Beneath the Surface: Thermal-Fluid Analysis of a Hot Bath

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Abstract

The overall temperature and uniformity of heat distribution are crucial to the enjoyment of a hot bath. Over time, a bath will lose heat to its surroundings; however, the temperature can be effectively controlled by adding a trickle of hot water. Quantifying the optimal method of restoring heat uniformly is our objective.

By making key assumptions, we reduce the problem to one spatial dimension with three-dimensional heat transfer considerations. We develop three independent generic mathematical models to describe the temperature distribution of the bathtub in space and time:

- **Steady-State Uniform Thermodynamic Model:** We characterize the energy and mass interactions between the system and the surroundings, deriving key physical constants used in the subsequent models.
- **Analytical Model**: Using *Fourier analysis* we derive an analytical solution to the one-dimensional heat equation with time dependent Neumann boundary conditions and temporally and spatially dependent heat generation/loss.
- **Numerical Model**: Using a *finite difference approximation* of the one-dimensional convection-diffusion equation, we implement an algorithm that estimates the temperature distribution over time based on bulk fluid motion and diffusion.

We analyze the results of each of these models, performing sensitivity analyses on the input parameters and demonstrating their flexibility. Ultimately, the numerical model proves the most physically accurate as it models convective fluid motion, which is found to dominate thermal diffusion. We then implement *closed-loop PID control* to regulate the total thermal energy of

The UMAP Journal 37 (3) (2016) 251–276.

the system. We also model an intelligent strategy for inlet water distribution to achieve temperature uniformity.

Ultimately, our model demonstrates that using well-tuned PID control of input water temperature and evenly distributing this water upstream of the bather quickly achieves a uniform objective temperature while conserving water.

Introduction

Overview

Our specific scenario is that our friend Joseph Fourier (hereafter Joe) decides to take a warm bath and fills his bathtub to the desired temperature. After a while, he notices that the water is cooler, and he wishes to increase the temperature by adding a trickle of warm water. What should he do so as to most effectively regain the initial temperature while keeping the temperature uniform throughout and not wasting too much water? What if he starts adding water as soon as he gets in the bath?

We attempt to answer these questions for Joe. We present mathematical models describing the temperature distribution in a bathtub in space and time. We describe the relevant physical mechanisms, mathematical development, implementation, and results. Then we evaluate the accuracy and explore the best solution for Joe. Finally, we present the strengths and weaknesses of our model.

Simplifying Assumptions

- The faucet of the bathtub is exactly at one end of the length of the bathtub with the overflow drain exactly at the opposite end.
- When Joe is in the bathtub, it is full. Any addition of water results immediately in drainage of the same volume.
- The bathtub is free-standing, that is, it is surrounded by air on all sides.
- The bathtub itself is in thermal equilibrium with the water at all times. Thus, the interface with the ambient air on all sides can be described without separate knowledge of the temperature distribution within the bathtub material or the heat capacity of the bathtub.
- The only mechanism of stationary heat transfer is convection. We ignore all radiation effects.
- The ambient air temperature is constant in space and time, that is, the surroundings can be modeled as a thermal reservoir.
- The ambient air can be modeled as a stationary ideal gas at 1 bar $\approx$ 1 atm.

Table 1. Nomenclature

Abbreviation	Description
a	Boundary condition at faucet
b	Boundary condition at overflow drain
c_p	Specific heat
E_{system}	Energy of a system
f	Controls forcing function
g	Gravitational acceleration
Gr_L	Grashof number
h	Enthalpy
h_b	Convection heat transfer coefficient through bottom surface
h_{conv}	Convection heat transfer coefficient
h_t	Convection heat transfer coefficient through top surface
h_w	Convection heat transfer coefficient through the walls
H	Height of bathtub
i	Spatial index variable
j	Temporal index variable
k	Thermal conductivity
K_d	Derivative gain constant
K_i	Integral gain constant
K_p	Proportional gain constant
L	Length of bathtub
L_c	Characteristic length
m	Mass
n	Summation index variable
Nu	Nusselt number
p	Perimeter
Pe	Péclet number
Pr	Prandtl number
Q	Heat generation
Ra_L	Rayleigh number
Re_L	Reynolds number
t	Time
u	Temperature
u_{avg}	Average temperature
u_{in}	Inlet water temperature
u_{obj}	Desired water temperature
u_s	Surface temperature
$\overline{u}$	Difference between average and desired temperature
u_∞	Air temperature
U	Shifted temperature distribution
v	Flow velocity
W	Width of bathtub
x	Length direction
y	Width direction
z	Height direction
α	Thermal diffusivity
β	Volume expansion coefficient
ϵ	Porosity
η	Fraction of input heat to be redistributed
ν	Kinematic viscosity
σ	Standard deviation from average temperature
ϕ	Initial condition
ρ	Density

- No water loss occurs due to evaporation, that is, the volume of water in the tub remains constant and the surface of the water interacts directly with dry air.
- The presence of Joe in the tub provides only thermal effects. He causes no disturbance to the flow.
- The flow is uniform, laminar, incompressible, inviscid, and quasi one-dimensional.
- The water has constant density and constant specific heat and remains in liquid form.
- All material properties are isotropic.

Geometric Considerations

The bathtub is considered a rectangular prism, with dimensions L, W, and H (**Figure 1**).

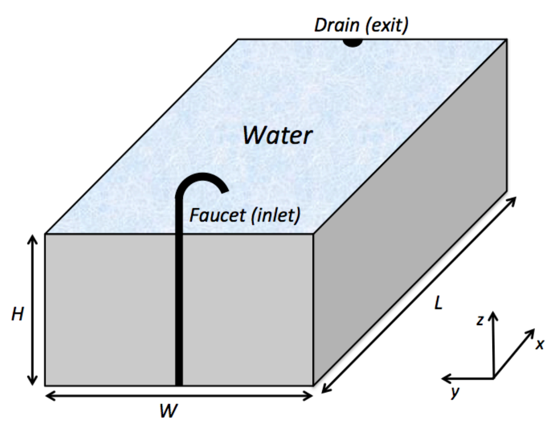


Figure 1. Bathtub geometry.

There are six rectangular faces: five composed of the bathtub material and one exposed to ambient air. The x-direction is along the length of the bathtub, and y and z are along the width and height. The faucet is at $x = 0$ and the overflow drain at $x = L$. When water is added through the faucet, an equal volume exits through the overflow drain.

The heat transfer and fluid flow within the water are assumed to be solely in the x-direction. The temperature within a cross-section parallel to the yz-plane is assumed to be uniform; it can be thought of as an average temperature over that cross-section. These two assumptions allow us to reduce the problem to one dimension.

Thermodynamic Considerations

We examine the key thermodynamic parameters of the problem: the specific enthalpy, specific heat, density, and thermal conductivity of water, as well as the convective heat transfer coefficients from the system to the surrounding air.

Properties of Water

We assume that the specific enthalpy of water varies linearly with temperature, with a slope equal to the specific heat. We validated this assumption using property values from the XSteam [1] Matlab function.

Enthalpy (h), the heat content, can be calculated from $h = c_p u$, where c_p is the specific heat and u is the temperature. We ignore the small y-intercept value and use a constant specific heat [2] of 4178 J/(kg-K).

The density and thermal conductivity of water are taken to be constant at 1000 kg/m^3 and 0.615 W/(mK) at an assumed average temperature of $30°$C [2].

Convective Heat Transfer

Because the bathtub itself is assumed to be in thermal equilibrium with the water, we consider only convective effects with the surrounding air. To appropriately estimate the heat transfer coefficient, this convection must be classified as natural (free), forced, or mixed. The situation depends on the ratio of the Grashof number (Gr_L) and the Reynolds number (Re_L). If $\mathrm{Gr}_L/\mathrm{Re}_L^2 \gg 1$, inertial forces prevail and natural convection dominates; if $\mathrm{Gr}_L/\mathrm{Re}_L^2 \ll 1$, the opposite is true and forced convection dominates; if $\mathrm{Gr}_L/\mathrm{Re}_L^2 \approx 1$, mixed convection must be considered [2].

[EDITOR'S NOTE: The authors calculate $\mathrm{Gr}_L/\mathrm{Re}_L^2$ for the system; the details are omitted.]

Since $\mathrm{Gr}_L/\mathrm{Re}_L^2 \approx 6 \times 10^5$, natural convection effects significantly dominate. Thus, we ignore forced convection.

Next, we estimate the convective heat transfer coefficients. We have six surfaces with three independent geometries. We model four of the surfaces (the sides) as vertical plates, one (the top) as a flat plate with the hotter side facing up, and one (the bottom) as a flat plate with the hotter side facing down. Due to the physics of natural convection, these three geometries have separate heat transfer coefficients. The coefficient depends on the respective Nusselt number, Nu, which in turn depends on the Rayleigh number, Ra_L, the product of the Grashof and Prandtl numbers. The Rayleigh number describes the ratio of buoyancy forces to thermal and momentum diffusivities. The Prandtl number, Pr, is a property of the air based on film temperature [2].

The Nusselt number can be determined as follows [2].

$$\text{Top:} \qquad \mathrm{Nu}_t = \begin{cases} 0.54\mathrm{Ra}_L^{1/4}, & 10^4 < \mathrm{Ra}_L < 10^7; \\ 0.15\mathrm{Ra}_L^{1/3}, & 10^7 < \mathrm{Ra}_L < 10^{11}. \end{cases}$$

$$\text{Bottom:} \qquad \mathrm{Nu}_b = 0.27\mathrm{Ra}_L^{1/4}, \quad 10^5 < \mathrm{Ra}_L < 10^{11}. \tag{1}$$

$$\text{Wall:} \qquad \mathrm{Nu}_w = \left(0.825 + \frac{0.387\mathrm{Ra}_L^{1/6}}{\left(1+(0.492/\mathrm{Pr})^{9/16}\right)^{8/27}}\right)^2, \quad \mathrm{Ra}_L > 0.$$

The convective heat transfer coefficient is

$$h_{\text{conv}} = \mathrm{Nu}\, k/L,$$

where k is the thermal conductivity of the air [2]. In practice, each thermodynamic property of air depends also upon the temperature, so there is no single explicit solution to these equations. Rather, a system temperature must be guessed and the validity of the guess evaluated. Assuming an air temperature of 20°C, the variation of h_t (the coefficient for the top surface) with water temperature is shown in **Figure 2**.

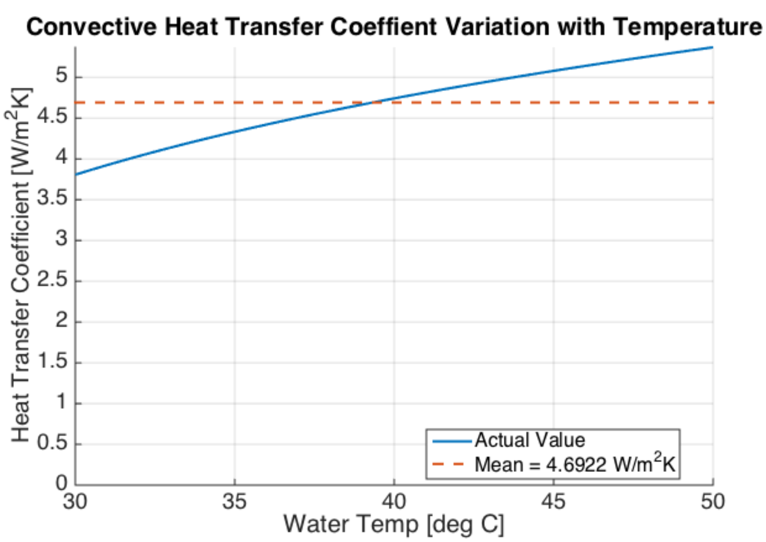


Figure 2. Variation of h_t with Water Temperature

The value of h_t rises slightly with temperature but very little over the range of temperatures that concern us. The same is true for h_b (for the bottom surface) and h_w (for the walls).

We have $h_t > h_w > h_b$, , demonstrating that natural convection is slightly more efficient from vertical walls than from face-down plates, but not as efficient as from face-up plates. For simplicity, we take the heat transfer coefficients to be constant at their mean values over the temperature range $30°$–$50°$ C.

Nominal Values for Parameters

Table 2 gives the nominal values for parameters that we use.

Table 2. Nominal values of parameters.

Parameter	Description	Nominal value
L	Bath Length	1.5 m
W	Bath Width	1 m
H	Bath Height	0.5 m
$\dot{m}$	Mass Flow Rate	0.05 kg/s
u_{in}	Input Water Temp.	45°C
L_{body}	Body Length	1 m
c_{body}	Body Circumference	0.5 m
u_{body}	Body Temperature	37°C
X_{body}	Position of Body Center	0.75 m
h_{body}	Heat Transfer Coeff. of Body	43 W/(m^{2}K) [3]
u_∞	Surroundings Temperature	20°C
u_0	Initial Water Temperature	37°C
k	Thermal Conductivity	0.615 W/(mK) [2]
ρ	Water Density	1000 kg/m^3 [2]
c_p	Specific Heat Capacity	4178 J/(kgK) [2]
h_t	Heat Transfer Coeff. of Top	4.692 W/(m^{2}K)
h_b	Heat Transfer Coeff. of Bottom	1.934 W/(m^{2}K)
h_w	Heat Transfer Coeff. of Wall	3.804 W/(m^{2}K)
t	Time of Bath	1 hr

Steady-State Uniform-Temperature Model

Theory

We move from defining the geometry and thermodynamic parameters of the problem to developing models for its solution.

For an initial model, we assume that the temperature in the bathtub is uniform and steady. This assumption allows the analysis to be simply derived from the First Law of Thermodynamics. The *system* is only the water in the bathtub, and the *surroundings* is composed of the inlet, the exit, the bather, and the ambient air.

Assuming steady state, the time derivatives of energy and mass in the system must be zero:

$$\frac{dE_{\text{system}}}{dt} = 0 \Rightarrow \dot{E}_{\text{in}} = \dot{E}_{\text{out}}, \qquad \frac{dm_{\text{system}}}{dt} = 0 \Rightarrow \dot{m}_{\text{in}} = \dot{m}_{\text{out}}. \tag{2}$$

Writing the first equation above using the First Law of Thermodynamics, we find [2]:

$$\dot{Q}_{\text{in}} + \dot{W}_{\text{in}} + \sum_{\text{inlets}} \dot{m}\left[h + \frac{v^2}{2} + gz\right] = \dot{Q}_{\text{out}} + \dot{W}_{\text{out}} + \sum_{\text{exits}} \dot{m}\left[h + \frac{v^2}{2} + gz\right]. \tag{3}$$

Physically, **(3)** is simply an energy balance. Energy transfer by heat, work, and mass are accounted for, with enthalpy, kinetic energy, and potential energy making up the mass transfer terms. We have only one inlet and one exit, which removes the summation from the equation. We also assume no work done on or by the system, no change in potential energy, and no change in kinetic energy. These are valid assumptions, since the inlet (faucet) and exit (overflow drain) should both be at the level of the water, and the kinetic energy terms will be negligible due to the assumed trickle of water. Furthermore, the mass flow entering and leaving should be the same, assuming $\dot{m}_{\text{in}} = \dot{m}_{\text{out}}$ from **(2)** above. Finally, we can combine $\dot{Q}_{\text{in}}$ and $\dot{Q}_{\text{out}}$ into one term representing the net heat transfer. Using these simplifications and the relationship $h = c_p u$, we get:

$$\dot{m}c_p u_{\text{in}} = \dot{m}c_p u_{\text{out}} + \dot{Q}_{\text{net,out}}. \tag{4}$$

Since we assume uniform temperature, the temperature at the exit is the temperature of the system ($u_{\text{out}} = u$).

The last piece of this model is to quantify $\dot{Q}_{\text{net,out}}$ using the heat transfer coefficients discussed in the previous section. We use Newton's Law of Cooling [2] to find an expression for the convective heat transfer:

$$\dot{Q}_{\text{conv,out}} = h_{\text{conv}} A(u - u_\infty). \tag{5}$$

Associating the appropriate surface areas with each coefficient, we arrive at the following expression for the total convective heat loss:

$$\dot{Q}_{\text{conv,out}} = (u - u_\infty)\left[(h_t + h_b)LW + h_w(2HW + 2HL)\right]. \tag{6}$$

The presence of Joe in the bathtub provides an additional heat transfer term with its own heat transfer coefficient. We assume the submerged part of the body is a perfect cylinder without end caps, with length L_{body}, circumference c_{body}, and external temperature u_{body}. The total heat loss from the system now includes a term to account for the body:

$$\dot{Q}_{\text{net,out}} = (u - u_\infty)\big[(h_t + h_b)LW + h_w(2HW + 2HL)\big] + (u - u_{\text{body}})h_{\text{body}}L_{\text{body}}c_{\text{body}}. \quad \textbf{(7)}$$

By substituting **(7)** into **(4)** and solving for u, we find the final expression for steady uniform temperature:

$$u = \frac{\dot{m}c_p u_{\text{in}} + u_\infty\left((h_t + h_b)LW + h_w(2HW + 2HL)\right) + u_{\text{body}}h_{\text{body}}L_{\text{body}}c_{\text{body}}}{(h_t + h_b)LW + h_w(2HW + 2HL) + h_{\text{body}}L_{\text{body}}c_{\text{body}}}. \quad \textbf{(8)}$$

Implementation and Results

We implemented **(8)** using Matlab. Using average values for the heat transfer coefficients and the nominal parameter values, we obtained the steady uniform temperature as a function of mass flow as shown in **Figure 3**.

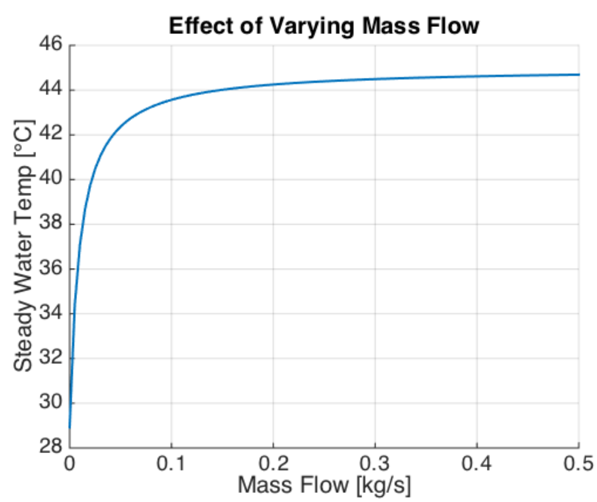


Figure 3. Sensitivity of final temperature to inlet mass flow.

The figure indicates that increasing the mass flow results in a final temperature that approaches the inlet temperature of 45°C. Also, as expected, the final steady temperature depends linearly on the inlet temperature.

When the inlet temperature is close to that of the ambient air (20°C), the steady temperature is higher than that of the inlet, due to the minimal convective heat losses and the effect of Joe at 37°C. However, when the inlet temperature is closer to body temperature, the steady temperature is slightly below the inlet temperature, due to the increased convective heat losses.

The final steady temperature depends linearly and relatively insensitively on the ambient air temperature. The model is clearly much less sensitive to this parameter, illustrating minimal heat losses to convection due to the relatively low heat transfer coefficients.

Ultimately, this model is too simple to capture the true dynamical nature of the bathtub system. Though more accurate models are described in the following sections, the qualitative results here serve as a good baseline for comparison with subsequent models.

Analytical Model of the Heat Equation

[EDITOR'S NOTE: This section is omitted, since the authors subsequently abandon the analytical model, which features only heat diffusion, in favor of a numerical model that includes also convection.]

Numerical Model of Convection-Diffusion

We use a finite difference approximation of the convection-diffusion equation. We briefly discuss the theoretical development and implementation, and then present initial results.

Theory

We introduce a convection term into the model, which allows the actual flow of the water—and not just diffusion of heat—to be modeled. Convection in this sense means the bulk motion of the water in the tub and not the heat loss to the surroundings. The governing PDE for a convection-diffusion problem is shown below, where the only new quantity is ϵ, the porosity of the medium:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} - \epsilon v \frac{\partial u}{\partial x} + \frac{q(x,t)}{\rho c_p} \tag{9}$$

In this equation,

- $\dfrac{\partial u}{\partial t}$ represents the transient,
- $\alpha \dfrac{\partial^2 u}{\partial x^2}$ represents diffusion,
- $\epsilon v \dfrac{\partial u}{\partial x}$ represents convection, and
- the final term represents heat generation or loss.

Eliminating the convection and generation terms yields the familiar heat equation. Prescribing an initial condition and enforcingappropriateboundary conditions fully defines this problem. Due to the difficulty in solving this equation analytically, we implemented a numerical solution using the finite difference method, as outlined by Majchrzak and Turcha [6].

The first step is to discretize the temporal and spatial variables. We let j index time and i index space, and Δt and Δx are the time step and grid spacing:

$$t_{j+1} = t_j + \Delta t, \qquad x_{i+1} = x_i + \Delta x. \tag{10}$$

Each derivative can be approximated using a Taylor series:

$$\frac{u_i^j - u_i^{j-1}}{\Delta t} = \alpha \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta x)^2} - \epsilon v \frac{u_{i+1}^{j-1} - u_{i-1}^{j-1}}{\Delta x} + \frac{q_i^{j-1}}{\rho c_p}. \tag{11}$$

Solving for the temperature at time j and location i yields:

$$\begin{aligned} u_i^j = \left(1 - \frac{2\alpha\Delta t}{(\Delta x)^2}\right) u_i^{j-1} + \left(\frac{\alpha \Delta t}{(\Delta x)^2} + \frac{\epsilon v \Delta t}{2\Delta x}\right) u_{i-1}^{j-1} \\ + \left(\frac{\alpha \Delta t}{(\Delta x)^2} - \frac{\epsilon v \Delta t}{2\Delta x}\right) + \frac{q_i^{j-1}}{c_p \rho}\,\Delta t. \end{aligned} \tag{12}$$

The porosity ϵ is the ratio of fluid volume to total volume; since the medium is water (entirely fluid), we take $\epsilon = 1$.

To complete this model, we must quantify q and enforce appropriate boundary conditions.

The fundamental source of heat transfer in this problem is convection from the water surface and from the bathtub surface. In **(12)** above, q is the heat transfer per unit volume, where the volume in consideration is a slice of the bathtub in the x direction. This slice has volume $WH\Delta x$. Thus,

$$q = \frac{Q}{WH\Delta x}, \tag{13}$$

where Q is the macroscopic heat loss of the slice. This heat loss is described by the convection terms and the conduction term as follows:

$$\begin{aligned} Q &= (u - u_\infty)\big[h_t A_t + h_w A_w + h_b A_b\big] \\ &= (u - u_\infty)\big[h_t (W\Delta x) + h_s(2H\Delta x) + h_b(W\Delta x)\big]. \end{aligned} \tag{14}$$

Plugging this into **(13)** gives

$$q = \frac{(u - u_\infty)}{WH}\ [W(h_t + h_b) + 2Hh_w]. \tag{15}$$

At location i at time step j, we use u_i^{j-1} to estimate u in the equation.

The same method can be used to model the presence of a person in the bathtub. [EDITOR'S NOTE: The details of the calculation are omitted.]

To enforce the boundary condition on the inlet side, we must specify the correct amount of heat flux into the system as a Neumann condition, increasing the energy content of the fluid element at the boundary based on the mass flow and temperature of the inlet.

The energy entering per unit volume, a q term, can be described as

$$q = \frac{Q}{WH\Delta x} = \frac{\dot{m}c_p(u_{\text{in}} - u)}{WH\Delta x}. \tag{16}$$

This can be translated directly into a change in temperature, as in **(12)** above, by dividing by ρc_p, multiplying by the time step Δt, and approximating u from the previous time step:

$$\Delta u_i^j = \frac{\dot{m}\left(u_{\text{in}} - u_i^{j-1}\right)}{\rho WH\Delta x}\Delta t \tag{17}$$

Equation **(17)** provides a direct method for changing the temperature of the boundary point in order to account for the heat entering the system.

At the other boundary point, we wish the water to exit the system naturally, which poses a challenge in prescribing a boundary condition. A realistic condition is that the flux out is equal to the flux between the two points adjacent to the boundary. Thus, the temperature at the end can be specified as follows:

$$\begin{aligned} u_{end}^j &= u_{end-1}^j + \frac{\partial u}{\partial x}\Delta x \\ &= u_{end-1}^j + \left(\frac{u_{end-1}^j - u_{end-2}^j}{\Delta x}\right)\Delta x = 2u_{end-1}^j - u_{end-2}^j. \end{aligned} \tag{18}$$

Now both boundary conditions have been specified and we can implement the finite difference method in Matlab.

Basic Results and Verification

We present basic results to demonstrate the functionality of our model.

First, we simulated the situation with no heat input (the faucet turned off) (**Figure 4**). The temperature in the tub remains uniform and slowly decreases due to heat loss to the surrounding air.

Next, we simulated the system with the faucet on with steady flow (**Figure 5**). In this case, the temperature away from the faucet (near $x = L$) drops steadily, as before, but the temperature near the faucet increases to the temperature of the incoming water. Through time, this wave of heat convects through the system at a rate determined by the mass flow and the geometry of the tub.

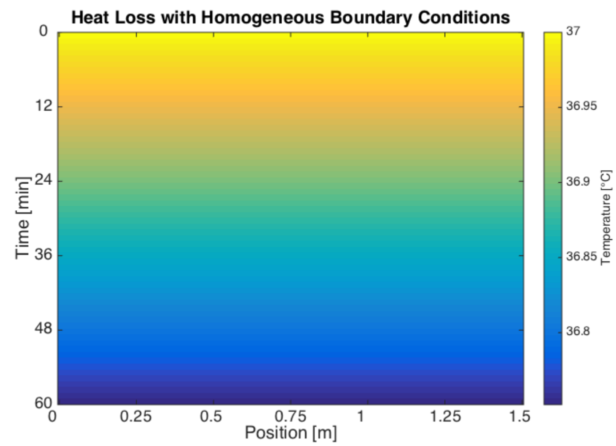


Figure 4. Heat loss over time with no heat input.

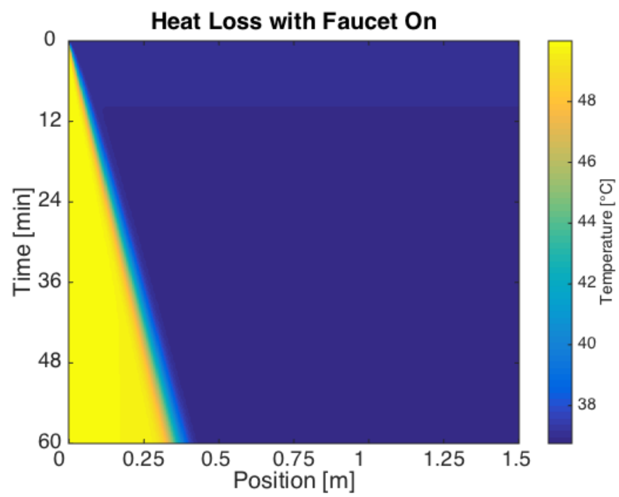


Figure 5. Heat map for steady flow from the faucet, together with heat loss.

We simulated adding bather Joe but found that the bather has almost no effect, which makes sense because both bath and body temperatures are nearly equal.

Modeling Bather Actions

We examine the effect of actions of the bather redistributing heat by moving, in the case of fixed inlet water temperature and mass flow.

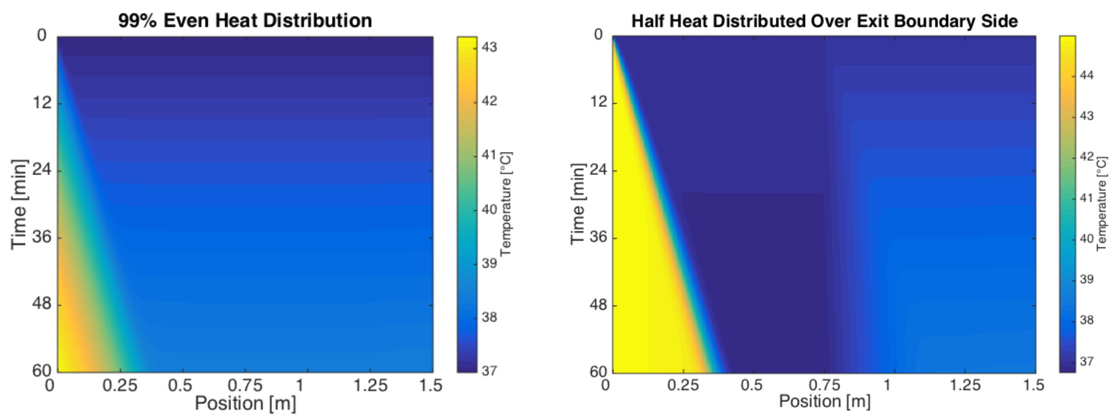
We extend the logic for the boundary condition on the inlet to the full domain. The input energy can be added to any locations in the domain, provided that the total energy is conserved. So we modify **(17)**, using a parameter η_i^j that is the fraction of available heat prescribed to enter grid point i at time j.

$$\Delta u_i^j = \eta_i^j \, \frac{\dot{m}\left(u_{in} - u_i^{j-1}\right)}{\rho W H \Delta x} \Delta t. \tag{19}$$

This temperature change term is in addition to that already calculated from the discretized derivatives and heat loss to the surroundings. Conservation of energy means

$$\sum_i \eta_i^j = 1. \tag{20}$$

Two interesting applications of bather effects are shown in **Figures 6a** and **6b**. In the left figure, 99% of the input water was evenly distributed over the whole domain and the remaining 1% was left to trickle through the $x = 0$ boundary. In the right figure, half of the heat was distributed evenly over the region from $x = 0.75$ to $x = 1.5$ and the other half trickled through the faucet.



(a) Nearly uniform heat redistribution by bather. **(b)** Partial redistribution to back half of tub.

Figure 6. Effects of Redistribution of Water

Model Comparison

Solution Metrics

First, we must develop a metric to evaluate our solutions. To do this, we return to the original objective: Joe wishes to attain a desired temperature uniformly distributed throughout the bath. We decouple the *desired temperature* constraint and the *uniform distribution* constraint and evaluate them separately.

Desired Temperature

The first metric evaluates the total heat within the bath, addressing the *desired temperature* constraint, with objective function $\overline{u}$:

$$\overline{u}(t) = u_{\text{avg}}(t) - u_{\text{obj}}. \tag{21}$$

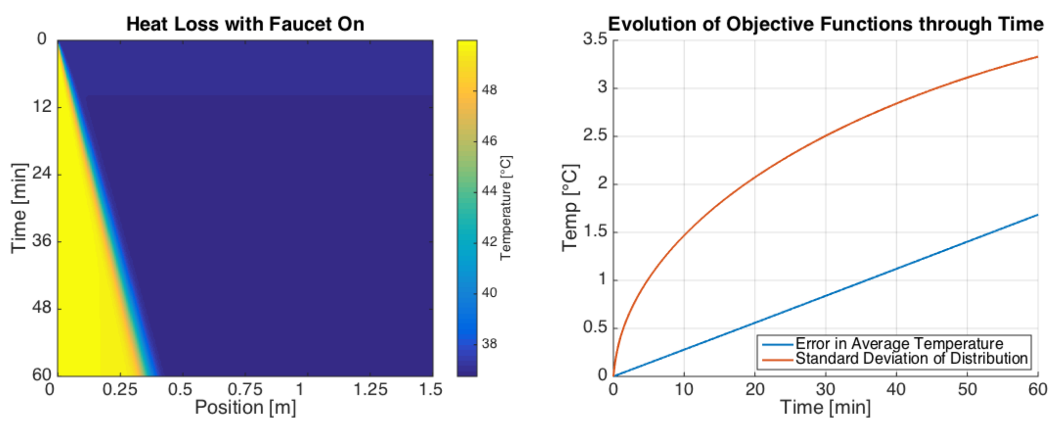
This metric will yield a time-dependent value proportional in magnitude to the difference between the average and objective, with a sign that denotes whether the bath is below or above the objective temperature. As a standard value, the objective temperature of the water will always be a comfortable 37°C.

Uniform Temperature

The second metric quantifies the *uniformity* of the temperature distribution over time. We use σ, the standard deviation of the temperature distribution at a given point in time:

$$\sigma(t) = \sqrt{\frac{1}{N}\sum_{n=1}^{N}\left(u_n(t) - u_{avg}\right)^2} \tag{22}$$

We show in **Figure 7** the evolution of the objective functions (metrics) through time for the model with nominal parameters, along with the heat map describing the temperature distribution.



(a) Heat map with faucet on. **(b)** Objective Functions From Simulation

Figure 7. Demonstration of the objective functions, $\overline{u}(t)$ and $\sigma(t)$ with nominal parameters.

The temperature becomes less uniform as we add too much heat to the system. For Joe to be happy with his bath, our goal is to drive both of these objective functions to zero. As a standard, we consider the temperature distribution at the end of a one-hour bath.

With metrics defined, we now run simulations to test the impact of each parameter. The following subsections will each describe the effect of a different parameter on the solution metrics.

Effect of Bathtub Geometry

Changing the dimensions of the tub changes the surface areas exposed to the air, the velocity of the water flow, and properties of the fluid element such as mass and volume. In **Figure 8**, we examine the effect of changing the length of the bathtub while holding other parameters constant.

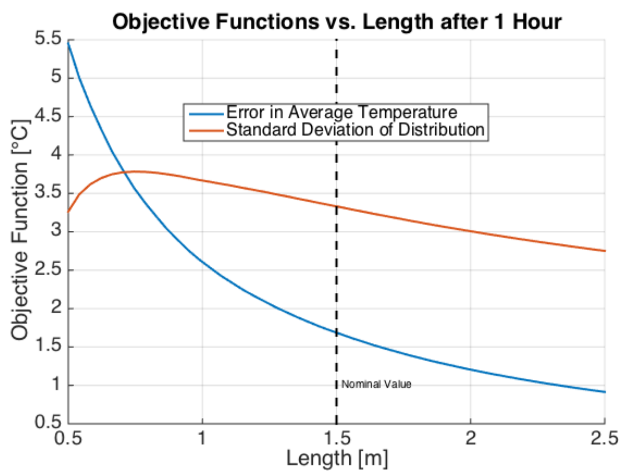


Figure 8. Effect of variation of length of tub.

Increasing the length of the tub decreases both the deviation from average temperature and the variability. The water is both more uniform and on average closer to the desired temperature. Since the flow rate is unchanged, a longer bathtub is less affected by the influx of the hot water.

We conducted similar simulations of the effect of tub width, tub height, and total volume, with similar results. [EDITOR'S NOTE: The details are omitted.]

Effect of Adding Soap

If Joe were to use a bubble bath additive or soap, the thermal diffusivity of the water would change. The primary ingredient of soap and bubble bath solutions is sodium stearate, whose density is nearly the same as that of water. Adding a solute such as sodium chloride to water reduces the thermal conductivity [9], so it is reasonable that sodium stearate has a similar effect.

To perform a sensitivity analysis of the diffusivity, we simulated a wide range of diffusivities—a full order of magnitude about the standard diffusivity of pure water, 1.472×10^{-7} m^2/s—to check for a noticeable change in solution behavior.

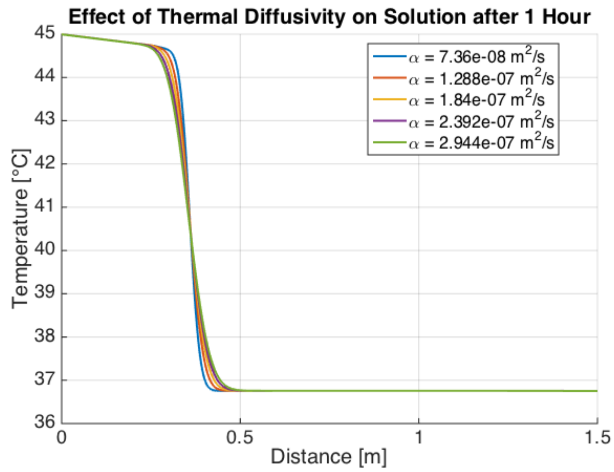


Figure 9. Effect of variation of diffusivity.

Final temperature distributions after one hour are shown in **Figure 9** for various values of thermal diffusivity. Thermal diffusivity has very little effect on the solution, supporting the hypothesis that the heat transfer is primarily due to bulk flow and not diffusion. By examining the Péclet number (Pe) [10], we can confirm this hypothesis. The Péclet number is the ratio of the convective transport rate to the diffusive transport rate. We calculate it with our nominal parameter values:

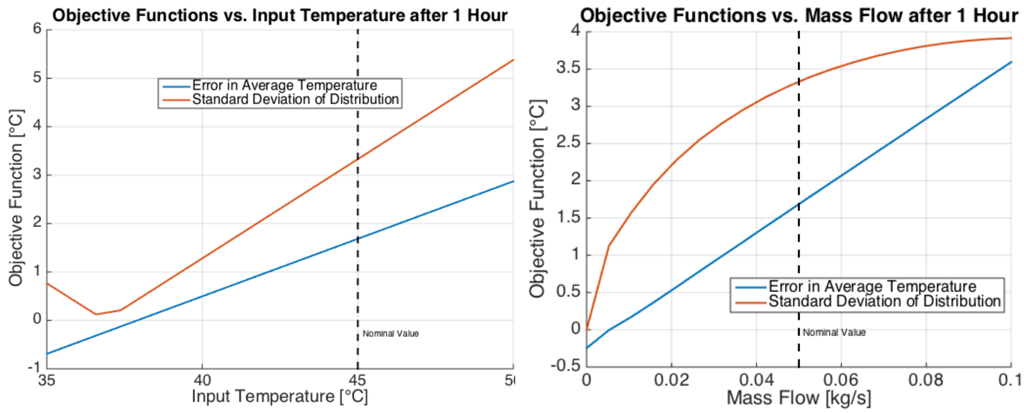
$$\mathrm{Pe} = \frac{vL}{\alpha} = \frac{(1.2 \times 10^{-4}\ \mathrm{m/s})(1.5\ \mathrm{m})}{1.472 \times 10^{-7}\ \mathrm{m^2/s}} \approx 1223 \gg 1. \tag{23}$$

The large ratio confirms that diffusion effects are negligible in comparison with bulk motion effects.

Effect of Input Temperature and Flow Rate

The effects of varying the input temperature and input rate are shown in **Figure 10**.

Increasing either parameter corresponds to increasing net energy in the system. A large flow rate results in a more uniform distribution than using instead a high temperature at the inlet, because increasing the mass flow rate increases the convective term, which is more efficient than the diffusion. The vertical lines in the plots are the nominal values, but the x-intercepts of the straight blue lines represent the correct amount of heat to the system so as to maintain the desired temperature.



(a) Variation of input temperature.

(b) Variation of mass flow rate.

Figure 10. Effects of varying input temperature and mass flow at input.

Effect of Varied Size and Position of Bather

Neither Joe's body length nor his circumference has a significant effect on the temperature distribution, because his body temperature is close to the temperature of the bath. The body temperature itself has a slight impact. [EDITOR'S NOTE: The details are omitted.]

Control Theory and Implementation

Now we consider the case where the temperature is uniform, but noticeably lower than desired, a potentially more realistic application.

PID Control Theory

Joe's ultimate goal is a bath that maintained at a uniform desired temperature. We develop a time-dependent strategy for Joe using Proportional-Integral-Derivative (PID) control.

The first portion of the control system is designed to achieve and maintain the target temperature by using the average temperature metric detailed previously. This segment of the control system focuses on optimization of the inlet temperature and flow rate of the water from the faucet.

PID control is implemented through the selection of the *gains* associated with each component of the control. Equation **(24)** shows the general gain equation of a PID controller as a function of the objective function $\overline{u}(t)$ with variable gains K for each stage, where $f(t)$ represents the calculated forcing (input).

$$f(t) = K_p\overline{u}(t) + K_i\int_0^t \overline{u}(\tau)\,d\tau + K_d\,\frac{\partial\overline{u}(t)}{\partial t}. \tag{24}$$

Each of the three components of a PID controller corresponds primarily to a specific response of the system:

- The proportional control element is based on the difference between actual temperature and objective temperature; this control is what physically drives the system to the objective.
- The integral component defines the response on accumulated error from the objective instead of current error, decreasing the rise time and removing the steady state error from the system.
- The derivative component predicts the response error based on the temporal derivative of the response, damping any oscillations.

Properly tuning each gain results in the optimal responses of a small rise time with minimal oscillation and no steady-state error.

Implementation of PID Control in Our Model

We implemented PID control by specifying the forcing as a function of $\overline{u}(t)$, as shown in **(24)**. In particular, we set

$$f(t) = \dot{m}[u_{\text{in}} - u(0,t)], \tag{25}$$

a term in the inlet Neumann condition.

We needed to determine the gain coefficients. They should weight the response with the ultimate goal of achieving a response with minimal rise time, little-to-no overshoot, and long-term stability. A rise time in the bath corresponds to the time it takes to raise the bath temperature from the initial temperature to the target temperature. The rise time should be short so that Joe does not have to wait long for his bath to reach the target temperature. Too much overshoot could result in Joe being scalded. Finally, long-term stability reflects maintaining the desired temperature over the course of the entire bath.

Methods for determining optimal gains include PID control software and algorithms such as the Ziegler-Nichols tuning process; but since we lack a well-defined transfer function, neither option is applicable and manual tuning is required.

We manually tuned our gains to obtain a roughly optimal response. **Figure 11** shows system response using only proportional control, starting at 30°C and attempting to reach 37°C.

There are noticeable oscillations about 0 (representing zero error in the temperature) in the objective function (the blue curve starting below at left), and there is an overshoot of about 4°C after one hour. Since the forcing is a function of both mass flow and inlet temperature, we fix mass flow and present the result as the required Δu (the red curve starting above at left). The Δu peaks at around 20°C, meaning that the inlet water temperature must be 20°C hotter than the current water temperature at the inlet.

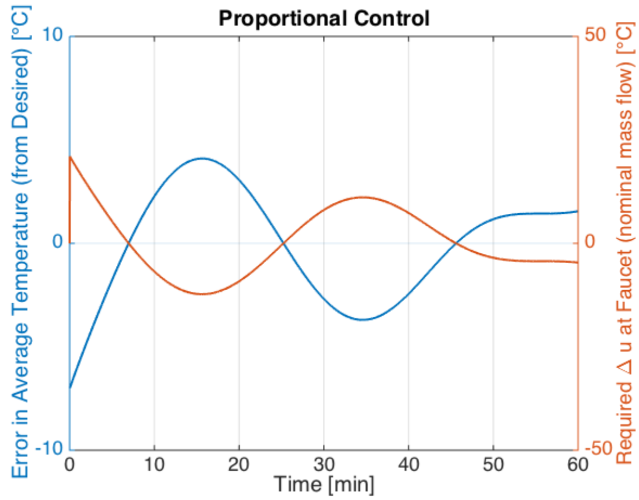


Figure 11. Effect of proportional control.

The results from using full PID are shown in **Figure 12**. The objective function is driven to nearly zero, as desired. More importantly, the response is achieved with a relatively short rise time, minimal overshoot, and maximal oscillation of less than $1°$C. The associated forcing function is more reasonable, requiring a maximum Δu of about $3°$C.

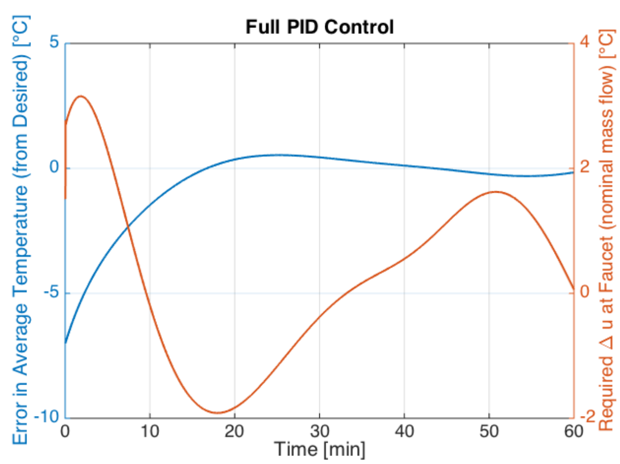


Figure 12. Optimal control of average bath temperature.

Control of Uniformity of Water Temperature Distribution

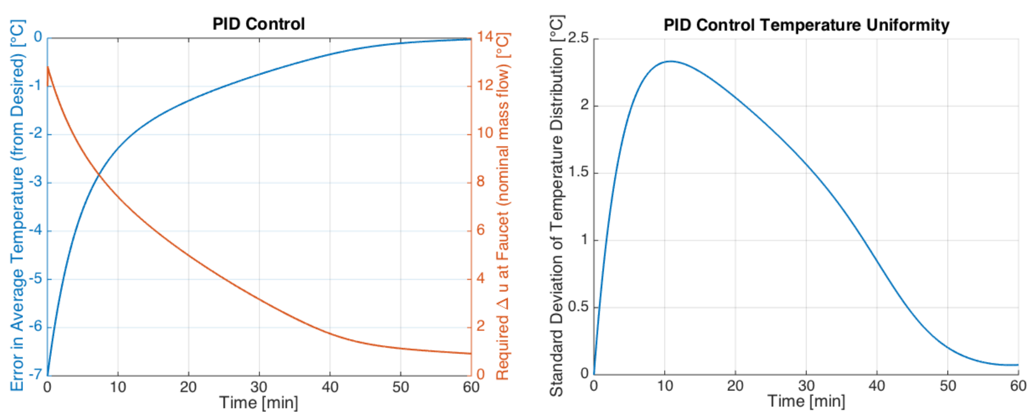
With the average temperature driven to the target temperature using PID control, the only remaining factor is the uniformity of the temperature. To maintain uniformity is very simple: Instead of adding heat from the faucet to the bathtub at $x = 0$, we evenly distribute the hot water over the whole bath. Physically, this method corresponds to Joe continuously redirecting the water from the faucet, using his hands, a cup, or another redistribution system. Our simulations confirm that uniform heat distribution is optimally effective and results in a perfect bath, using either the optimal input temperature or the optimal mass flow rate. [EDITOR'S NOTE: The details are omitted.]

Final PID Control with With Manual Heat Distribution

We close with a simultaneous consideration of average temperature (PID) control and uniformity control. Up to this point, we have analyzed these control methods separately and under different initial conditions; but to complete our solution to the problem, we need to implement them simultaneously. The initial condition is a 30°C uniform bath (Joe realizes that his bath has become cold) with target 37°C.

We use PID control to determine the required temperature of the input water (given a constant nominal mass flow rate) and using intelligent redistribution to maintain uniformity. We determined that distributing inlet water over the region from $x = 0$ (the faucet) to where the body starts, $x = 0.25$, is optimal. Doing so allows the body heat to convect downstream and the inlet heat to warm the water upstream of the body.

After manually tuning the gains and implementing full PID in addition to intelligent inlet water distribution, we realized the results shown in **Figure 13**.



(a) PID Control with Manual Heat Distribution

(b) Uniformity over Time

Figure 13. System response using PID and intelligent heat distribution.

On the left, the average temperature objective function rises to zero relatively quickly with no overshoot, an optimal result. Additionally, the required Δu with a nominal flow rate is physically reasonable. On the right, the temperature uniformity initially increases (due to Joe's body) but settles to zero as the inlet water distribution begins to have an effect.

The heat map corresponding to this control is shown in **Figure 14**.

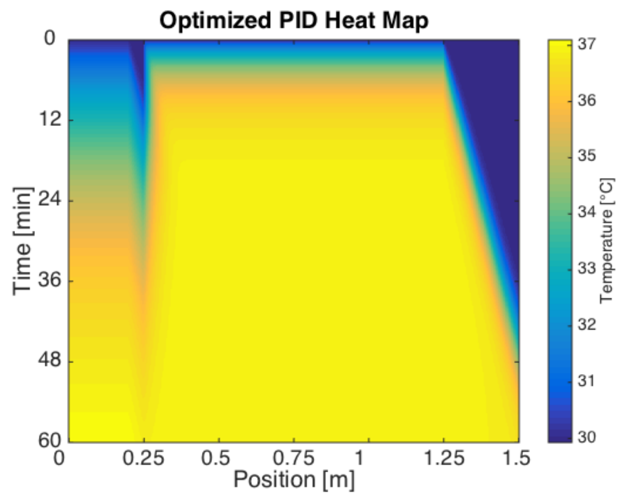


Figure 14. Heat map of system response using PID and intelligent heat distribution.

Joe should use PID control to vary the input temperature and simultaneously use a cup or other redirecting device to distribute the inlet water over the region of the bathtub upstream from his body.

Final Remarks

Strengths and Weaknesses of Solution Model

Strengths

- The greatest strength is generality, allowing for a wide range of parameter customization, from dimensions of the bath, types of boundaries enforced on the system, heat transfer coefficients, effects of Joe on the system, and final control application.
- The model incorporates effects of both convection and diffusion, yielding a realistic fluid thermodynamic simulation.
- Time-dependent boundaries and a spatially- and temporally-dependent heat source can be customized.
- The straightforward implementation of simulations allows for a variety of tests of parameter sensitivity as well as the implementation of control.
- The numerical approach allows for the possibility of expanding the simulations into a multidimensional model.

Weaknesses

- The main weakness is that the model considers only one-dimensional heat flow.

- The mathematics involved in this numerical implementation is highly sophisticated.

Future Model Development

As discussed in the weaknesses of the models, many possibilities exist for the development of a more precise model. In the future, a more comprehensive and definitive model would be developed in the following ways:

- Expand the models to incorporate more detailed information through increasing the dimensionality of each model, more accurately reflecting a physical three-dimensional system.
- Conduct experimental trials and extensive research to ensure the accuracy of certain physical parameters, such as heat transfer coefficients for humans or the precise effects of soluble materials on water's thermal properties.
- Apply geometry and coordinate transformations to explore the effect of various geometries on the system, such as cylindrical and trapezoidal geometries.
- Analyze the steady-state distribution of the thermal energy with high accuracy across three dimensions through implementation of Laplace's equation, spatially dependent boundary conditions, and realistic heat loss.

Conclusions

We modeled the temperature distribution in a bathtub in space and time so as to determine the most effective method of maintaining the temperature. Three models were developed to describe this temperature distribution: a steady-state uniform model, an analytical heat equation model, and a numerical convection-diffusion equation model. Ultimately, the numerical model proved most accurate because it alone took into account bulk fluid motion, the dominant heat transfer effect. Using this model, we conducted sensitivity analyses on different parameters to explore the effects of different bathtub geometries, the addition of soap, and other variations to the problem at hand. We also implemented PID control of the inlet water temperature and intelligent distribution of inlet water. By optimizing this process, we developed a strategy for quickly re-heating the bath while maintaining uniformity.

Overall, our model shows that well-tuned PID control combined with distributing the inlet water upstream of the bather is the most effective strategy.

References

[1] International Association for the Properties of Water and Steam (IAPWS). 2008. IAPWS Industrial Formulation 1997 for the Thermodynamic Properties of Water and Steam. In *International Steam Tables: Properties of Water and Steam Based on the Industrial Formulation IAPWS-IF97*, 7–150.

[2] Cengel, Yunus A, and Robert H. Turner. 2001. *Fundamentals of Thermal-Fluid Sciences*. Boston, MA: McGraw-Hill.

[3] Boutelier, C., L. Bougues, and J. Timbal. 2016. Experimental study of convective heat transfer coefficient for the human body in water. *Journal of Applied Psychology* 42 (1): 93–100.

[4] Grigoryan, Viktor. 2012. Inhomogeneous boundary value problems. `http://www.math.ucsb.edu/~grigoryan/124B/lecs/lec7.pdf`.

[5] Haberman, Richard. 2013. *Applied Partial Differential Equations: With Fourier Series and Boundary Value Problems*. 5th ed. Upper Saddle River, NJ: Pearson Prentice Hall.

[6] Majchrzak, Ewa, and Lukasz Turchan. 2012. The finite difference method for transient convection-diffusion problems. *Scientific Research of the Institute of Mathematics and Computer Science* 11 (1): 63–72. `http://amcm.pcz.pl/get.php?article=2012_1/art_07.pdf`.

[7] Burden, Richard L., and J.D. Faires. 2015. *Numerical Analysis*. 10th ed. Belmont, CA: Thomson Brooks/Cole.

[8] Santa Cruz Biotechnology. 2007. Sodium stearate (CAS 822-16-2). `http://www.scbt.com/datasheet-215884-sodium-stearate.html`.

[9] Martínez, Isidoro. 2016. Properties of some particular solutions. `http://webserver.dmt.upm.es/~isidoro/bk3/c07sol/Solution%20properties.pdf`.

[10] MIT OpenCourseWare. n.d.. 1.061 / 1.61 Transport Processes in the Environment. Chapter 5. Advection and Diffusion of an Instantaneous Release. `http://web.mit.edu/1.061/www/dream/FIVE/fivepage.htm`.

[11] Wikipedia. 2015. PID Controller. `https://en.wikipedia.org/wiki/PID_controller`.

Superior Products*: Applied Mathematical Consulting Firm

Superior Products, LLC[1]
Letter to International Society for the Promotion of Comfort and Cleanliness
2/1/16

As many of you are likely aware, there has been a singular problem facing avid bath users for many generations. Though we have come quite far as a society with the invention of running water, synthetic bathtub materials, and even advances in soap and shampoo solutions, the issue of keeping your bath at a comfortable temperature without pockets of cold or hot water has not been improved upon for generations.

Solutions range from self-circulating baths, expensive water jets, or even the deferring of a magnificent bath to a common shower, but an explanation has slowly presented itself to the difficulty shrouding this dilemma. The underlying factor of the cooling bath problem is a combination of being able to maintain heat within the bath over the course of your soak and also to distribute any hot water you introduce to the system. This is difficult due to the fact that water diffuses heat very poorly. Your relaxing afternoon bath is in reality a complex dynamical system, controlled by the physical phenomena of thermodynamics, heat transfer, and fluid mechanics.

Fortunately for the bathing community, a small percentage of our fellow bathers are not only aspiring to the ideals of cleanliness, but also ardently pursuing an understanding of applied mathematics. This question of the fluid dynamics of the bath tub has intrigued your friends here at Superior Products, and we have attempted to find a solution to the age old question of how to achieve the perfect bath.

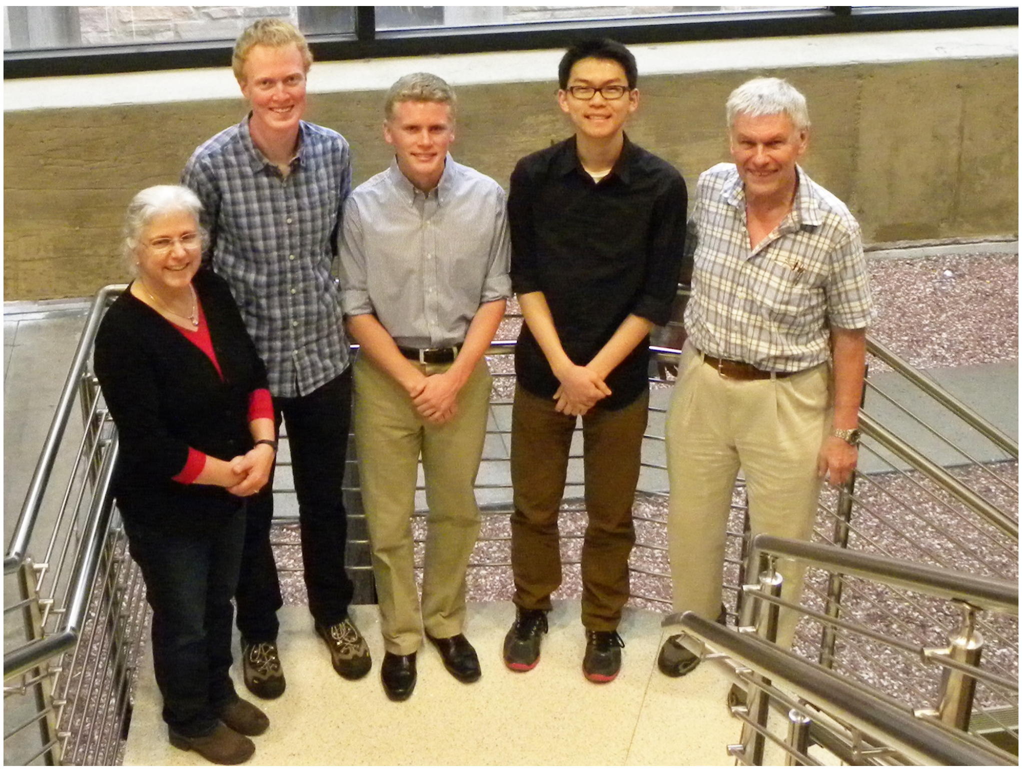
Despite the daunting complexities of the common bathtub, the advent of modern computational methods and simulation tools has allowed for an extensive understanding of heat flow within the bath. Through careful studying of the heat loss from the bath, diffusion of energy from warm to cold regions, and the bulk flow of water from faucet to drain, a final solution to your issues has been found.

Ultimately, the solution involves precision control theory to define the optimal temperature of the incoming water at every given time of your bath. This temperature is chosen to achieve your ideal temperature quickly, without the risk of scalding yourself. The second fundamental portion of the strategy is to spread the incoming water continuously over the surface of your bath

[1] This company is purely a work of fiction.

as it enters through the faucet, possibly through the use of a cup or customized bathwater distribution device. Our generic solution can take into account any uniqueness in your specific bath, such as your desired bath temperature and your bathtub shape and size.

This simple yet refined strategy will ensure that your bath is always at the perfect, uniform temperature. Feel free to use whatever soaps, shampoos, or even enhanced bubble solutions to improve your bathing experience.



Anne Dougherty (Associate Mathematics Department Chair), team members Matthew Hurst, Jordan Deitsch, and Nathan Yeo, and team advisor Bengt Fornberg.