P1)

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Proponemos:

$$P(\mu, \sigma^2; \mu_0, \nu, \alpha, \beta) =$$

$$\frac{\sqrt{\nu}}{\sqrt{2\pi\sigma^2}} \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\sigma^2}\right)^{\alpha+1} \exp\left(-\frac{2\beta + \nu(\mu-\mu_0)^2}{2\sigma^2}\right)$$

$$= \exp\left(-\frac{(\mu-\mu_0)^2}{2\sigma^2\nu^{-1}}\right) \frac{1}{\sqrt{2\pi\sigma^2\nu^{-1}}}$$

$$\times \exp\left(-\frac{\beta}{\sigma^2}\right) \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\sigma^2}\right)^{\alpha+1}$$

$$=: NG(\mu, \sigma^2; \mu_0, \nu, \alpha, \beta).$$

Vamos a desarrollar $p(D \mid \mu, \sigma^2)\, p(\mu, \sigma^2)$ hasta llegar a la forma de una $NG(\mu, 1/\sigma^2)$ multiplicado por una cte C independiente de μ, σ^2.

$$p(D|\mu,\sigma^2)\, p(\mu,\sigma^2)$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2\right)$$

$$\times \frac{\sqrt{\nu}}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\nu(\mu-\mu_0)^2}{2\sigma^2}\right) \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\sigma^2}\right)^{\alpha-1} \exp\left(-\frac{\beta}{\sigma^2}\right)$$

Trabajaremos sólo los exp que contengan μ.

$$\sim \exp\left(\frac{-1}{2\sigma^2}\sum(x_i^2-2x_i\mu+\mu^2)\right) \exp\left(\frac{-\nu}{2\sigma^2}(\mu^2-2\mu\mu_0+\mu_0^2)\right)$$

$$= \exp\left[\frac{-1}{2\sigma^2}\left(\sum(x_i^2-2x_i\mu+\mu^2) + \nu(\mu^2-2\mu\mu_0+\mu_0^2)\right)\right]$$

$$= \exp\left[\frac{-1}{2\sigma^2}\left(\sum x_i^2 - 2\mu n\bar{x} + \mu^2 n + \nu\mu^2 - 2\mu\nu\mu_0 + \nu\mu_0^2\right)\right]$$

$$= \exp\left[\frac{-1}{2\sigma^2}\left(\mu^2(n+\nu) - 2\mu(n\bar{x}+\nu\mu_0) + \sum x_i^2 + \nu\mu_0^2\right)\right]$$

$$= \exp\left[\frac{-1}{2\sigma^2}\left(\mu^2(n+\nu) - 2\mu(n\bar{x}+\nu\mu_0)\right)\right] \underbrace{\exp\left(-\frac{(\sum x_i^2+\nu\mu_0^2)}{2\sigma^2}\right)}_{\%}$$

$$= \exp\left[\frac{-(n+\nu)}{2\sigma^2}\left(\mu^2-2\mu\frac{(n\bar{x}+\nu\mu_0)}{n+\nu}\right)\right] \quad \%$$

$$= \exp\left(-\frac{(n+\nu)}{2\sigma^2}(\mu^2-2\mu\varepsilon)\right) \underbrace{\exp\left(-\frac{[\sum x_i^2+\nu\mu_0^2]}{2\sigma^2}\right)}_{(\%)}$$

donde $\varepsilon := \dfrac{n\bar{x}+\nu\mu_0}{n+\nu}$.

$$= \exp\left(\frac{-(n+\nu)}{2\sigma^2}\left(\mu^2 - 2\mu\varepsilon + \underbrace{\varepsilon^2 - \varepsilon^2}_{+0}\right)\right) \quad (\checkmark)$$

$$= \exp\left(-\frac{(n+\nu)}{2\sigma^2}\left(\mu^2 - 2\mu\varepsilon + \varepsilon^2\right) + \frac{\varepsilon^2 (n+\nu)}{2\sigma^2}\right) \quad (\checkmark)$$

$$= \exp\left(-\frac{(n+\nu)}{2\sigma^2}(\mu-\varepsilon)^2\right) \exp\left(\frac{\varepsilon^2 (n+\nu)}{2\sigma^2}\right) \quad (\checkmark)$$

$$= \exp\left(-\frac{(n+\nu)}{2\sigma^2}(\mu-\varepsilon)^2\right) \exp\left(-\frac{1}{2\sigma^2}\left(\sum x_i^2 + \nu\mu_0^2 - (n+\nu)\varepsilon^2\right)\right)$$

Reemplazamos:

$$p(D|\mu,\sigma^2)\,p(\mu,\sigma^2)$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(\frac{-(n+\nu)}{2\sigma^2}(\mu-\varepsilon)^2\right) \exp\left(-\frac{1}{2\sigma^2}\left[\sum x_i^2 + \nu\mu_0^2 - (n+\nu)\varepsilon^2\right]\right)$$

$$\times \frac{1}{\sqrt{2\pi\sigma^2\nu^{-1}}} \exp\left(-\frac{\beta}{\sigma^2}\right) \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\sigma^2}\right)^{\alpha+1}$$

$$= \exp\left(\frac{-(\mu-\varepsilon)^2}{2\sigma^2(n+\nu)^{-1}}\right) \exp\left(-\frac{\left(\sum x_i^2 + \nu\mu_0^2 - (n+\nu)\varepsilon^2 + 2\beta\right)}{2\sigma^2}\right)$$

$$\times \frac{1}{(2\pi\sigma^2)^{n/2}} \frac{1}{\sqrt{2\pi\sigma^2\nu^{-1}}} \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\sigma^2}\right)^{\alpha+1}$$

Vamos a armar una expresión proporcional a $p(\mu, \sigma^2)$ con parámetros $(\tilde{\mu}_0, \tilde{\nu}, \tilde{\alpha}, \tilde{\beta})$

Tomemos $\varepsilon = \dfrac{n\bar{x} + \mu_0 \nu}{n + \mu}$, $\tilde{\nu} = n + \nu$

$$\tilde{\beta} = \beta + \frac{\sum x_i^2 + \nu \mu_0^2 - (n+\nu)\varepsilon^2}{2}, \quad \tilde{\alpha} = \alpha + \frac{n}{2}$$

Reemplazando estas variables y haciendo aparecer las constantes necesarias:

$$= \sqrt{n+\nu}\, \exp\left(\frac{-(\mu-\varepsilon)^2}{2\sigma^2 (n+\nu)^{-1}}\right) \frac{\beta^{\alpha}}{\Gamma(\alpha + \frac{n}{2})} \left(\frac{1}{\sigma^2}\right)^{\alpha+1+\frac{n}{2}}$$

$$\times \exp\left(\frac{-(\sum x_i^2 + \nu\mu_0^2 - (n+\nu)\varepsilon^2 + 2\beta)}{2\sigma^2}\right)$$

$$\times \frac{\Gamma(\alpha + \frac{n}{2})}{\sqrt{n+\nu}} \frac{1}{(2\pi)^{n/2} \sqrt{\nu^{-1}}} \frac{1}{\Gamma(\alpha)}$$

$$= \sqrt{\tilde{\nu}}\, \exp\left(\frac{-(\mu - \tilde{\mu}_0)^2 \tilde{\nu}}{2\sigma^2}\right) \frac{\tilde{\beta}^{\tilde{\alpha}}}{\Gamma(\tilde{\alpha})} \left(\frac{1}{\sigma^2}\right)^{\tilde{\alpha}+1}$$

$$\times \underbrace{\left(\exp\left(\frac{-\tilde{\beta}}{\sigma^2}\right) \frac{\Gamma(\alpha+\frac{n}{2})}{\sqrt{n+\nu}\ \tilde{\beta}^{\tilde{\alpha}}} \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\nu^{-1}}} \frac{1}{\Gamma(\alpha)}\right)}_{=: C,\ \text{que es indep de } \mu, \sigma^2}$$

$$= NG(\mu, 1/\sigma^2; \tilde{\mu}_0, \tilde{\nu}, \tilde{\alpha}, \tilde{\beta})\ C$$

Como $p(\theta|D) = \frac{p(D|\theta)\,p(\theta)}{p(D)} \quad \Big/ \int_{\Theta} d\theta$

$$p(D)\int_{\Theta} p(\theta|D)\,d\theta = \int_{\Theta} p(D|\theta)\,p(\theta)$$

$$\Rightarrow \quad p(D) = \int_{\Theta} p(D|\theta)\,p(\theta)$$

Aplicando esto:

$$p(D) = \iint_{\mathbb{R}_+ \mathbb{R}} NG\left(\mu, \frac{1}{\sigma^2}; \tilde{\mu}_0, \tilde{\nu}, \tilde{\alpha}, \tilde{\beta}\right) c \, d\mu \, d\sigma^2$$

$$= c \quad \Rightarrow \quad p(D) = c$$

$$\Rightarrow p(\mu, \sigma^2|D) = NG\left(\mu, \frac{1}{\sigma^2}; \tilde{\mu}_0, \tilde{\nu}, \tilde{\alpha}, \tilde{\beta}\right)$$

Es decir, NG es prior conjugado para el modelo