

MA377 Rings and modules :: Lecture notes

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1 Introduction

1.1 Definitions

Definition 1.1.1. A *ring* is ...

A ring R is *commutative* if $xy = yx \forall x, y \in R$.

R is a *division ring* if $(R \setminus \{0\}, \cdot)$ is a group.

R is a *field* if it's a commutative division ring.

Definition 1.1.2. A *left R -module* is an abelian group M and an action map $R \times M \rightarrow M$ such that $1_R m = m$, $(x + y)m = xm + ym$, $x(m + n) = xm + xn$, $x(ym) = (xy)m \forall m \in M, x, y \in R$. A *right R -module* is similar except the last axiom reads $x(ym) = (yx)m$, also written $(my)x = m(yx)$, with element of R written on the right.

Example 1.1.3. Each R is a left/right module over itself by left/right multiplication, denoted ${}_R R$ and R_R .

$M_n(R)$ is a ring with usual addition and multiplication of matrices. Column/row vectors form a left/right $M_n(R)$ -module.

Definition 1.1.4. A *ring homomorphism* is a function $f : R \rightarrow S$ such that $f(x + y) = f(x) + f(y)$, $f(xy) = f(x)f(y)$, $f(1_R) = 1_S$. An *isomorphism* is a bijective homomorphism.

Notation. $R \times S := \{(r, s) : r \in R, s \in S\}$. This is a ring with the obvious trivial addition and multiplication.

Example 1.1.5. $i_1 : R \rightarrow R \times S : r \mapsto (r, 0)$ is not a homomorphism since $i_1(1_R) = (1_R, 0_S) \neq (1_R, 1_S) = 1_{R \times S}$, but it satisfies the first two conditions.

$\pi_1 : R \times S \rightarrow R : (r, s) \mapsto r$ is.

Week 1, lecture 2 starts here

Definition 1.1.6. $A \subseteq R$ is a *subring* of R if A is a ring under the same operations, i.e. $1_R \in A$, $xy, x - y \in A \forall x, y \in A$.

Example 1.1.7. Centre of R : $Z(R) := \{x \in R : xy = yx \forall y \in R\}$.

Centraliser of $X \subseteq R$ in R : $C_R(X) := \{y \in R : xy = yx \forall x \in X\}$.

Definition 1.1.8. A left (or right) *ideal* of R is an additive subgroup $L \leq R$ such that xa (or ax) $\in L \forall a \in L, x \in R$, denoted $L \leq^l R$ or $L \leq^r R$. L is a two-sided ideal (or simply ideal) of R if it's both a left and right ideal, denoted $L \leq R$.

If $I \leq R$ then $R/I = \{x + I : x \in R\}$ is a ring, called the *quotient ring*, with the following definitions:

$$\begin{aligned} (x + I) + (y + I) &= (x + y) + I \\ (x + I)(y + I) &= xy + I \\ 1_{R/I} &= 1_R + I \end{aligned}$$

Example 1.1.9. For $x_1, \dots, x_n \in R$, one can generate an ideal

$$(x_1, \dots, x_n) = Rx_1R + \dots + Rx_nR = \{r_1x_1s_1 + \dots + r_nx_ns_n : r_i, s_i \in R\}.$$

If R is commutative, then

$$(x_1, \dots, x_n) = Rx_1 + \dots + Rx_n = \{r_1x_1 + \dots + r_nx_n : r_i \in R\}.$$

Lemma 1.1.10. Let S be a ring and $R = M_n(S)$ with E_{ij} , a matrix with 1 on the i, j position and 0 elsewhere. Then $(E_{ij}) = R$.

Proof. Let $I = (E_{ij})$. One has

$$\begin{aligned} E_{RR} &= E_{Ri}E_{ij}E_{jR} \in I \\ 1_R &= E_{11} + \dots + E_{nn} \in I \\ x &= x1_R \in I \forall x \in R \end{aligned}$$

□

Definition 1.1.11. A *principal ideal domain* is ...

A *unique factorisation domain* is ...

Every PID is a UFD.

Lemma 1.1.12. If R is a UFD and $x_1, \dots, x_n \in R$ with $m = \text{lcm}(x_i)$, then

$$(x_1) \cap \dots \cap (x_n) = (m).$$

Proof.

$$(x_1) \cap \dots \cap (x_n) = \{a : x_i \mid a \ \forall i\} = \{a : m \mid a\} = (m).$$

□

Lemma 1.1.13. If R is a PID and $x_1, \dots, x_n \in R$ with $d = \text{gcd}(x_1, \dots, x_n)$, then

$$(x_1) + \dots + (x_n) = (d).$$

Proof. $\subseteq: d \mid x_i \ \forall i \implies d \mid (a_1x_1 + \dots + a_nx_n).$

$\supseteq:$ Since R is a PID, $\exists z \in R : (x_1) + \dots + (x_n) = (z)$. We want to show $(z) \supseteq (d) \iff z \mid d$. But $(z) \supseteq (x_i)$, so $z \mid x_i \implies z \mid \text{gcd}(x_i) = d$.

□

Remark. This indeed fails for UFDs. Consider $R = \mathbb{C}[x, y]$, then $\text{gcd}(x, y) = 1$, but $(x) + (y) = (x, y) \neq (1) = R$.

Theorem 1.1.14 (Isomorphism theorems for rings). If $f : R \rightarrow S$ is a ring homomorphism, then

1. $\ker f \trianglelefteq R$
2. $\text{im } f \leq S$
3. f decomposes as

$$R \twoheadrightarrow R/\ker f \xrightarrow{\bar{f}} \text{im } f \hookrightarrow S$$

Week 1, lecture 3 starts here

1.2 Chinese remainder theorem

Theorem 1.2.1 (Elementary form of Chinese remainder). The system

$$\begin{aligned} x &\equiv k_1 \pmod{n_1} \\ &\vdots \\ x &\equiv k_t \pmod{n_t} \end{aligned}$$

where $n_1, \dots, n_t \in \mathbb{Z}$ relatively prime and $k_1, \dots, k_t \in \mathbb{Z}$, has a solution, and any two solutions differ by a multiple of $n_1 \cdots n_t$.

Proof. Consider

$$\begin{aligned} f : \mathbb{Z} &\rightarrow \mathbb{Z}/(n_1) \times \dots \times \mathbb{Z}/(n_t) \\ x &\mapsto (x + (n_1), \dots, x + (n_t)). \end{aligned}$$

By Lemma 1.1.12, $\ker f = (n_1) \cap \dots \cap (n_t) = (n_1 \cdots n_t)$. By the isomorphism theorems,

$$\mathbb{Z}/(n_1 \cdots n_t) \xrightarrow{\bar{f}} \text{im } f \hookrightarrow \mathbb{Z}/(n_1) \times \dots \times \mathbb{Z}/(n_t),$$

but both $\mathbb{Z}/(n_1 \cdots n_t)$ and $\mathbb{Z}/(n_1) \times \dots \times \mathbb{Z}/(n_t)$ has $|n_1 \cdots n_t|$ elements, so it's an isomorphism. Therefore $\exists x \in \mathbb{Z} : f(x) = (k_1, \dots, k_t)$.

If y is another solution, then $f(x - y) = f(x) - f(y) = 0$, i.e. $x - y \in \ker f = (n_1 \cdots n_t)$.

□

Example 1.2.2. Consider the system

$$\begin{aligned}x &\equiv 1 \pmod{7} \\x &\equiv 7 \pmod{9} \\x &\equiv 3 \pmod{11}\end{aligned}$$

Note that by f in the proof,

$$\begin{aligned}7 \times 9 = 63 &\mapsto (0, 0, 8) \\7 \times 11 = 77 &\mapsto (0, 5, 0) \\9 \times 11 = 99 &\mapsto (1, 0, 0),\end{aligned}$$

and one needs $f(x) = (1, 7, 3)$, but

$$\begin{aligned}(1, 7, 3) &= (1, 0, 0) + (0, 7, 0) + (0, 0, 3) \\&= (1, 0, 0) + 5 \times (0, 5, 0) - (0, 0, 8) \\&= f(99) + 5 \times f(77) - f(63) \\&= f(99 + 5 \times 77 - 63) \\&= f(421).\end{aligned}$$

Definition 1.2.3. Let $I, J \trianglelefteq R$. I and J are *coprime* if $I + J = R$.

Lemma 1.2.4. If $I_1, \dots, I_n \trianglelefteq R$, then

$$\begin{aligned}f : R &\rightarrow R/I_1 \times \dots \times R/I_n \\x &\mapsto (x + I_1, \dots, x + I_n)\end{aligned}$$

is a ring homomorphism with kernel $I_1 \cap \dots \cap I_n$.

Theorem 1.2.5. If I_1, \dots, I_n are pairwise coprime then

$$\bar{f} : R/(I_1 \cap \dots \cap I_n) \rightarrow R/I_1 \times \dots \times R/I_n$$

is an isomorphism.

Proof. It suffices to find, for each i , $a_i \in R : f(a_i) = e_i$, since then f would be surjective:

$$\begin{aligned}(x_1 + I_1, \dots, x_n + I_n) &= (x_1 + I_1)e_1 + \dots + (x_n + I_n)e_n \\&= f(x_1)f(a_1) + \dots + f(x_n)f(a_n) = f(x_1a_1 + \dots + x_na_n).\end{aligned}$$

Let's now find a_i . Note that $\forall j \neq i$, $I_i + I_j = R \ni 1$, so $\exists b_j \in I_i$, $c_j \in I_j : b_j + c_j = 1$. We claim $a_i = \prod_{j \neq i} c_j$. Indeed, $c_j = 0$ in I_j and 1 in I_i . \square

Example 1.2.6. In the same example as above, note that $7 \times 9 \times 11 = 693$ and we can write

$$28 - 27 = 45 - 44 = -21 + 22 = 1$$

where $28, -21 \in (7)$, $-27, 45 \in (9)$ and $-44, 22 \in (11)$. Hence

$$\begin{aligned}a_1 &= (-27)(22) = -594 \equiv 99 \pmod{693} \\a_2 &= (28)(-44) = -1232 \equiv 154 \pmod{693} \\a_3 &= (-21)(45) = -945 \equiv 441 \pmod{693}\end{aligned}$$

Week 2, lecture 1 starts here

1.3 Isomorphism theorems

With a left/right R -module we can convert R into its opposite R^{op} by swapping the multiplication. Then a right R -module is a left R^{op} -module, and vice versa.

Definition 1.3.1. For a R -module ${}_R M$, $N \leq M$ is a *submodule* if it's an abelian subgroup and $\forall r \in R, x \in N : rx \in N$.

Note for ${}_R R$ and R_R , submodules are precisely left/right ideals.

Definition 1.3.2. For ${}_R M \geq_R N$, the abelian quotient group M/N is called the *quotient module*, with multiplication defined $r(x + N) = rx + N$. This is well-defined since

$$\begin{aligned} x + N = y + N &\implies x - y \in N \\ &\implies r(x + N) = rx + N = r(y + (x - y))N = ry + r(x - y) + N = ry + N = r(y + N). \end{aligned}$$

Other axioms follow from those for ${}_R M$.

Example 1.3.3. If $L \leq R$ then R/L is a left R -module.

Definition 1.3.4. A *homomorphism* of R -modules $\varphi : {}_R M \rightarrow {}_R N$ is a homomorphism of abelian groups and $\varphi(rm) = r\varphi(m) \forall r \in R, m \in M$.

For left R -modules, we write homomorphism on the right: $(rm)\varphi = r(m\varphi) = rm\varphi$ to keep in line with the can-get-rid-of-bracket perspective of associativity. For right R -modules we then simply write $\varphi(mr) = \varphi(m)r = \varphi mr$.

Theorem 1.3.5 (1st isomorphism theorem). If R -modules $\varphi : {}_R M \rightarrow {}_R N$ is a homomorphism of modules, then

1. $\ker \varphi \leq_R M$
2. $\text{im } \varphi \leq_R N$
3. φ decomposes as

$$\begin{array}{ccc} M & \xrightarrow{\pi} & M/\ker \varphi & \xrightarrow[\cong]{\cong} & \text{im } \varphi & \xrightarrow{\iota} & N \\ m & \mapsto & m + \ker \varphi & \mapsto & m\varphi & & \\ & & & & & & x \mapsto x \end{array}$$

Proof. All statements hold on the level of abelian groups by isomorphism theorems for groups. It remains to see the R -module structure through.

1. Let $m \in \ker \varphi$, $r \in R$. Then $(rm)\varphi = r(m\varphi) = r0_M = 0_M$, so $rm \in \ker \varphi$, so indeed $\ker \varphi \leq_R M$.
2. Let $x \in \text{im } \varphi$, $r \in R$. Then $\exists m \in M : m\varphi = x$. Then $rx = r(m\varphi) = (rm)\varphi \in \text{im } \varphi$, so indeed $\ker \varphi \leq_R N$.
3. We need to check all 3 maps are homomorphism of R -modules.
 - $(rm)\pi = rm + \ker \varphi = r(m + \ker \varphi) = r(m\pi)$.
 - $(r(m + \ker \varphi))\bar{\varphi} = (rm + \ker \varphi)\bar{\varphi} = (rm)\varphi = r(m\varphi) = r((m + \ker \varphi)\bar{\varphi})$.
 - $(rx)\iota = rx = r(x\iota)$.

□

Proposition 1.3.6 (2nd isomorphism theorem). If ${}_R M, K \leq_R N$ then

$$\frac{M + K}{M} \cong \frac{K}{M \cap K}.$$

Proposition 1.3.7 (3rd isomorphism theorem). If ${}_R K \leq_R M \leq_R N$ then

$$\frac{N/K}{M/K} \cong \frac{N}{M}.$$

Proposition 1.3.8 (Correspondence theorem). Let ${}_R M \leq_R N$. Denote the set of all submodules of N by $S(N)$ and the set of all submodules of N containing M by $S(N, M)$. Then

$$\begin{aligned} \pi : N &\rightarrow N/M \\ n &\mapsto n + m \end{aligned}$$

gives a bijection

$$\begin{aligned} S(N, M) &\leftrightarrow S(N/M) \\ {}_R M \leq_R A \leq_R N &\mapsto \pi(A) \\ \pi^{-1}(B) &\leftarrow_R B \leq_R N/M \end{aligned}$$

Notation. $\text{Hom}({}_R M, {}_R N) = \{\text{homomorphisms } \varphi : M \rightarrow N\}$. This is an abelian group.
 $\text{End}_R M = \{\text{homomorphisms } \varphi : M \rightarrow M\}$. This is a ring.

Week 2, lecture 2 starts here

Example 1.3.9. Let R be a (noncommutative) ring, $A = M_a(R)$, $B = M_b(R)$, two rings and $V = R^{a \times b}$, which is just an abelian group. Then ${}_A V$ is a left module and V_B is a right module, and there's no natural choice for V to be a right A -module or a left B -module.

Now consider $E = \text{End}_A V$. Our convention turns V into a right E -module, and there is a ring homomorphism

$$\begin{aligned} \varphi : B &\rightarrow E \\ y &\mapsto (\gamma \mapsto \gamma y) \end{aligned}$$

Similarly, if $F = \text{End}_B V$ then V is a left F -module and there is a ring homomorphism $\psi : A \rightarrow F$. In fact they are isomorphisms, the proof is left as an exercise.

Lemma 1.3.10 (The $a = b = 1$ special case). $\text{End}_R R \cong R$.

Proof. Consider

$$\begin{aligned} \varphi : R &\rightarrow \text{End}_R R \\ x &\mapsto \varphi_x : r \mapsto rx \end{aligned}$$

φ is well-defined since φ_x is well-defined. Also, $(sr)\varphi_x = srx = s(r\varphi_x)$, so indeed $\varphi_x \in \text{End}_R R$. Also, $r\varphi_{x+y} = r(x+y) = rx+ry = r\varphi_x + r\varphi_y = r(\varphi_x + \varphi_y)$, $r\varphi_{xy} = rxy = (r\varphi_x)\varphi_y = r(\varphi_x\varphi_y)$, and $r\varphi_{1_R} = r1 = r = r1_{\text{End}_R R}$, so φ is indeed a homomorphism.

Suppose $\varphi_x = 0$, i.e. $r\varphi_x = 0 \forall r \in R$. Then for $r = 1$, $0 = 1\varphi_x = 1x = x$, so $\ker \varphi = \{0\}$, i.e. φ is injective.

Now pick $f \in \text{End}_R R$ and let $x = 1_R f$. Then $\forall r \in R$, $r\varphi_x = rx = r1_R f = rf$. So $f = \varphi_x$, and φ is surjective. \square

2 Basis

2.1 Free module

Notation. Let ${}_R M$ be a left module and X a subset of M . Then

$$\text{Fun}(X, M) := \{\text{functions } X \rightarrow M\}.$$

This is a left R -module, with a submodule

$$\text{Fun}_f(X, M) := \{f : f(x) = 0 \text{ } \forall \text{ but finitely many } x \in X\}.$$

Definition 2.1.1. A subset $X \subseteq {}_R M$ spans M if $\forall m \in M$,

$$\exists f \in \text{Fun}_f(X, R) : m = \sum_{a \in X} f(a)a.$$

X is linearly independent if $\forall f \in \text{Fun}_F(X, R)$,

$$\sum_{a \in X} f(a)a = 0 \implies f(a) = 0 \forall a \in X.$$

X is a *basis* for M if it spans M and is linearly independent.

Definition 2.1.2. ${}_R M$ is *free* if it admits a basis.

Example 2.1.3. 1. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}/n\mathbb{Z}$. Then $\{1 + n\mathbb{Z}\}$ spans M but M is not free, since $nx = 0 \forall x \in M$.

2. $\emptyset \subseteq M$ is linearly independent for any M , since $\text{Fun}(\emptyset, R)$ only has one element $\widehat{0}$ which is identically zero, and summing over nothing gives zero.

3. Let R be a ring, $M = {}_R R$, and $X = \{a\}$. Then

$$\begin{aligned} X \text{ is linearly independent} &\iff (ba = 0 \implies b = 0) \\ X \text{ spans } {}_R R &\iff (\exists b : ba = 1_R) \end{aligned}$$

Lemma 2.1.4. \forall set X and $\forall R$, \exists a free R -module M with a basis of cardinality $|X|$.

Proof. Let $M = \text{Fun}_f(X, {}_R R)$. Then $\forall a \in X$, $\delta_a \in M$ where $\delta_a(b) := \begin{cases} 1_R & a = b \\ 0_R & a \neq b \end{cases}$. This gives us a basis. Indeed,

- For $f \in M$, list all $x_1, \dots, x_n \in X : f(x_i) \neq 0_R$. Then

$$f = f(x_1)\delta_{x_1} + \dots + f(x_n)\delta_{x_n}.$$

So it spans M .

- If $r_1\delta_{x_1} + \dots + r_n\delta_{x_n} = 0_M$ then

$$0_R = (r_1\delta_{x_1} + \dots + r_n\delta_{x_n})(x_i) = r_i\delta_{x_i}(x_i) = r_i \quad \forall i,$$

so $\{\delta_{x_1}, \dots, \delta_{x_n}\}$ is linearly independent. □

Lemma 2.1.5. Every ${}_R M$ is isomorphic to a quotient of a free module.

Proof. Pick $M \subseteq M$ that spans M (e.g. $X = M$). Then

$$\begin{aligned} \varphi : \text{Fun}_F(X, R) &\rightarrow M \\ f &\mapsto \sum_{a \in X} f(a)a \end{aligned}$$

is surjective. By lemma above, $\text{Fun}_F(X, R)$ is free, and by 1st isomorphism theorem,

$$M \cong \text{Fun}_f(X, R) / \ker \varphi. \quad \square$$

Definition 2.1.6. A *partially ordered set* (or *poset*) is denoted (\mathcal{P}, \preceq) where \preceq can be viewed as a subset of $\mathcal{P} \times \mathcal{P}$. If $(x, y) \in \preceq$ we denote it as $x \preceq y$. The \preceq satisfies that it's reflexive, antisymmetric ($x \preceq y, y \preceq x \implies x = y$) and transitive.

A partial order \preceq is *linear order* if $\forall x, y \in \mathcal{P}$, either $x \preceq y$ or $y \preceq x$.

A *chain* is a subset $X \subset \mathcal{P}$ such that (X, \preceq) is a linearly ordered set.

$a \in \mathcal{P}$ is a *maximal element* if $\forall b \in \mathcal{P}, a \preceq b \implies a = b$.

$a \in \mathcal{P}$ is an *upper bound* of a chain X if $\forall b \in X, b \preceq a$.

Lemma 2.1.7 (Zorn's). Let \mathcal{P} be a nonempty poset. If every chain in \mathcal{P} has an upper bound then \mathcal{P} contains a maximal element.

Theorem 2.1.8. Let D be a division ring and ${}_D M$ a module. Then

1. M is free
2. \forall linearly independent $X \subseteq M$, \exists basis $B \supseteq X$
3. \forall spanning $Q \subseteq M$, \exists basis $B \subseteq Q$

Proof. 1. This follows from 2 by taking $X = \emptyset$.

2. Consider poset $\mathcal{P} = \{Z \subseteq M : Z \supseteq X \text{ and } Z \text{ is linearly independent}\}$ with $\preceq = \subseteq$. Then $X \in \mathcal{P}$. Pick a chain $C \subseteq \mathcal{P}$ and consider $Z = \bigcup_{Y \in C} Y$. If $Z \in \mathcal{P}$ then it's obviously an upper bound of C . Now by construction, $Z \supseteq X$. Now if $a_1, \dots, a_n \in Z$, clearly $\exists Y \in C : a_i \in Y$, so $r_1 a_1 + \dots + r_n a_n = 0_M$ would imply $a_i = 0$. Thus, by Zorn's lemma, there is a maximal element $Z \in \mathcal{P}$. We claim Z spans M , and therefore is a basis. Suppose for contradiction $\exists a \in M : a \notin \text{span}(Z)$. Then $\{a\} \cup Z \supsetneq Z$ and is linearly independent. Indeed, if

$$ra + \underbrace{r_1 a_1 + \dots + r_n a_n}_{\in Z} = 0 \text{ and } r \neq 0,$$

then $a \in \text{span}(Z)$, a contradiction, so $r = 0$ and $r_1 a_1 + \dots + r_n a_n = 0$. Since Z is linearly independent, $a_i = 0$. So $\{a\} \cup Z \in \mathcal{P}$, contradicting maximality of Z .

3. Consider poset $\mathcal{P} = \{Z \subseteq M : Z \subseteq Q \text{ and } Z \text{ is linear independent}\}$ with $\preceq = \subseteq$. It's nonempty since $\emptyset \in \mathcal{P}$. Similarly to above, a chain C in \mathcal{P} has an upper bound $X = \bigcup_{A \in C} A$, which spans M by the same argument.

□

2.2 Embark on Artin–Wedderburn theory

Definition 2.2.1. ${}_R M$ is *simple* if $M \neq 0$ and $\forall {}_R N \leq {}_R M$, either $N = 0$ or $N = M$. i.e. Simple modules have exactly two submodules.

Example 2.2.2. 1. $\mathbb{Z}/m\mathbb{Z}$ as a \mathbb{Z} -module is simple iff m is prime.

2. ${}_R R$ is simple iff R is a division ring.

Proof. \Leftarrow : Let ${}_R L \leq {}_R R$ such that ${}_R L \neq 0$. Then $\forall 0 \neq x \in L$, $1_R = x^{-1}x \in L$, so $r = r \cdot 1_R \in L \forall r \in R$, i.e. $L = R$.

\Rightarrow : Let $x \in R$, $x \neq 0$. Then $Rx = \{rx : r \in R\} \leq {}^l R$, so ${}_R Rx \leq {}_R R$, and since $Rx \neq 0$ and ${}_R R$ is simple, one has $Rx = R$, and since $1_R \in R$, $\exists y \in R : yx = 1$. Similarly, $Ry = R$ so $\exists z \in R : zy = 1$, so $x = (zy)x = z(yx) = z$ and y is both left and right inverse of x .

□

Notation. $\mathcal{L}(R) = \{L : L \leq {}^l R\}$. This is a poset under \subseteq . Maximal left ideal is then a maximal element in $(\mathcal{L}(R) \setminus \{R\})$ and minimal left ideal is a minimal element in $(\mathcal{L}(R) \setminus \{0\})$.

Lemma 2.2.3. $L \leq {}^l R$ is maximal iff R/L is a simple left R -module.

Proof. By correspondence theorem,

$$\{L, R\} = \{M : L \subsetneq M \leq {}^l R\} \leftrightarrow \text{nonzero submodules of } R/L.$$

□

Remark. Given ${}_R M \ni m$, we have a homomorphism of R -modules $\varphi_m : {}_R R \rightarrow M : r \mapsto rm$. Indeed, $\varphi_m(sr) = srm = s\varphi_m(r)$. We call the kernel $\ker \varphi_m = \{x \in R : xm = 0\}$ the *annihilator* of m , denoted $\text{Ann}(m)$. 1st isomorphism theorem says $\text{Ann}(m) \leq {}^l R$, and $\text{im } \varphi_m = Rm \cong R/\text{Ann}(m)$.

Lemma 2.2.4. If ${}_R M$ is simple with $x \in M$, $x \neq 0$, then $\text{Ann}(x)$ is a maximal left ideal and $M \cong R/\text{Ann}(x)$.

Proof. One has $x \in \text{im } \varphi_x$, so $\text{im } \varphi_x \neq 0$. By simplicity of M , $\text{im } \varphi_x = M$. $M \cong R/\text{Ann}(x)$ then follows from 1st isomorphism theorem. Maximality of $\text{Ann}(x)$ follows from correspondence theorem.

□

Theorem 2.2.5. A nonzero ring has a maximal left ideal.

Proof. Let R be a nonzero ring and consider poset $\mathcal{P} = \{L \leq {}^l R : L \neq R\}$ with $\preceq = \subseteq$. One has $0 \in \mathcal{P}$ so $\mathcal{P} \neq \emptyset$. Let $C \subseteq \mathcal{P}$ be a chain. Define $I = \bigcup_{L \in C} L$. Clearly I is an additive abelian subgroup, since for $x, y \in I$ then $x \in L_1$ and $y \in L_2$, but C is chain so WLOG $L_1 \supseteq L_2$, so $x, y \in L_1 \implies x - y \in L_1 \implies x - y \in I$. We claim I is in fact a left ideal. Indeed, for $x \in I$, one knows $x \in L \in C$, and $\forall r \in R$, $rx \in L$, so $rx \in I$. Note that $I \neq R$ since $1_R \notin L \forall L \in C$. Therefore I is an upper bound for C , and by Zorn's lemma \mathcal{P} has a maximal element J , which by definition is a maximal left ideal.

□

Corollary 2.2.6. A nonzero ring admits a simple module.

Proof. Let $I \leq {}^l R$ be a maximal ideal of a nonzero ring R , which is guaranteed by theorem above. Then R/I is a simple R -module by 2.2.3.

□

Proposition 2.2.7 (Schur lemma I). If $\varphi : {}_R M \rightarrow {}_R N$ is a homomorphism of simple modules, then either $\varphi = 0$ or φ is an isomorphism.

Proof. Note $\ker \varphi \leq {}_R M$ and $\text{im } \varphi \leq {}_R N$. By simplicity, $\ker \varphi \in \{0, M\}$ and $\text{im } \varphi \in \{0, N\}$, i.e. there are 4 possible cases.

(0, 0) This is impossible, since $\text{im } \varphi = 0 \implies \ker \varphi = M$.

(0, N) This implies precisely φ is an isomorphism.

(M , 0) It follows $\varphi = 0$.

(M , N) This is impossible, since $\ker \varphi = M \implies \text{im } \varphi = 0$.

□

Corollary 2.2.8 (Schur lemma II). If ${}_R M$ is simple then $\text{End}_R M$ is a division ring.

Proof. By Schur lemma I, if ${}_R M$ is simple then every $\varphi \in \text{End}_R M = \{\text{homomorphisms } \varphi : {}_R M \rightarrow {}_R M\}$ either is 0 or has an inverse. □

Example 2.2.9. $R = \mathbb{R}[x]$, $M = \mathbb{R}^2$, $X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. M is an R -module with $f(x)v := f(X)v$. Consider a submodule $N \leq M$, then for $\forall \alpha \in R$, $\alpha 1 \in R$, so $\alpha N \subseteq N$, hence N is a vector subspace. But $\dim N = 1$ is impossible, so M is simple. Suppose it is, then $\forall v \in N : v \neq 0$, $xv = \alpha v$, i.e. v is an eigenvector of X , which has no real eigenvalues, an absurdity. Now we have $\text{End}_R M$ is a division ring, and note that

$$\begin{aligned} \text{End}_R M &= \{f : M \rightarrow M : f(xv) = xf(v)\} = \{Y \in M_2(\mathbb{R}) : XY = YX\} = C_{M_2(\mathbb{R})}(X) \\ &= \left\{ aI + \frac{1}{2}bX^2 : a, b \in \mathbb{R} \right\} \cong \mathbb{C} \text{ via } X \mapsto 1 + i. \end{aligned}$$

Theorem 2.2.10 (baby Artin–Wedderburn). The following are equivalent for a nonzero ring R .

1. Every left R -module is free.
2. R is a division ring.

Proof. $2 \Rightarrow 1$: This is Theorem 2.1.8.1.

$1 \Rightarrow 2$: By Corollary 2.2.6, \exists a simple R -module M , which is free by assumption, i.e. admits a basis $B \subseteq M$. Pick $x \in B$, then $Rx \leq M$ by simplicity has to be M , so $M = Rx \cong R/\text{Ann}(x)$ by Lemma 2.2.4. But $rx = 0_M \implies r = 0_R$ since x is in a basis, so $\text{Ann}(x) = 0$, hence by Lemma 1.3.10, $M \cong R \cong \text{End}_R R \cong \text{End}_R M$ which is a division ring by 2.2.8. □

Week 3, lecture 3 starts here

2.3 Algebra

Definition 2.3.1. An *algebra* is a pair (A, \mathbb{F}) where A is a ring and a \mathbb{F} -vector space such that

1. $\underbrace{x+y}_{\text{in ring}} = \underbrace{x+y}_{\text{in vector space}} \quad \forall x, y \in A$
2. $(\alpha x)y = \alpha(xy) = x(\alpha y) \quad \forall x, y \in A, \alpha \in \mathbb{F}$

Remark. Notions about a ring are extended to algebras like so:

- An ideal of (A, \mathbb{F}) is an ideal of A that is also an \mathbb{F} -vector subspace
- A subalgebra of (A, \mathbb{F}) is a subring of R that is also an \mathbb{F} -vector subspace
- A homomorphism $(A, \mathbb{F}) \rightarrow (B, \mathbb{F})$ is a ring homomorphism $A \rightarrow B$ with \mathbb{F} -linearity
- A module over (A, \mathbb{F}) is a module over A with the action being \mathbb{F} -linear
- A submodule of a module over (A, \mathbb{F}) is a submodule of the module over A and a \mathbb{F} -vector subspace
- A homomorphism of modules over (A, \mathbb{F}) is a module homomorphism with \mathbb{F} -linearity

Lemma 2.3.2. Let R be a ring and \mathbb{F} a field. Then there is a bijection

$$\{\text{algebras } (R, \mathbb{F})\} \leftrightarrow \{\text{ring homomorphisms } \mathbb{F} \rightarrow Z(R)\}.$$

Proof. For an algebra (R, \mathbb{F}) , define $\varphi : \mathbb{F} \rightarrow Z(R) : \alpha \mapsto \alpha 1_R$. (Verify this is indeed a ring homomorphism.) Then by definition, $(\alpha 1_R)x = \alpha x = \alpha(x1) = x(\alpha 1_R) \forall x \in R$, so $\text{im } \varphi \subseteq Z(R)$.

For a ring homomorphism $\varphi : \mathbb{F} \rightarrow Z(R)$, define $\mathbb{F} \times R \rightarrow R : (\alpha, x) \mapsto \varphi(\alpha)x =: \alpha x$. Then $(\alpha\beta)(x) = \varphi(\alpha\beta)x = \varphi(\alpha)(\varphi(\beta)x) = \alpha(\beta x)$ (verify similar statements for $(\alpha + \beta)(x)$ and $\alpha(x + y)$) and $\alpha(xy) = \varphi(\alpha)xy = (\varphi(\alpha)x)y = (\alpha x)y$ and since $\varphi(\alpha) \in Z(R)$ it's also $x(\alpha y)$.

It remains to verify they are indeed inverse bijections:

$$\begin{aligned} & (R, \mathbb{F}) \\ & \rightarrow \varphi : \mathbb{F} \rightarrow Z(R) : \alpha \mapsto \alpha 1_R \\ & \rightarrow \alpha x := \varphi(\alpha)x = \alpha 1_R x = \alpha x \end{aligned}$$

and

$$\begin{aligned} & \varphi : \mathbb{F} \rightarrow Z(R) \\ & \rightarrow \alpha x := \varphi(\alpha)x \\ & \rightarrow \varphi(\alpha) = \alpha 1_R = \varphi(\alpha) \cdot 1 = \varphi(\alpha). \end{aligned}$$

□

Remark. 1. By the structure of a field, the following ring things are automatically algebra things: ideals, modules, submodules, module homomorphisms (ingredients in 1st isomorphism theorem). e.g. Suppose M is a module over algebra (A, \mathbb{F}) and N is a submodule of M for the ring A . Then $\forall \alpha \in \mathbb{F}, n \in N, \alpha n = (\alpha 1_A)n \in N$ since $\alpha 1_A \in Z(A)$. So N is a subspace and hence a submodule of the algebra (A, \mathbb{F}) .

2. Subrings and ring homomorphisms are different. Consider the algebra (\mathbb{C}, \mathbb{Q}) , then $\mathbb{Z}[i] \leq \mathbb{C}$ is not a subalgebra. Also, for the algebra $A = (\mathbb{C}, \mathbb{C})$, $\varphi : A \rightarrow A : x \mapsto \bar{x}$ is a ring homomorphism $\mathbb{C} \rightarrow \mathbb{C}$ but not an algebra homomorphism since it's not \mathbb{C} -linear.

Definition 2.3.3. Let (A, \mathbb{F}) be an algebra with a \mathbb{F} -basis of A e_1, \dots, e_n . Then one can write $\forall i, j = 1, \dots, n$

$$e_i \cdot e_j = \sum_k c_{ij}^k e_k,$$

where $c_{ij}^k \in \mathbb{F}$, called *structure constants*, determine and are determined by the algebra structure of (A, \mathbb{F}) .

Example 2.3.4. The quaternions $\mathbb{H} = \mathbb{R}^4$ with basis $1, i, j, k$ has the structure constants table:

	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

Week 4, lecture 1 starts here

2.3.1 Polynomial

The video recording was completely black! See notes given by Dmitriy. The following is the best I can manage:

Proposition 2.3.5. If $n \geq 1$ then $\dim_{\mathbb{F}} \mathbb{F} \langle x_1, \dots, x_n \rangle$ is countable.

Proposition 2.3.6 (Universal property). Let (A, \mathbb{F}) be an algebra. Then $\forall a_1, \dots, a_n \in A, \exists!$ homomorphism of algebras $\varphi : \mathbb{F} \langle x_1, \dots, x_n \rangle \rightarrow A : \varphi(x_i) = a_i \forall i$.

Proof. Define φ by $x_1 \cdots x_n \mapsto a_1 \cdots a_n$ and extend by \mathbb{F} -linearity, so that it's an algebra homomorphism. Suppose $\psi : \mathbb{F} \langle x_1, \dots, x_n \rangle \rightarrow A$ is another such homomorphism, then $\varphi(x_i) = \psi(x_i) = a_i$ and by properties of homomorphism and linearity they must then be the same map. □

2.3.2 Noncommutative Nullstellensatz

Definition 2.3.7. Let (A, \mathbb{F}) be an algebra with $\alpha \in A$. Consider the algebra homomorphism $\varphi_{\alpha} : \mathbb{F}[x] \rightarrow A : x \mapsto \alpha$. Since $\mathbb{F}[x]$ is a PID, $\ker f$ is generated by one element $\mu_{\alpha}(x)$, called the *minimal polynomial* of α . One says α is *transcendental* if $\mu_{\alpha} \equiv 0$ and *algebraic* if $\mu_{\alpha} \not\equiv 0$.

Example 2.3.8. $A = M_n(\mathbb{F}) \ni \alpha$, then all α are algebraic by Cayley–Hamilton theorem.

If $\dim_{\mathbb{F}} A < \infty$ then $1, \alpha, \alpha^2, \dots$ are linearly dependent, so all α are algebraic.

Lemma 2.3.9. If (D, \mathbb{F}) is a division algebra, then $\forall \alpha \in D \setminus \{0\}$, $\mu_{\alpha}(x) \in \mathbb{F}[x]$ is irreducible.

Proof. Suppose $\mu_{\alpha}(x) = g(x)h(x)$ with $0 < \deg g < \deg \mu_{\alpha}$, but then since $\mu_{\alpha}(\alpha) = 0$ and D is a division ring, WLOG $g(\alpha) = 0$, contradicting minimality of μ_{α} . \square

Week 4, lecture 2 starts here

Theorem 2.3.10 (Amitsur–Schur lemma). If (A, \mathbb{F}) is an algebra with $\dim_{\mathbb{F}} A < |\mathbb{F}|$ and M is simple A -module, then any $d \in D = \text{End}_A M$ (also an \mathbb{F} algebra) is algebraic over \mathbb{F} .

Proof. Note that $\dim_{\mathbb{F}} D \leq \dim_{\mathbb{F}} M \leq \dim_{\mathbb{F}} A < |\mathbb{F}|$. Indeed, since M is simple, $\forall m \in M, m \neq 0$, $M \cong A/\text{Ann}(m)$ (Lemma 2.2.4), so $\dim_{\mathbb{F}} M \leq \dim_{\mathbb{F}} A$; now pick $m \in M, m \neq 0$ and consider $\alpha_m : D \rightarrow M : x \mapsto mx$. This is injective: suppose $\alpha_m(x) = 0$, but $M = Am$ by simplicity, so $\forall \tilde{m} \in M, \exists a \in A : \tilde{m} = am$. Then $\tilde{m}x = a(mx) = a\alpha_m(x) = 0$, so $x = 0_D$.

Now let $d \in D$. Note $\mathbb{F} = \mathbb{F}1_D \leq Z(D)$, and if $d \in \mathbb{F}$ then $d = \alpha 1_D$ for some $\alpha \in \mathbb{F}$, so minimal polynomial of d is simply $z - \alpha$, hence algebraic. Suppose now $d \notin \mathbb{F}$. Then $d - \alpha \notin \mathbb{F} \forall \alpha \in \mathbb{F}$. This implies $(d - \alpha)^{-1}$ are linearly dependent over \mathbb{F} , hence $\exists \gamma_1, \dots, \gamma_n$ all $\neq 0$ such that

$$\gamma_1 \frac{1}{d - \alpha_1} + \dots + \gamma_n \frac{1}{d - \alpha_n} = 0.$$

Now note that $\alpha_i \in \mathbb{F}$, so all $(d - \alpha_i)$ commute, hence $(d - \alpha_i)^{-1}$ commute as well, since

$$xy = yx \implies y = x^{-1}xy = x^{-1}yx \implies yx^{-1} = x^{-1}yx x^{-1} = x^{-1}y$$

and doing the same trick for y one yields $x^{-1}y^{-1} = y^{-1}x^{-1}$. We can therefore multiply $(d - \alpha_1)(d - \alpha_2) \cdots (d - \alpha_n)$ on both sides and get

$$\gamma_1(d - \alpha_2) \cdots (d - \alpha_n) + \gamma_2(d - \alpha_1)(d - \alpha_3) \cdots (d - \alpha_n) + \dots + \gamma_n(d - \alpha_1) \cdots (d - \alpha_{n-1}) = 0.$$

In other words, if we let

$$f(z) = \sum_{i=1}^n \gamma_i \frac{\prod_{k=1}^n (z - \alpha_k)}{z - \alpha_i}$$

then $f(d) = 0$. One has d is algebraic as long as $f \neq 0$. And indeed $f \neq 0$, since

$$f(\alpha_1) = \gamma_1(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \cdots (\alpha_1 - \alpha_n) \neq 0.$$

\square

Corollary 2.3.11 (Noncommutative Nullstellensatz). If (A, \mathbb{C}) is an algebra with A finitely generated and M is a simple A -module, then $\text{End}_A M = \mathbb{C}$.

Proof. Suppose A is generated by a_1, \dots, a_n . Then $\mathbb{C}\langle x_1, \dots, x_n \rangle \rightarrow A : x_i \mapsto a_i$ is surjective. By 2.3.5, $\dim_{\mathbb{C}} A$ is at most countable, so by theorem above, any $d \in \text{End}_A M$ is algebraic over \mathbb{C} and let $f_d(z) \in \mathbb{C}[z]$ be its minimal polynomial. By 2.3.9, it's irreducible, but since \mathbb{C} is algebraically closed, $f_d(z)$ must be of the form $\alpha z - \beta$ where $\alpha \neq 0$. It follows that $d \in \mathbb{C}$. \square

Corollary 2.3.12 (Weak Nullstellensatz). Let $I \triangleleft \mathbb{C}[x_1, \dots, x_n]$ be a proper ideal. Then $\exists (a_i) \in \mathbb{C}^n : \forall f \in I, f(a_1, \dots, a_n) = 0$.

Proof. Adapt proof of Theorem 2.2.5 with \mathcal{P} now being the poset of all left ideals $J \trianglelefteq R$ such that $J \supseteq I$ and $J \neq R$. The maximal element L the argument produces gives a simple $\mathbb{C}[x_1, \dots, x_n]$ -module $M = \mathbb{C}[x_1, \dots, x_n]/L$ (2.2.3). Now each x_i defines $\widehat{x}_i : f + L \mapsto x_i f + L \in \text{End}_{\mathbb{C}[x_1, \dots, x_n]} M$, and by corollary above, $\text{End}_{\mathbb{C}[x_1, \dots, x_n]} M = \mathbb{C}$, so let $\widehat{x}_i = a_i \in \mathbb{C}$. Let $h(x_1, \dots, x_n) \in I \subseteq L$ and consider $\widehat{h} : f + L \mapsto hf + L$. Since $h \in L$, \widehat{h} is identically zero, i.e. $\widehat{h} = 0$, but on the other hand,

$$\widehat{h} = h(\widehat{x}_1, \dots, \widehat{x}_n) = h(a_1, \dots, a_n) \in \mathbb{C},$$

the desired is thus proven. \square

Week 4, lecture 3 starts here

3 Division

3.1 Quaternion

By writing down the fundamental formula for quaternions $i^2 = j^2 = k^2 = ijk = -1$, Sir William Rowan Hamilton defined, in modern language, the quotient algebra

$$\mathbb{H} = \mathbb{R} \langle x_1, x_2, x_3 \rangle / I \text{ where } I = (1 + x_1^2, 1 + x_2^2, 1 + x_3^2, 1 + x_1x_2x_3),$$

and i, j, k are then $x_1 + I, x_2 + I, x_3 + I$.

Proposition 3.1.1. Products of i, j, k are as the table in 2.3.4.

Proof. The diagonal is immediate from the formula. Now

$$\begin{aligned} -i &= -iijk = -jk & \implies & \quad jk = i \\ -k &= ijkk = -ij & \implies & \quad ij = k \end{aligned}$$

and similarly for the rest. □

Proposition 3.1.2. $1, i, j, k$ is a basis for (\mathbb{H}, \mathbb{R}) .

Proof. Clearly $1, i, j, k$ generate \mathbb{H} and any product is a linear combination of $1, i, j, k$. It remains to show they are linearly independent. Consider an algebra homomorphism $f : \mathbb{R} \langle x_1, x_2, x_3 \rangle \rightarrow M_2(\mathbb{C})$ given by

$$\begin{aligned} x_1 &\mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = A_1 \\ x_2 &\mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = A_2 \\ x_3 &\mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = A_3. \end{aligned}$$

We claim $I \subseteq \ker f$. Indeed $A_1^2 = A_2^2 = A_3^2 = -1_{M_2(\mathbb{C})}$ so $1 + x_i^2 \in \ker f$, and $A_1A_2A_3 = -1_{M_2(\mathbb{C})}$ so $1 + x_1x_2x_3 \in \ker f$. Hence $\bar{f} : \mathbb{H} \rightarrow M_2(\mathbb{C})$ given by $i \mapsto A_1, j \mapsto A_2, k \mapsto A_3$ is a well-defined algebra homomorphism. Since I, A_1, A_2, A_3 are linearly independent over \mathbb{R} , so are $1, i, j, k$. □

3.1.1 Quaternions form a division ring

Definition 3.1.3. Similar to complex numbers, quaternions can be divided into their *real part* and *imaginary part*, i.e. one can write $X = \alpha + x$ where $\alpha \in \mathbb{R}$ and $x \in \text{span}(i, j, k) = \mathbb{H}_0$. *Conjugation* is defined similarly as well: $X^* := \alpha - x$, e.g. $(3 + 5i - 77j)^* = 3 - 5i + 77j$. One also has

$$\Re X = \frac{q + q^*}{2}, \quad \Im X = \frac{q - q^*}{2}.$$

Define and notate the *norm* as $q(X) = XX^*$. Notate the usual Euclidean distance by $\|x\| = \sqrt{q(x)}$.

Theorem 3.1.4. If $\alpha, \beta \in \mathbb{R}$ and $x, y \in \mathbb{H}_0$ then

$$(\alpha + x)(\beta + y) = \underbrace{\alpha\beta - x \cdot y}_{\in \mathbb{R}} + \underbrace{\alpha y + \beta x + x \times y}_{\in \mathbb{H}_0}.$$

Proof. One has

$$(\alpha + x)(\beta + y) = \alpha\beta + \alpha y + \beta x + xy,$$

so it remains to show $xy = x \times y - x \cdot y$. Write $x = \alpha i + \beta j + \gamma k$ and $y = \hat{\alpha} i + \hat{\beta} j + \hat{\gamma} k$, then

$$\begin{aligned} xy &= -(\alpha\hat{\alpha} + \beta\hat{\beta} + \gamma\hat{\gamma}) + (\beta\hat{\gamma} - \hat{\beta}\gamma)i + (\gamma\hat{\alpha} - \alpha\hat{\gamma})j + (\alpha\hat{\beta} - \beta\hat{\alpha})k \\ &= -x \cdot y + x \times y. \end{aligned}$$

□

Corollary 3.1.5. $q(X) = q(\alpha + \beta i + \gamma j + \delta k) = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$.

Proof. Write $X = \alpha + \nu$. Then by definition,

$$q(X) = (\alpha + \nu)(\alpha - \nu) = \alpha^2 - \nu \cdot (-\nu) - \alpha\nu + \alpha\nu - \nu \times \nu = \alpha^2 + \nu \cdot \nu,$$

which is what's desired. \square

Corollary 3.1.6. $(qp)^* = p^*q^*$.

Proof. Write $p = \alpha + x$ and $q = \beta + y$. Then

$$(qp)^* = (\alpha\beta - x \cdot y + \beta x + \alpha y + y \times x)^* = \alpha\beta - x \cdot y - \beta x - \alpha y - y \times x$$

and

$$(\alpha - x)(\beta - y) = \alpha\beta - (-x) \cdot (-y) - \alpha y - \beta x + (-x) \times (-y),$$

the desired then follows from $(-x) \times (-y) = -y \times x = x \times y$ (the other parts don't care about orders). \square

Corollary 3.1.7. $\|pq\| = \|p\|\|q\|$.

Proof. $\|pq\| = (pq)(pq)^* = pqq^*p^* = p\|q\|p^* = pp^*\|q\| = \|p\|\|q\|$. \square

Proposition 3.1.8. \mathbb{H} is a division algebra.

Proof. Let $q \in \mathbb{H}$, $q \neq 0$. Then $\|q\| \neq 0$, and since $qq^* = \|q\|$, one has $q^{-1} = \frac{1}{\|q\|}q^*$. \square

Week 5, lecture 1 starts here

3.1.2 Multiplicative group of quaternions

The group $\mathbb{H}^\times = (\mathbb{H} \setminus \{0\}, \cdot)$ has subgroups $\mathbb{R}_+^\times = \{\alpha : \alpha > 0\}$ and $U(\mathbb{H}) = \{x \in \mathbb{H} : \|x\| = 1\}$ (the 3-sphere).

Proposition 3.1.9 (Polar representation of quaternions). $\mathbb{H}^\times \cong \mathbb{R}_+^\times \times U(\mathbb{H})$.

Proof. Define $f(\alpha, X) = \alpha X$. This is a group homomorphism:

$$f((\alpha, X), (\beta, Y)) = f(\alpha\beta, XY) = \alpha\beta XY = \alpha X \beta Y = f(\alpha, X)f(\beta, Y).$$

f is injective: indeed, let $(\alpha, X) \in \ker f$. Then $\alpha X = 1$ and $X = \alpha^{-1} \in \mathbb{R}$, and since $\|x\| = 1$, $x = \pm 1$, but $\alpha > 0$, so $(\alpha, X) = (1, 1)$.

f is surjective: indeed, pick $X \in \mathbb{H}^\times$ and one can write $X = \|X\| \cdot \|X\|^{-1}X$ where $\|X\| \in \mathbb{R}_+$ and $\| \|X\|^{-1}X \| = \|X\|^{-1}\|X\| = 1$, i.e. $\|X\|^{-1}X \in U(\mathbb{H})$. \square

Proposition 3.1.10. For $X \in \mathbb{H}^\times$, the following hold:

1. $X^2 \in \mathbb{R} \iff X \in \mathbb{R} \cup \mathbb{H}_0$
2. $X^2 \in \mathbb{R}_{>0} \iff X \in \mathbb{R}$
3. $X^2 \in \mathbb{R}_{<0} \iff X \in \mathbb{H}_0$
4. $|X| = 2 \iff X = -1$
5. $|X| = 4 \iff X \in \mathbb{H}_0$ and $\|X\| = 1$

Proof. 1. Write $X = \alpha + x$. Then $X^2 = (\alpha^2 - x \cdot x) + 2\alpha x + \underbrace{x \times x}_0$, hence $\Im X = 2\alpha x$, so $\Im X = 0 \iff \alpha = 0$ or $x = 0 \iff X \in \mathbb{H}_0$ or $X \in \mathbb{R}$.

2, 3. Now suppose $X \in \mathbb{R} \cup \mathbb{H}_0$, then $X^2 = \alpha^2 - x \cdot x$. Note $\alpha = 0$ or $x = 0$. So $X^2 > 0 \iff x = 0 \iff X \in \mathbb{R}$ and $X^2 < 0 \iff \alpha = 0 \iff X \in \mathbb{H}_0$.

4. $X^2 = 1 \iff x = 0$ and $\alpha^2 = 1$, so $\alpha = \pm 1$, but $|1| = 1$ so $\alpha = -1$.

5. By above, $|X| = 4 \implies X^2 = -1$ and this is equivalent to $\alpha = 0$ and $\|x\| = 1$. \square

Proposition 3.1.11 (Quaternionic Euler formula). Write $X = \alpha + \beta x$ where $\alpha, \beta \in \mathbb{R}$ and $x \in U(\mathbb{H}) \cap \mathbb{H}_0$. Then

$$e^X = e^\alpha (\cos \beta + x \sin \beta).$$

Proposition 3.1.12 (de Moivre's formula). If $x \in \mathbb{H}_0 \cap U(\mathbb{H})$ and $n \in \mathbb{N}$ then

$$(\cos \alpha + x \sin \alpha)^n = \cos n\alpha + x \sin n\alpha$$

Proof. $(e^{\alpha x})^n = e^{n\alpha x}$. □

3.1.3 Orthogonal matrix and transformation

Recall that for $(c_1 \ \cdots \ c_n) = A \in \mathbb{R}^{n \times n}$, the following are equivalent:

1. $A^T A = I_n$
2. c_1, \dots, c_n is an orthonormal basis
3. $x \mapsto Ax$ preserves dot product, i.e. $(Ax) \cdot (Ay) = x \cdot y \ \forall x, y \in \mathbb{R}^n$
4. $x \mapsto Ax$ preserves distances, i.e. $\|Ax\| = \|x\| \ \forall x \in \mathbb{R}^n$

We are going to see that \mathbb{C} gives nice description of orthogonal transformations on \mathbb{R}^2 and \mathbb{H} gives these of those on \mathbb{R}^3 and \mathbb{R}^4 . Specifically, a unit vector $v_\alpha = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$ (which can also be described as a complex number) determines two orthogonal transformations of \mathbb{R}^2 : $R_\alpha = (v_\alpha \ v_{\alpha+\pi/2}) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$ and $S_\alpha = (v_\alpha \ v_{\alpha-\pi/2}) = \begin{pmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{pmatrix}$ which have determinants ± 1 respectively.

Proposition 3.1.13. $\{S_\alpha, R_\alpha : \alpha \in \mathbb{R}\}$ is precisely the set of 2×2 orthogonal matrices.

Proposition 3.1.14. Rotations on \mathbb{R}^2 are given by left multiplication of $z \in \mathbb{C}$, $\|z\| = 1$.

Proof. This is clear by writing such z as $\cos \alpha + i \sin \alpha$. □

3.1.4 3D rotation

To specify a 3D rotation, we need a directional axis and an angle and use Euler's angle-axis notation $R_{(\alpha, v)}$.

Week 5, lecture 2 starts here

Lemma 3.1.15. If $f \in \mathbb{R}[x]$ is monic and irreducible, then either $f(x) = x - \alpha$ or $x^2 + \alpha x + \beta$ with $\mathcal{D} = \alpha^2 - 4\beta < 0$.

Proof. One has $\exists \lambda \in \mathbb{C} : f(\lambda) = 0$. If $\lambda \in \mathbb{R}$, then $(x - \lambda) \mid f$ so $f = x - \lambda$ by irreducibility. If $\lambda \notin \mathbb{R}$, then $f(\bar{\lambda}) = 0$ and $(x - \lambda)(x - \bar{\lambda}) \mid f(x)$ where $(x - \lambda)(x - \bar{\lambda}) = x^2 + \alpha x + \beta$ with $\mathcal{D} < 0$ and again by irreducibility $f(x) = x^2 + \alpha x + \beta$. □

Corollary 3.1.16. Let $V_{\mathbb{R}}$ be a vector space with $\dim_{\mathbb{R}} V$ odd and $L : V \rightarrow V$ a linear operator. Then L admits a real eigenvalue.

Proof. Write the characteristic polynomial $\chi_L(z)$ of L as $\pm f_1 \cdots f_n$ where f_i are all monic and irreducible, but $\deg \chi$ is odd, so there must be one $f_i = x - \alpha$, where α is the desired eigenvalue. □

Recall Sylvester's theorem from MA251.

Lemma 3.1.17. If $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is special orthogonal ($\det L = 1$), then \exists orthonormal basis in which the matrix of L is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix}.$$

Proof. L admits eigenvalue $\alpha \in \mathbb{R}$ by previous lemma, so $Lx = \alpha x$ for some $x \in \mathbb{R}^3 \setminus \{0\}$. Since $\|x\| = \|Lx\| = |\alpha| \|x\|$, $\alpha = \pm 1$. Now $Lx^\perp \subseteq x^\perp$. Indeed, let $y \in x^\perp$, then $x \cdot y = 0$, and $0 = x \cdot y = Lx \cdot Ly = \pm x \cdot Ly$, so $Ly \in x^\perp$. Consider the two cases.

1. $\alpha = 1$, then $L|_{x^\perp} : x^\perp \rightarrow x^\perp$ is orthogonal of $\det = 1$, so $L|_{x^\perp} = R_\alpha$ and in an orthonormal basis $\frac{1}{\|x\|}x, y, z$, L has the desired form.
2. $\alpha = -1$, then $L|_{x^\perp} : x^\perp \rightarrow x^\perp$ is orthogonal of $\det = -1$, so $L|_{x^\perp}$ is reflection $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and one has orthonormal basis $y, \frac{1}{\|x\|}x, z$ such that

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\text{where } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = R_\pi.$$

□

We now bring quaternions in by identifying $\mathbb{R}^3 \cong \mathbb{H}_0 \ni x$ and rotation as $R_{x,\alpha}$.

Lemma 3.1.18. $\forall w \in \mathbb{H}_0$,

$$R_{x,\alpha}(w) = e^{\frac{\alpha}{2}x} w e^{-\frac{\alpha}{2}x}.$$

Proof. Pick any $y : x \cdot y = 0$ and $\|y\| = 1$. Define $z := x \times y$. Then x, y, z behave exactly like i, j, k , so it suffices to check the lemma on the basis x, y, z . Now a priori one has

$$R_{x,\alpha}(x) = x, \quad R_{x,\alpha}(y) = y \cos \alpha + z \sin \alpha, \quad R_{x,\alpha}(z) = -y \sin \alpha + z \cos \alpha,$$

and let's check the case for z :

$$\begin{aligned} e^{\frac{\alpha}{2}x} z e^{-\frac{\alpha}{2}x} &= \left(\cos \frac{\alpha}{2} + x \sin \frac{\alpha}{2} \right) z \left(\cos \frac{\alpha}{2} - x \sin \frac{\alpha}{2} \right) \\ &= \left(z \cos \frac{\alpha}{2} - y \sin \frac{\alpha}{2} \right) \left(\cos \frac{\alpha}{2} - x \sin \frac{\alpha}{2} \right) \\ &= z \cos^2 \frac{\alpha}{2} - y \cos \frac{\alpha}{2} \sin \frac{\alpha}{2} - y \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} - z \sin^2 \frac{\alpha}{2} \\ &= z \left(\cos^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2} \right) - 2y \cos \frac{\alpha}{2} \sin \frac{\alpha}{2} \\ &= z \cos \alpha - y \sin \alpha. \end{aligned}$$

The remaining two are left as enjoyment.

□

Theorem 3.1.19.

$$\begin{aligned} \varphi : U(\mathbb{H}) &\rightarrow SO(\mathbb{H}_0) \cong SO_3(\mathbb{R}) \\ x &\mapsto (z \mapsto xzx^{-1}) \end{aligned}$$

is a surjective 2-to-1 group homomorphism.

Proof. Check φ is indeed a group homomorphism:

- $\varphi(x) \in SO(\mathbb{H}_0)$ since $\|xzx^{-1}\| = \|x\|\|z\|\|x^{-1}\| = \|z\| \forall z \in \mathbb{H}_0$.
- $\varphi(xy)(z) = (xy)z(xy)^{-1} = x(yzy^{-1})x^{-1} = \varphi(x)(\varphi(y)(z))$.

Now 3.1.17 says $L = R_{x,\alpha}$ and 3.1.18 says $L = \varphi(e^{\frac{\alpha}{2}x}) \in \text{im } \varphi$, so φ is surjective.

If $x \in \ker \varphi$ then $xzx^{-1} = z$, i.e. $z \in Z(\mathbb{H}) = \mathbb{R}$ so $z = \pm 1$, hence in particular $|\ker \varphi| = 2$.

□

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3.1.5 4D scroll

Rotations in 4D can be understood by identifying $\mathbb{R}^4 \cong \mathbb{H}$. For $x \in U(\mathbb{H})$, define $L_x : z \mapsto xz$ and $R_x : z \mapsto zx$, called *left scroll* and *right scroll*, which are clearly orthogonal. They are also special orthogonal (see Lemma 3.1.19 in Dmitriy's notes). Analogously,

Theorem 3.1.20.

$$\begin{aligned}\varphi : U(\mathbb{H}) \times U(\mathbb{H}) &\rightarrow SO(\mathbb{H}) \cong SO_4(\mathbb{R}) \\ (x, y) &\mapsto L_x R_{y^{-1}}\end{aligned}$$

is a surjective 2-to-1 group homomorphism.

Example 3.1.21. Consider $f : 1 \mapsto i \mapsto j \mapsto k \mapsto -1 \in SO(\mathbb{H})$. Write it in the form as in previous theorem:

1. We need to fix 1 by

$$L_{-i}f : 1 \mapsto (-i)i = 1, \quad i \mapsto (-i)j = -k, \quad j \mapsto (-i)k = j, \quad k \mapsto (-i)(-1) = i.$$

2. Identify the axis of $L_{-i}f|_{\mathbb{H}_0}$, i.e. the vector that's fixed, which in this case is j .

3. Find the angle: let $(k, i, j) \cong (x, y, z)$ be the positively oriented basis in \mathbb{R}^3 and one can see it's a rotation by $\pi/2$, hence

$$L_{-i}f(w) = e^{\frac{\pi}{4}j} w e^{-\frac{\pi}{4}j}, \quad \text{i.e. } L_{-i}f = L_{e^{\frac{\pi}{4}j}} R_{e^{-\frac{\pi}{4}j}}$$

4. Assemble:

$$f = L_i L_{e^{\frac{\pi}{4}j}} R_{e^{-\frac{\pi}{4}j}} = L_{ie^{\frac{\pi}{4}j}} R_{e^{-\frac{\pi}{4}j}},$$

where $ie^{\frac{\pi}{4}j} = i \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j \right) = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}k$ and $e^{-\frac{\pi}{4}j} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j$. Let's check this on j :

$$\begin{aligned}\left(\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}k \right) j \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j \right) &= \frac{1}{2}(i+k)(j+1) \\ &= \frac{1}{2}(ij + i + kj + k) = \frac{1}{2}(k + i - i + k) = k.\end{aligned}$$

3.2 Division algebra over \mathbb{R}, \mathbb{C}

Proposition 3.2.1. \mathbb{C} is the only finite dimensional division algebra over \mathbb{C} .

Proof. Let D be such algebra and $a \in D$. Lemma 2.3.9 says $\mu_a(z) \in \mathbb{C}[z]$ is irreducible, but then $\mu_a(z) = z - \alpha$ where $\alpha \in \mathbb{C}$, so $a \in \mathbb{C}$. \square

Proposition 3.2.2. If D is a division algebra over \mathbb{R} and $\dim_{\mathbb{R}} D$ is odd, then $D = \mathbb{R}$.

Proof. Pick $a \in D$, and left multiplication $L_a : D \rightarrow D$ admits a real eigenvalue α , so $L_a(x) = \alpha x$ for some $x \in D$, $x \neq 0$, but then $ax = \alpha x \implies (a - \alpha)x = 0 \implies a - \alpha = (a - \alpha)xx^{-1} = 0x^{-1} = 0$, so $a = \alpha \in \mathbb{R}$. \square

Definition 3.2.3. For a finite dimensional algebra (A, \mathbb{F}) , define the (*algebraic*) trace as

$$\text{Tr}_A : A \rightarrow \mathbb{F} : a \mapsto \text{Tr}(L_a),$$

the trace of matrix of left multiplication.

Example 3.2.4. $x + yi \in \mathbb{C}$, then $(x + yi)1 = x + yi$ and $(x + yi)i = -y + xi$, so in the basis $1, i$, L_{x+yi} is given by $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, so $\text{Tr}_{\mathbb{C}}(x + iy) = 2x$.

Similarly $\text{Tr}_{\mathbb{H}}(\alpha + x) = 4\alpha$.

Lemma 3.2.5. If (A, \mathbb{F}) is a finite dimensional algebra, then

1. $\text{Tr}_A : A \rightarrow \mathbb{F}$ is a linear map
2. $\text{Tr}_A(\alpha 1_A) = \alpha \dim_{\mathbb{F}} A \quad \forall \alpha \in \mathbb{F}$

Proof. 1. This is clear after writing $\text{Tr}_A : A \rightarrow \text{End}_{\mathbb{F}} A \rightarrow \mathbb{F}$ where the two arrow are linear

2. Also trivial since $L_{\alpha} = \alpha \text{id}_A$.

\square

Corollary 3.2.6. $A = \mathbb{F} \oplus A_0$ where $A_0 := \ker \text{Tr}_A$.

Lemma 3.2.7. If $a \in A$ then $\mu_a(z)$ is the minimal polynomial of L_a .

Proof. Note that

$$L_{a^n}(x) = a^n x = \underbrace{a \cdots a}_n x = (L_a)^n(x),$$

so for any polynomial $f(z)$, $f(L_a) = L_{f(a)}$. Now

$$f(a) = 0 \implies f(L_a) = L_0 = 0$$

and

$$f(L_a) = 0 \implies 0 = f(L_a)(1_A) = L_{f(a)} = f(a)1 = f(a),$$

so L_a and a satisfy the same polynomials. \square

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Lemma 3.2.8. Let D be a finite division algebra over \mathbb{R} and $a \in D_0 = \ker \text{Tr}_D$. Then $a^2 \in \mathbb{R}$, $a^2 \leq 0$ and $a^2 = 0 \iff a = 0$.

Proof. 1. By 3.1.15 and 2.3.9, the minimal polynomial of a is $\mu_a(x) = x^2 + \alpha x + \beta$ with $\mathcal{D} = \alpha^2 - 4\beta < 0$. Also $\mu_a = \mu_{L_a}$, where $L_a : D \rightarrow D$ is a linear map with eigenvalues the roots of $\mu_a(x)$ and $\chi_{L_a}(x) = \mu_a(x)^{\frac{1}{2} \dim D}$. Denote $n = \dim D$ (which is even), then one can write

$$\chi_{L_a}(x) = x^n - \text{Tr}(L_a)x^{n-1} + \cdots = x^n + \frac{n}{2}dx^{n-1} + \cdots,$$

so $-\text{Tr}(L_a) = \frac{n}{2}\alpha$. But $\text{Tr}(L_a) = \text{Tr}_D(a) = 0$ since $a \in D_0$. It follows $\alpha = 0$, $a^2 + \beta = 0$ and $-4\beta = \mathcal{D} \leq 0$, so $a^2 = -\beta \in \mathbb{R}$ and $a^2 \leq 0$.

2. Obvious since D is a division ring. \square

Definition 3.2.9. Equip D_0 with euclidean form

$$\begin{aligned} q : D_0 &\rightarrow \mathbb{R} \\ a &\mapsto -a^2 \geq 0 \end{aligned}$$

and

$$\begin{aligned} \tau : D_0 \times D_0 &\rightarrow \mathbb{R} \\ (a, b) &\mapsto \frac{1}{2}(q(a+b) - q(a) - q(b)) \\ &= \frac{1}{2}(-(a+b)^2 + a^2 + b^2) = -\frac{1}{2}(ab + ba) \end{aligned}$$

Lemma 3.2.10. (D_0, τ) is a finite dimensional euclidean space.

Proof. Note $\tau(a, b) = -\frac{1}{2}(ab + ba)$ is symmetric bilinear and

$$a \neq 0 \implies \tau(a, a) = q(a) = -a^2 \in \mathbb{R}_{>0}.$$

\square

Lemma 3.2.11. If e_1, \dots, e_n is an orthonormal basis of D_0 then $e_i^2 = -1$ and if $i \neq j$ then $e_i e_j = -e_j e_i$.

Proof. First note $e_i^2 = -q(e_i) = -1$. Then

$$0 = \tau(e_i, e_j) = -\frac{1}{2}(e_i e_j + e_j e_i),$$

so $e_i e_j = -e_j e_i$. \square

Corollary 3.2.12. Suppose $i < j < k$, then $e_k = \pm(e_i e_j)^{-1}$.

Proof. Let $u = e_i e_j e_k$, then $u^2 = e_i e_j \underbrace{e_k e_i}_{-e_i e_k} \underbrace{e_j e_k}_{-e_k e_j} = \underbrace{e_i e_j}_{-e_j e_i} e_i \underbrace{e_k e_k}_{-1} e_j = e_j e_i e_i e_j = -e_j e_j = 1$. Then $u^2 - 1 = (u-1)(u+1) = 0$, and since D is division, $u = \pm 1$, i.e. $e_i e_j e_k = \pm 1$, which gives the desired after rearranging. \square

Theorem 3.2.13 (Frobenius). A finite dimensional division algebra over \mathbb{R} is isomorphic to \mathbb{R}, \mathbb{C} or \mathbb{H} .

Proof. Consider values of $n = \dim_{\mathbb{R}} D$.

1. $n = 1$, then $D = \mathbb{R}$.
2. $n = 2$, then e_1 is a basis of D_0 with $e_1^2 = -1$, so $D \cong \mathbb{C}$ via $i \mapsto e_1$.
3. $n = 3$, then $D = \mathbb{R}$ by 3.2.2.
4. $n = 4$, then e_1, e_2, e_3 is a basis of D_0 , so $D \cong \mathbb{H}$ via $i \mapsto e_1, j \mapsto e_2, k \mapsto e_1 e_2$.
5. $n \geq 5$, then $\exists e_1, e_2, e_3, e_4$, but $e_3 = \pm(e_1 e_2)^{-1}$ and $e_4 = \pm(e_1 e_2)^{-1}$, so $e_3 = \pm e_4$, contradicting linear independence of a basis.

□

Theorem 3.2.14. A countably generated division algebra over \mathbb{R} is isomorphic to \mathbb{R}, \mathbb{C} or \mathbb{H} .

Proof. Consider such D . The Amitsur trick (2.3.10) tells us any $d \in D$ is algebraic over \mathbb{R} . But since D is division, $\forall d \in D \setminus \mathbb{R}$, $\mu_d(x) = x^2 + \alpha x + \beta$ with $\mathcal{D} < 0$ again by 3.1.15 and 2.3.9. So now suppose $D \neq \mathbb{R}$ and pick $a \in D \setminus \mathbb{R}$, then $a^2 = -\alpha_a a - \beta_a$, so $\mathbb{R}(a) \cong \mathbb{C}$. If $\mathbb{R}(a) = D$ we are done, so suppose $\mathbb{R}(a) \neq D$ and pick $b \in D \setminus \mathbb{R}(a)$. One has

$$\mu_{a+b}(x) = (a+b)^2 + \alpha_{a+b}(a+b) + \beta_{a+b} = a^2 + ab + ba + b^2 + \dots = 0,$$

so

$$ba = -(a^2 + b^2 + ab + \alpha_{a+b}(a+b) + \beta_{a+b}).$$

This implies $\mathbb{R}\langle a, b \rangle$, the subalgebra generated by a, b , is spanned by $1, a, b, ab$, so

$$3 \leq \dim \mathbb{R}\langle a, b \rangle \leq 4,$$

but $\mathbb{R}\langle a, b \rangle$ is a division algebra since $\forall d \in D$,

$$d^{-1} = \beta_d^{-1}(d + \alpha_d),$$

so $\mathbb{R}\langle a, b \rangle = \mathbb{H}$ by Frobenius. If $\mathbb{R}\langle a, b \rangle = D$ we are done, so pick $c \in D \setminus \mathbb{R}\langle a, b \rangle$ and consider $\mathbb{R}\langle a, b, c \rangle$. Similarly, it is division and is spanned by $1, a, b, c, ab, bc, ac$, so

$$5 \leq \dim \mathbb{R}\langle a, b, c \rangle \leq 7,$$

contradicting Frobenius.

□

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3.3 Finite division ring

Proposition 3.3.1. If R is a commutative ring and $I \trianglelefteq R$ then I is maximal iff R/I is a field.

Proof. \Rightarrow Pick $0 \neq x + I \in R/I$, then $x \notin I$ and $J := Rx + I \not\subseteq I$, so maximality of I tells us $J = R \ni 1$, i.e. $\exists y \in R, z \in I : 1 = xy + z$, but then $1 + I = (x + I)(y + I)$, hence $y + I$ is the inverse of $x + I$.

\Leftarrow It follows 0 and R/I are the only ideals and in particular they are the only R -submodules of R/I . Correspondence theorem gives us a bijection between submodules of R/I and submodules of R containing I . Hence there are only two submodules of R containing I and they can only be R and I , which is equivalent to that I is maximal.

□

Corollary 3.3.2. If R is a PID and $I = (r) \trianglelefteq R$, then the following are equivalent:

1. r is irreducible
2. I is maximal
3. R/I is a field

Proof. • $2 \iff 3$: This is 3.3.1.

- $2 \implies 1$: We write $r = xy$ and we want to show x or y is a unit. Note (x) contains I , so by maximality either
 1. $(x) = R \ni 1$, hence $\exists z \in R : xz = 1$ so x is a unit; or
 2. $(x) = I \ni r$, hence $\exists z : x = rz$ so $r = xy = rzy$ and since R is a domain $zy = 1$ so y is a unit
- $1 \implies 2$: Pick $J \trianglelefteq R : J \supseteq I$. Then $J = (x) \ni r$, so $\exists y : r = xy$. Since r is irreducible, either
 1. x is a unit, hence $J = R$.
 2. y is a unit, hence $x = ry^{-1}$ so $J = (x) = (r) = I$.

□

Recall that if \mathbb{F} is a field then $\mathbb{F}[x]$ is a PID and $R = \mathbb{F}[x]/I$ where $I = (f(x))$ is a field iff f is irreducible.

Lemma 3.3.3. If \mathbb{F} is a field and $\deg f = n$ then for any $z \in \mathbb{F}[x]/(f(x))$,

$$\exists! h(x) \in \mathbb{F}[x]_{\leq n-1} : z = h + I.$$

Proof. Write $z = g(x) + I$, then $g(x) = q(x)f(x) + r(x)$ where $\deg r \leq n-1$, so $z = r + I$. Now suppose $z = r + I = s + I$, then $r - s \in I$ with $\deg(r - s) \leq n-1$, so $r - s = 0 \implies r = s$. □

Example 3.3.4. Consider $A = \mathbb{Q}[x]/I$ where $I = (x^3 - 2x^2 + 1)$. By Eisenstein's criterion $x^3 - 2x^2 + 2$ is irreducible, so A is a field. x^3 is now $2x^2 - 2$ and by previous lemma $1, x, x^2$ is a \mathbb{Q} -basis of A . For example,

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1 = 2x^2 - 2 + 3x^2 + 3x + 1 = 5x^2 + 3x - 1,$$

$$x^4 = x(2x^2 - 2) = 2x^3 - 2x = 2(2x^2 - 2) - 2x = 4x^2 - 2x - 4,$$

$$x^5 = x(4x^2 - 2x - 4) = 4x^3 - 2x^2 - 4x = 4(2x^2 - 2) - 2x^2 - 4x = 6x^2 - 4x - 8,$$

and

$$x^6 = x^3 x^3 = (2x^2 - 2)^2 = \dots$$

In general, one has the multiplication table

	1	x	x^2
1	1	x	x^2
x	x	x^2	$2x^2 - 2$
x^2	x^2	$2x^2 - 2$	$4x^2 - 2x - 4$

and the left multiplication by x and x^2 are

$$L_x = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad L_{x^2} = \begin{pmatrix} 0 & -2 & -4 \\ 0 & 0 & -2 \\ 1 & 2 & 4 \end{pmatrix}$$

with traces

$$\text{Tr}_A(x) = 2, \quad \text{Tr}_A(x^2) = 4$$

and $\text{Tr}_A(1) = \dim A = 3$.

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Example 3.3.5. $\mathbb{F}_3 = \mathbb{Z}/(3)$ is a field of 3 elements.

Note that $\mathbb{Z}/(9)$ is not a field since $3 \cdot 3 = 0_{\mathbb{Z}/(9)}$. So how do we get a field of 9 elements? It is $\mathbb{F}_9 = \mathbb{F}_3[x]/(f(x))$ where f is monic, quadratic and irreducible, so that $1, x$ is a \mathbb{F}_3 basis of \mathbb{F}_9 . Since $f(x)$ is of the form $x^2 + \dots$ and one needs $f(0), f(1), f(2) \neq 0$ for f to be irreducible, so f can only be $x^2 + x + 2$, $x^2 + 1$ or $x^2 + 2x + 2$. The 9 elements of \mathbb{F}_9 can therefore be explicitly written down as: $0, 1, 2$, two roots of $x^2 + x + 2$, two roots of $x^2 + 1$, and two roots of $x^2 + 2x + 2$.

Lemma 3.3.6. If \mathbb{F} is a field and $G \leq \mathbb{F}^\times$ with $|G| < \infty$, then G is cyclic.

Proof. Suppose $|G| = n$. By the fundamental theorem of finitely generated abelian groups, $G \cong C_{k_1} \times C_{k_2} \times \cdots \times C_{k_m}$ where $k_m \mid k_{m-1} \mid \cdots \mid k_1$, $k_m > 1$, and $n = k_1 \cdots k_m$. Then $\forall g \in G$, $g^{k_m} = 1$, i.e. every $g \in G$ satisfies $f(g) = 0$ where $f(x) = x^{k_m} - 1$, so

$$\prod_{g \in G} (x - g) \mid f(x)$$

since $\mathbb{F}[x]$ is a UFD, so

$$n = \deg \prod_{g \in G} (x - g) \leq k_m$$

hence $m = 1$. □

Proposition 3.3.7. Any finite field is isomorphic (as a ring) to $\mathbb{F}_p[x]/(f)$ where p is prime and $f(x) \in \mathbb{F}_p[x]$ is irreducible.

Proof. Let \mathbb{F} be a finite field. Consider $\varphi : \mathbb{Z} \rightarrow \mathbb{F} : n \mapsto n1_{\mathbb{F}}$. Note $\ker \varphi = (p)$ and so $\text{im } \varphi = \mathbb{Z}/\ker \varphi = \mathbb{F}_p \leq \mathbb{F}$ by 1st isomorphism theorem. In particular, \mathbb{F} is an \mathbb{F}_p algebra. By , \mathbb{F}^\times is cyclic, so let $z \in \mathbb{F} : \langle z \rangle = \mathbb{F}^\times$. One has a \mathbb{F}_p algebra homomorphism $\psi : \mathbb{F}_p[x] \rightarrow \mathbb{F} : f(x) \mapsto f(z)$. Since powers of z span \mathbb{F} , ψ is surjective, so $\mathbb{F} \cong \mathbb{F}_p[x]/\ker \psi$, and since $\mathbb{F}_p[x]$ is a PID one can write $\ker \psi = (h)$. By 3.3.2, since \mathbb{F} is a field, h is irreducible. □

Summary:

1. For any prime power $q = p^n$, \exists a field of size q
2. Such field is unique up to isomorphism
3. This field is $\mathbb{F}_p[x]/(f)$ where $\deg f = n$ but such f is not unique

Proposition 3.3.8 (Chinese remainder theorem for $\mathbb{F}[x]$). Write $f = h_1^{a_1} \cdots h_n^{a_n} \in \mathbb{F}[x]$ where $a_i \in \mathbb{N}$ and h_i distinct irreducibles. Then $\mathbb{F}[x]/(f) \cong \mathbb{F}[x]/(h_1^{a_1}) \times \cdots \times \mathbb{F}[x]/(h_n^{a_n})$.

Lemma 3.3.9. If R is a division ring then

1. $Z(R)$ is a field
2. R is a vector space over $Z(R)$
3. $(R, Z(R))$ is an algebra

Proof. 1. $Z(R)$ is a subring so it suffices to show it's division. Let $x \in Z(R)$, then $\exists x^{-1} \in R$, and for $y \in R$ one has $xy = yx$, so $yx^{-1} = x^{-1}xyx^{-1} = x^{-1}yxx^{-1} = x^{-1}y$, hence $x^{-1} \in Z(R)$.

2. follows from 3.
3. $\text{id} : Z(R) \rightarrow Z(R)$ gives the algebra structure.

□

Corollary 3.3.10. If D is a finite division ring then

1. $Z(D) = \mathbb{F}_q$ for prime power q
2. $n = \dim_{\mathbb{F}} D$ is finite
3. $|D| = q^n$

Proof. 1. Note $Z(D)$ is a finite field

2. since D is finite
- 3.

$$|\mathbb{F}_q^n| = \left| \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} : a_i \in \mathbb{F}_q \right\} \right| = q^n.$$

□

Lemma 3.3.11. If D is a division ring then each centraliser $C(x) = \{a \in D : ax = xa\}$ is a $Z(D)$ -subalgebra.

Proof. First note $0, 1 \in C(x)$. Now if $a, b \in C(x)$ then $(a - b)x = ax - bx = xa - xb = x(a - b)$ and $abx = a(xb) = (xa)b$ so $ab, a - b \in C(x)$, hence $C(x)$ is a subring. Also $Z(D) \subseteq C(x)$ so $C(x)$ is closed under scalar multiplication by $\alpha \in Z(D)$. Finally if $a \in C(x)$ then $ax = xa \implies xa^{-1} = a^{-1}axa^{-1} = a^{-1}xaa^{-1} = a^{-1}x$, i.e. $a^{-1} \in C(x)$, hence $C(x)$ is division; so it is a $Z(D)$ -subalgebra. \square

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3.3.1 Finite group action

Recall

Definition 3.3.12. One says a finite group G acts on a finite set X if one can specify a map $G \times X \rightarrow X : (g, x) \mapsto {}^gx$ such that ${}^1x = x$ and ${}^{gh}x = g^h x$.

For $x \in X$ one has the orbit of x : $\text{orb}(x) = {}^Gx = \{{}^gx : g \in G\}$ and the stabiliser of x : $\text{stab}(x) = G_x = \{g : {}^gx = x\}$.

Proposition 3.3.13 (Orbit–Stabiliser formula).

$$|\text{orb}(x)| = |G : \text{stab}(x)| = \frac{|G|}{|\text{stab}(x)|}.$$

Proof. There exists a bijection $\text{orb}(x) \leftrightarrow G/\text{stab}(x)$. \square

Proposition 3.3.14 (Class equation I). Let G act on X and x_1, \dots, x_n representatives of different orbits. Then

$$|X| = \sum_{i=1}^n |\text{orb}(x_i)| = \sum_{i=1}^n \frac{|G|}{|\text{stab}(x_i)|}.$$

Proof. It follows from that $X = \text{orb}(x_1) \sqcup \dots \sqcup \text{orb}(x_n)$ and 3.3.13. \square

Definition 3.3.15. The *fixed point set* is $X^G := \{x : {}^gx = x \ \forall g\} = \{x : |\text{orb}(x)| = 1\}$.

Corollary 3.3.16 (Class equation II). Let y_1, \dots, y_k be representatives of orbits of size ≥ 2 , then

$$|X| = |X^G| + \sum_{i=1}^k \frac{|G|}{|\text{stab}(y_i)|}.$$

We already know if D is a finite division ring then $Z = Z(D)$ is a field of size $q = p^n$ where p is prime and $|D| = q^m$ where $m = \dim_Z D$.

Now consider $G = D^\times$ (so $|G| = q^m - 1$) and let G act on D (called an *inner automorphism*) by conjugation: ${}^gd = gdg^{-1}$. This is indeed an action: ${}^1d = 1d1^{-1} = d$ and ${}^{gh}d = g(hdh^{-1}) = gh dh^{-1} g^{-1} = (gh)d(gh)^{-1} = ({}^{gh}d)$.

The stabiliser of x is

$$\text{stab}(x) = \{g \in D^\times : gxg^{-1} = x\} = C(x)^\times$$

and note that the fixed point set is $D^G = Z(D) = Z$.

Proposition 3.3.17. In the notation above, $\exists d_1, \dots, d_k \in \mathbb{Z}^+ : d_i \mid m, d_i < m \ \forall i$ and

$$q^m = q + \sum_{i=1}^k \frac{q^m - 1}{q^{d_i} - 1}.$$

Proof. If $m = 1$ then $D = Z$ and we take $k = 0$ (empty set of d_i 's). The desired is then a tautology: $q = q$.

Now suppose $m > 1$ and let y_1, \dots, y_k be representatives of G -orbits of size ≥ 2 . By 3.3.16,

$$|D| = |D^G| + \sum_{i=1}^k \frac{|G|}{|\text{stab}(y_i)|}$$

and by previous observation, this implies

$$q^m = q + \sum_{i=1}^k \frac{q^m - 1}{|C(y_i)^\times|},$$

where $C(y_i)$ is a division algebra over Z by 3.3.11, hence $|C(y_i)| = q^{d_i}$ where $d_i \geq 1$. Also $|\text{orb}(y_i)| \geq 2 \implies C(y_i) \subsetneq D \implies d_i < m$. Finally, since D is a vector space over $C(y_i)$, define $C(y_i) \times D \rightarrow D : (a, b) \mapsto ab$ and let $a_i = \dim_{C(y_i)} D$, then

$$|D| = |C(y_i)|^{a_i} \implies q^m = (q^{d_i})^{a_i} \implies d_i a_i = m,$$

and in particular $d_i \mid m$. □

Lemma 3.3.18. If $d \mid n$ then $(x^d - 1) \mid (x^n - 1)$ in $\mathbb{Z}[x]$.

Proof. Write $z = x^d$, then

$$\frac{x^n - 1}{x^d - 1} = \frac{z^{n/d} - 1}{z - 1} = z^{n/d-1} + z^{n/d-2} + \dots + 1.$$

In $\mathbb{C}[x]$, let $\alpha_k = e^{\frac{2\pi k}{n}i}$ so that $\alpha_0, \dots, \alpha_{n-1}$ are all n th roots of 1 and one can write

$$x^n - 1 = (x - \alpha_0) \cdots (x - \alpha_{n-1}).$$

Lemma 3.3.19. Let $d_k = \gcd(n, k)$. Then

1. $|\alpha_k| = \frac{n}{d_k}$,
2. α_k is $\frac{n}{d_k}$ -th primitive root of unity
3. If $d_k = 1$ then α_k is n th primitive root of unity

Proof. 1 implies 2 which trivially implies 3, so let's prove 1.

$$(\alpha_k)^{n/d_k} = \alpha_1^{\frac{kn}{d_k}} = (\alpha_1^n)^{\frac{k}{d_k}} = 1,$$

so $|\alpha_k| \mid \frac{n}{d_k}$. Now suppose $|\alpha_k| = m < \frac{n}{d_k}$, then

$$\alpha_k^m = 1 \implies \alpha_1^{km} = 1 \implies n \mid km \implies \frac{n}{d_k} \mid \frac{k}{d_k} m \implies \frac{n}{d_k} \mid m.$$

So $|\alpha_k| = \frac{n}{d_k}$. □

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3.3.2 Cyclotomic polynomial

Definition 3.3.20 (Cyclotomic polynomial).

$$\phi_n(x) = \prod_{k=1, \gcd(k, n)=1}^n (x - \alpha^k)$$

where $\alpha = e^{\frac{2\pi}{n}i}$.

Proposition 3.3.21.

$$x^n - 1 = \prod_{d \mid n} \phi_d(x) \in \mathbb{C}[x].$$

Proof. $(x - \alpha^k)$ appears once in both sides since $x^n - 1 = \prod_{k=1}^n (x - \alpha^k)$ and $(x - \alpha^k)$ appears in $\phi_d(x)$ where $d = |\alpha^k|$ in \mathbb{C}^\times . □

Example 3.3.22. If p is prime then

$$\phi_p(x) = \frac{x^p - 1}{\phi_1(x)} = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \dots + 1.$$

Proposition 3.3.23. $\phi_n(x) \in \mathbb{Z}[x]$ and is monic.

Proof. One proves by induction on n using 3.3.21. If $n = 1$ then $\phi_1(x) = x - 1$ so done. Now suppose the statement is true for all values $< n$. Then

$$x^n - 1 = \phi_n(x) \cdot \underbrace{\prod_{d|n, d < n} \phi_d(x)}_{:=f(x)}$$

where $f(x) \in \mathbb{Z}[x]$ and is monic by inductive hypothesis. Now from the above one can write

$$(x^n + \dots) = (\alpha x^a + \dots)(x^b + \dots)$$

so $x^n = \alpha x^{a+b}$ hence $\alpha = 1$, i.e. monic. Now the division

$$\phi_n(x) = \frac{x^n - 1}{f(x)}$$

can be thought of as the rewriting rule $x^b \rightsquigarrow x^b - f(x) \in \mathbb{Z}[x]_{\leq b-1}$ applied repeatedly to $x^n - 1$. The fact that the result is $\in \mathbb{Z}[x]$ simply follows from that $x^b - f(x)$ is integer-valued. \square

3.3.3 Unabomber theorem

Theorem 3.3.24. A finite division ring is a field.

Proof. Suppose such D is not a field. $Z(D)$ is a field, $|Z(D)| = q$ and $|D| = q^m$ where $m \geq 2$. Rewrite 3.3.17 as

$$q - 1 = q^m - 1 + \sum_{i=1}^k \frac{q^m - 1}{q^{d_i} - 1} \quad (*)$$

and consider $\phi_m(q) \in \mathbb{Z}$. Since $\phi_m(z) \mid z^m - 1$ by 3.3.21 one has $\phi_m(q) \mid q^m - 1$. Also $\phi_m(z) \nmid z^{d_i} - 1$ so $\phi_m(z) \mid \frac{z^m - 1}{z^{d_i} - 1}$, hence $\phi_m(q) \mid \frac{q^m - 1}{q^{d_i} - 1}$, i.e. $\phi_m(q)$ divides the RHS of $*$, so $\phi_m(q) \mid q - 1$. Now

$$\phi_m(q) = \prod_{k|m, \gcd(k,m)=1} \left(q - e^{\frac{2\pi k}{m}i} \right)$$

but note that

$$\left| q - e^{\frac{2\pi k}{m}i} \right| > |q - 1| \quad \forall k$$

since $m \geq 2$, an absurdity. \square

3.4 Laurent series

Definition 3.4.1. Given a ring R one has new rings $R[x] \leq R[[x]] \leq R((x))$ where the last one is defined as

$$R((x)) := \left\{ \sum_{k=N}^{\infty} a_k x^k \right\}$$

where N is allowed to be negative, called the *Laurent series*. (The series infinite in both directions $R[[x, x^{-1}]]$ do not form a ring.)

Addition is defined by

$$\sum_{k=N}^{\infty} a_k x^k + \sum_{k=M}^{\infty} b_k x^k = \sum_{k=\min(N,M)}^{\infty} (a_k + b_k) x^k$$

and multiplication is defined by

$$ax^k \cdot bx^m = abx^{k+m}$$

extended by “infinite transitivity”:

$$\sum_{k=N}^{\infty} a_k x^k \cdot \sum_{k=M}^{\infty} b_k x^k = \sum_{k=N+M}^{\infty} c_k x^k$$

where

$$c_k = \sum_{i+j=k} a_i b_j.$$

Note that although $R[x][y] = R[y][x]$ naively, it's not true that $R((x))((y)) = R((y))((x))$:

$$\underbrace{\sum_{k=-\infty}^0 (x^{-k})(y^k)}_{\notin R((x))((y))} = \sum_{n=0}^{\infty} x^n y^{-n} = \underbrace{\sum_{n=0}^{\infty} (y^{-n})x^n}_{\in R((y))((x))}$$

since you are not allowed to sum from $-\infty$.

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Lemma 3.4.2. $t = a_n x^n + \dots \in R((x))$ where $a_n \neq 0$ is invertible in $R((x))$ iff a_n is invertible in R .

Proof. \Leftarrow : Write $t^{-1} = z_{-n}x^{-n} + z_{-n+1}x^{-n+1} + \dots$ and solve $t \cdot t^{-1} = 1$:

$$\begin{cases} a_n z_{-n} = 1 \\ a_n z_{-n+1} + a_{n+1} z_{-n} = 0 \\ a_n z_{-n+2} + a_{n+1} z_{-n+1} + a_{n+2} z_{-n} = 0 \\ \vdots \end{cases}$$

which can be solved recursively if a_n^{-1} exists:

$$\begin{aligned} z_{-n} &= a_n^{-1} \\ z_{-n+1} &= -a_n^{-1} a_{n+1} z_{-n} = -a_n^{-1} a_{n+1} a_n^{-1} \\ z_{-n+2} &= -a_n^{-1} a_{n+1} z_{-n+1} - a_n^{-1} a_{n+2} z_{-n} \\ &= a_n^{-1} a_{n+1} a_n^{-1} a_{n+1} a_n^{-1} - a_n^{-1} a_{n+2} a_n^{-1} \\ &\vdots \end{aligned}$$

□

Corollary 3.4.3. If R is division then $R((x))$ is division.

This gives us division algebras $\mathbb{H}((x))$, $\mathbb{H}((x))((y))$ and so on.

Consider $\mathbb{C}((z, \sigma))$ which is equal to $\mathbb{C}((z))$ as abelian groups but with extra rule $z\alpha = \bar{\alpha}z$ where $\alpha \in \mathbb{C}$, i.e.

$$\alpha z^n \cdot \beta z^m = \begin{cases} \alpha \beta z^{n+m} & n \text{ is even} \\ \alpha \bar{\beta} z^{n+m} & n \text{ is odd} \end{cases}$$

extended by infinite transitivity. It's also a division ring. Note that

$$Z(\mathbb{H}((x))) = \mathbb{R}((x)), \quad Z(\mathbb{C}((z, \sigma))) = \mathbb{R}((z^2))$$

which are isomorphic via $x \mapsto z^2$, but $\mathbb{H}((x)) \not\cong \mathbb{C}((z, \sigma))$ as rings.

4 Semisimplicity

4.1 Direct sum

Definition 4.1.1. For R -modules M_i , $i \in I$, their *direct product* is

$$\prod M_i = \{(m_i) : m_i \in M_i\} = \left\{ f : I \rightarrow \prod M_i : f(i) \in M_i \right\}$$

and their *direct sum* is

$$\bigoplus M_i = \left\{ (m_i) \in \prod M_i : \text{for all but finitely many } i, m_i = 0 \right\} = \left\{ f : I \rightarrow \bigcup M_i : |\text{supp}(f)| < \infty \right\}$$

where

$$\text{supp}(f) = \{i : f(i) \neq 0\}.$$

It follows that if $|I| < \infty$, $\bigoplus_{i \in I} M_i = \prod_{i \in I} M_i$.

Example 4.1.2. Let $M_i = \mathbb{R}$ be a \mathbb{Q} -module and $I = \mathbb{N}$. Then

$$\prod M_i = \{(a_0, a_1, \dots)\} \quad \text{all sequences}$$

and

$$\bigoplus M_i = \{(a_0, a_1, \dots)\} \quad \text{eventually 0 sequences, i.e. } \exists N : \forall n > N, a_n = 0.$$

These are characterised as “external”: producing new modules from existing ones. On the other hand, if M is a R -module with $M_i < M$, $i \in I$, the question of when we can say M is a direct sum of its submodules is characterised as an “internal” one. In this situation we have a homomorphism of R -modules:

$$\begin{aligned} \varphi : \bigoplus_{i \in I} M_i &\rightarrow M \\ (m_i) &\mapsto \sum_{i \in I} m_i \end{aligned}$$

which is well defined since the sum $\sum_{i \in I} m_i$ is finite.

Definition 4.1.3. Define the *sum* $\sum_{i \in I} M_i := \text{im } \varphi$ in the above notation.

In particular, if φ is surjective then $M = \sum_{i \in I} M_i$. If φ is injective then $\bigoplus_{i \in I} M_i \cong \text{im } \varphi$. In this case we identify $\sum M_i$ with $\bigoplus M_i$ and call $\sum M_i$ the *internal direct sum*.

If φ is bijective then $\bigoplus M_i \cong M$. In this case M is a direct sum of its submodules M_i .

4.1.1 Peirce decomposition

In this section we consider how to decompose M into $M_1 \oplus \dots \oplus M_n$.

Example 4.1.4. Let $M = V$ be a 2-dimensional vector space over \mathbb{F} . How do we get $V = U \oplus W$? If we have we have 2 projection operators $p : V \rightarrow U \rightarrow V : u + w \mapsto u \mapsto u$ and $q : V \rightarrow W \rightarrow V : u + w \mapsto w \mapsto w$. Both $p, q \in \text{End}_{\mathbb{F}} V$. Note that $p + q = \text{id}_V = 1_{\text{End}_{\mathbb{F}} V}$, $p^2 = p$, $q^2 = q$ and $pq = qp = 0$. This is a system of orthogonal idempotents.

Claim: idempotents $e \in \text{End}_{\mathbb{F}} V$ are projection operators.

Indeed, $e^2 - e = 0 \implies \mu_e(x) \mid x(x - 1) \implies e$ is diagonalisable with 1, 0 on the diagonal \implies one can let V be the 1-eigenspace of e (i.e. $\text{im } e$) and W be the 0-eigenspace (i.e. $\ker e$).

Therefore, in the previous example, $U = \text{im } p = \ker q$ and $W = \ker p = \text{im } q$.

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Let's define properly.

Definition 4.1.5. $R \ni e$ is *idempotent* if $e^2 = e$.

Idempotent e, f are *orthogonal* if $ef = fe = 0$.

e_1, \dots, e_n is a *full system of orthogonal idempotents* if $\begin{cases} \forall i, e_i^2 = e_i \\ \forall i \neq j, e_i e_j = e_j e_i = 0 \\ e_1 + \dots + e_n = 1 \end{cases}$.

Example 4.1.6. 1. For $R = R_1 \times \dots \times R_n$, $e_i := (0, \dots, \underbrace{1}_{i\text{th position}}, \dots, 0)$ form such system.

2. If $e \in R$ is idempotent then $f = 1 - e$ is as well since $f^2 = (1 - e)^2 = 1 - 2e + e^2 = 1 - e = f$, and $ef = e(1 - e) = 0$ and $fe = 0$, so e, f form such system.

Proposition 4.1.7. If M is a R -module then there is a bijection between

$$\{\text{decompositions of } R\text{-modules } M = M_1 \oplus M_2 \oplus \dots \oplus M_n \text{ with all } M_i \neq 0\}$$

and

$$\{\text{full systems of orthogonal idempotents in } \text{End}_R M\}.$$

These are called Peirce decompositions.

Proof. 1→2 Define $e_i : M \twoheadrightarrow M_i \hookrightarrow M$, i.e. $m = \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ \vdots \\ m_i \\ \vdots \\ 0 \end{pmatrix}$. Then it's trivial that

- i. $e_i \in \text{End}_R M$
- ii. $e_i^2 = e_i$
- iii. $e_i e_j = 0$ for $i \neq j$
- iv. $e_1 + \cdots + e_n = 1_{\text{End}_R M}$

2→1 Define $M_i = \text{im } e_i = M e_i$. Since e_i is a homomorphism of R -modules, $\text{im } e_i$ is a submodule. It remains to check $\psi : \bigoplus_{i=1}^n M_i \rightarrow M$ is bijective:

- i. ψ is surjective: let $m \in M$ so that $m e_i \in M_i$, and

$$\begin{pmatrix} m e_1 \\ \vdots \\ m e_n \end{pmatrix} \xrightarrow{\psi} m e_1 + \cdots + m e_n = m(e_1 + \cdots + e_n) = m 1 = m$$

- ii. ψ is injective: let $x = \begin{pmatrix} m_1 e_1 \\ \vdots \\ m_n e_n \end{pmatrix} \in \ker \psi$, then $0 = \psi(x) = m_1 e_1 + \cdots + m_n e_n$. Multiplying this by e_i gives

$$0 = m_1 e_1 e_i + \cdots + m_n e_n e_i = m_i e_i$$

by orthogonality, hence $x = 0$.

Finally, they are inverse bijections by construction. □

4.1.2 Primary decomposition (example of Peirce decomposition)

Let A be an abelian group under $+$ such that $\exists N : \forall x \in A, |x| < N$, i.e. order of an element is bounded. Let $n = \text{lcm } \{|x| : x \in A\}$. Note that A is a \mathbb{Z} -module with

$$E = \text{End}_{\mathbb{Z}} A \geq \mathbb{Z}/(n) = \{x \mapsto kx\}$$

where k is the natural image of quotient map $\mathbb{Z} \rightarrow \mathbb{Z}/(n)$. Now if one decomposes n into $p_1^{a_1} \cdots p_k^{a_k}$ where p_i are distinct primes, then Chinese remainder theorem gives

$$\mathbb{Z}/(n) \cong \mathbb{Z}/(p_1^{a_1}) \times \cdots \times \mathbb{Z}/(p_k^{a_k}) \leq E$$

which gives a full system of orthogonal idempotents

$$e_i = (0, \dots, 1 + (p_i^{a_i}), \dots, 0) \in E$$

and the Peirce decomposition of the group

$$A = A e_1 \oplus \cdots \oplus A e_k,$$

called the *primary decomposition*.

Claim 4.1.8. $A e_i = \{x \in A : |x| = p_i^{b_i} \text{ where } b_i \leq a_i\}$.

Proof. \subseteq : Write $x = y e_i$ and note that $p_i^{a_i} x = p_i^{a_i} y e_i = y(p_i^{a_i} e_i) = y 0_E = 0$, so $|x| \mid p_i^{a_i}$.

\supseteq : Write $x = x 1_E = x e_1 + \cdots + x e_k$ and note that

$$0 = p_i^{b_i} x = p_i^{b_i} x e_1 + \cdots + p_i^{b_i} x e_k, \tag{*}$$

since $|x e_j| = p_j^{b_j}$, one has for $j \neq i$, $x e_j \neq 0 \implies p_i^{b_i} x e_j \neq 0$, i.e.

$$\text{for } j \neq i, p_i^{b_i} x e_j = 0 \implies x e_j = 0.$$

But $*$ is a direct sum decomposition, so all $p_i^{b_i} e_j = 0$, hence $x e_j = 0 \forall j \neq i$, therefore $x = x e_i \in A e_i$. □

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4.1.3 Primary decomposition on a vector space

Let V be a finite dimensional vector space over \mathbb{F} and $T : V \rightarrow V$ a linear operator. Suppose $\chi_T(z) = \pm(z - \alpha_1) \cdots (z - \alpha_n)$ with $\alpha_i \in \mathbb{F}$. Consider the minimal polynomial $\mu_T(z) = (z - \beta_1)^{a_1} \cdots (z - \beta_k)^{a_k}$ with $i \neq j \implies \beta_i \neq \beta_j$ and $a_i \geq 1$. Let $R = \mathbb{F}[x]$ so that V is a left R -module via $x \cdot v = T(v)$. We then have a homomorphism $\varphi : \mathbb{F}[x] \rightarrow \text{End}_R V : x \mapsto (v \mapsto T(v))$ with $\ker \varphi = (\mu_T(z))$. Therefore by 1st isomorphism and Chinese remainder theorems

$$\text{im } \varphi \cong \mathbb{F}[z]/(\mu_T(z)) \cong \mathbb{F}[z]/((z - \beta_1)^{a_1}) \times \cdots \times \mathbb{F}[z]/((z - \beta_k)^{a_k})$$

and one gets a full system of orthogonal idempotents $e_1, \dots, e_k \in \text{End}_R V$ where

$$e_i = (0, \dots, 1 + ((z - \beta_i)^{a_i}), \dots, 0)$$

with a corresponding Peirce decomposition

$$V = Ve_1 \oplus \cdots \oplus Ve_k,$$

called the *primary decomposition* of V with respect to T . See Dmitriy's notes for a proof of

$$Ve_i = \{v \in V : \exists a \geq 1 : (T - \beta_i)^a(v) = 0\},$$

where the right hand side is called the *generalised eigenspace* with eigenvalue β_i . This implies generalised eigenvectors for distinct eigenvalues are linearly independent.

4.1.4 Peirce decomposition and matrix

Let R be any ring. One has $\text{End}_R R \cong R$ (1.3.10) and submodules of ${}_R R$ are left ideals. One therefore has

Proposition 4.1.9 (4.1.7 where $M = R$). There is a bijection between

$$\{\text{full systems of orthogonal idempotents in } R\}$$

and

$$\{\text{decompositions } R = L_1 \oplus \cdots \oplus L_n\}$$

where L_i are left ideals.

Now for a full system $e_1, \dots, e_r \in R$ and ${}_R M$ a left R -module, one can write

$$M = \bigoplus_{i=1}^n e_i M = \begin{pmatrix} e_1 M \\ e_2 M \\ \vdots \\ e_n M \end{pmatrix}$$

and with R itself one has

$$R = \bigoplus_{i,j=1}^n e_i R e_j = \begin{pmatrix} e_1 R e_1 & \cdots & e_1 R e_n \\ \vdots & e_i R e_j & \vdots \\ e_n R e_1 & \cdots & e_n R e_n \end{pmatrix}$$

where $e_i R e_j$ are distinct abelian groups. This is called the *double Peirce decomposition*.

Theorem 4.1.10. 1. If R is a \mathbb{F} -algebra, all $e_i R e_j$ and $e_i M$ are vector spaces over \mathbb{F} .

2. Each $e_i R e_i$ is a nonzero ring.

3. $e_i M$ is a $e_i R e_i$ -module.

4. Multiplication in R and R -action on M satisfy standard “matrix rules”:

$$\begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & & \vdots \\ r_{n1} & \cdots & r_{nn} \end{pmatrix} \begin{pmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & & \vdots \\ s_{n1} & \cdots & s_{nn} \end{pmatrix} = \left(\sum_R r_{iR} s_{Rj} \right)$$

where $r_{ij}, s_{ij} \in e_i R e_j$, and

$$\begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & & \vdots \\ r_{n1} & \cdots & r_{nn} \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \left(\sum_{i=1}^n r_{ik} m_k \right).$$

Proof. 1. Let $\alpha \in \mathbb{F}$, $x \in e_i Re_j$. Then one can write $x = e_i y e_j$ with $y \in R$, and

$$\alpha x = \alpha e_i y e_j = e_i (\alpha y) e_j \in e_i Re_j,$$

so $e_i Re_j$ is a \mathbb{F} -vector subspace. Similar for $e_i M$.

2. Note $(e_i x e_i)(e_i y e_i) = e_i (x e_i y) e_i \in e_i Re_i$, so it's closed under product. Also $1_{e_i Re_i} = e_i \neq 0$, so nonzero ring (but not a subring of R).

3. One has

$$(e_i r e_i) e_i m = e_i (r e_i m) \in e_i M, \quad \text{and} \quad 1_{e_i Re_i} e_i m = e_i^2 m = e_i m$$

4. By definition,

$$(r_{ij})(s_{ij}) = \left(\sum r_{ij} \right) \left(\sum s_{ij} \right) = \sum_{i,j,k,m} r_{ij} s_{km}$$

where

$$r_{ij} s_{km} = e_i r e_j e_k s e_m = \begin{cases} 0 & \text{if } j \neq k \\ e_i r e_j s e_m & \text{if } j = k \end{cases},$$

so

$$\sum_{i,j,k,m} r_{ij} s_{km} = \sum_{i,j,m} r_{ij} s_{jm} = \sum_{i,m} \left(\sum_j r_{ij} s_{jm} \right).$$

Similar for $R \times M \rightarrow M$.

□

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Lemma 4.1.11. Let $e, f, g \in R$ be 3 idempotents.

1. $eRf \cong \text{Hom}_R(Re, Rf)$ as abelian groups.
2. This \cong commutes with compositors, i.e.

$$\alpha, \beta \longmapsto \alpha\beta$$

$$\text{Hom}_R(Re, Rf) \times \text{Hom}_R(Rf, Rg) \longrightarrow \text{Hom}_R(Re, Rg)$$

$$\cong \qquad \qquad \qquad \cong$$

$$eRf \times fRg \longrightarrow eRg$$

$$a, b \longmapsto ab$$

is commutative.

This is a generalisation of the ring isomorphism $\text{End}_R R \cong R$ (which is the special case $e = f = 1$).

Proof. 1. Consider homomorphism of abelian groups

$$\begin{aligned} \psi : eRf &\rightarrow \text{Hom}(Re, Rf) \\ exf &\mapsto (se \mapsto sexf). \end{aligned}$$

This is

injective: let $exf \in \ker \psi$, then $e\psi(exf) = e^2 xf = exf = 0$.

surjective: consider $\varphi : Re \rightarrow Rf$. Then

$$\varphi(re) = \varphi(re^2) = \varphi((re)e) = re\varphi(e)$$

and

$$\varphi(e) = \varphi(e^2) = e\varphi(e)$$

so $\varphi(e) = eRf$ and $\psi(\varphi(e)) = \varphi$.

2. Let $(a, b) \in eRf \times fRg$ and write $a = exf$. Then $a = e^2xf^2 = e(exf)f = eaf$ and similarly $b = fbg$. So one can see

$$\begin{array}{ccc} (\alpha : x \mapsto x e a f, \beta : y \mapsto y f b g) & \longmapsto & \alpha \beta : x \mapsto x e a f b g \\ \cong \uparrow & & \cong \uparrow \\ (e a f, f b g) & \longmapsto & e a f b g. \end{array}$$

□

4.2 Semisimple module

Definition 4.2.1. M is *semisimple* if M is a direct sum of simple (sub-)modules.

Remark. 1. The sum is not necessarily finite.

2. The sum can be empty. This gives a zero module, which is semisimple.
3. If $R = \mathbb{F}$ is a field then ${}_{\mathbb{F}}\mathbb{F}$ is the only simple left R -module, and since every vector space has a basis, every R -module is semisimple.
4. If $R = \mathbb{F}[x]$, then a simple R -module is R/L where L is a maximal left ideal by 2.2.3, and we know L is of the form $(f(x))$ where f is irreducible. In particular, if \mathbb{F} is algebraically closed, then all simple modules have the form $R/(x - \alpha)$, i.e. 1-dimensional.
5. In the case of the considered object in section 4.1.3, V as a R -module is semisimple iff T is diagonalisable.

Definition 4.2.2. For ${}_R M$, the *socle* of M is

$$\text{soc } M := \sum_{S \leq M, S \text{ is simple}} S.$$

Example 4.2.3. Consider an abelian group A as a \mathbb{Z} -module. The simple \mathbb{Z} -modules are $\mathbb{Z}/(p)$ where p is prime, and the simple submodules of A are $\{\mathbb{Z}x : x \in A, |x| = p, p \text{ prime}\}$, so

$$\text{soc } A = \sum_{|x| \text{ is prime}} \mathbb{Z}x = \{x \in A : |x| \text{ is square free}\}.$$

Example 4.2.4. Let \mathbb{F} be an algebraically closed field and V a $\mathbb{F}[x]$ -module. Simple submodules are then $\{\mathbb{F}v : v \text{ is an eigenvector of } T\}$ and $\text{soc } V = \text{span}\{\text{eigenvectors}\}$.

Lemma 4.2.5. 1. M is semisimple iff $M = \text{soc } M$.

2. More precisely, if $M = \sum_{i \in I} S_i$ where S_i are all simple, then $\exists J \subseteq I : M = \bigoplus_{i \in J} S_i$.

Proof. 1. \Rightarrow : trivial since

$$M = \bigoplus_{i \in X, L_i \text{ simple}} L_i \implies \text{soc } M \supseteq \sum L_i = M.$$

\Leftarrow : follows from 2.

2. Consider the poset $\mathcal{P} := \{J \subseteq I : \sum_{i \in J} S_i = \bigoplus_{i \in J} S_i\}$ under \subseteq . Since $\emptyset \in \mathcal{P}$, one has $\mathcal{P} \neq \emptyset$ and so can apply Zorn's lemma. Consider the chain $\mathcal{C} : J_1 \subseteq J_2 \subseteq \dots \subseteq J_\infty \subseteq \dots$ in \mathcal{P} and define $Y = \bigcup_{J \in \mathcal{C}} J$. It's clear that once $Y \in \mathcal{P}$, it is an upper bound of \mathcal{C} and thus by Zorn's \mathcal{P} has a maximal element J . Examine the map

$$\begin{aligned} \varphi_Y : \bigoplus_{i \in Y} S_i &\rightarrow \sum_{i \in Y} S_i \\ (s_i) &\mapsto \sum s_i \end{aligned}$$

which is clearly surjective, and it's injective iff $\sum_{i \in Y} S_i$ is direct iff $Y \in \mathcal{P}$. Let $x \in \ker \varphi$, and write $x = (x_1, x_2, \dots, x_n, 0, \dots, 0)$. Then $1, 2, \dots, n \in Y$, and since there are only finitely many positions, $\exists J \in \mathcal{C} : 1, \dots, n \in J$. But φ_J is an isomorphism by construction, so $x_1 = \dots = x_n = 0$, hence $x = 0$.

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Remark. If V is a \mathbb{F} -vector space, then there exists a basis $\{e_i : i \in I\}$ which gives a decomposition into 1-dimensional subspaces ${}_F V = \bigoplus_{i \in I} \mathbb{F}e_i$. Now note that $\mathbb{F}e_i \cong \mathbb{F}$: this leads to the idea of a free module. Also, $\mathbb{F}e_i$ is simple, so this also leads to the idea of semisimple module. The proof of 4.2.5 now proceeds.

Now let $N = \sum_{i \in J} S_i = \bigoplus_{i \in J} S_i$ where J is the maximal element the argument above yields. If $N = M$ then we are done. If not, $\exists 0 \in I : S_0 \not\subseteq N$ (so $0 \notin J$) and since S_0 is simple one has $S_0 \cap N = \{0\}$. Let $\hat{J} := J \cup \{0\}$. Consider $\psi : \bigoplus_{i \in \hat{J}} S_i \rightarrow \sum_{i \in \hat{J}} S_i = S_0 + N$ and let $x \in \ker \psi$. Write $x = (x_0, x_1, \dots, x_n, 0, \dots, 0)$ where $x_0 \in S_0$. Then $0 = \psi(x) = x_0 + \dots + x_n$ so $x_0 = -(x_1 + x_2 + \dots + x_n) \in S_0 \cap N = \{0\}$, hence $x_0 = x_1 + \dots + x_n = 0$. But $\sum_{i \in J} S_i = \bigoplus_{i \in J} S_i$, so $x_1 = \dots = x_n = 0$. Therefore ψ is injective and hence an isomorphism, and thus $\hat{J} \in \mathcal{P}$, which contradicts maximality of J . \square

Corollary 4.2.6. A quotient module of a semisimple module is semisimple.

Proof. Suppose M is semisimple and write $M = \bigoplus_{i \in I} S_i$. For a submodule $N \leq M$, consider M/N and the quotient map $\varphi : M \rightarrow M/N$. Then $M/N = \sum_{i \in I} \varphi(S_i)$, and since S_i is simple, $\varphi(S_i) = S_i$ or 0 , so

$$M/N = \sum_{i \in I, \varphi(S_i) = S_i} \varphi(S_i)$$

and by 4.2.5 one has M/N is semisimple. \square

Comparing with quotient modules, submodules are harder: e.g. $\mathbb{R}^2 = \mathbb{R}e_1 \oplus \mathbb{R}e_2 = \bigoplus_{i \in I} S_i$, but $\mathbb{R} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \bigoplus_{i \in J} S_i$ for any $J \subseteq I$. We need something more.

Definition 4.2.7. ${}_R M$ is *completely reducible* if $\forall N \leq M, \exists K \leq M : {}_R M = {}_R N \oplus {}_R K$. Such K is the *direct complement* to N .

Lemma 4.2.8. If $N \leq M$, then any direct complement K is isomorphic to M/N as modules.

Proof. Consider quotient map $\varphi : M \rightarrow M/N$ and restrict to K : $\varphi|_K : K \rightarrow M/N$, which is injective since if $x \in \ker \varphi|_K \subseteq \ker \varphi = N$ then $x \in N \cap K = \{0\}$ and surjective since if $m + N \in M/N$ then $m = n + k$ where $n \in N, k \in K$, so $\varphi|_K(k) = \varphi|_K(m - n) = m - n + N = m + N$. \square

Lemma 4.2.9. A submodule of a completely reducible module is completely reducible.

Proof. Let $N \leq M$ with M being completely reducible and let $K \leq N$. We need to find a direct complement for K . By assumption $M = K \oplus P$ for some P . Consider $\pi : M \rightarrow K$, projection along P . This induces a restriction $\hat{\pi} := \pi|_N : N \rightarrow K$ with $\text{im } \hat{\pi} \subseteq \text{im } \pi = K$, but $\pi(k) = k \ \forall k \in K$ so $K \subseteq \text{im } \hat{\pi}$, hence $\text{im } \hat{\pi} = K$ and by the 1st isomorphism theorem one can write $N = \text{im } \hat{\pi} \oplus \ker \hat{\pi} = K \oplus \ker \hat{\pi}$ where $\ker \hat{\pi}$ is the direct complement we are looking for. \square

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Lemma 4.2.10. A nonzero completely reducible module contains a simple submodule.

Proof. Let M be such a R -module and $x \in M$ with $x \neq 0$. Consider homomorphism

$$\begin{aligned} \varphi_x : {}_R R &\rightarrow M \\ r &\mapsto rx \end{aligned}$$

and note that $Rx \cong R/\text{Ann}(x) \leq M$ by remark before 2.2.4, so Rx is completely reducible by 4.2.9. Now $\text{Ann}(x) \subseteq L$, the maximal left ideal, so one can consider the surjection

$$\begin{aligned} \psi : Rx &\twoheadrightarrow R/L \\ r + \text{Ann}(x) &\mapsto r + L \end{aligned}$$

where R/L is simple by 2.2.3. Let P be the direct complement of $\ker \psi \leq Rx$, i.e. $Rx = \ker \psi \oplus P$. But $Rx = \ker \psi \oplus \text{im } \psi$ where $\text{im } \psi = R/L$, so P is simple. \square

Theorem 4.2.11. M is semisimple iff M is completely reducible.

Proof. \Leftarrow : By 4.2.10 one has $\text{soc } M \neq 0$. If $M = \text{soc } M$ we are done, so suppose $M \neq \text{soc } M$, then $\exists P \leq M : M = \text{soc } M \oplus P$ with $P \neq 0$. But P is completely reducible, so again by 4.2.10 there is a simple $S \leq P$, but this means $S \not\leq \text{soc } M$, an absurdity.

\Rightarrow : Write $M = \bigoplus_{i \in I} S_i \geq N$ and we need a direct complement for N . Consider quotient map $\varphi : M \rightarrow M/N$. Since S_i is simple,

$$\varphi(S_i) \cong S_i / (S_i \cap N) = \begin{cases} 0 \\ \cong S_i \end{cases},$$

so

$$M/N = \sum_{i \in I, \varphi(S_i) \neq 0} \varphi(S_i),$$

and by 4.2.5 one has $\exists J \subseteq I : M/N = \bigoplus_{i \in J} \varphi(S_i)$ and $\varphi(S_i) \cong S_i$ for $i \in J$. Then

$$M = N \oplus \left(\sum_{i \in J} S_i \right).$$

Indeed, consider

$$\psi : N \oplus \left(\sum_{i \in J} S_i \right) \rightarrow M.$$

ψ is surjective: let $m \in M$ then $M/N \ni m + N = \varphi(m) = \varphi(x_1) + \cdots + \varphi(x_n)$ where $x_i \in S_i, i \in J$, so

$$m - x_1 - \cdots - x_n \in N$$

and hence

$$m = y + x_1 + \cdots + x_n \in \text{im } \psi$$

for some $y \in N$.

ψ is injective: let $(m, x_1 + \cdots + x_n) \in \ker \psi$ where $m \in N, x_i \in S_i, i \in J$, then

$$m + x_1 + \cdots + x_n = 0$$

and so

$$\varphi(x_1) + \cdots + \varphi(x_n) = 0$$

since $\varphi(m) = 0$, which follows from that $\sum_{i \in J} \varphi(s_i)$ is direct, so $x_1 = \cdots = x_n = 0$ and hence $m = 0$ and therefore $(m, x_1 + \cdots + x_n) = 0$. □

Corollary 4.2.12. A submodule of a semisimple module is semisimple.

4.2.1 Radical

Definition 4.2.13. A submodule P of M is *cosimple* if M/P is simple.

The *radical* of M is

$$\text{rad } M := \bigcap_{P \leq M, P \text{ is cosimple}} P.$$

Recall for M/N one has the bijective correspondence

$$\{P \leq M : P \supseteq N\} \leftrightarrow \{Q \leq M/N\},$$

and for M/N to be simple it means both sets only have two elements, N, M and $0, M/N$, so N is maximal.

Example 4.2.14. ${}_Z\mathbb{Z}$ has no simple submodules, and the simple \mathbb{Z} -modules are $\mathbb{Z}/(p)$ where p is prime, so

$$\text{soc } \mathbb{Z} = \sum_{\emptyset} = 0$$

and

$$\text{rad } \mathbb{Z} = \bigcap_p \mathbb{Z}/(p) = \{n : n \mid n \ \forall p\} = 0.$$

Example 4.2.15. Consider $M = \mathbb{Z}/(n)$ and $R = \mathbb{Z}$. For $n \in \mathbb{N}$, recall we also had a definition for radical of n : $\text{rad } n = p_1 \cdots p_k$ with $n = p_1^{a_1} \cdots p_k^{a_k}$ where $a_i \geq 1$ and p_i are primes, e.g.

$$\text{rad } 12000 = \text{rad } 3 \times 2^5 \times 5^3 = 3 \times 2 \times 5 = 30.$$

A submodule Rx of M is simple when $|x| = p_i$, so $x = \frac{n}{p_i}$ and

$$\text{soc } M = \mathbb{Z} \frac{n}{p_1} + \cdots + \mathbb{Z} \frac{n}{p_k} = \mathbb{Z} \frac{n}{p_1 \cdots p_k} = \mathbb{Z} \frac{n}{\text{rad } n},$$

which also gives

$$\text{soc } M \cong \mathbb{Z}/(p_1) \oplus \cdots \oplus \mathbb{Z}/(p_k) \cong \mathbb{Z}/(\text{rad } n),$$

and by 4.2.5 M is semisimple iff $n = \text{rad } n$, i.e. n is squarefree.

Similarly, a submodule Rx is cosimple if $M/Rx \cong \mathbb{Z}/(p_i)$, where an obvious choice for x is p_i , and

$$\text{rad } M = \bigcap_{p_i} \mathbb{Z}(p_i + (n)) = \{x \in M : \forall i, p_i \mid x\} = \mathbb{Z}p_1 \cdots p_k = \mathbb{Z} \text{rad } n,$$

so $M/\text{rad } M \cong \mathbb{Z}/(\text{rad } n) \cong \text{soc } M$, which is semisimple. This implies if $\text{rad } M = 0$ then M is semisimple. Let's see this in more generality.

Lemma 4.2.16. If M is semisimple then $\text{rad } M = 0$.

Proof. Write $M = \bigoplus_{i \in I} S_i$. For i , let

$$P_i := \bigoplus_{k \in I \setminus \{i\}} S_k,$$

so that $M/P_i \cong S_i$ is simple, i.e. P_i is cosimple. But then $\text{rad } M \subseteq \bigcap_i P_i = 0$. □

Definition 4.2.17. ${}_R M$ is *artinian* if any descending chain of submodules terminates, i.e. for any chain $P_1 \geq P_2 \geq \cdots \geq P_k \geq \cdots$, $\exists N : P_N = P_{N+1} = \cdots$.

A ring is *left artinian* if ${}_R R$ is artinian.

Theorem 4.2.18. If ${}_R M$ is artinian then M is semisimple iff $\text{rad } M = 0$.

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Proof. By 4.2.16, it remains to prove the \Rightarrow direction. Since $\text{rad } M = 0$, \exists cosimple submodules

$$P_1, \dots, P_n, \dots : P_1 \cap \cdots \cap P_n \cap \cdots = \text{rad } M = 0.$$

This induces a descending chain

$$P_1 \supseteq P_1 \cap P_2 \supseteq P_1 \cap P_2 \cap P_3 \supseteq \cdots$$

which, by assumption, must terminate at some $P_1 \cap \cdots \cap P_n = 0$. Consider

$$\begin{aligned} \psi : M &\rightarrow \underbrace{M/P_1 \oplus \cdots \oplus M/P_n}_{\text{semisimple}} \\ m &\mapsto (m + P_1, \dots, m + P_n), \end{aligned}$$

whose kernel is precisely $P_1 \cap \cdots \cap P_n = 0$, hence ψ is injective and M is a submodule of $M/P_1 \oplus \cdots \oplus M/P_n$, therefore M is semisimple by 4.2.12. □

We are finally strong enough.

4.3 Semisimple ring

4.3.1 Artin–Wedderburn theorem

Theorem 4.3.1 (Artin–Wedderburn). The following are equivalent for a ring R .

1. Every left R -module is semisimple.
2. ${}_R R$ is semisimple.

3. \exists division rings $D_1, \dots, D_k : R \cong M_{n_1}(D_1) \times \dots \times M_{n_k}(D_k)$.

Proof. 1 \Rightarrow 2: trivial.

2 \Rightarrow 1: Let ${}_R M$ be a left R -module and $X \subseteq M$ a generating set. Consider

$$\varphi : \overbrace{\bigoplus_X {}_R R}^{\text{semisimple}} \rightarrow M$$

$$(a_i)_{i \in X} \mapsto \sum_{i \in X} a_i i,$$

so M is a quotient of a semisimple module, hence by 4.2.6 M is semisimple.

3 \Rightarrow 2: Note that $D_i^{n_i}$ is a simple R -module, since $M_{n_i}(D_i)$ acts on it by matrix multiplication, so that every nonzero vector can be mapped to another. Now

$$M_{n_i}(D_i) = \underbrace{\begin{pmatrix} * & 0 & \cdots & 0 \\ * & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 0 \end{pmatrix}}_{\cong D_i^{n_i}} \oplus \underbrace{\begin{pmatrix} 0 & * & \cdots & 0 \\ 0 & * & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & * & \cdots & 0 \end{pmatrix}}_{\cong D_i^{n_i}} \oplus \dots \cong (D_i^{n_i})^{n_i}$$

so ${}_R M_{n_i}(D_i)$ is semisimple, hence ${}_R R$ is semisimple as well.

2 \Rightarrow 3: Write ${}_R R = \bigoplus_{i \in I} S_i$ where S_i is simple. Then

$$1_R = x_1 + \dots + x_n \quad x_i \in S_i, \text{ all } x_i \neq 0$$

(note that n is finite) and any $r \in R$ can be written as

$$r = r1 = rx_1 + \dots + rx_n,$$

so effectively ${}_R R = S_1 \oplus \dots \oplus S_n$. Therefore \exists idempotents $e_1, \dots, e_n \in \text{End}_R R \cong R$ yielding this decomposition, i.e. $S_i = Re_i$. We now change the order:

$$\begin{aligned} {}_R R &= S_1 \oplus \dots \oplus S_{a_1} \oplus \\ &\quad S_{a_1+1} \oplus \dots \oplus S_{a_1+a_2} \oplus \\ &\quad \vdots \\ &\quad S_{a_1+\dots+a_{k-1}+1} \oplus \dots \oplus S_{a_1+\dots+a_k} \end{aligned}$$

so that every module in a line are isomorphic and modules in different lines are not. Now apply double Peirce decomposition

$$R = \bigoplus_{i,j=1}^n e_i R e_j$$

and let $D_i := \text{End} S_i$, which is a division ring by 2.2.8, and by 4.1.11

$$e_i R e_j \cong \text{Hom}(Re_i, Re_j) = \begin{cases} 0 & \text{if } i, j \text{ are in different lines} \\ D_i \psi_{i,j} & \text{if } i, j \text{ are in the same line} \end{cases}$$

for some fixed isomorphism $\psi_{i,j}$ by construction, and hence

$$R = \left(\begin{array}{c|c|c|c|c} D_1 & 0 & 0 & \cdots & 0 \\ \hline 0 & D_2 & 0 & \cdots & 0 \\ \hline 0 & 0 & \ddots & & \\ \hline \end{array} \right) \cong M_{n_1}(D_1) \times \dots \times M_{n_k}(D_k).$$

□

Note that the 3rd statement does not mention any sides but 1st and 2nd are left. The corollary is then

Corollary 4.3.2. ${}_R R$ is semisimple iff R_R is semisimple. In this case one says the ring R is *semisimple*.

4.3.2 Semisimple algebra

If (R, \mathbb{F}) is an algebra and a semisimple ring, then $R = M_{n_1}(\mathbb{F}) \times \cdots \times M_{n_k}(\mathbb{F})$ where all D_i are \mathbb{F} -algebras. Our knowledge so far (recall 3.2.1, 3.2.14, 3.3.10) allows us to write the following.

Proposition 4.3.3. 1. A countable dimensional semisimple \mathbb{C} -algebra is isomorphic to

$$\prod_{i=1}^k M_{n_i}(\mathbb{C}).$$

2. A countable dimensional semisimple \mathbb{R} -algebra is isomorphic to

$$\prod_{i=1}^k M_{n_i}(D_i) \quad \text{where } D_i \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}.$$

3. A finite dimensional semisimple \mathbb{F}_q -algebra is isomorphic to

$$\prod_{i=1}^k M_{n_i}(\mathbb{F}_{q^{a_i}}).$$

4.3.3 Maschke's theorem

Let G be a group and \mathbb{F} a field of characteristic p . Define the group algebra

$$\mathbb{F}G := \left\{ \sum_{g \in G} \alpha_g g : \alpha_g \in \mathbb{F} \right\} \quad \text{with multiplication } \alpha g \beta h := \alpha \beta gh$$

Theorem 4.3.4. The following are equivalent for a group G and a field \mathbb{F} of characteristic p .

1. $\mathbb{F}G$ is semisimple.
2. G is finite and $p \nmid |G|$.

Remark. $2 \Rightarrow 1$ is called Maschke's theorem.

Proof. $1 \Rightarrow 2$: Let $R = \mathbb{F}G$. Consider \mathbb{F} as a trivial R -module with $\forall \alpha \in \mathbb{F}, g\alpha = \alpha \forall g \in G$. So \exists surjective homomorphism

$$\begin{aligned} \psi : {}_R R &\rightarrow \mathbb{F} \\ g &\mapsto 1 \end{aligned}$$

Since R is semisimple and $\ker \psi \leq {}_R R$, one has ${}_R R = \ker \psi \oplus P$ for some P and hence ${}_R P \cong {}_R \mathbb{F}$. So $\exists x \in P : P = \mathbb{F}x$. Write $x = \sum_{g \in G} \alpha_g g$. Since $P \cong \mathbb{F}$, $hx = x \forall h \in G$, i.e.

$$\sum_{g \in G} \alpha_g hg = \sum_{g \in G} \alpha_g g \quad \forall h \in G,$$

it follows that all α_g are equal and $\neq 0$. Therefore G has to be finite because if it's not then $x = \sum_{g \in G} \alpha_g$ which is not well defined. Now suppose $|G| = n$ and $p \mid n$, then $x \in \mathbb{F}G$ and $\psi(x) = n\alpha = 0$, i.e. $x \in \ker \psi$, a contradiction to the direct sum.

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$2 \Rightarrow 1$: We will show every $\mathbb{F}G$ -module is completely reducible and then apply 4.2.11, 4.3.1 and 4.3.2. Let ${}_G M > {}_G N$ and the goal is to find a direct complement for N . One can write $M = N \oplus K$ as \mathbb{F} -vector spaces. Consider the corresponding projection $p : M \rightarrow N \hookrightarrow M$ which is idempotent. Let $\alpha \in \mathbb{F}$ satisfy $|G|\alpha = 1_{\mathbb{F}}$ (one can think of α as $\frac{1}{|G|}$). Define $\hat{p} \in \text{End}_{\mathbb{F}} M$ by $x \mapsto \alpha \sum_{g \in G} g(p(g^{-1}x))$. Since N is a submodule, $\text{im } \hat{p} \subseteq N$. Now for any $x \in N$, $g^{-1}x \in N$ and so

$$\hat{p}(x) = \alpha \sum_{g \in G} g(p(g^{-1}x)) = \alpha \sum_{g \in G} g(g^{-1}x) = \alpha |G|x = x,$$

so $\text{im } \hat{p} = N$ and $\hat{p}^2 = \hat{p}$, i.e. \hat{p} is idempotent. Moreover, for $g \in G$ and $y \in M$,

$$\begin{aligned}\hat{p}(gy) &= \alpha \sum_{h \in G} h(p(h^{-1}gy)) = \alpha \sum_{k_1, k_2 \in G: k_1 k_2 = g} k_1(p(k_2 y)) \\ &= \alpha \sum_{h \in G} gh(p(h^{-1}y)) = g\hat{p}(y),\end{aligned}$$

so $\hat{p} \in \text{End}_R M$, hence one can write $M = \text{im } \hat{p} \oplus \ker \hat{p} = N \oplus \ker \hat{p}$, where $\ker \hat{p}$ is the direct complement we are looking for. \square

Example 4.3.5. Consider $\mathbb{F}C_n$ where $C_n = \langle x \mid x^n = 1 \rangle$, which can be written as $\mathbb{F}[y]/(y^n - 1)$. If one writes $y^n - 1 = f_1^{a_1} \cdots f_1^{a_1}$ where $f_i \in \mathbb{F}[y]$ are irreducible and $a_i \geq 1$, then using Chinese remainder theorem one has

$$\mathbb{F}C_n \cong \mathbb{F}[y]/(f_1^{a_1}) \times \cdots \times \mathbb{F}[y]/(f_n^{a_n}),$$

which is semisimple iff

$$\begin{aligned}a_1 &= \cdots = a_n = 1 \\ \iff z^n - 1 &\text{ has no multiple factors} \\ \iff \gcd((z^n - 1), (z^n - 1)'') &= 1 \\ \iff p \nmid n,\end{aligned}$$

which is what Maschke's theorem tells us as well.

If $\mathbb{F} = \mathbb{C}$ then

$$z^n - 1 = \prod_{k=0}^{n-1} \left(z - e^{\frac{2\pi k}{n}i} \right)$$

so

$$\mathbb{C}C_n \cong \prod_{k=0}^{n-1} \mathbb{C}[z]/\left(z - e^{\frac{2\pi k}{n}i} \right) \cong \mathbb{C}^n.$$

If $\mathbb{F} = \mathbb{Q}$ then $z^n - 1 = \prod_{d|n} \phi_d(z)$ where ϕ_d is the cyclotomic polynomial. So

$$\mathbb{Q}C_n \cong \prod_{d|n} \mathbb{Q}[z]/(\phi_d) \cong \prod_{d|n} \mathbb{Q} \left(\sqrt[d]{1} \right).$$

Example 4.3.6. Consider $\mathbb{R}Q_8$ where $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \leq \mathbb{H}^\times$. 4.3.3.2 applies. Now note that for each Artin-Wedderburn factor $M_n(\mathbb{F})$ there is a different surjective \mathbb{R} -algebra homomorphism $\mathbb{R}Q_8 \rightarrow M_n(\mathbb{F})$ given by projection

$$\begin{aligned}\eta : \mathbb{R}Q_8 &\twoheadrightarrow \mathbb{H} \\ \pm i &\mapsto \pm i \\ \pm j &\mapsto \pm j\end{aligned}$$

or

$$\begin{aligned}\theta_{\epsilon, \delta} : \mathbb{R}Q_8 &\twoheadrightarrow \mathbb{R} \\ i &\mapsto \epsilon \\ j &\mapsto \delta\end{aligned}$$

where $\epsilon, \delta \in \{\pm 1\}$. Since there can be $2 \times 2 = 4$ different $\theta_{\epsilon, \delta}$ and just one η , we conclude

$$\mathbb{R}Q_8 \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{H}.$$

Proposition 4.3.7. If

$${}_R M = \bigoplus_{i=1}^n S_i = \bigoplus_{j=1}^m N_j$$

where S_i, N_j are simple, then $n = m$ and $\exists \sigma \in \text{Sym}_n : S_i \cong N_{\sigma(j)}$.

Proof. We prove by induction on n . If $n = 0$ then $M = 0$ so $m = 0 = n$. If $n = 1$ then $M = S_1$ is simple so $m = 1$ and $S_1 = N_1$. Now suppose the statement is true for values $\leq n - 1$ and consider projection $\pi : M \twoheadrightarrow S_n$ along $\bigoplus_{i=1}^{n-1} S_i$. Then

$$S_n = \pi(M) = \sum_{j=1}^m \pi(N_j) \quad \text{where } \pi(N_j) \text{ is either } 0 \text{ or } N_j$$

but S_n is simple, so it has to be that $S_n \cong N_{j_0}$ for some $j_0 \in \{1, \dots, m\}$. One then has that $\bigoplus_{j \neq j_0} N_j$ is a direct complement of S_n , so

$$\bigoplus_{j \neq j_0} N_j \cong \bigoplus_{i=1}^{n-1} S_i$$

and by inductive hypothesis, $n-1 = m-1$, so $n = m$; and $\exists \widehat{\delta} \in \text{Sym}_{n-1} : S_i \cong N_{\widehat{\delta}(i)}$. Together with $S_n \cong N_{j_0}$ this completes the proof. \square

Corollary 4.3.8. For a semisimple ring $R \cong \prod M_{a_i}(D_i)$, the division rings D_i and a_i are unique up to permutation.

4.4 Jacobson radical

Definition 4.4.1. $x \in R$ is *nilpotent* if $\exists n : x^n = 0$, *quasiregular* if $1+x$ is invertible.

Example 4.4.2. Let \mathbb{F} is a field and $x \in M_n(\mathbb{F})$, then x is nilpotent iff 0 is the only eigenvalue, and quasiregular iff -1 is not an eigenvalue of x . In particular, nilpotent implies quasiregular in this case.

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Notation. $J(R) = \text{rad } R$.

Definition 4.4.3. An ideal I is *nilpotent* if $\exists n : I^n = 0$, *nil* if every $x \in I$ is nilpotent and *quasiregular* if every $x \in I$ is quasiregular.

Lemma 4.4.4. Nilpotent ideals \subseteq nil ideals \subseteq quasiregular ideals.

Proof. That nilpotent ideals \subseteq nil ideals is obvious ($\exists n : I^n = 0$ means $\exists n$: any product of n elements of I is 0).

It remains to show that a nilpotent element is quasiregular, but

$$x^n = 0 \implies (1+x)(1-x+x^2-\dots+(-1)^{n-1}x^{n-1}) = 1.$$

\square

Example 4.4.5. $R = \mathbb{C}[[x]] \leq \mathbb{C}((x))$. Set $J := (x) = \{\alpha_1 x + \dots + \alpha_n x^n + \dots\}$. Then J is quasiregular: write

$$J \ni z = \alpha_n x^n + \dots \quad \text{where } \alpha_n \neq 0, n \geq 1$$

then

$$(1+z)^{-1} = \sum_{k=0}^{\infty} (-1)^k z^k.$$

J is also maximal since $R/J \cong \mathbb{C}$, a field. We will later see that this implies $J = J(R)$.

Note that J is not nil; in fact R is a domain.

Example 4.4.6. $S = \mathbb{C}[x_1, x_2, \dots]$, $I = (x_1^2, x_2^2, \dots)$, $R = S/I$, $\overline{x_i} = x_i + I$, $J = (\overline{x_1}, \overline{x_2}, \dots)$. Then J is trivially nil, so quasiregular. Again $R/J \cong \mathbb{C}$ so J is maximal, hence $J = J(R)$.

Note that J is not nilpotent since $\overline{x_1} \overline{x_2} \dots \overline{x_n} \neq 0$.

Proposition 4.4.7. If $I, J \trianglelefteq R$ and $I^n = J^m = 0$, then $(I+J)^{n+m} = 0$. In particular, the sum of two nilpotent ideals is nilpotent.

Proof. $(I+J)^a$ is the \mathbb{R} -span of elements of the form

$$\prod_{i=1}^a (x_i + y_i) = \prod_{i=1}^a x_i + \text{terms with } y_i$$

where $x_i \in I$, $y_i \in J$, hence $(I+J)^a \subseteq I^a + J$, and so

$$(I+J)^{n+m} = ((I+J)^n)^m \subseteq (I^n + J)^m \subseteq J^m = 0.$$

\square

Conjecture (Köthe). If $I, J \leq^l R$ and I, J are nil, then $I + J$ is nil.

Theorem 4.4.8. For a ring R , $J_1 = \dots = J_7$ where

- $J_1 = \text{rad}_R R$
- $J_2 = \text{rad}_R R$
- $J_3 = \bigcap_{L \leq_{\max}^l R} L$
- $J_4 = \bigcap_{I \leq_{\max}^r R} I$
- $J_5 = \{x \in R : \forall \text{ simple } {}_R M, xM = 0\}$
- $J_6 = \{x \in R : \forall \text{ simple } M_R, xM = 0\}$
- J_7 is the largest 2-sided quasiregular ideal

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Proof. 1. $J_1 \subseteq J_5$: let $x \in J_1$ and ${}_R M$ a simple left R -module. $\forall m \in M$, $\text{Ann}_R(m)$ is maximal, so $x \in \text{Ann}_R(m)$, hence $xm = 0 \implies xM = 0 \implies x \in J_5$.

2. $J_5 \subseteq J_3$: let $x \in J_5$ and $L \leq_{\max}^l R$. Then R/L is a simple R -module, so $xR/L = 0$ and in particular $x(1+L) = 0+L$, so $x \in L$ and hence $x \in J_3$.

3. J_3 is quasiregular: let $x \in J_3$. Note that $R(1+x) = R$, since if $R(1+x) \neq R$, then $\exists L \leq_{\max}^l R$ which contains $R(1+x)$ and in particular $1+x \in L$ and since $x \in \bigcap_{L \leq_{\max}^l R} L$ one has $x \in L$ as well, therefore $1 \in L$ and so $L = R$, a contradiction. Hence $1+x$ has a left inverse $1+z$, and

$$\begin{aligned} (1+z)(1+x) &= 1 \\ z+x+zx &= 0 \\ z &= -(1+z)x \in J_3 \end{aligned}$$

so z also has a left inverse. Denote it t , then

$$t = t1 = t(1+z)(1+x) = 1+x$$

so

$$1 = t(1+z) = (1+x)(1+z),$$

hence $1+z$ is also the right inverse of $1+x$.

4. J_1 contains every left quasiregular ideal: suppose $\exists I \leq_{\text{quasiregular}}^l R : I \not\subseteq J_1$, so $\exists L \leq_{\max}^l R$ and $x \in I : x \notin L$. This implies $L + Rx = R$ and in particular $a + bx = 1$ for some $a \in L, b \in R$. Since $-bx \in I$ which is quasiregular, $a = 1 - bx$ has a left inverse t , but then $1 = ta \in L$ so $L = R$, a contradiction.

5. J_5 is a 2-sided ideal: we already know J_5 is a left ideal. Now pick $x \in J_5$, $r \in R$ and let ${}_R M$ be a simple left R -module. Then $(xr)M \subseteq x(rM) \subseteq xM = 0$, so $xr \in J_5$ and hence J_5 is also a right ideal.

The 5 steps prove $J_1 = J_3 = J_5 = J_7$. The proof for $J_2 = J_4 = J_6 = J_7$ is analogous. \square

Remark. 1. Radical property: $J(R/J(R)) = 0$. The philosophy is: radical is the bad stuff we can get rid off.

2. A ring R with $J(R) = 0$ are also called semisimple in literature. This watershed between classical semisimplicity and Jacobson semisimplicity is presented in the following proposition.

Proposition 4.4.9. The following are equivalent.

1. R is semisimple.
2. R is left artinian and $J(R) = 0$.

Theorem 4.4.10. If R is left artinian then $J(R)$ is nilpotent.

Proof. Denote $J = J(R)$. Consider descending chain

$$J \supseteq J^2 \supseteq \cdots \supseteq J^n \supseteq \cdots$$

since R is artinian, $\exists n : J^n = J^{n+1} = \cdots$. We claim $J^n = 0$. Let

$$I = \text{Ann}_R(J_R^n) = \{x \in R : J^n x = 0\}.$$

Note that I is a 2-sided ideal: let $x \in I, y \in R$, then $J^n xy \subseteq 0y \subseteq 0$ and $J^n yx \subseteq J^n x = 0$, so $xy, yx \in I$. If $I \supseteq J^n$ then we are done since $J^n = J^{2n} = J^n J^n \subseteq J^n I = 0$ by construction, so suppose $I \not\supseteq J^n$ and consider quotient homomorphism $\psi : R \rightarrow R/I =: S$. Then $\psi(J^n) \neq 0$. Since $J^n \subseteq J = J(R)$, (see HW4 P4) $\psi(J^n) \subseteq \psi(J) \subseteq J(S)$. Since R is artinian, so is S , hence $\exists L \trianglelefteq_{\min}^l S : L \subseteq \psi(J^n)$. Then L is a simple S -module, so $\psi(J^n)L \subseteq J(S)L = 0$ by 4.4.8. Apply ψ^{-1} and one has $J^n \psi^{-1}(L) \subseteq I$, and

$$J^n \psi^{-1}(L) = J^{2n} \psi^{-1}(L) = J^n (J^n \psi^{-1}(L)) \subseteq J^n I = 0,$$

so $\psi^{-1}L \subseteq I$ and hence $L = 0$, a contradiction. □

Corollary 4.4.11. For a left artinian ring R , $J(R)$ is the largest nilpotent 2-sided/left/right ideal of R .

Proof. R being nilpotent follows from 4.4.10. Let $I \triangleleft R$ be nilpotent. Then it's quasiregular so $I \subseteq J(R)$ by 4.4.8.

Now let $L \triangleleft^l R$ with $L^n = 0$, then $LR \trianglelefteq R$ and $(LR)^n = L(RL)^{n-1}R \subseteq L^n R = 0$, so by above $L \subseteq LR \subseteq J(R)$. Similar for right. □