Jieying Gong 1009388066.

1. (a). Base Case: Let $n=8$.

$2^n = 2^8 = 256 > 192 = 3\times 8^2 = 3n^2$ as wanted.

Therefore $P(8)$ holds as wanted.

(b). Ind$^{\underline{n}}$ Step: for $n \geq 8$.

Suppose $P(n)$ holds i.e. $2^n > 3n^2$.

WTP: $P(n+1)$ holds i.e. $2^{n+1} > 3(n+1)^2$

(c). Then $2^{n+1} = 2\cdot 2^n$

$= 2^n + 2^n$

$> 3n^2 + 3n^2$ [IH]

$\geq 3n^2 + 3\cdot 8n$ since $n \geq 8$, $n^2 = n\cdot n \geq 8n$

$> 3n^2 + 3(2n+1)$ since $8n = 2n + 6n > 2n+1$

$= 3n^2 + 6n + 3$

$= 3(n^2+2n+1)$

$= 3(n+1)^2$ as wanted.

Therefore, by PSI, $P(n)$ holds $\forall n > 7$, $n \in \mathbb{Z}$. $\boxed{\text{III}}$

2. (a) Base Cases: Consider three cases: $n = 0, 1, 2$

Let $n=0$: $f(n) = 0 \leq 1 = 3^0 = 3^n$ [defn of f] as wanted

$\therefore P(0)$ holds.

Let $n=1$: $f(n) = 1+2 = 3 = 3 = 3^1 = 3^n$ [defn of f] as wanted

$\therefore P(1)$ holds

Let $n=2$: $f(n) = 4+4 = 8 < 9 = 3^2 = 3^n$. [defn of f] as wanted.

$\therefore P(2)$ holds.

continue $\rightarrow$

Therefore, $P(0), P(1), P(2)$ holds. as wanted

(b). Ind$^{\underline{n}}$ Step: Let $n \geq 3$

Suppose $P(j)$ holds whenever $0 \leq j < n$ [IH]

i.e. $f(j) < 3^j$ whenever $0 \leq j < n$.

WTP: $P(n)$ holds i.e. $f(n) < 3^n$

(c). Since $n \geq 3$, we have $0 \leq n-3 < n$ and

$0 \leq n-2 < n$

$\therefore$ By IH, $P(n-3)$ and $P(n-2)$ holds.

i.e. $f(n-3) \leq 3^{n-3}$ and $f(n-2) \leq 3^{n-2}$.

$$
\begin{aligned}
\text{Then } f(n) &= 7f(n-2) + 6f(n-3). && [\text{defn of } f;\ n \geq 3] \\
&\leq 7 \cdot 3^{n-2} + 6 \cdot 3^{n-3} && [\text{IH}] \\
&= \left(\tfrac{7}{9}\right) 3^n + \left(\tfrac{6}{27}\right) 3^n && [\text{express using common term } 3^n] \\
&= \left(\tfrac{7}{9} + \tfrac{6}{27}\right) 3^n \\
&= 3^n \quad \text{as wanted.}
\end{aligned}
$$

Therefore, by PCI, $P(n)$ holds $\forall n \in \mathbb{N}$.

$\blacksquare$