1.7. Let $\lambda \vdash n$ and $\{t_1\}, \ldots, \{t_k\}$ a complete list of λ-tabloids. Define

$$M^\lambda := \operatorname{linspan}\{\{t_1\}, \ldots, \{t_k\}\},$$

the *permutation module* corresponding to λ.

Example 9.1.8. Consider $\lambda = (n)$, giving one-row Young tableaux. Then $M^{(n)} = \mathbb{C}\left\{\begin{array}{c}\hline 1\ 2\ \cdots\ n \\ \hline\end{array}\right\}$ with the trivial action.
Now consider $\lambda = (1^n)$, giving one-column Young tableaux. Then $M^{(1^n)} \sim \mathbb{C}S_n$.
Let $\lambda = (n-1, 1)$. Then each tabloid is uniquely defined by the entry at position $(2, 1)$, hence $M^{(n-1,1)}$ is isomorphic to the permutation representation of S_n on the set $\{1, 2, \ldots, n\}$ defined via $\pi \cdot i = \pi(i)$.

Proposition 9.1.9. $M^\lambda \sim V^\lambda$.

Proof. Fix the Young tableau t^λ that has row-wise consecutive increasing numbers from left to right, e.g.

$$t^{(4,2,1)} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}$$

and let $\pi_1, \ldots, \pi_k$ be a transversal for S_λ. Define $\theta : V^\lambda \to M^\lambda : \pi_i S_\lambda \mapsto \pi_i t^\lambda$. It is easy to verify that θ is an isomorphism of S_n-representations. $\square$

Week 8, lecture 2 starts here

9.2 Dominance and lexicographic ordering

Definition 9.2.1. A *partial order* on a set A is a relation $\leq$ such that

1. $\forall a \in A,\ a \leq a$ — reflexivity
2. $\forall a, b \in A,\ a \leq b, b \leq a \implies a = b$ — antisymmetry
3. $\forall a, b, c \in A,\ a \leq b, b \leq c \implies a \leq c$ — transitivity

and one says A is a *partially ordered set*, or *poset.* If in addition $\forall a, b \in A$ either $a \leq b$ or $b \leq a$, then $\leq$ is a *total order.*

Definition 9.2.2. Let $\lambda, \mu \vdash n$. Then λ *dominates* μ, denoted $\lambda \trianglerighteq \mu$, if

$$\forall k,\ \sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i.$$

For example, $(3, 3) \trianglerighteq (2, 2, 1, 1)$. Note it's not a total order, e.g. $(3, 3)$ and $(4, 1, 1)$ are incomparable.

Definition 9.2.3. Let A be a poset. For $b, c \in A$, one says c *covers* b if $c > b$ (meaning $c \geq b$ and $c \neq b$) and $\nexists d \in A : b < d < c$.
The *Hasse diagram* consists of

- a vertex for each $a \in A$
- an arrow from b to c if c covers b

For example,

$$\begin{tikzcd}
 & (6) & \\
 & (5,1) \arrow[u] & \\
 & (4,2) \arrow[u] & \\
(3^2) \arrow[ur] & & (4,1^2) \arrow[ul] \\
 & (3,2,1) \arrow[ul] \arrow[ur] & \\
(3,1^3) \arrow[ur] & & (2^3) \arrow[ul] \\
 & (2^2,1^2) \arrow[ul] \arrow[ur] & \\
 & (2,1^4) \arrow[u] & \\
 & (1^6) \arrow[u] &
\end{tikzcd}$$

Lemma 9.2.4 (Dominance lemma for partitions)**.** Let $\lambda, \mu \vdash n$ and t^λ and s^μ be Young tableaux of shape λ and μ respectively. If for each i the elements of row i of s^μ are all in different columns in t^λ, then $\lambda \trianglerighteq \mu$.

Proof. We can sort the entries in each column of t^λ so that the elements of the rows $1, 2, \ldots, i$ of s^μ all occur in the first i rows of t^λ. Let $E_i(t)$ denote the set of elements in the first i rows of t. Then

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i = |E_i(t^\lambda)| \geq |E_i(t^\lambda) \cap E_i(s^\mu)| = |E_i(s^\mu)| = \mu_1 + \mu_2 + \cdots + \mu_i,$$

i.e. $\lambda \trianglerighteq \mu$. □

Definition 9.2.5. Let $\lambda, \mu \vdash n$. One writes $\lambda < \mu$ if one has for some i

1. $\forall j < i,\ \lambda_j = \mu_j$
2. $\lambda_i < \mu_i$

This is the *lexicographic order*, which is a total order.
For example, $(1^6) < (2,1^4) < (2^2,1^2) < (2^3) < (3,1^3) < (3,1,2) < (3,3) < (4,1^2) < (4,2) < (5,1) < (6)$.

Proposition 9.2.6 (Lexicographic order is a refinement of dominance)**.** Let $\lambda, \mu \vdash n$. If $\lambda \trianglerighteq \mu$ then $\lambda \geq \mu$.

Proof. If $\lambda = \mu$ then we are done, so suppose $\lambda \neq \mu$ and find the smallest i with $\lambda_i \neq \mu_i$, so in particular $\forall k < i,\ \sum_{j=1}^k \lambda_j = \sum_{j=1}^k \mu_j$ and since $\lambda \trianglerighteq \mu$ one has $\sum_{j=1}^i \lambda_j > \sum_{j=1}^i \mu_j$, so $\lambda_i > \mu_i$ and hence $\lambda > \mu$. □

9.3 Specht module

Definition 9.3.1. For a tableaux t with rows $R_1, \ldots, R_l$ and columns $C_1, \ldots, C_k$, define the *row-stabiliser*

$$R_t := S_{R_1} \times S_{R_2} \times \cdots \times S_{R_l}$$

and the *column-stabiliser*

$$C_t := S_{C_1} \times \cdots \times S_{C_k}.$$

Example 9.3.2. For $t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}$, one has $R_t = S_{\{1,2,4\}} \times S_{\{3,5\}}$ and $C_t = S_{\{3,4\}} \times S_{\{1,5\}} \times S_{\{2\}}$.

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Remark. Note that we can identify the tabloid $\{t\}$ with the right coset $R_t t$.

Notation. For any subset $H \subseteq S_n$, define the elements in the group algebra

$$H^+ := \sum_{\pi \in H} \pi, \qquad H^- := \sum_{\pi \in H} \operatorname{sgn}(\pi)\pi,$$

in particular, define $\kappa_t := C_t^-$.

Observe that if t has columns $C_1, \dots, C_k$, then $\kappa_t = \kappa_{C_1}\kappa_{C_2}\cdots\kappa_{C_k}$.

Definition 9.3.3. For a tableau t of shape λ, the *associated polytabloid* $e_t \in M^\lambda$ is $e_t := \kappa_t\{t\}$.

Example 9.3.4. For $t = \begin{array}{|c|c|c|}\hline 4&1&2\\\hline 3&5\\\cline{1-2}\end{array}$, one has

$$\kappa_t = (\mathrm{id} - (3,4))(\mathrm{id} - (1,5)) = \mathrm{id} - (3,4) - (1,5) + (3,4)(1,5),$$

so

$$\begin{aligned} e_t &= \begin{array}{ccc}\hline 4&1&2\\\hline 3&5\\\cline{1-2}\end{array} - \begin{array}{ccc}\hline 3&1&2\\\hline 4&5\\\cline{1-2}\end{array} - \begin{array}{ccc}\hline 4&5&2\\\hline 3&1\\\cline{1-2}\end{array} + \begin{array}{ccc}\hline 3&5&2\\\hline 4&1\\\cline{1-2}\end{array} \\ &= \begin{array}{ccc}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array} - \begin{array}{ccc}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} - \begin{array}{ccc}\hline 2&4&5\\\hline 1&3\\\cline{1-2}\end{array} + \begin{array}{ccc}\hline 2&3&5\\\hline 1&4\\\cline{1-2}\end{array}. \end{aligned}$$

Definition 9.3.5. For any partition λ, the *Specht module* S^λ is defined as the submodule of M^λ spanned by the polytabloids e_t where $\operatorname{sh}(t) = \lambda$.

Lemma 9.3.6. Let t be a tableau and π a permutation. Then

1. $R_{\pi t} = \pi R_t \pi^{-1}$
2. $C_{\pi t} = \pi C_t \pi^{-1}$
3. $\kappa_{\pi t} = \pi \kappa_t \pi^{-1}$
4. $e_{\pi t} = \pi e_t$

Proof. 1. $\sigma \in R_{\pi t} \iff \sigma\{\pi t\} = \{\pi t\} \iff \sigma\pi\{t\} = \pi\{t\} \iff \pi^{-1}\sigma\pi\{t\} = \{t\} \iff \pi^{-1}\sigma\pi \in R_t \iff \sigma \in \pi R_t \pi^{-1}$.

2, 3. Similar.

4. $e_{\pi t} = \kappa_{\pi t}\{\pi t\} = \pi\kappa_t\pi^{-1}\{\pi t\} = \pi\kappa_t\{t\} = \pi e_t$.

$\square$

Example 9.3.7. $S^{(n)} \subseteq M^{(n)}$ is the trivial representation.

Example 9.3.8. Let $\lambda = (1^n)$ and $t = \begin{array}{|c|}\hline 1\\\hline 2\\\hline \vdots\\\hline n\\\hline\end{array}$. Then $\kappa_t = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\sigma$. For $\pi \in S_n$, by Lemma 9.3.6 one has

$$e_{\pi t} = \pi e_t = \sum_{\sigma \in G} \operatorname{sgn}(\sigma)\pi\sigma\{t\},$$

replacing $\pi\sigma$ by τ one has

$$e_{\pi t} = \sum_{\tau \in S_n} \operatorname{sgn}(\pi^{-1}\tau)\tau\{t\} = \operatorname{sgn}(\pi^{-1}) \sum_{\tau \in S_n} \operatorname{sgn}(\tau)\tau\{t\} = \operatorname{sgn}(\pi)e_t,$$

thus every polytabloid is a multiple of e_t, hence $S^{(1^n)} = \mathbb{C}e_t$ and $\pi e_t = \operatorname{sgn}(\pi)e_t$ (therefore this is the sign representation).

Example 9.3.9. Let $\lambda = (n-1, 1)$, $t_k = \begin{array}{|c|c|c|}\hline i&\cdots&j\\\hline k\\\cline{1-1}\end{array}$ and $v_k = \{t_k\}$. Then $e_t = v_k - v_i$ and the span of all such vectors is

$$S^{(n-1,1)} = \{\alpha_1 v_1 + \cdots + \alpha_n v_n : \alpha_1 + \cdots + \alpha_n = 0, \alpha_i \in \mathbb{C}\}.$$

This is the kernel of Example 6.4.4.

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9.4 The submodule theorem

Definition 9.4.1. Define inner product on M^λ via

$$\langle\{t\},\{s\}\rangle := S_{\{t\},\{s\}}.$$

Note that $\forall\pi\in S_n$ one has $\langle\{t\},\{s\}\rangle = \langle\pi\{t\},\pi\{s\}\rangle$ and hence $\forall u,v\in M^\lambda, \langle u,v\rangle = \langle\pi u,\pi v\rangle$.

Notation. $\pi^- := \{\pi\}^- = \operatorname{sgn}(\pi)\pi$.

Lemma 9.4.2 (Sign)**.** Let $H\le S_n$ be a subgroup. Then

1. If $\pi\in H$ then $\pi H^- = H^-\pi = \operatorname{sgn}(\pi)H^-$, i.e. $\pi^-H^- = H^-$.
2. $\forall u,v\in M^\lambda,\ \langle H^-u,v\rangle = \langle u,H^-v\rangle$.
3. If $(b,c)\in H$ then one can factor $H^- = k\cdot(\mathrm{id}-(b,c))$ for some $k\in\mathbb{C}S_n$.
4. If t is a tableau with b,c in the same row and $(b,c)\in H$ then $H^-\{t\}=0$.

Proof. 1. Similar to $\pi e_t = \operatorname{sgn}(\pi)e_t$ in 9.3.8:

$$\pi H^- = \sum_{\sigma\in H}\operatorname{sgn}(\sigma)\pi\sigma = \sum_{\tau\in H}\operatorname{sgn}(\pi^{-1}\tau)\tau = \operatorname{sgn}(\pi^{-1})\sum_{\tau\in H}\operatorname{sgn}(\tau)\tau = \operatorname{sgn}(\pi)H^-.$$

2.

$$\begin{aligned}\left\langle H^-u,v\right\rangle &= \sum_{\pi\in H}\langle\operatorname{sgn}(\pi)\pi u,v\rangle = \sum_{\pi\in H}\left\langle\operatorname{sgn}(\pi)u,\pi^{-1}v\right\rangle\\ &= \sum_{\pi\in H}\left\langle u,\operatorname{sgn}(\pi^{-1}),\pi^{-1}v\right\rangle = \sum_{\pi\in H}\langle u,\operatorname{sgn}(\pi)\pi v\rangle = \left\langle u,H^-,v\right\rangle.\end{aligned}$$

3. Consider the subgroup $\{\mathrm{id},(b,c)\}\le H$. Take a transversal

$$k_1\{\mathrm{id},(b,c)\}\sqcup k_2\{\mathrm{id},(b,c)\}\sqcup\cdots\sqcup\cdots$$

Observe

$$\left(\sum_i k_i^-\right)(\mathrm{id}-(b,c)) = H^-$$

as desired.

4. By assumption, $(b,c)\{t\}=\{t\}$, so

$$H^-\{t\} = k\cdot(\mathrm{id}-(b,c))\{t\} = 0.$$

□

Corollary 9.4.3. Let $\lambda,\mu\vdash n$ and t a λ-tableau and s a μ-tableau. If $\kappa_t\{s\}\ne 0$ then $\lambda\trianglerighteq\mu$ and if $\lambda=\mu$ then $\kappa_t\{s\}\in\{-e_t,e_t\}$.

Proof. Let b,c be two elements in the same row of s. If they are also in the same column of t then by 9.4.2.4 $\kappa_t\{s\}=0$. If not then 9.2.4 gives $\lambda\trianglerighteq\mu$.
If additionally $\lambda=\mu$ then by the same argument one can reorder within columns of t, i.e. $\exists\pi\in C_t:\{s\}=\pi\{t\}$, and 9.4.2.1 gives $\kappa_t\{s\}=\kappa_t\pi\{t\}=\operatorname{sgn}(\pi)\kappa_t\{t\}\in\{\pm e_t\}$. □

Corollary 9.4.4. If $u\in M^\mu$ and $\operatorname{sh}(t)=\mu$ then $\kappa_t u$ is a multiple of e_t.

Proof. Write $u=\sum_i\alpha_i\{s_i\}$ where $\{s_i\}$ are μ-tabloids. Corollary 9.4.3 gives

$$\kappa_t u = \kappa_t\sum_i\alpha_i\{s_i\} = \sum_i\alpha_i\kappa_t\{s_i\} = \left(\sum_i\pm\alpha_i\right)e_t.$$

□

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Notation. For a linear subspace $U \subseteq M^\mu$, define

$$U^\perp := \{v \in M^\mu : \langle u, v\rangle = 0 \ \forall u \in U\}.$$

Theorem 9.4.5 (Submodule)**.** If $U \subseteq M^\mu$ is a submodule then $S^\mu \subseteq U$ or $U \subseteq (S^\mu)^\perp$.

Proof. For all $u \in U$ and a μ-tableau t we know $\exists \alpha_{u,t} \in \mathbb{C} : \kappa_t u = \alpha_{u,t} e_t$ by 9.4.4.

Case 1: $\exists u, t : \alpha_{u,t} \neq 0$. Since $u \in U$ one has $\alpha_{u,t} e_t = \kappa_t u \in U$, hence $e_t = \alpha_{u,t}^{-1} \kappa_t u \in U$. Therefore $\forall \pi \in S_n,\ e_{\pi t} = \pi e_t \in U$ and so $S^\mu \subseteq U$.

Case 2: $\alpha_{u,t} = 0\ \forall u, t$. The e_t with $\mathrm{sh}(t) = \mu$ spans S^μ. Let $u \in U$, then

$$\begin{aligned} \langle u, e_t\rangle &= \langle u, \kappa_t\{t\}\rangle \\ &= \langle \kappa_t u, \{t\}\rangle \qquad \text{by 9.4.2.2} \\ &= \langle 0, \{t\}\rangle = 0. \end{aligned}$$

□

Proposition 9.4.6. If $0 \neq f \in \mathrm{Hom}_{S_n}(S^\lambda, M^\mu)$ then $\lambda \trianglerighteq \mu$. If $\lambda = \mu$ then f is multiplication by a scalar.

Proof. Since $f \neq 0$ and S^λ is generated by the e_t, there must be an $e_t : f(e_t) \neq 0$. Now $M^\lambda = S^\lambda \oplus (S^\lambda)^\perp$. Thus we can extend f to an element of $\mathrm{Hom}_{S_n}(M^\lambda, M^\mu)$ by setting $f(v) = 0\ \forall v \in (S^\lambda)^\perp$. Now

$$\begin{aligned} 0 \neq f(e_t) = f(\kappa_t\{t\}) = \kappa_t f(\{t\}) &= \kappa_t \sum_i \alpha_i \{s_i\} \\ &= \sum_i \alpha_i \kappa_t \{s_i\} \qquad \text{for some } \alpha_i \in \mathbb{C} \text{ and } s_i \text{ are } \mu\text{-tableaux} \end{aligned}$$

and $\lambda \trianglerighteq \mu$ by 9.4.3.

If $\lambda = \mu$ then by 9.4.4 $f(e_t) = \sum_i \alpha_i \kappa_t \{s_i\} = \alpha e_t$ for some $\alpha \in \mathbb{C}$, so for every $\pi \in S_n,\ f(e_{\pi t}) = f(\pi e_t) = \pi f(e_t) = \pi \alpha e_t = \alpha e_{\pi t}$. □

Theorem 9.4.7. The S^λ for $\lambda \vdash n$ form a complete list of irreducible S_n-representations.

Proof. Let $U \subseteq S^\lambda$ be a subrepresentation. By Theorem 9.4.5, either $S^\lambda \subseteq U$ or $U \subseteq (S^\lambda)^\perp$, so either $U = S^\lambda$ or $U \subseteq S^\lambda \cap (S^\lambda)^\perp = \{0\}$, i.e. S^λ is irreducible.

Since we have the correct number of irreducible representations, it remains to show that they are pairwise nonisomorphic. Suppose $S^\lambda \sim S^\mu$, then there is a nonzero $f \in \mathrm{Hom}_{S_n}(S^\lambda, S^\mu)$ which can be interpreted as $f \in \mathrm{Hom}_{S_n}(S^\lambda, M^\mu)$ since $S^\mu \subseteq M^\mu$. Then by 9.4.6 $\lambda \trianglerighteq \mu$. Symmetrically $\mu \trianglerighteq \lambda$, so $\lambda = \mu$. □

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Corollary 9.4.8.

$$M^\mu \sim \bigoplus_{\lambda \trianglerighteq \mu} (S^\lambda)^{\oplus m_{\lambda,\mu}},$$

with $m_{\mu,\mu} = 1\ \forall \mu$.

Proof. If S^λ appears in M^μ with nonzero multiplicity (i.e. $m_{\lambda,\mu} \geq 1$) then there exists an injective S_n-homomorphism $f : S^\lambda \to M^\mu$, so by 9.4.6 $\lambda \trianglerighteq \mu$.

Now $m_{\mu,\mu} \geq 1$ by definition of $S^\mu \subseteq M^\mu$. Suppose for contradiction $m_{\mu,\mu} \geq 2$. Then one can take any decomposition of M^μ into irreducibles

$$M^\mu = \bigoplus_{\lambda \vdash n,\ \lambda \trianglerighteq \mu} \left(V_{\lambda,1} \oplus V_{\lambda,2} \oplus \cdots \oplus V_{\lambda, m_{\lambda,\mu}}\right) \qquad \text{where } \forall i,\ V_{\lambda,i} \sim S^\lambda.$$

Take the isomorphism $f_1 : S^\mu \to V_{\mu,1}$ and $f_2 : S^\mu \to V_{\mu,2}$, then

$$\forall \alpha, \beta \in \mathbb{C},\ \alpha f_1 + \beta f_2 \in \mathrm{Hom}_{S_n}(S^\mu, M^\mu)$$

and in particular, $\dim \mathrm{Hom} S^n(S^\mu, M^\mu) \geq 2$. But $\dim \mathrm{Hom}_{S_n}(S^\mu, M^\mu) = 1$ by 9.4.6. □

9.5 Standard tableaux and basis for S^λ: linear independence

Definition 9.5.1. A tableau is *standard* if the rows are increasing from left to right and the columns are increasing from top to bottom. In this case, the corresponding is tabloid and polytabloid are also *standard.*

e.g. $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&6\\\cline{1-2} 5\\\cline{1-1}\end{array}$ is standard but $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 5&4\\\cline{1-2} 6\\\cline{1-1}\end{array}$ is not.

Theorem 9.5.2. The set $\{e_t : t \text{ is a standard } \lambda\text{-tableau}\}$ is a basis of S^λ.

Example 9.5.3. $S_3,\ \lambda = (2,1)$. Then

$$e_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} = \begin{array}{c}\hline 1\ 2\\\hline 3\\\hline\end{array} - \begin{array}{c}\hline 3\ 2\\\hline 1\\\hline\end{array} = \begin{array}{c}\hline 1\ 2\\\hline 3\\\hline\end{array} - \begin{array}{c}\hline 2\ 3\\\hline 1\\\hline\end{array},$$

$$e_{\begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array}} = \begin{array}{c}\hline 2\ 1\\\hline 3\\\hline\end{array} - \begin{array}{c}\hline 3\ 1\\\hline 2\\\hline\end{array} = \begin{array}{c}\hline 1\ 2\\\hline 3\\\hline\end{array} - \begin{array}{c}\hline 1\ 3\\\hline 2\\\hline\end{array},$$

and

$$e_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} = \begin{array}{c}\hline 1\ 3\\\hline 2\\\hline\end{array} - \begin{array}{c}\hline 2\ 3\\\hline 1\\\hline\end{array}.$$

Now notice that

$$e_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} - e_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} = e_{\begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array}}$$

and indeed that $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ are standard.

Definition 9.5.4. A *composition* of n is a sequence of nonnegative integers $(\lambda_1, \dots, \lambda_l)$ such that $\sum_{i=1}^{l} \lambda_i = n$. Every partition is a composition.

One extend the notions of Young diagrams/tableaux/tabloids and dominance order to compositions with verbatim definitions, e.g. $(5,3,4,4) \trianglerighteq (4,4,3,5)$.

Given $\{t\}$ with $\mathrm{sh}(t) = \lambda,\ \lambda \vdash n$, for each $i \in \{1, \dots, n\}$ define

$$\{t^i\} := \text{the tabloid formed by all elements} \le i \text{ in } \{t\}$$

and

$$\lambda^i := \text{the composition that is the shape of } \{t^i\},$$

e.g. for $\{t\} = \begin{array}{c}\hline 2\ 4\\\hline 1\ 3\\\hline\end{array}$,

$$\{t^1\} = \begin{array}{c}\hline \\\hline 1\\\hline\end{array}, \quad \{t^2\} = \begin{array}{c}\hline 2\\\hline 1\\\hline\end{array}, \quad \{t^3\} = \begin{array}{c}\hline 2\\\hline 1\ 3\\\hline\end{array}, \quad \{t^4\} = \begin{array}{c}\hline 2\ 4\\\hline 1\ 3\\\hline\end{array}$$

and

$$\lambda^1 = (0,1), \quad \lambda^2 = (1,1), \quad \lambda^3 = (1,2), \quad \lambda^4 = (2,2),$$

which is called a *composition sequence.*

Definition 9.5.5. For two tabloids $\{s\}, \{t\}$ with composition sequences λ^i and μ^i respectively. One say $\{s\}$ *dominates* $\{t\}$, denoted $\{s\} \trianglerighteq \{t\}$, if $\forall i,\ \lambda^i \trianglerighteq \mu^i$.

Example 9.5.6. The Hasse diagram for $(2,2)$-tabloids:

$$\begin{tikzcd}
 & & \begin{array}{c}\hline 1\ 2\\\hline 3\ 4\\\hline\end{array} & (1,0),(2,0),(2,1),(2,2) & \\
 & & \begin{array}{c}\hline 1\ 3\\\hline 2\ 4\\\hline\end{array} \arrow[u] & (1,0),(1,1),(2,1),(2,2) & \\
(0,1),(1,1),(2,1),(2,2) & \begin{array}{c}\hline 2\ 3\\\hline 1\ 4\\\hline\end{array} \arrow[ur] & & \begin{array}{c}\hline 1\ 4\\\hline 2\ 3\\\hline\end{array} \arrow[ul] & (1,0),(1,1),(1,2),(2,2) \\
 & & \begin{array}{c}\hline 2\ 4\\\hline 1\ 3\\\hline\end{array} \arrow[ul] \arrow[ur] & (0,1),(1,1),(1,2),(2,2) & \\
 & & \begin{array}{c}\hline 3\ 4\\\hline 1\ 2\\\hline\end{array} \arrow[u] & (0,1),(0,2),(1,2),(2,2) &
\end{tikzcd}$$

Lemma 9.5.7 (Dominance lemma for tabloids)**.** If $k < l$ and k appears in a lower row than l in $\{t\}$, then $\{t\} \lhd (k,l)\{t\}$.

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Proof. Let λ^i be the composition sequence of $\{t\}$ and μ^i that of $(k,l)\{t\}$. Then for $i < k$ and $i \geq l$ one has $\lambda^i = \mu^i$, so consider $k \leq i < l$. Let r be the row of $\{t\}$ in which k appears and q be that of $\{t\}$ in which l does. Note that $q < r$ by assumption. Then $\lambda^i = \mu^i$ with the q-th part decreased by 1 and r-th part increased by 1. Since $q < r$, one has $\lambda^i \lhd \mu^i$. $\square$

Definition 9.5.8. For $v = \sum_i \alpha_i \{t_i\} \in M^\mu$, one says $\{t_i\}$ *appears* in v if $\alpha_i \neq 0$.

Corollary 9.5.9. If t is standard and $\{s\}$ appears in e_t, then $\{t\} \unrhd \{s\}$.

Proof. Let $s = \pi t$ for some $\pi \in C_t$ so $\{s\}$ appears in e_t. We prove by induction on number of pairs $k < l$ in the same column of s such that k is in a lower row than l. Such a pair is called a *column inversion*. Given any such pair, Lemma 9.5.7 implies $\{s\} \lhd (k,l)\{s\}$. But $(k,l)\{s\}$ has fewer column inversions than $\{s\}$: to prove this, note that only the entries between k and l must be considered, and for each of those, the number of inversions they are involved in cannot increase. Hence, by induction, $(k,l)\{s\} \unlhd \{t\}$. $\square$

Corollary 9.5.10. $\{t\}$ is the maximum tabloid that appears in e_t.

Definition 9.5.11. Let $(A, \leq)$ be a poset. Then an element $b \in A$ is <u>the</u> *maximum* if $\forall c \in A,\ b \geq c$, and an element $b \in A$ is <u>a</u> *maximal element* if $\forall c \in A,\ b \not< c$. Minimum and minimality are defined analogously.

Proposition 9.5.12. The set $\{e_t : t \text{ is a standard } \lambda\text{-tableau}\}$ is linearly independent.

Proof. Distinct standard tableaux $s \neq t$ have distinct tabloids $\{s\} \neq \{t\}$. By 9.5.10, $\{t\}$ is the maximum tabloid in e_t. Sort the standard λ-tableaux $t_1, \ldots, t_m$ so that $\{t_1\}$ is the maximal among the $\{t_i\}$. Hence, $\{t_1\}$ only appears in e_{t_1} and not in any other e_{t_i}. Hence, every zero combination $\alpha_1 e_{t_1} + \cdots + \alpha_m e_{t_m} = 0$ must have $\alpha_i = 0$ because otherwise the coefficients for $\{t_1\}$ do not cancel. Remove t_1 from the list and continue inductively with the next maximal tabloid. $\square$

It is also true that $\{e_t : t \text{ is a standard tableau}\}$ spans S^λ but we will not prove it in class. A proof can be found in Sagan's book *The symmetric group*, 2nd ed., Section 2.6. This proves Theorem 9.5.2.

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10 More examples

10.1 Alternating group A_4

Recall $A_4 = \{\pi \in S_4 : \operatorname{sgn}(\pi) = 1\}$, which is isomorphic to group of rotations $\mathbb{R}^3$ that stabilises a regular tetrahedron with barycentre the origin, and $|A_4| = 12 = |S_4|/2$.

Let $x = (1,2)(3,4),\ y = (1,3)(2,4),\ z = (1,4)(2,3)$ and $t = (1,2,3)$. Now $K := \{\mathrm{id}, t, t^2\}$ is clearly a subgroup of A_4, but $H := \{\mathrm{id}, x, y, z\}$ is as well since

$$xy = z = yx,\ xz = y = zx,\ yz = x = zy. \tag{G.1}$$

Recall 1.1.9 and note that

$$txt^{-1} = z,\ tzt^{-1} = y,\ tyt^{-1} = x, \tag{G.2}$$

and hence H is normal.

Every element of A_4 can be written as hk where $h \in H, k \in K$ by shifting via G.2. The presentation is unique since $|H| \cdot |K| = |A_4|$.

Claim 10.1.1. The conjugacy classes in A_4 are $\{\mathrm{id}\},\ \{x, y, z\},\ \{t, tx, ty, tz\},\ \{t^2, t^2x, t^2y, t^2z\}$.

Proof. Indeed all 4 sets are closed under conjugation with t by G.2. Similarly, conjugation with x, y or z does not change exponent of t in the unique representation hk.

Define $s : H \to H : h \mapsto tht^{-1}$. Then $\forall i \in \{0,1,2\},\ s(t^i h) = t(t^i h)t^{-1} = t^i tht^{-1} = t^i s(h)$ and $\forall i \in \{1,2\},\ xt^i x^{-1} = xt^i x = t^i s^i(x) x = \begin{cases} ty \text{ if } i = 1 \\ t^2 z \text{ if } i = 2 \end{cases}$. $\square$

For the 1-dimensional representations of A_4, let $\zeta = e^{2\pi i/3}$ and one obtains 3 non-isomorphic 1-dimensional irreducible characters of A_4 via $\forall h \in H,\ \chi_i(ht^j) = \zeta^{ij}$. Now $\chi_i : A_4 \to GL_1(\mathbb{C})$ is indeed a group homomorphism since the conversion to normed form hk does not change the exponent of t, which implies

$$\forall h_1, h_2 \in H,\ \exists h \in H : \chi_i(h_1 t^{j_1} h_2 t^{j_2}) = \chi_i(h t^{j_1+j_2}) = \zeta^{i(j_1+j_2)} = \zeta^{ij_1}\zeta^{ij_2} = \chi_i(h_1 t^{j_1})\chi_i(h_1 t^{j_2}).$$

Now by 7.4.5 and 7.5.1, there must be one remaining 3-dimensional irreducible representation. One can try and check if $S^{3,1} \downarrow^{S_4}_{A_4}$ is irreducible: $\dim S^{(3,1)} = \#$ standard tableaux of shape $(3,1)$:

$$\ytableaushort{123,4},\ \ytableaushort{124,3},\ \ytableaushort{134,2}$$

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Now

$$\begin{aligned}
e_{\ytableaushort{134,2}} &= \begin{array}{l}\hline 1\ 3\ 4\\\hline 2\\\hline\end{array} - \begin{array}{l}\hline 2\ 3\ 4\\\hline 1\\\hline\end{array}\\
e_{\ytableaushort{124,3}} &= \begin{array}{l}\hline 1\ 2\ 4\\\hline 3\\\hline\end{array} - \begin{array}{l}\hline 3\ 2\ 4\\\hline 1\\\hline\end{array} = \begin{array}{l}\hline 1\ 2\ 4\\\hline 3\\\hline\end{array} - \begin{array}{l}\hline 2\ 3\ 4\\\hline 1\\\hline\end{array}\\
e_{\ytableaushort{123,4}} &= \begin{array}{l}\hline 1\ 2\ 3\\\hline 4\\\hline\end{array} - \begin{array}{l}\hline 4\ 2\ 3\\\hline 1\\\hline\end{array} = \begin{array}{l}\hline 1\ 2\ 3\\\hline 4\\\hline\end{array} - \begin{array}{l}\hline 2\ 3\ 4\\\hline 1\\\hline\end{array}
\end{aligned}$$

so

$$\begin{aligned}
xe_{\ytableaushort{134,2}} &= e_{x\ytableaushort{134,2}} = e_{\ytableaushort{243,1}} = \begin{array}{l}\hline 2\ 4\ 3\\\hline 1\\\hline\end{array} - \begin{array}{l}\hline 1\ 4\ 3\\\hline 2\\\hline\end{array} = -e_{\ytableaushort{134,2}}\\
xe_{\ytableaushort{124,3}} &= e_{x\ytableaushort{124,3}} = e_{\ytableaushort{213,4}} = \begin{array}{l}\hline 2\ 1\ 3\\\hline 4\\\hline\end{array} - \begin{array}{l}\hline 4\ 1\ 3\\\hline 2\\\hline\end{array} = e_{\ytableaushort{123,4}} - e_{\ytableaushort{134,2}}\\
xe_{\ytableaushort{123,4}} &= e_{x\ytableaushort{123,4}} = e_{\ytableaushort{214,3}} = \begin{array}{l}\hline 2\ 1\ 4\\\hline 3\\\hline\end{array} - \begin{array}{l}\hline 3\ 1\ 4\\\hline 2\\\hline\end{array} = e_{\ytableaushort{124,3}} - e_{\ytableaushort{134,2}}
\end{aligned}$$

which gives us the representation matrix of x

$$\begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

with respect to the basis

$$e_{\ytableaushort{134,2}},\ e_{\ytableaushort{124,3}},\ e_{\ytableaushort{123,4}}$$

with trace -1. One continues and calculates $\psi = \chi_{(3,1)} \downarrow^{S_4}_{A_4}$:

$$\psi(\mathrm{id}) = 3, \quad \psi(x) = -1, \quad \psi(t) = 0, \quad \psi(t^2) = 0$$

One verifies with Lemma 7.4.9 that ψ is irreducible:

$$\langle \psi, \psi \rangle = \frac{1}{12}(1 \cdot 3^2 + 3 \times (-1)^2 + 0) = 1.$$

The character table is

A_4	$\mathrm{id}_{(1)}$	$x_{(3)}$	$t_{(4)}$	$t^2_{(4)}$
χ_0	1	1	1	1
χ_1	1	1	ζ	ζ^2
χ_2	1	1	ζ^2	ζ
ψ	3	-1	0	0

where ζ is the cubic root of unity.

10.2 Dihedral group

Recall that $D_{2n} = \langle r, s \mid r^n = 1, s^2 = 1, srs^{-1} = r^{-1} \rangle$ and $|D_{2n}| = 2n$. With the 1-dimensional representations $\phi : D_{2n} \to GL_1(\mathbb{C})$,

$$(\phi(r), \phi(s)) \in \begin{cases} \{(1,1), (1,-1)\} & \text{if } n \text{ is odd} \\ \{(1,1), (1,-1), (-1,1), (-1,-1)\} & \text{if } n \text{ is even} \end{cases}$$

Let $\zeta = e^{2\pi i/n}$ and for $h \in \mathbb{Z}$ define the representation

$$\begin{aligned} \rho^h : D_{2n} &\to GL_2(\mathbb{C}) \\ r^k &\mapsto \begin{pmatrix} \zeta^{hk} & 0 \\ 0 & \zeta^{-hk} \end{pmatrix} \\ sr^k &\mapsto \begin{pmatrix} 0 & \zeta^{hk} \\ \zeta^{-hk} & 0 \end{pmatrix} \end{aligned}$$

(Verify that $\rho^h = \rho_{\zeta^h} \uparrow_{C_n}^{D_n}$.)

Claim 10.2.1. For $0 < h < \frac{n}{2}$, ρ^h is irreducible. (Check common eigenvectors of the two matrices.)

The characters χ_h of ρ^h:

$$\chi_h(r^k) = 2\cos\frac{2\pi hk}{n}, \qquad \chi_h(sr^k) = 0$$

Verify Lemma 7.5.1: if n is even,

$$4 \cdot 1^2 + \left(\frac{n}{2} - 1\right) \cdot 2^2 = 2n = |D_{2n}|,$$

and if n is odd

$$2 \cdot 1^2 + \left(\frac{n-1}{2}\right) \cdot 2^2 = 2n = |D_{2n}|.$$

10.3 Quaternion group Q_8

Recall that $Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} \rangle$ and $|Q_8| = 8$. We found (see HW2 Q1) that there are 4 1-dimensional representations and there is 1 2-dimensional representation

$$\begin{aligned} \phi : Q_8 &\to GL_2(\mathbb{C}) \\ a &\mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ b &\mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

The two matrices similarly have no common eigenvectors so the representation is irreducible. Applying 7.5.1:

$$1 \cdot 2^2 + 4 \cdot 1^2 = 8 = |Q_8|$$

and as a corollary we get that there are 5 conjugacy classes in Q_8 for free; in fact the character table is

Q_8	$\mathrm{id}_{(1)}$	$a_{(2)}$	$ab_{(2)}$	$b_{(2)}$	$a^2_{(1)}$
$\chi_{1,1}$	1	1	1	1	1
$\chi_{1,-1}$	1	1	-1	-1	1
$\chi_{-1,1}$	1	-1	-1	1	1
$\chi_{-1,-1}$	1	-1	1	-1	1
ϕ	2	0	0	0	-2