Solutions and Additional Hints to Some of the Exercises

2.4. a) Put $A = \{\omega; |X(\omega)| \geq \lambda\}$. Then

$$\int_{\Omega} |X(\omega)|^p dP(\omega) \geq \int_{A} |X(\omega)|^p dP(\omega) \geq \lambda^p P(A).$$

□

b) By a) we have (choosing $p = 1$)

$$\begin{aligned} P[|X| \geq \lambda] &= P[\exp(k|X|) \geq \exp(k\,\lambda)] \\ &\leq \exp(-k\,\lambda) E[\exp(k|X|)]\,. \end{aligned}$$

□

2.6. $$P\Big(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k\Big) = \lim_{m\to\infty} P\Big(\bigcup_{k=m}^{\infty} A_k\Big) \leq \overline{\lim_{m\to\infty}} \sum_{k=m}^{\infty} P(A_k) = 0$$

since $\sum_{k=1}^{\infty} P(A_k) < \infty$.

Hence

$P(\{\omega; \omega$ belongs to infinitely many A_k's$\})$

$$= P(\{\omega; (\forall m)(\exists k \geq m) \text{ s.t. } \omega \in A_k\}) = P\Big(\Big\{\omega; \omega \in \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k\Big\}\Big) = 0\,.$$

□

2.7. a) We must verify that
(i) $\phi \in \mathcal{G}$
(ii) $F \in \mathcal{G} \Rightarrow F^C \in \mathcal{G}$
(iii) $F_1, \ldots, F_k \in \mathcal{G} \Rightarrow F_1 \cup F_2 \cup \cdots \cup F_k \in \mathcal{G}$
Since each element F of $\mathcal{G}$ consists of a union of some of the G_i's, these statements are easily verified.

b) Let $\mathcal{F}$ be a *finite* σ-algebra of subsets of Ω. For each $x \in \Omega$ define

$$F_x = \bigcap \{F \in \mathcal{F}; x \in F\}$$

Since $\mathcal{F}$ is finite we have $F_x \in \mathcal{F}$ and clearly F_x is the smallest set in $\mathcal{F}$ which contains x. We claim that for given $x, y \in \Omega$ the following holds:

(i) Either $F_x \cap F_y = \emptyset$ or $F_x = F_y$.

To prove this we argue by contradiction: Assume that we both have

(ii) $F_x \cap F_y \neq \emptyset$ and $F_x \neq F_y$, e.g. $F_x \setminus F_y \neq \emptyset$.

Then there are at most two possibilities:

a) $x \in F_x \cap F_y$
Then $F_x \cap F_y$ is a set in $\mathcal{F}$ containing x and $F_x \cap F_y$ is strictly smaller than F_x. This is impossible.

b) $x \in F_x \setminus F_y$
Then $F_x \setminus F_y$ is a set in $\mathcal{F}$ containing x and $F_x \setminus F_y$ is strictly smaller than F_x. This is again impossible.

Hence the claim (i) is proved. Therefore there exist $x_1, \ldots, x_m \in \Omega$ such that

$$F_{x_1}, F_{x_2}, \ldots, F_{x_m}$$

forms a partition of Ω. Hence any $F \in \mathcal{F}$ is a disjoint union of some of these F_{x_i}'s. Therefore $\mathcal{F}$ is of the form $\mathcal{G}$ described in a). □

c) Let $X : \Omega \to \mathbf{R}$ be $\mathcal{F}$-measurable. Then for all $x \in \mathbf{R}$ we have

$$\{\omega \in \Omega; X(\omega) = c\} = X^{-1}(\{c\}) \in \mathcal{F}$$

Therefore X has the constant value c on a finite (possibly empty) union of the F_i's, where $F_i = F_{x_i}$ is as in b). □

2.9. With

$$X_t(\omega) = \begin{cases} 1 & \text{if } \quad t = \omega \\ 0 & \text{otherwise} \end{cases}$$

and $Y_t(\omega) = 0$ for all $(t, \omega) \in [0, \infty) \times [0, \infty)$ we have

$$P[X_t = Y_t] = P[X_t = 0] = P(\{\omega; \omega \neq t\}) = 1 \,.$$

Hence X_t is a version of Y_t.

2.13. Define $D_\rho = \{x \in \mathbf{R}^2; |x| < \rho\}$. Then by (2.2.2) with $n = 2$ we have

$$P^0[B_t \in D_\rho] = (2\pi t)^{-1} \iint\limits_{D_\rho} e^{-\frac{x^2+y^2}{2t}} \, dx \, dy \,.$$

Introducing polar coordinates

$$\begin{aligned} x &= r\cos\theta \,; \\ y &= r\sin\theta \,; \quad 0 \le r \le \rho, \quad 0 \le \theta \le 2\pi \end{aligned}$$

we get

$$\begin{aligned} P^0[B_t \in D_\rho] &= (2\pi t)^{-1} \int_0^{2\pi} \int_0^{\rho} r e^{-\frac{r^2}{2t}} \, dr \, d\theta \\ &= (2\pi t)^{-1} \cdot 2\pi t \Big|_0^{\rho} -e^{-\frac{r^2}{2t}} = \underline{\underline{1 - e^{-\frac{\rho^2}{2t}}}} \,. \end{aligned}$$

2.14. The expected total length of time that B_t spends in the set K is given by

$$E^x\Big[\int_0^\infty \mathcal{X}_K(B_t)dt\Big] = \int_0^\infty P^x[B_t \in K]dt = \int_0^\infty (2\pi t)^{-n/2} \int_K e^{-\frac{(x-y)^2}{2t}} dy = 0 \,,$$

since K has Lebesgue measure 0.

2.15. $P[\widetilde{B}_{t_1} \in F_1, \ldots, \widetilde{B}_{t_k} \in F_k] = P[B_{t_1} \in U^{-1}F_1, \ldots, B_{t_k} \in U^{-1}F_k]$

$$= \int\limits_{U^{-1}F_1 \times \cdots \times U^{-1}F_k} p(t_1, 0, x_1)p(t_2 - t_1, x_1, x_2) \cdots p(t_k - t_{k-1}, x_{k-1}, x_k)dx_1 \cdots dx_k$$

$$= \int\limits_{F_1 \times \cdots \times F_k} p(t_1, 0, y_1)p(t_2 - t_1, y_1, y_2) \cdots p(t_k - t_{k-1}, y_{k-1}, y_k)dy_1 \cdots dy_k$$

$= P[B_{t_1} \in F_1, \ldots, B_{t_k} \in F_k]$,

by (2.2.1), using the substitutions $y_j = Ux_j$ and the fact that $|Ux_j - Ux_{j-1}|^2 = |x_j - x_{j-1}|^2$.

2.17. a) $E[(Y_n(t,\cdot) - t)^2] = E\left[\left(\sum_{k=0}^{2^n-1}(\Delta B_k)^2 - \sum_{k=0}^{2^n-1} 2^{-n}t\right)^2\right]$

$$= E\left[\left\{\sum_{k=0}^{2^n-1}((\Delta B_k)^2 - 2^{-n}t)\right\}^2\right]$$

$$= E\left[\sum_{j,k=0}^{2^n-1}((\Delta B_j)^2 - 2^{-n}t)((\Delta B_k)^2 - 2^{-n}t)\right]$$

$$= \sum_{k=0}^{2^n-1} E[((\Delta B_k)^2 - 2^{-n}t)^2]$$

$$= \sum_{k=0}^{2^n-1} E[(\Delta B_k)^4 - 2 \cdot 2^{-2n}t^2 + 2^{-2n}t^2]$$

$$= \sum_{k=0}^{2^n-1} 2 \cdot 2^{-2n}t^2 = 2 \cdot 2^{-n}t^2 \to 0 \qquad \text{as } n \to \infty .$$

b) This follows from the following general result: If the quadratic variation of a real function over an interval is positive, then the total variation of the function over that interval is infinite.

3.1. Using the notation $\Delta X_j = X_{t_j+1} - X_{t_j}$ we have

$$tB_t = \sum_j \Delta(t_j B_j) = \sum_j (t_{j+1}B_{t_j+1} - t_j B_{t_j})$$

$$= \sum_j t_j \,\Delta B_j + \sum_j B_{j+1}\Delta t_j \to \int_0^t s\,dB_s + \int_0^t B_s ds \quad \text{as} \quad \Delta t_j \to 0 .$$

3.3. a) Suppose X_t is a martingale with respect to some filtration $\{\mathcal{N}_t\}_{t \geq 0}$. Then $\mathcal{H}_t^{(X)} \subseteq \mathcal{N}_t$ and hence, for $s > t$,

$$E[X_s|\mathcal{H}_t^{(X)}] = E[E[X_s|\mathcal{N}_t]|\mathcal{H}_t^{(X)}] = E[X_t|\mathcal{H}_t^{(X)}] = X_t .$$

b) If X_t is a martingale with respect to $\mathcal{H}_t^{(X)}$ then

$$E[X_t|\mathcal{H}_0^{(X)}] = X_0$$

i.e.

$$E[X_t] = E[X_0] \quad \text{for all} \quad t \geq 0 .$$

c) The process $X_t := B_t^3$ satisfies (*), but it is not a martingale. To see this, choose $s > t$ and consider

$$\begin{aligned} E[B_s^3|\mathcal{F}_t] &= E[(B_s - B_t)^3 + 3B_s^2 B_t - 3B_s B_t^2 + B_t^3|\mathcal{F}_t] \\ &= 0 + 3B_t E[B_s^2|\mathcal{F}_t] - 3B_t^2 E[B_s|\mathcal{F}_t] + B_t^3 \\ &= 3B_t E[(B_s - B_t)^2 + 2B_s B_t - B_t^2|\mathcal{F}_t] - 2B_t^3 \\ &= 3B_t(s-t) + 6B_t^3 - 3B_t^3 - 2B_t^3 = 3B_t(s-t) + B_t^3 \neq B_t^3 \,. \end{aligned}$$

3.4. (i) If $X_t = B_t + t$ then $E[X_t] = B(0) + t \neq E[X_0]$, so X_t does not satisfy (*) of Exercise 3.3 b). Hence X_t cannot be a martingale.

(ii) If $X_t = B_t^2$ then $E[X_t] = nt + B_0^2 \neq E[X_0]$, so X_t cannot be a martingale.

(iii) If $X_t = t^2 B_t - 2\int_0^t r\, B_r dr$ then, for $s > t$,

$$\begin{aligned} E[X_s|\mathcal{F}_t] &= E[s^2 B_s|\mathcal{F}_t] - 2\int_0^t r\, B_r dr - 2\int_t^s r\, E[B_r|\mathcal{F}_t] dr \\ &= s^2 B_t - 2\int_0^t r\, B_r dr - 2B_t \int_t^s r\, dr \\ &= s^2 B_t - 2\int_0^t r\, B_r dr - B_t(s^2 - t^2) = t^2 B_t - 2\int_0^t r\, B_r dr = X_t \,. \end{aligned}$$

Hence X_t is a martingale. □

(iv) If $X_t = B_1(t) B_2(t)$ then

$$\begin{aligned} E[X_s|\mathcal{F}_t] &= E[B_1(s) B_2(s)|\mathcal{F}_t] \\ &= E[(B_1(s) - B_1(t))(B_2(s) - B_2(t))|\mathcal{F}_t] \\ &\quad + E[B_1(t)(B_2(s) - B_2(t))|\mathcal{F}_t] \\ &\quad + E[B_2(t)(B_1(s) - B_1(t))|\mathcal{F}_t] + E[B_1(t) B_2(t)|\mathcal{F}_t] \\ &= E[(B_1(s) - B_1(t)) \cdot (B_2(s) - B_2(t))] + 0 + 0 + B_1(t) B_2(t) \\ &= E[B_1(s) - B_1(t)] \cdot E[B_2(s) - B_2(t)] + B_1(t) B_2(t) \\ &= B_1(t) B_2(t) = X_t \,. \end{aligned}$$

Hence X_t is a martingale. □

3.5. To prove that $M_t := B_t^2 - t$ is a martingale, choose $s > t$ and consider

$$\begin{aligned} E[M_s \mid \mathcal{F}_t] &= E[B_s^2 - s \mid \mathcal{F}_t] = E[(B_s - B_t)^2 + 2B_s B_t - B_t^2 \mid \mathcal{F}_t] - s \\ &= s - t + 2B_t^2 - B_t^2 - s = B_t^2 - t = M_t \,. \end{aligned}$$

□

3.6. To prove that $N_t := B_t^3 - 3tB_t$ is a martingale, choose $s > t$ and consider

$$\begin{aligned} E[N_s \mid \mathcal{F}_t] &= E[(B_s - B_t)^3 + 3B_s^2 B_t - 3B_s B_t^2 + B_t^3 \mid \mathcal{F}_t] - 3sB_t \\ &= 3B_t E[B_s^2 \mid \mathcal{F}_t] - 3B_t^2 B_t + B_t^3 - 3sB_t \\ &= 3B_t E[(B_s - B_t)^2 + 2B_s B_t - B_t^2 \mid \mathcal{F}_t] - 2B_t^3 - 3sB_t \\ &= 3B_t(s-t) + B_t^3 - 3sB_t = B_t^3 - 3tB_t = N_t \,. \end{aligned}$$

□

3.8. a) If $M_t = E[Y \mid \mathcal{F}_t]$ then for $s > t$,

$$E[M_s \mid \mathcal{F}_t] = E[E[Y \mid \mathcal{F}_s] \mid \mathcal{F}_t] = E[Y \mid \mathcal{F}_t] = M_t\,.$$

□

3.9. $\int\limits_0^T B_t \circ dB_t = \frac{1}{2}B_T^2 \quad \text{if } B_0 = 0\,.$

3.12. (i) a) $dX_t = (\gamma + \frac{1}{2}\alpha^2)X_t dt + \alpha X_t dB_t\,.$

b) $dX_t = \frac{1}{2}\sin X_t[\cos X_t - t^2]dt + (t^2 + \cos X_t)dB_t\,.$

(ii) a) $dX_t = (r - \frac{1}{2}\alpha^2)X_t dt + \alpha X_t \circ dB_t\,.$

b) $dX_t = (2e^{-X_t} - X_t^3)dt + X_t^2 \circ dB_t\,.$

3.15. Suppose

$$C + \int\limits_S^T f(t,\omega)dB_t(\omega) = D + \int\limits_S^T g(t,\omega)dB_t(\omega)$$

By taking expectation we get

$$C = D\,.$$

Hence

$$\int\limits_S^T f(t,\omega)dB_t(\omega) = \int\limits_S^T g(t,\omega)dB_t(\omega)$$

and therefore, by the Itô isometry,

$$0 = E\Big[\Big(\int\limits_S^T (f(t,\omega) - g(t,\omega))dB(t)\Big)^2\Big] = E\Big[\int\limits_S^T (f(t,\omega) - g(t,\omega))^2 dt\Big],$$

which implies that

$$f(t,\omega) = g(t,\omega) \qquad \text{for a.a. } (t,\omega) \in [S,T] \times \Omega.$$

□

4.1. a) $dX_t = 2B_t dB_t + dt\,.$

b) $dX_t = (1 + \frac{1}{2}eB_t)dt + e^{B_t}dB_t\,.$

c) $dX_t = 2dt + 2B_1 dB_1(t) + 2B_2 dB_2(t)\,.$

d) $dx_t = \begin{bmatrix} dt \\ dB_t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dB_t\,.$

e) $dX_1(t) = dB_1(t) + dB_2(t) + dB_3(t)$
$dX_2(t) = dt - B_3(t)dB_1(t) + 2B_2(t)dB_2(t) - B_1(t)dB_3(t)$
or

$$dX_t = \begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt + \begin{bmatrix} 1 & 1 & 1 \\ -B_3(t) & 2B_2(t) & -B_1(t) \end{bmatrix} \begin{bmatrix} dB_1(t) \\ dB_2(t) \\ dB_3(t) \end{bmatrix}.$$

4.2. By Itô's formula we get

$$d\left(\tfrac{1}{3}B_t^3\right) = B_t^2 dB_t + B_t dt .$$

Hence

$$\tfrac{1}{3}B_t^3 = \int_0^t B_s^2 dB_s + \int_0^t B_s ds .$$

□

4.4. If we apply the Itô formula with $g(x,y) = x \cdot y$ we get

$$\begin{aligned} d(X_tY_t) &= d(g(X_t,Y_t)) = \frac{\partial g}{\partial x}(X_t,Y_t)dX_t + \frac{\partial g}{\partial y}(X_t,Y_t)dY_t \\ &+ \tfrac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t,Y_t)\cdot(dX_t)^2 + \frac{\partial^2 g}{\partial x \partial y}(X_t,Y_t)dX_tdY_t + \tfrac{1}{2}\frac{\partial^2 g}{\partial y^2}(X_t,Y_t)\cdot(dY_t)^2 \\ &= Y_t dX_t + X_t dY_t + dX_t dY_t . \end{aligned}$$

From this we obtain

$$X_tY_t = X_0Y_0 + \int_0^t Y_s dX_s + \int_0^t X_s dY_s + \int_0^t dX_s dY_s .$$

□

4.5. $E[B_t^6] = 15t^3$ if $B_0 = 0$.

4.11. b) By the Itô formula we have

$$\begin{aligned} d\left(e^{\frac{1}{2}t}\sin B_t\right) &= \tfrac{1}{2}e^{\frac{1}{2}t}\sin B_t dt + e^{\frac{1}{2}t}\cos B_t dB_t + \tfrac{1}{2}e^{\frac{1}{2}t}(-\sin B_t)dt \\ &= e^{\frac{1}{2}t}\cos B_t dB_t . \end{aligned}$$

Since

$$f(t) := e^{\frac{1}{2}t}\cos B_t \in \mathcal{V}(0,T) \qquad \text{for all } T,$$

we conclude that $X_t = e^{\frac{1}{2}}\sin B_t$ is a martingale. □

c) By the Itô formula (or Exercise 4.3) we get

$$\begin{aligned} d\left((B_t+t)\exp(-B_t - \tfrac{1}{2}t)\right) &= (B_t+t)\exp(-B_t - \tfrac{1}{2}t)(-1)dB_t \\ &+ \exp(-B_t - \tfrac{1}{2}t)(dB_t + dt) + \exp(-B_t - \tfrac{1}{2}t)(-1)dt \\ &= \exp(-B_t - \tfrac{1}{2}t)(1 - t - B_t)dB_t , \end{aligned}$$

where we have used that

$$d(\exp(-B_t - \tfrac{1}{2}t)) = -\exp(-B_t - \tfrac{1}{2}t)dB_t .$$

Since

$$f(t) := \exp(-B_t - \tfrac{1}{2}t)(1 - t - B_t) \in \mathcal{V}(0,T)$$

for all $T > 0$, we conclude that $X_t = (B_t + t)\exp(-B_t - \frac{1}{2}t)$ is a martingale.

□

4.14. a) $B_T(\omega) = \int_0^T 1 \cdot dB_t .$ □

b) By integration by parts we have

$$\int_0^T B_t dt = T B_T - \int_0^T t\,dB_t = \int_0^T (T-t)dB_t\,.$$

□

c) By the Itô formula we have $d(B_t^2) = 2B_t dB_t + dt$. This gives

$$B_T^2 = T + \int_0^T 2B_t dB_t\,.$$

□

d) By the Itô formula we have

$$d(B_t^3) = 3B_t^2 dB_t + 3B_t dt\,.$$

Combined with 4.14 b) this gives

$$B_T^3 = \int_0^T 3B_t^2 dB_t + 3\int_0^T B_t dt = \int_0^T (3B_t^2 + 3(T-t))dB_t\,.$$

□

e) Since $d\left(e^{B_t - \frac{1}{2}t}\right) = e^{B_t - \frac{1}{2}t} dB_t$ we have

$$e^{B_T - \frac{1}{2}T} = 1 + \int_0^T e^{B_t - \frac{1}{2}t} dB_t$$

or

$$e^{B_T} = e^{\frac{1}{2}T} + \int_0^T e^{B_t + \frac{1}{2}(T-t)} dB_t\,.$$

□

f) By Exercise 4.11 b) we have

$$d\left(e^{\frac{1}{2}t}\sin B_t\right) = e^{\frac{1}{2}t}\cos B_t dB_t$$

or

$$e^{\frac{1}{2}T}\sin B_T = \int_0^T e^{\frac{1}{2}t}\cos B_t dB_t\,.$$

Hence

$$\sin B_T = \int_0^T e^{\frac{1}{2}(t-T)}\cos B_t dB_t\,.$$

□

5.3. $X_t = X_0 \cdot \exp\left(\left(r - \frac{1}{2}\sum_{k=1}^{n}\alpha_k^2\right)t + \sum_{k=1}^{n}\alpha_k B_k(t)\right)$ (if $B(0)=0$).

5.4. (i) $X_1(t) = X_1(0) + t + B_1(t)$,

$X_2(t) = X_2(0) + X_1(0)B_2(t) + \int_0^t s\,dB_2(s) + \int_0^t B_1(s)dB_2(s)$, assuming (as usual) that $B(0) = 0$.

(ii) $X_t = e^t X_0 + \int\limits_0^t e^{t-s} dB_t$.

(iii) $X_t = e^{-t} X_0 + e^{-t} B_t$ (assuming $B_0 = 0$).

5.6. By the Itô formula we have

$$\begin{aligned} dF_t &= F_t(-\alpha\, dB_t + \tfrac{1}{2}\alpha^2 dt) + \tfrac{1}{2} F_t \alpha^2 dt \\ &= F_t(-\alpha\, dB_t + \alpha^2 dt) . \end{aligned}$$

Therefore, by Exercise 4.3,

$$\begin{aligned} d(F_t Y_t) &= F_t dY_t + Y_t dF_t + dF_t dY_t \\ &= F_t dY_t + Y_t F_t(-\alpha\, dB_t + \alpha^2 dt) + (-\alpha\, F_t dB_t)(\alpha\, Y_t dB_t) \\ &= F_t(dY_t - \alpha\, Y_t dB_t) = F_t r\, dt . \end{aligned}$$

Integrating this we get

$$F_t Y_t = F_0 Y_0 + \int\limits_0^t r\, F_s ds$$

or

$$\begin{aligned} Y_t &= Y_0 F_t^{-1} + F_t^{-1} \int\limits_0^t r\, F_s ds \\ &= \underline{\underline{Y_0 \exp(\alpha\, B_t - \tfrac{1}{2}\alpha^2 t) + r \int\limits_0^t \exp(\alpha(B_t - B_s) - \tfrac{1}{2}\alpha^2 (t-s)) ds.}} \end{aligned}$$

5.7. a) $X_t = m + (X_0 - m)e^{-t} + \sigma \int\limits_0^t e^{s-t} dB_s$.

b) $E[X_t] = m + (X_0 - m)e^{-t}$.
$\operatorname{Var}[X_t] = \frac{\sigma^2}{2}[1 - e^{-2t}]$.

5.8. $X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \exp(tJ)X(0) + \exp(tJ) \int\limits_0^t \exp(-sJ) M dB(s)$, where

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad dB(s) = \begin{bmatrix} dB_1(s) \\ dB_2(s) \end{bmatrix}$$

and

$$\exp(tJ) = I + tJ + \frac{t^2}{2} J^2 + \cdots + \frac{t^n}{n!} J^n + \cdots \in \mathbf{R}^{2\times 2} .$$

Using that $J^2 = -I$ this can be rewritten as

$$X_1(t) = X_1(0)\cos(t) + X_2(0)\sin(t) + \int\limits_0^t \alpha \cos(t-s) dB_1(s)$$
$$+ \int\limits_0^t \beta \sin(t-s) dB_2(s) ,$$

$$X_2(t) = -X_1(0)\sin(t) + X_2(0)\cos(t) - \int\limits_0^t \alpha \sin(t-s) dB_1(s)$$
$$+ \beta \int\limits_0^t \cos(t-s) dB_2(s) .$$

5.11. Hint: To prove that $\lim_{t\to 1}(1-t)\int_0^t \frac{dB_s}{1-s}=0$ a.s., put $M_t=\int_0^t \frac{dB_s}{1-s}$ for $0\le t<1$ and apply the martingale inequality to prove that

$$P[\sup\{(1-t)|M_t|; t\in[1-2^{-n},1-2^{-n-1}]\}>\epsilon]\le 2\epsilon^{-2}\cdot 2^{-n}\,.$$

Hence by the Borel-Cantelli lemma we obtain that for a.a. ω there exists $n(\omega)<\infty$ such that

$$n\ge n(\omega)\Rightarrow \omega\notin A_n\,,$$

where

$$A_n=\{\omega;\sup\{(1-t)|M_t|; t\in[1-2^{-n},1-2^{-n-1}]\}>2^{-\frac{n}{4}}\}\,.$$

5.12. a) If we introduce

$$x_1(t)=y(t),\quad x_2(t)=y'(t)\quad\text{and}\quad X(t)=\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

then the equation

$$y''(t)+(1+\epsilon W_t)y(t)=0$$

can be written

$$X'(t):=\begin{bmatrix}x_1'(t)\\ x_2'(t)\end{bmatrix}=\begin{bmatrix}y'(t)\\ y''(t)\end{bmatrix}=\begin{bmatrix}x_2(t)\\ -(1+\epsilon W_t)x_1(t)\end{bmatrix}$$

which is intepreted as the stochastic differential equation

$$\begin{aligned}dX(t)&=\begin{bmatrix}x_2(t)dt\\ -x_1(t)dt-\epsilon x_1(t)dB(t)\end{bmatrix}\\ &=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}dt-\epsilon\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}dB_t\\ &=K\,X(t)dt-\epsilon\,L\,X(t)dB_t\,,\end{aligned}$$

where

$$K=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\qquad\text{and}\qquad L=\begin{bmatrix}0&0\\1&0\end{bmatrix}.$$

b) By the above interpretation we have

$$y'(t)=y'(0)+\int_0^t y''(s)ds=y'(0)-\int_0^t y(s)ds-\epsilon\int_0^t y(s)dB_s$$

Hence, if we apply a stochastic Fubini theorem,

$$\begin{aligned}y(t)&=y(0)+\int_0^t y'(s)ds\\ &=y(0)+y'(0)t-\int_0^t\Big(\int_0^s y(r)dr\Big)ds-\epsilon\int_0^t\Big(\int_0^s y(r)dB_r\Big)ds\\ &=y(0)+y'(0)t-\int_0^t\Big(\int_r^t y(r)ds\Big)dr-\epsilon\int_0^t\Big(\int_r^t y(r)ds\Big)dB_r\\ &=y(0)+y'(0)t+\int_0^t(r-t)y(r)dr+\epsilon\int_0^t(r-t)y(r)dB_r\,.\end{aligned}$$

5.16. c) $X_t = \exp(\alpha B_t - \frac{1}{2}\alpha^2 t)\left[x^2 + 2\int_0^t \exp(-2\alpha B_s + \alpha^2 s)ds\right]^{1/2}$.

6.15. We have

a)
$$X_t = X_0 \exp(\sigma\, dB_t + (\mu - \tfrac{1}{2}\sigma^2)t) = x\exp(\xi_t - \tfrac{1}{2}\sigma^2 t)\ .$$

Therefore
$$\mathcal{M}_t \subseteq \mathcal{N}_t\ .$$

On the other hand, since
$$\xi_t = \tfrac{1}{2}\sigma^2 t + \ln\frac{X_t}{x}$$
we see that $\mathcal{N}_t \subseteq \mathcal{M}_t$. Hence $\mathcal{M}_t = \mathcal{N}_t$.

b) Consider the filtering problem

(system) $\quad d\mu = 0$, $\bar{\mu} = E[\mu]$, $a^2 = E[(\mu - \bar{\mu})^2] = \theta^{-1}$
(observations) $\quad d\xi_t = \mu\, dt + \sigma\, dB_t;\ \ \xi_0 = 0$

By Example 6.2.9 the solution is
$$\begin{aligned}\widehat{\mu} &= E[\mu|\mathcal{N}_t] = \frac{\sigma^2\bar{\mu}}{\sigma^2 + a^2 t} + \frac{a^2}{\sigma^2 + a^2 t}\xi_t \\ &= (\theta + \sigma^{-2}t)^{-1}(\bar{\mu}\theta + \sigma^{-2}\xi_t)\ .\end{aligned}$$

c) The innovation process N_t of the filtering problem in b) is
$$N_t = \xi_t - \int_0^t \widehat{\mu}(s)ds\ .$$

Therefore, by a)
$$\begin{aligned}\widetilde{B}_t &= \int_0^t \sigma^{-1}(\mu - E[\mu|\mathcal{M}_s])ds + B_t \\ &= \int_0^t \sigma^{-1}(\mu - E[\mu|\mathcal{M}_s])ds + B_t = \sigma^{-1}N_t\end{aligned}$$

is a Brownian motion by Lemma 6.2.6.

d) Since $\widetilde{B}_t = \sigma^{-1}(\xi_t - \int_0^t \widehat{\mu}(s)ds)$ we see that $\widetilde{B}_t$ is $\mathcal{N}_t$-measurable and hence $\mathcal{M}_t$-measurable by a).

e) We have
$$\sigma\, d\widetilde{B}_t = d\xi_t - \widehat{\mu}(t)dt = d\xi_t - \frac{\bar{\mu}\theta + \sigma^{-2}\xi_t}{\theta + \sigma^{-2}t}dt\ .$$

Hence
$$d\xi_t - \frac{1}{\sigma^2\theta + t}\xi_t dt = \frac{\bar{\mu}\theta}{\theta + \sigma^{-2}t}dt + \sigma\, d\widetilde{B}_t$$

which gives

$$d\Big(\xi_t \exp\Big(-\int_0^t \frac{ds}{\sigma^2\theta + s}\Big)\Big) = \exp\Big(-\int_0^t \frac{ds}{\sigma^2\theta + s}\Big)\Big[\frac{\bar{\mu}\theta}{\theta + \sigma^{-2}t}dt + \sigma\, d\widetilde{B}_t\Big]$$

or

$$d\Big(\frac{\xi_t}{\sigma^2\theta + t}\Big) = \frac{1}{\sigma^2\theta + t}\Big[\frac{\bar{\mu}\theta}{\theta + \sigma^{-2}t}dt + \sigma\, d\widetilde{B}_t\Big].$$

We conclude that

$$\xi_t = \bar{\mu} - \frac{\bar{\mu}\sigma^2\theta}{\sigma^2\theta + t} + \sigma \int_0^t \frac{d\widetilde{B}_s}{\sigma^2\theta + s},$$

which shows that ξ_t is $\widetilde{\mathcal{F}}_t$-measurable.

f) By combining (6.3.20), (6.3.24) and a) we have

$$\begin{aligned} dX_t &= X_t(\mu\, dt + \sigma\, dB_t) = X_t d\xi_t \\ &= X_t(\widehat{\mu}(t)dt + \sigma\, d\widetilde{B}_t) \\ &= E[\mu|\mathcal{M}_t]X_t dt + \sigma\, X_t\, d\widetilde{B}_t\,. \end{aligned}$$

□

7.1. a) $Af(x) = \mu x f'(x) + \frac{1}{2}\sigma^2 f''(x);\quad f \in C_0^2(\mathbf{R})$.

b) $Af(x) = rxf'(x) + \frac{1}{2}\alpha^2 x^2 f''(x);\quad f \in C_0^2(\mathbf{R})$.

c) $Af(y) = rf'(y) + \frac{1}{2}\alpha^2 y^2 f''(y);\quad f \in C_0^2(\mathbf{R})$.

d) $Af(t,x) = \frac{\partial f}{\partial t} + \mu x\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2};\quad f \in C_0^2(\mathbf{R}^2)$.

e) $Af(x_1,x_2) = \frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \frac{1}{2}e^{2x_1}\frac{\partial^2 f}{\partial x_2^2};\quad f \in C_0^2(\mathbf{R}^2)$.

f) $Af(x_1,x_2) = \frac{\partial f}{\partial x_1} + \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2} + \frac{1}{2}x_1^2\frac{\partial^2 f}{\partial x_2^2};\quad f \in C_0^2(\mathbf{R}^2)$.

g) $Af(x_1,\dots,x_n) = \sum\limits_{k=1}^n r_k x_k \frac{\partial f}{\partial x_k} + \frac{1}{2}\sum\limits_{i,j=1}^n x_i x_j\big(\sum\limits_{k=1}^n \alpha_{ik}\alpha_{jk}\big)\frac{\partial^2 f}{\partial x_i \partial x_j};$

$f \in C_0^2(\mathbf{R}^n)$.

7.2. a) $dX_t = dt + \sqrt{2}\, dB_t\,.$

b) $dX(t) = \begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ cX_2(t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ \alpha X_2(t) \end{bmatrix} dB_t\,.$

c) $dX(t) = \begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} 2X_2(t) \\ \ln(1+X_1^2(t)+X_2^2(t)) \end{bmatrix} dt + \begin{bmatrix} X_1(t) & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}.$

(Several other diffusion coefficients are possible.)

7.4. a), b). Let $\tau_k = \inf\{t > 0; B_t^x = 0 \text{ or } B_t^x = k\};\ k > x > 0$ and put

$$\rho_k = P^x[B_{\tau_k} = k]\,.$$

Then by Dynkin's formula applied to $f(y) = y^2$ for $0 \le y \le k$ we get

$$E^x[\tau_k] = k^2 p_k - x^2\,. \tag{S1}$$

On the other hand, Dynkin's formula applied to $f(y) = y$ for $0 \le y \le k$ gives

$$kp_k = x\,. \tag{S2}$$

Combining these two identities we get that

$$E^x[\tau] = \lim_{k\to\infty} E^x[\tau_k] = \lim_{k\to\infty} x(k-x)) = \infty \,. \tag{S3}$$

Moreover, from (S2) we get

$$P^x[\exists t<\infty \text{ with } B_t=0] = \lim_{k\to\infty} P^x[B_{\tau_k}=0] = \lim_{k\to\infty}(1-p_k)=1\,, \tag{S4}$$

so $\tau < \infty$ a.s. P^x.

7.15. By the Markov property (7.2.5) we have

$$\begin{aligned}
E^x[\mathcal{X}_{[K,\infty)}(B_T) \mid \mathcal{F}_t] &= E^x[\theta_t \mathcal{X}_{[K,\infty)}(B_{T-t}) \mid \mathcal{F}_t] \\
&= E^{B_t}[\mathcal{X}_{[K,\infty)}(B_{T-t})] = E^y[f(B_{T-t})]_{y=B_t} \\
&= \Big[\frac{1}{\sqrt{2\pi(T-t)}} \int_{\mathbf{R}} f(z) e^{-\frac{(z-y)^2}{2(T-t)}}\,dz\Big]_{y=B_t} \\
&= \frac{1}{\sqrt{2\pi(T-t)}} \int_K^\infty e^{-\frac{(z-y)^2}{2(T-t)}}\,dz\,,
\end{aligned}$$

where

$$f(x) = \mathcal{X}_{[K,\infty)}(x)\,.$$

7.18. a) Define

$$w(x) = \frac{f(x)-f(a)}{f(b)-f(a)}\,; \qquad x \in [a,b].$$

Then

$$Lw(x) = 0 \qquad \text{in } D := (a,b)$$

and

$$w(a) = 1, \quad w(b) = 1\,.$$

Hence by the Dynkin formula

$$1\cdot P^x[X_\tau = b] + 0\cdot P^x[X_\tau = a] = w(x) + E^x\Big[\int_0^\tau Lw(X_t)dt\Big] = w(x)\,,$$

i.e.

$$w = P^x[X_\tau = b] = \frac{f(x)-f(a)}{f(b)-f(a)}\,.$$

c) $p = \frac{\exp(-\frac{2bx}{\sigma^2})-\exp(-\frac{2ab}{\sigma^2})}{\exp(-\frac{2b^2}{\sigma^2})-\exp(-\frac{2ab}{\sigma^2})}$.

8.1. a) $g(t,x) = E^x[\phi(B_t)]$.

b) $u(x) = E^x[\int_0^\infty e^{-\alpha t}\psi(B_t)dt]$.

8.11. For $T > 0$ define the measure Q_T on $\mathcal{F}_T$ by

$$dQ_T(\omega) = \exp(-B(T) - \tfrac{1}{2}T)dP(\omega) \qquad \text{on } \mathcal{F}_T .$$

Then by the Girsanov theorem $Y(t) := t + B(t);\ 0 \le t \le T$ is a Brownian motion with respect to Q_T. Since

$$Q_T = Q_S \qquad \text{on } \mathcal{F}_t \text{ for all } t \le \min(S, T)$$

there exists a measure Q on $\mathcal{F}_\infty$ such that

$$Q = Q_T \qquad \text{on } \mathcal{F}_T \text{ for all } T < \infty$$

(See Theorem 1.14 in Folland (1984).)
By the law of iterated logarithm (Theorem 5.1.2) we know that

$$P[N] = 1 ,$$

where

$$N = \{\omega;\ \lim_{t\to\infty} Y(t) = \infty\}.$$

On the other hand, since $Y(t)$ is a Brownian motion w.r.t. Q we have that

$$Q[N] = 0 .$$

Hence P is not absolutely continuous w.r.t. Q.

8.12. Here the equation

$$\sigma(t,\omega)u(t,\omega) = \beta(t,\omega)$$

has the form

$$\begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which has the solution

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} .$$

Hence we define the measure Q on $\mathcal{F}_T^{(2)}$ by

$$dQ(\omega) = \exp(-B_1(T,\omega) + 3B_2(T,\omega) - 5T)dP(\omega) .$$

9.1. a) $dX_t = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ \beta \end{bmatrix} dB_t$.

b) $dX_t = \begin{bmatrix} a \\ b \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} dB_t$.

c) $dX_t = \alpha X_t dt + \beta dB_t$.

d) $dX_t = \alpha dt + \beta X_t dB_t$.

e) $dX_t = \begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} \ln(1+X_1^2(t)) \\ X_2(t) \end{bmatrix} dt + \sqrt{2} \begin{bmatrix} X_2(t) & 0 \\ X_1(t) & X_1(t) \end{bmatrix} \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}$.

9.2. (i) In this case we put

$$dX_t = \begin{bmatrix} dt \\ dB_t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dB_t ; \qquad X_0 = \begin{bmatrix} s \\ x \end{bmatrix}$$

and

$$D = \{(s, x) \in \mathbf{R}^2;\ s < T\} .$$

Then

$$\tau_D = \inf\{t > 0; (s+t, B_t) \notin D\} = T - s$$

and we get

$$\begin{aligned} u(s,x) &= E^{s,x}\Big[\psi(B_{\tau_D}) + \int_0^{\tau_D} g(X_t)dt\Big] \\ &= E\Big[\psi(B^x_{T-s}) - \int_0^{T-s} \phi(s+t, B^x_t)dt\Big], \end{aligned}$$

where B^x_t is Brownian motion starting at x.

(ii) Define X_t by

$$dX_t = \alpha X_t dt + \beta X_t dB_t\ ; \qquad X_0 = x > 0\,,$$

and put

$$D = (0, x_0)\,.$$

If $\alpha \geq \frac{1}{2}\beta^2$ then $\tau_D = \inf\{t > 0; X_t \notin D\} < \infty$ a.s. and

$$X_{\tau_D} = x_0 \quad \text{a.s.} \quad \text{(see Example 5.1.1).}$$

Therefore the only bounded solution is

$$u(x) = E^x[(X_{\tau_D})^2] = x_0^2 \quad \text{(constant).}$$

(iii) If we try a solution of the form

$$u(x) = x^\gamma \qquad \text{for some constant } \gamma$$

we get

$$\alpha\, x u'(x) + \tfrac{1}{2}\beta^2 x^2 u''(x) = (\alpha + \tfrac{1}{2}\beta^2(\gamma - 1))x^\gamma,$$

so $u(x)$ is a solution iff

$$\gamma = 1 - \frac{2\alpha}{\beta^2}\,.$$

With this value of γ we get that the general solution of

$$\alpha\, x u'(x) + \tfrac{1}{2}\beta^2 x^2 u''(x) = 0$$

is

$$u(x) = C_1 + C_2 x^\gamma$$

where C_1, C_2 are arbitrary constants. If $\alpha < \frac{1}{2}\beta^2$ then all these solutions are bounded on $(0, x_0)$ and we need an additional boundary value at $x = 0$ to get uniqueness. In this case $P[\tau_D = \infty] > 0$. If $\alpha > \frac{1}{2}\beta^2$ then $u(x) = C_1$ is the only bounded solution on $(0, x_0)$, in agreement with Theorem 9.1.1 (and (ii) above).

9.3. a) $u(t,x) = E^x[\phi(B_{T-t})]$.
b) $u(t,x) = E^x[\psi(B_t)]$.

9.8. a) Let $X_t \in \mathbf{R}^2$ be uniform motion to the right, as described in Example 9.2.1. Then each one-point set $\{(x_1, x_2)\}$ is thin (and hence semipolar) but not polar.

b) With X_t as in a) let $H_k = \{(a_k, 1)\}$; $k = 1, 2, \ldots$ where $\{a_k\}_{k=1}^{\infty}$ is the set of rational numbers. Then each H_k is thin but $Q^{(x_1,1)}[T_H = 0] = 1$ for all $x_1 \in \mathbf{R}$.

9.10. Let $Y_t = Y_t^{s,x} = (s+t, X_t^x)$ for $t \geq 0$, where $X_t = X_t^x$ satisfies

$$dX_t = \alpha X_t dt + \beta X_t dB_t \,; \qquad t \geq 0,\; X_0 = x > 0\,.$$

Then the generator $\widehat{A}$ of Y_t is given by

$$\widehat{A} f(s,x) = \frac{\partial f}{\partial s} + \alpha x \frac{\partial f}{\partial x} + \tfrac{1}{2}\beta^2 x^2 \frac{\partial^2 f}{\partial x^2}\,; \qquad f \in C_0^2(\mathbf{R}^2)\,.$$

Moreover, with $D = \{(t,x); x > 0 \text{ and } t < T\}$ we have

$$\tau_D := \inf[t > 0; Y_t \notin D\} = \inf\{t > 0; s + t > T\} = T - s\,.$$

Hence

$$Y_{\tau_D} = (T, X_{T-s})\,.$$

Therefore, by Theorem 9.3.3 the solution is

$$f(s,x) = E\left[e^{-\rho T}\phi(X_{T-s}^x) + \int_0^{T-s} e^{-\rho(s+t)} K(X_t^x)dt\right].$$

9.15. a) If we put $w(s,x) = e^{-\rho s}h(x)$ we get

$$\frac{1}{2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial s} = e^{-\rho s}\big(\tfrac{1}{2}h''(x) - \rho\, h(x)\big),$$

and the boundary value problem reduces to

(1) $\frac{1}{2}h''(x) - \rho\, h(x) = -\theta x^2$; $\quad a < x < b$

(2) $h(a) = \psi(a), \quad h(b) = \psi(b)$

The general solution of (1) is

$$h(x) = C_1 e^{\sqrt{2\rho}\,x} + C_2 e^{-\sqrt{2\rho}\,x} + \frac{\theta}{\rho}x^2 - \frac{\theta}{\rho^2}\,,$$

where C_1, C_2 are arbitrary constants. The boundary values (ii) will determine C_1, C_2 uniquely.

b) To find

$$g(x,\rho) = g(x) := E^x\left[e^{-\rho\tau_D}\right]$$

we apply the above with $\psi = 1$, $\theta = 0$ and get

$$g(x) = C_1 e^{\sqrt{2\rho}\,x} + C_2 e^{-\sqrt{2\rho}\,x},$$

where the constants C_1, C_2 are determined by

$$g(a) = 1, \quad g(b) = 1\,.$$

After some simplifications this gives

$$g(x) = \frac{\sinh(\sqrt{2\rho}(b-x)) + \sinh(\sqrt{2\rho}(x-a))}{\sinh(\sqrt{2\rho}(b-a))}\,; \qquad a \leq x \leq b\,.$$

10.1. a) $g^*(x) = \infty$, τ^* does not exist.
b) $g^*(x) = \infty$, τ^* does not exist.
c) $g^*(x) = 1$, $\tau^* = \inf\{t > 0; B_t = 0\}$.
d) If $\rho < \frac{1}{2}$ then $g^*(s,x) = \infty$ and τ^* does not exist.
If $\rho \geq \frac{1}{2}$ then $g^*(s,x) = g(s,x) = e^{-\rho s}\cosh x$ and $\tau^* = 0$.

10.2. a) Let W be a closed disc centered at y with radius $\rho < |x-y|$. Define

$$\tau_\rho = \inf\{t > 0; B_t^x \in W\} .$$

In $\mathbf{R}^2$ we know that $\tau < \infty$ a.s. (see Example 7.4.2). Suppose there exists a nonnegative superharmonic function u such that

$$u(x) < u(y) . \tag{1}$$

Then

$$u(x) \geq E^x\big[u(B_{\tau_\rho})\big]. \tag{2}$$

Since u is lower semicontinuous there exists $\rho > 0$ such that

$$\inf\{u(z); z \in W_\rho\} > u(x) . \tag{3}$$

Combining this with (2) we get the contradiction

$$u(x) \geq E^x\big[u(B_{\tau_\rho})\big] > u(x) .$$

□

b) The argument in a) also works for $\mathbf{R}$. Therefore

$$g^*(x) = \sup\{g(x); x \in \mathbf{R}\} = \sup\{x\,e^{-x}; x > 0\} = 1 .$$

c) For $x \neq 0$ we have

$$\Delta(|x|^\gamma) = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}\Big(\Big(\sum_{j=1}^n x_j^2\Big)^{\gamma/2}\Big) = \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big(\frac{\gamma}{2}\Big(\sum_{j=1}^n x_j^2\Big)^{\frac{\gamma}{2}-1} 2x_i\Big)$$
$$= \gamma(\gamma + n - 2)|x|^\gamma.$$

So $|x|^\gamma$, and hence $f_\gamma(x) = \min(1, |x|^\gamma)$, is superharmonic iff $\gamma(\gamma + n - 2) \leq 0$, i.e.

$$2 - n \leq \gamma \leq 0 .$$

10.3. $x_0 > 0$ is given implicitly by the equation

$$x_0 = \sqrt{\frac{2}{\rho}} \cdot \frac{e^{2\sqrt{2\rho}\,x_0} + 1}{e^{2\sqrt{2\rho}\,x_0} - 1} ,$$

and $g^*(s,x) = e^{-\rho s}x_0^2 \frac{\cosh(\sqrt{2\rho}\,x)}{\cosh(\sqrt{2\rho}\,x_0)}$ for $-x_0 \leq x \leq x_0$, where $\cosh\xi = \frac{1}{2}(e^\xi + e^{-\xi})$.

10.9. If $0 < \rho \leq 1$ then $\gamma(x) = \frac{1}{\rho}x^2 + \frac{1}{\rho^2}$ but τ^* does not exist. If $\rho > 1$ then

$$\gamma(x) = \begin{cases} \frac{1}{\rho}x^2 + \frac{1}{\rho^2} + C\cosh(\sqrt{2\rho}\,x) & \text{for } |x| \leq x^* \\ x^2 & \text{for } |x| > x^* \end{cases}$$

where $C > 0$, $x^* > 0$ are the unique solutions of the equations

$$C\cosh(\sqrt{2\rho}\,x^*) = \left(1 - \frac{1}{\rho}\right)(x^*)^2 - \frac{1}{\rho^2}$$

$$C\sqrt{2\rho}\sinh(\sqrt{2\rho}\,x^*) = 2\left(1 - \frac{1}{\rho}\right)x^* .$$

10.12. If $\rho > r$ then $g^*(s,x) = e^{-\rho s}(x_0 - 1)^+(\frac{x}{x_0})^\gamma$ and $\tau^* = \inf\{t > 0; X_t \ge x_0\}$, where

$$\gamma = \alpha^{-2}\left[\tfrac{1}{2}\alpha^2 - r + \sqrt{(\frac{1}{2}\alpha^2 - r)^2 + 2\alpha^2\rho}\right]$$

and

$$x_0 = \frac{\gamma}{\gamma - 1} \qquad (\gamma > 1 \Leftrightarrow \rho > r) .$$

10.13. If $\alpha \le \rho$ then $\tau^* = 0$.
If $\rho < \alpha < \rho + \lambda$ then

$$G^*(s,p,q) = \begin{cases} e^{-\rho s}pq \quad ; & \text{if } 0 < pq < y_0 \\ e^{-\rho s}(C_1(pq)^{\gamma_1} + \frac{\lambda}{\rho+\lambda-\alpha}\cdot pq - \frac{K}{\rho}); & \text{if } pq \ge y_0 \end{cases}$$

where

$$\gamma_1 = \beta^{-2}\left[\tfrac{1}{2}\beta^2 + \lambda - \alpha - \sqrt{(\frac{1}{2}\beta^2 + \lambda - \alpha)^2 + 2\rho\beta^2}\right] < 0 ,$$

$$y_0 = \frac{(-\gamma_1)K(\rho + \lambda - \alpha)}{(1-\gamma_1)\rho(\alpha - \rho)} > 0$$

and

$$C_1 = \frac{(\alpha - \rho)y_0^{1-\gamma_1}}{(-\gamma_1)(\rho + \lambda - \alpha)} .$$

The continuation region is

$$D = \{(s,p,q); pq > y_0\} .$$

If $\rho + \lambda \le \alpha$ then $G^* = \infty$.

10.14. First assume that
Case 1: $\rho > \alpha$.
Note that

$$\int_\tau^\infty e^{-\rho(s+t)}P_t dt = e^{-\rho s}\left[\int_0^\infty e^{-\rho t}P_t dt - \int_0^\tau e^{-\rho t}P_t dt\right]$$

and

$$E\left[\int_0^\infty e^{-\rho t}P_t dt\right] = \int_0^\infty e^{-\rho t}E[p\exp(\beta B_t + (\alpha - \tfrac{1}{2}\beta^2)t)]dt$$

$$= \int_0^\infty p e^{(\alpha-\rho)t}dt = \frac{p}{\rho - \alpha} .$$

Therefore
$$\Phi(s,p) = \frac{p\,e^{-\rho s}}{\rho - \alpha} + \Psi(s,p)\ ,$$
where
$$\Psi(s,p) = \sup_{\tau} E^{(s,p)}\Big[\int_0^{\tau} (-e^{-\rho(s+t)}P_t)dt - C\,e^{-\rho(s+\tau)}\Big].$$
This is a problem of the form discussed in Section 10.4, with
$$Y(t) = \begin{bmatrix} s+t \\ P_t \end{bmatrix}, \quad Y(0) = \begin{bmatrix} s \\ p \end{bmatrix} = y \in \mathbf{R}^2$$
and
$$f(y) = f(s,x) = -e^{-\rho s}p\,, \quad g(s,x) = -C\,e^{-\rho s}.$$
To get an indication of where the continuation region D is situated, we consider the set
$$U = \{y; Lg(y) + f(y) > 0\} \qquad \text{(see (10.3.7))}.$$
In this case the generator L is given by
$$L\phi(s,p) = \frac{\partial \phi}{\partial s} + \alpha\,p\frac{\partial \phi}{\partial p} + \tfrac{1}{2}\beta^2 p^2 \frac{\partial^2 \phi}{\partial p^2}$$
and so
$$U = \{(s,p); (-\rho)(-C) - p > 0\} = \{(s,p); p < \rho C\}.$$
In view of this we try a continuation D of the form
$$D = \{(s,p); 0 < p < p^*\}$$
for some $p^* > 0$ (to be determined).
We try a value function candidate of the form
$$\phi(s,p) = e^{-\rho s}\psi(p)$$
where, by Theorem 10.4.1, the function ψ is required to satisfy the following conditions:

(1) $L_0\psi(p) := -\rho\psi(p) + \alpha\,p\,\psi'(p) + \frac{1}{2}\beta^2 p^2 \psi''(p) - p = 0;\ 0 < p < p^*$
(2) $L_0\psi(p) \le 0;\quad p > p^*$
(3) $\psi(p) = -C;\quad p > p^*$
(4) $\psi(p) > -C;\quad 0 < p < p^*$.

The general solution of (1) is
$$\psi(p) = K_1\,p^{\gamma_1} + K_2\,p^{\gamma_2} + \frac{p}{\alpha - \rho}$$
where
$$(5) \qquad \gamma_i = \beta^{-2}\Big[\tfrac{1}{2}\beta^2 - \alpha \pm \sqrt{\left(\tfrac{1}{2}\beta^2 - \alpha\right)^2 + 2\rho\beta^2}\,\Big]; \qquad i = 1,2$$
with
$$\gamma_2 < 0 < \gamma_1$$
and K_1, K_2 are arbitrary constants.

Since $\psi(p)$ must be bounded near $p = 0$ we must have $K_2 = 0$. Hence we put

$$(6) \qquad \psi(p) = \begin{cases} K_1 p^{\gamma_1} + \frac{p}{\alpha-\rho} & ; \ 0 \le p < p^* \\ -C & ; \ p \ge p^* \end{cases}$$

If we require $\psi(p)$ to be continuous at $p = p^*$ we get the equation

$$(7) \qquad K_1 (p^*)^{\gamma_1} + \frac{p^*}{\alpha - \rho} = -C$$

If $\psi(p)$ is also C^1 at $p = p^*$ we get

$$(8) \qquad K_1 \gamma_1 (p^*)^{\gamma_1 - 1} + \frac{1}{\alpha - \rho} = 0$$

Combining these two equations we get

$$(9) \qquad x^* = \frac{C(\rho - \alpha)\gamma_1}{\gamma_1 - 1}$$

and

$$(10) \qquad K_1 = \frac{(p^*)^{1-\gamma_1}}{\gamma_1(\rho - \alpha)}$$

It is easy to see that

$$\gamma_1 > 1 \Leftrightarrow \rho > \alpha$$

Since we have assumed $\rho > \alpha$ we get by (9) that $p^* > 0$ and by (10) that $K_1 > 0$.

It remains to verify that with these values of p^* and K_1 the function

$$\phi(s, p) = e^{-\rho s} \psi(p) \ ,$$

with ψ given by (6), satisfies all the conditions of Theorem 10.4.1. Many of these verifications are straightforward. However, to avoid false solutions, it is important to check (ii) and (vi), which correspond to (4) and (2) above:

Verification of (4):
Define $\quad h(p) = \psi(p) + C$
Then

$$h(p^*) = h'(p^*) = 0 \qquad \text{by (7) and (8) above.}$$

Moreover, if $0 < p < p^*$ then

$$h''(p) = K_1 \gamma_1 (\gamma_1 - 1) p^{\gamma_1 - 2} > 0$$

which implies that $h'(p) < 0$ and hence $h(p) > 0$ for $0 < p < p^*$, as required.

Verification of (2):
For $p > p^*$ we have $\psi(p) = -C$ and hence

$$L_0 \psi(p) = (-\rho)(-C) - p = \rho C - p \ ,$$

so

$$L_0 \psi(p) \le 0 \qquad \text{for all } \ p > p^*$$

if and only if

$$p^* \geq \rho C ,$$

which holds because $U = \{(s,p); p < \rho C\}$ and $U \subset D$.

The remaining cases,
Case 2: $\rho = \alpha$
Case 3: $\rho < \alpha$
are left to the reader.

11.6. $u^* = \frac{a_1 - a_2 + \sigma_2^2(1-\gamma)}{(\sigma_1^2+\sigma_2^2)(1-\gamma)}$ (constant),

$$\Phi(s,x) = e^{\lambda(t-t_1)} x^\gamma \qquad \text{for } t < t_1,\ x > 0$$

where

$$\lambda = \tfrac{1}{2}\gamma(1-\gamma)[\sigma_1^2(u^*)^2 + \sigma_2^2(1-u^*)^2] - \gamma[a_1 u^* + a_2(1-u^*)] .$$

11.7. Define

$$dY(t) = \begin{bmatrix} dt \\ dX_t \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha\, u(t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ u(t) \end{bmatrix} dB_t\,; \qquad Y(0) = \begin{bmatrix} s \\ x \end{bmatrix}$$

and

$$G = \{(s,x); x > 0 \text{ and } s < T\}.$$

Then

$$\Phi(s,x) = \Phi(y) = \sup_{u(\cdot)} E^y[g(Y_{\tau_G})],$$

where

$$g(y) = g(s,x) = x^\gamma$$

and

$$\tau_G = \inf\{t > 0; Y(t) \notin G\} = \widehat{\tau} .$$

We apply Theorem 11.2.2 and hence look for a function ϕ such that

(1) $$\sup_{v \in \mathbf{R}} \{f^v(y) + (L^v\phi)(y)\} = 0 \qquad \text{for all } y \in G ,$$

where in this case $f^v(y) = 0$ and

$$L^v\phi(y) = L^v\phi(s,x) = \frac{\partial\phi}{\partial s} + av\frac{\partial\phi}{\partial x} + \tfrac{1}{2}v^2\frac{\partial^2\phi}{\partial x^2} .$$

If we guess that $\frac{\partial^2\phi}{\partial x^2} < 0$ then the maximum of the function $v \to L^v\phi(s,x)$ is attained at

(2) $$v = v^*(s,x) = -\frac{a\frac{\partial\phi}{\partial x}(s,x)}{\frac{\partial^2\phi}{\partial x^2}(s,x)} .$$

We try a function ϕ of the form

(3) $$\phi(s,x) = f(s)x^\gamma$$

for some function f (to be determined). Substituted in (2) this gives

(4) $$v^*(s,x) = -\frac{af(s)\gamma x^{\gamma-1}}{f(s)\gamma(\gamma-1)x^{\gamma-2}} = \frac{ax}{1-\gamma}$$

and (1) becomes

$$f'(s)x^\gamma + \frac{a^2x}{1-\gamma}f(s)\gamma x^{\gamma-1} + \tfrac{1}{2}\left(\frac{ax}{1-\gamma}\right)^2 f(s)\gamma(\gamma-1)x^{\gamma-2} = 0$$

or

$$f'(s) + \frac{a^2\gamma}{2(1-\gamma)}f(s) = 0\,. \tag{5}$$

Combined with the terminal condition

$$\phi(y) = g(y) \qquad \text{for } y \in \partial G$$

i.e.

$$f(T) = 1 \tag{6}$$

the equation (5) has the solution

$$f(s) = \exp\left(\frac{a^2\gamma}{2(1-\gamma)}(T-s)\right)\,; \qquad s \le T\,. \tag{7}$$

With this value of f it is now easily verified that

$$\phi(s,x) = f(s)x^\gamma$$

satisfies all the conditions of Theorem 11.2.2, and we conclude that the value function is

$$\Phi(s,x) = \phi(s,x) = f(s)x^\gamma$$

and that

$$u^*(s,x) = v^*(s,x) = \frac{ax}{1-\gamma}$$

is an optimal Markov control.

11.11. Additional hints:
For the solution of the unconstrained problem try a function $\phi_\lambda(s,x)$ of the form

$$\phi_\lambda(s,x) = a_\lambda(s)x^2 + b_\lambda(s)\,,$$

for suitable functions $a_\lambda(s), b_\lambda(s)$ with $\lambda \in \mathbf{R}$ fixed. By substituting this into the HJB equation we arrive at the equations

$$a'_\lambda(s) = \frac{1}{\theta}a^2_\lambda(s) - 1 \qquad \text{for } s < t_1$$
$$a_\lambda(t_1) = \lambda$$

and

$$b'_\lambda(s) = -\sigma^2 a_\lambda(s) \qquad \text{for } s < t_1$$
$$b_\lambda(t_1) = 0\,,$$

with optimal control $u^*(s,x) = -\frac{1}{\theta}a_\lambda(s)x$.
Now substitute this into the equation for $X_t^{u^*}$ and use the terminal condition to determine λ_0.
If we put $s = 0$ for simplicity, then $\lambda = \lambda_0$ can be chosen as any solution of the equation

$$A\lambda^3 + B\lambda^2 + C\lambda + D = 0\,,$$

where

$$\begin{aligned}
A &= m^2(e^{t_1} - e^{-t_1})^2 ,\\
B &= m^2(e^{2t_1} + 2 - 3e^{-2t_1}) - \sigma^2(e^{t_1} - e^{-t_1})^2 ,\\
C &= m^2(-e^{2t_1} + 2 + 3e^{-2t_1}) - 4x^2 - 2\sigma^2(1 - e^{-2t_1})\\
D &= -m^2(e^{t_1} + e^{-t_1})^2 + 4x^2 + \sigma^2(e^{2t_1} - e^{-2t_1}) .
\end{aligned}$$

11.12. If we introduce the process

$$dY(t) = \begin{bmatrix} dt \\ dX_t \end{bmatrix} = \begin{bmatrix} 1 \\ u_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ \sigma \end{bmatrix} dB_t ; \qquad Y(0) = y = \begin{bmatrix} s \\ x \end{bmatrix}$$

then the problem can be written

$$\Psi(s,x) = \inf_u E^y\Big[\int_0^\infty f(Y(t),u(t))dt\Big],$$

where

$$f(y,u) = f^u(y) = f^u(s,x) = e^{-\rho s}(x^2 + \theta u^2).$$

In this case we look for a function ψ such that

$$(1) \quad \inf_{v\in\mathbf{R}} \Big\{e^{-\rho s}(x^2+\theta v^2) + \frac{\partial\psi}{\partial s} + v\frac{\partial\psi}{\partial x} + \tfrac{1}{2}\sigma^2\frac{\partial^2\psi}{\partial x^2}\Big\} = 0; \quad (s,x)\in\mathbf{R}^2.$$

If we guess that ψ has the form

$$\psi(s,x) = e^{-\rho s}(ax^2 + b)$$

then (1) gets the form

$$(2) \quad \inf_{v\in\mathbf{R}} \{x^2 + \theta v^2 - \rho(ax^2 + b) + v\,2ax + \tfrac{1}{2}\sigma^2 2a\} = 0; \qquad x \in \mathbf{R} .$$

The minimum is attained when

$$(3) \qquad v = v^*(s,x) = -\frac{ax}{\theta}$$

and substituting this into (2) we get

$$x^2\Big[1 - \rho a - \frac{a^2}{\theta}\Big] + \sigma^2 a - \rho b = 0 \qquad \text{for all } x \in \mathbf{R} .$$

This is only possible if

$$(4) \qquad a^2 + \rho\theta a - \theta = 0$$

and

$$(5) \qquad b = \frac{\sigma^2 a}{\rho}$$

The positive root of (4) is

$$a = \tfrac{1}{2}\left[-\rho\,\theta + \sqrt{\rho^2\theta^2 + 4\theta}\right]. \tag{6}$$

We can now verify that with the values of a and b given by (6) and (5), respectively, the function

$$\psi(s,x) = e^{-\rho s}(ax^2 + b)$$

satisfies all the requirements of Theorem 11.2.2 and we conclude that the value function is

$$\Psi(s,x) = \psi(s,x) = e^{-\rho s}(ax^2 + b)$$

and that

$$u^*(s,x) = v^*(s,x) = -\frac{ax}{\theta}$$

is an optimal control.

11.13. If we introduce

$$dY(t) = \begin{bmatrix} dt \\ dX_t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 - u_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ \sigma \end{bmatrix} dB_t\,; \qquad Y(0) = \begin{bmatrix} s \\ x \end{bmatrix}$$

and

$$G = \{(s,x) \in \mathbf{R}^2; x > 0\},$$

then the problem gets the form

$$\Phi(s,x) = \sup_u E^y\Big[\int_0^{\tau_G} f(Y(t), u_t)dt\Big],$$

where

$$f(y,u) = f(s,x,u) = e^{-\rho s}u\,.$$

The corresponding HJB equation is

$$\sup_{v\in[0,1]}\left\{e^{-\rho s}v + \frac{\partial\phi}{\partial s} + (1-v)\frac{\partial\phi}{\partial x} + \tfrac{1}{2}\sigma^2\frac{\partial^2\phi}{\partial x^2}\right\} = 0\,. \tag{1}$$

If we substitute

$$\phi(s,x) = e^{-\rho s}\frac{1}{\rho}\left(1 - \exp\left(-\sqrt{\tfrac{2}{\sigma^2}}\,x\right)\right) \tag{2}$$

then (1) gets the form

$$\begin{aligned}\sup\nolimits_{v\in[0,1]}\Big\{-1 + \sqrt{\tfrac{2}{\rho\sigma^2}}\exp\left(-\sqrt{\tfrac{2\rho}{\sigma^2}}\,x\right)\\ +v\left[1 - \sqrt{\tfrac{2}{\rho\sigma^2}}\exp\left(-\sqrt{\tfrac{2\rho}{\sigma^2}}\,x\right)\right]\Big\} = 0\,.\end{aligned} \tag{3}$$

If $\rho \geq \frac{2}{\sigma^2}$ then $1 - \sqrt{\frac{2}{\rho\sigma^2}}\exp\left(-\sqrt{\frac{2\rho}{\sigma^2}}\,x\right) \geq 0$ for all x and hence the supremum in (3) is attained for

$$v = u^*(s,x) = 1\,.$$

Moreover, we see that the corresponding supremum in (2) is 0. Hence by Theorem 11.2.2 we conclude that $\Phi = \phi$ and $u_t^* = 1$.

12.6. a) no arbitrage
b) no arbitrage
c) $\theta(t) = (0, 1, 1)$ is an arbitrage
d) no arbitrage
e) arbitrages exist
f) no arbitrage.

12.7. a) complete
b) not complete. For example, the claim

$$F(\omega) = \int_0^T B_3(t) dB_3(t) = \tfrac{1}{2} B_3^2(T) - \tfrac{1}{2} T$$

cannot be hedged.
c) (arbitrages exist)
d) not complete
e) (arbitrages exist)
f) complete.

12.9. For the n-dimensional Brownian motion $B(t) = (B_1(t), \dots, B_n(t))$ the representation formula (12.3.33) gets the form

$$h(B(T)) = E[h(B(T))] + \int_0^T \sum_{j=1}^n \frac{\partial}{\partial z_j} E^z \left[h(B(t-t))\right]_{z=B(t)} dB_j(t).$$

(i) If $h(x) = x^2 = x_1^2 + \cdots + x_n^2$ we get

$$E^z[h(B(T-t))] = E[(B^z(T-t))^2] = z^2 + n(T-t).$$

Hence

$$\frac{\partial}{\partial z_j} E^z [h(B(T-t))] = 2z_j$$

and we conclude that

$$B^2(T) = x^2 + nT + \int_0^T 2B_j(t) dB_j(t), \quad \text{with} \quad B(0) = x.$$

(ii) If $h(x) = \exp(x_1 + \cdots + x_n)$ we get

$$E^z [h(B(T-t))] = E[\exp(z_1 + B_1(T-t) + \cdots + z_n + B_n(T-t))]$$
$$= \exp\left(\frac{n}{2}(T-t) + \sum_{i=1}^n z_i\right).$$

Hence

$$\frac{\partial}{\partial z_j} E^z[h(B(T-t))] = \exp\left(\frac{n}{2}(T-t) + \sum_{i=1}^n z_i\right)$$

and we conclude that if $B(0) = x$ then

$$\exp\Big(\sum_{i=1}^n B_i(T)\Big) = \exp\Big(\frac{n}{2}T + \sum_{i=1}^n x_i\Big)$$
$$+ \int_0^T \exp\Big(\frac{n}{2}(T-t) + \sum_{i=1}^n B_i(t)\Big)(dB_1(t) + \cdots + dB_n(t))$$

12.12. c) $E_Q[\xi(T)F] = \sigma^{-1}x_1(1-\frac{\alpha}{\rho})(1-e^{-\rho T})$. The replicating portfolio is $\theta(t) = (\theta_0(t), \theta_1(t))$, where

$$\theta_1(t) = \sigma^{-1}\Big[1 - \frac{\alpha}{\rho}(1-e^{\rho(t-T)})\Big]$$

and $\theta_0(t)$ is determined by (12.1.14).

12.15.

$$\Phi(s,x) = \begin{cases} e^{-\rho s}(K-x) & \text{for} \quad x \le x^* \\ e^{-\rho s}(K-x^*)(\frac{x}{x^*})^\gamma & \text{for} \quad x > x^* \end{cases}$$

where

$$\gamma = \beta^{-2}\Big[\tfrac{1}{2}\beta^2 - \alpha - \sqrt{\big(\tfrac{1}{2}\beta^2-\alpha\big)^2 + 2\rho\beta^2}\Big] < 0$$

and

$$x^* = \frac{K\gamma}{\gamma - 1} \in (0,K)\,.$$

Hence it is optimal to stop the first time $X(t) \le x^*$.
If $\alpha = \rho$ this simplifies to

$$\gamma = -\frac{2\rho}{\beta^2} \qquad \text{and} \qquad x^* = \frac{2\rho K}{\beta^2 + 2\rho}\,.$$

12.16. Recall that (see Theorem 12.3.5)

$$V^\theta(T) = V^\theta(0) + \int_0^T \theta(s)dX(s)$$

iff

$$\bar{V}^\theta(T) = V^\theta(0) + \int_0^T \theta(s)d\bar{X}(s) = V^\theta(0) + \int_0^T \xi(s)\widehat{\theta}(s)\sigma(s)d\widetilde{B}(s)\,.$$

Therefore, if we seek a portfolio θ such that

$$V^\theta(T) = F \text{ a.s.}\,,$$

we first find ϕ such that

(1) $$\bar{V}^\theta(T) = \xi(T)F = V^\theta(0) + \int_0^T \phi(s)d\widetilde{B}(s)$$

and then put

(2) $$\widehat{\theta}(s) = X_0(s)\phi(s)\Lambda(s),$$

where $\Lambda(s)$ is the left inverse of $\sigma(s)$.

a) $F(\omega) = (K - X_1(T,\omega))^+$

In this case (1) gets the form

$$e^{-\rho T}(K - X_1(T,\omega))^+ = E_Q[e^{-\rho T}(K - X_1(T,\omega))^+] + \int_0^T \phi(s)d\widetilde{B}(s)$$

or

$$(3) \qquad (K - X_1(T,\omega))^+ = E_Q[(K - X_1(T,\omega))^+] + \int_0^T \phi_0(s)d\widetilde{B}(s),$$

where

$$\phi_0(s) = e^{\rho T}\phi(s).$$

To find ϕ_0 we use Theorem 12.3.3:
In this case we have

$$dY(t) = dX_1(t) = \alpha X_1(t)dt + \beta X_1(t)dB(t) = \rho X_1(t)dt + \beta X_1(t)d\widetilde{B}(t),$$

with

$$d\widetilde{B}(t) = dB(t) + \frac{\alpha - \rho}{\beta}dt\,.$$

Moreover

$$h(y) = (K - y)^+$$

and

$$\begin{aligned} E_Q^y[h(Y(T-t))] &= E_Q^{x_1}[(K - X_1(T-t))^+] \\ &= E_Q\big[(K - x_1\exp\{\beta\,\widetilde{B}(T-t) + (\rho - \tfrac{1}{2}\beta^2)(T-t)\})^+\big]. \end{aligned}$$

From this we deduce that

$$\begin{aligned} (4) \quad \frac{d}{dx_1}E_Q^{x_1}[(K - X_1(T-t))^+] &= -E_Q\big[\mathcal{X}_{[0,K]}(x_1\exp\{\beta\,\widetilde{B}(T-t) \\ &\quad + (\rho - \tfrac{1}{2}\beta^2)(T-t)\})\cdot X_1^{(1)}(T-t)\big], \end{aligned}$$

where $X_1^{(1)}(T-t) = \exp\{\beta\,\widetilde{B}(T-t) + (\rho - \frac{1}{2}\beta^2)(T-t)\}$.
Hence

$$\phi_0(t) = \frac{d}{dx_1}E_Q^{x_1}\big[(K - X_1(T-t))^+\big]_{x_1 = X_1(t)}\beta X_1(t) = e^{\rho T}\phi(s).$$

Substituting this into (2) we get

$$\begin{aligned} \widehat{\theta}(t) &= \theta_1(t) \\ &= -e^{-\rho(T-t)}E_Q\big[\mathcal{X}_{[0,K]}(X_1(t)\exp\{\beta\,\widetilde{B}(T-t) \\ &\qquad + (\rho - \tfrac{1}{2}\beta^2)(T-t)\})X_1^{(1)}(T-t)\big] \\ &= -(2\pi(T-t))^{-\frac{1}{2}}\int_{\mathbf{R}} \mathcal{X}_{[0,K]}(X_1(t)\exp\{\beta y + (\rho - \tfrac{1}{2}\beta^2)(T-t)\}) \\ (5) \qquad &\qquad \cdot\exp\left\{\beta y - \tfrac{1}{2}\beta^2(T-t) - \frac{y^2}{2(T-t)}\right\}dy\,. \end{aligned}$$

Note that $\theta_1(t) < 0$ for all $t \in [0,T]$. Hence it is necessary to ***shortsell*** at all times in order to replicate the European put option.

1 Problems in Oksendal's book

3.2.

Proof. WLOG, we assume $t = 1$, then

$$\begin{aligned} B_1^3 &= \sum_{j=1}^{n}(B_{j/n}^3 - B_{(j-1)/n}^3) \\ &= \sum_{j=1}^{n}[(B_{j/n} - B_{(j-1)/n})^3 + 3B_{(j-1)/n}B_{j/n}(B_{j/n} - B_{(j-1)/n})] \\ &= \sum_{j=1}^{n}(B_{j/n} - B_{(j-1)/n})^3 + \sum_{j=1}^{n} 3B_{(j-1)/n}^2(B_{j/n} - B_{(j-1)/n}) \\ &\quad + \sum_{j=1}^{n} 3B_{(j-1)/n}(B_{j/n} - B_{(j-1)/n})^2 \\ &:= I + II + III \end{aligned}$$

By Problem EP1-1 and the continuity of Brownian motion.

$$I \leq [\sum_{j=1}^{n}(B_{j/n} - B_{(j-1)/n})^2] \max_{1\leq j\leq n} |B_{j/n} - B_{(j-1)/n}| \to 0 \quad a.s.$$

To argue $II \to 3\int_0^1 B_t^2 dB_t$ as $n \to \infty$, it suffices to show $E[\int_0^1 (B_t - B_t^{(n)})^2 dt] \to 0$, where $B_t^{(n)} = \sum_{j=1}^{n} B_{(j-1)/n}^2 1_{\{(j-1)/n < t \leq j/n\}}$. Indeed,

$$E[\int_0^1 |B_t - B_t^{(n)}|^2 dt] = \sum_{j=1}^{n} \int_{(j-1)/n}^{j/n} E(B_{(j-1)/n}^2 - B_t^2)^2 dt$$

We note $(B_t^2 - B_{\frac{j-1}{n}}^2)^2$ is equal to

$$(B_t - B_{\frac{j-1}{n}})^4 + 4(B_t - B_{\frac{j-1}{n}})^3 B_{\frac{j-1}{n}} + 4(B_t - B_{\frac{j-1}{n}})^2 B_{\frac{j-1}{n}}^2$$

so $E(B_{(j-1)/n}^2 - B_t^2)^2 = 3(t - (j-1)/n)^2 + 4(t - (j-1)/n)(j-1)/n$, and

$$\int_{\frac{j-1}{n}}^{\frac{j}{n}} E(B_{\frac{j-1}{n}}^2 - B_t^2)^2 dt = \frac{2j+1}{n^3}$$

Hence $E\int_0^1 (B_t - B_t^{(n)})^2 dt = \sum_{j=1}^{n} \frac{2j-1}{n^3} \to 0$ as $n \to \infty$.

To argue $III \to 3\int_0^1 B_t dt$ as $n \to \infty$, it suffices to prove

$$\sum_{j=1}^{n} B_{(j-1)/n}(B_{j/n} - B_{(j-1)/n})^2 - \sum_{j=1}^{n} B_{(j-1)/n}(\frac{j}{n} - \frac{j-1}{n}) \to 0 \quad a.s.$$

By looking at a subsequence, we only need to prove the L^2-convergence. Indeed,

$$
\begin{aligned}
& E\left(\sum_{j=1}^{n} B_{(j-1)/n}[(B_{j/n}-B_{(j-1)/n})^2-\frac{1}{n}]\right)^2 \\
= & \sum_{j=1}^{n} E\left(B_{(j-1)/n}^2[(B_{j/n}-B_{(j-1)/n})^2-\frac{1}{n}]^2\right) \\
= & \sum_{j=1}^{n} \frac{j-1}{n} E\left[(B_{j/n}-B_{(j-1)/n})^4-\frac{2}{n}(B_{j/n}-B_{(j-1)/n})^2+\frac{1}{n^2}\right] \\
= & \sum_{j=1}^{n} \frac{j-1}{n}(3\frac{1}{n^2}-2\frac{1}{n^2}+\frac{1}{n^2}) \\
= & \sum_{j=1}^{n} \frac{2(j-1)}{n^3} \to 0
\end{aligned}
$$

as $n \to \infty$. This completes our proof. □

3.18.

Proof. If $t > s$, then

$$E\left[\frac{M_t}{M_s}|\mathcal{F}_s\right] = E\left[e^{\sigma(B_t-B_s)-\frac{1}{2}\sigma^2(t-s)}|\mathcal{F}_s\right] = \frac{E[e^{\sigma B_{t-s}}]}{e^{\frac{1}{2}\sigma^2(t-s)}} = 1$$

The second equality is due to the fact $B_t - B_s$ is independent of $\mathcal{F}_s$. □

4.4.

Proof. For part a), set $g(t,x) = e^x$ and use Theorem 4.12. For part b), it comes from the fundamental property of Ito integral, i.e. Ito integral preserves martingale property for integrands in $\mathcal{V}$. □

Comments: The power of Ito formula is that it gives martingales, which vanish under expectation.

4.5.

Proof.

$$B_t^k = \int_0^t kB_s^{k-1}dB_s + \frac{1}{2}k(k-1)\int_0^t B_s^{k-2}ds$$

Therefore,

$$\beta_k(t) = \frac{k(k-1)}{2}\int_0^t \beta_{k-2}(s)ds$$

This gives $E[B_t^4]$ and $E[B_t^6]$. For part b), prove by induction. □

4.6. (b)

Proof. Apply Theorem 4.12 with $g(t,x) = e^x$ and $X_t = ct + \sum_{j=1}^n \alpha_j B_j$. Note $\sum_{j=1}^n \alpha_j B_j$ is a BM, up to a constant coefficient. $\square$

5.1. (ii)

Proof. Set $f(t,x) = x/(1+t)$, then by Ito's formula, we have

$$dX_t = df(t, B_t) = -\frac{B_t}{(1+t)^2}dt + \frac{dB_t}{1+t} = -\frac{X_t}{1+t}dt + \frac{dB_t}{1+t}$$

$\square$

(iv)

Proof. $dX_t^1 = dt$ is obvvious. Set $f(t,x) = e^t x$, then

$$dX_t^2 = df(t, B_t) = e^t B_t dt + e^t dB_t = X_t^2 dt + e^t dB_t$$

$\square$

5.9.

Proof. Let $b(t,x) = \log(1+x^2)$ and $\sigma(t,x) = 1_{\{x>0\}}x$, then

$$|b(t,x)| + |\sigma(t,x)| \le \log(1+x^2) + |x|$$

Note $\log(1+x^2)/|x|$ is continuous on $\mathbb{R} - \{0\}$, has limit 0 as $x \to 0$ and $x \to \infty$. So it's bounded on $\mathbb{R}$. Therefore, there exists a constant C, such that

$$|b(t,x)| + |\sigma(t,x)| \le C(1+|x|)$$

Also,

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le \frac{2|\xi|}{1+\xi^2}|x-y| + |1_{\{x>0\}}x - 1_{\{y>0\}}y|$$

for some ξ between x and y. So

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le |x-y| + |x-y|$$

Conditions in Theorem 5.2.1 are satisfied and we have existence and uniqueness of a strong solution. $\square$

5.11.

Proof. First, we check by integration-by-parts formula,

$$dY_t = \left(-a + b - \int_0^t \frac{dB_s}{1-s}\right) dt + (1-t)\frac{dB_t}{1-t} = \frac{b - Y_t}{1-t}dt + dB_t$$

Set $X_t = (1-t)\int_0^t \frac{dB_s}{1-s}$, then X_t is centered Gaussian, with variance

$$E[X_t^2] = (1-t)^2 \int_0^t \frac{ds}{(1-s)^2} = (1-t) - (1-t)^2$$

So X_t converges in L^2 to 0 as $t \to 1$. Since X_t is continuous a.s. for $t \in [0,1)$, we conclude 0 is the unique a.s. limit of X_t as $t \to 1$. □

7.8

Proof.

$$\{\tau_1 \wedge \tau_2 \le t\} = \{\tau_1 \le t\} \cup \{\tau_2 \le t\} \in \mathcal{N}_t$$

And since $\{\tau_i \ge t\} = \{\tau_i < t\}^c \in \mathcal{N}_t$,

$$\{\tau_1 \vee \tau_2 \ge t\} = \{\tau_1 \ge t\} \cup \{\tau_2 \ge t\} \in \mathcal{N}_t$$

□

7.9. a)

Proof. By Theorem 7.3.3, A restricted to $C_0^2(\mathbb{R})$ is $rx\frac{d}{dx} + \frac{\alpha^2 x^2}{2}\frac{d^2}{dx^2}$. For $f(x) = x^\gamma$, Af can be calculated by definition. Indeed, $X_t = xe^{(r-\frac{\alpha^2}{2})t+\alpha B_t}$, and $E^x[f(X_t)] = x^\gamma e^{(r-\frac{\alpha^2}{2}+\frac{\alpha^2\gamma}{2})\gamma t}$. So

$$\lim_{t\downarrow 0} \frac{E^x[f(X_t)] - f(x)}{t} = (r\gamma + \frac{\alpha^2}{2}\gamma(\gamma-1))x^\gamma$$

So $f \in D_A$ and $Af(x) = (r\gamma + \frac{\alpha^2}{2}\gamma(\gamma-1))x^\gamma$. □

b)

Proof. We choose ρ such that $0 < \rho < x < R$. We choose $f_0 \in C_0^2(\mathbb{R})$ such that $f_0 = f$ on (ρ, R). Define $\tau_{(\rho,R)} = \inf\{t > 0 : X_t \notin (\rho, R)\}$. Then by Dynkin's formula, and the fact $Af_0(x) = Af(x) = \gamma_1 x^{\gamma_1}(r + \frac{\alpha^2}{2}(\gamma_1 - 1)) = 0$ on (ρ, R), we get

$$E^x[f_0(X_{\tau_{(\rho,R)}\wedge k})] = f_0(x)$$

The condition $r < \frac{\alpha^2}{2}$ implies $X_t \to 0$ a.s. as $t \to 0$. So $\tau_{(\rho,R)} < \infty$ a.s.. Let $k \uparrow \infty$, by bounded convergence theorem and the fact $\tau_{(\rho,R)} < \infty$, we conclude

$$f_0(\rho)(1 - p(\rho)) + f_0(R)p(\rho) = f_0(x)$$

where $p(\rho) = P^x\{X_t \text{ exits } (\rho, R) \text{ by hitting R first}\}$. Then

$$\rho(p) = \frac{x^{\gamma_1} - \rho^{\gamma_1}}{R^{\gamma_1} - \rho^{\gamma_1}}$$

Let $\rho \downarrow 0$, we get the desired result. □

c)

Proof. We consider $\rho > 0$ such that $\rho < x < R$. $\tau_{(\rho,R)}$ is the first exit time of X from (ρ, R). Choose $f_0 \in C_0^2(\mathbb{R})$ such that $f_0 = f$ on (ρ, R). By Dynkin's formula with $f(x) = \log x$ and the fact $Af_0(x) = Af(x) = r - \frac{\alpha^2}{2}$ for $x \in (\rho, R)$, we get

$$E^x[f_0(X_{\tau_{(\rho,R)}\wedge k})] = f_0(x) + (r - \frac{\alpha^2}{2})E^x[\tau_{(\rho,R)} \wedge k]$$

Since $r > \frac{\alpha^2}{2}$, $X_t \to \infty$ a.s. as $t \to \infty$. So $\tau_{(\rho,R)} < \infty$ a.s.. Let $k \uparrow \infty$, we get

$$E^x[\tau_{(\rho,R)}] = \frac{f_0(R)p(\rho) + f_0(\rho)(1 - p(\rho)) - f_0(x)}{r - \frac{\alpha^2}{2}}$$

where $p(\rho) = P^x(X_t$ exits (ρ, R) by hitting R first$)$. To get the desired formula, we only need to show $\lim_{\rho\to 0} p(\rho) = 1$ and $\lim_{\rho\to 0} \log\rho(1 - p(\rho)) = 0$. This is trivial to see once we note by our previous calculation in part b),

$$p(\rho) = \frac{x^{\gamma_1} - \rho^{\gamma_1}}{R^{\gamma_1} - \rho^{\gamma_1}}$$

□

7.18 a)

Proof. The line of reasoning is exactly what we have done for 7.9 b). Just replace x^γ with a general function $f(x)$ satisfying certain conditions. □

b)

Proof. The characteristic operator $\mathcal{A} = \frac{1}{2}\frac{d^2}{dx^2}$ and $f(x) = x$ are such that $\mathcal{A}f(x) = 0$. By formula (7.5.10), we are done. □

c)

Proof. $\mathcal{A} = \mu\frac{d}{dx} + \frac{\sigma^2}{2}\frac{d^2}{dx^2}$. So we can choose $f(x) = e^{-\frac{2\mu}{\sigma^2}x}$. Therefore

$$p = \frac{e^{-\frac{2\mu x}{\sigma^2}} - e^{-\frac{2\mu a}{\sigma^2}}}{e^{-\frac{2\mu b}{\sigma^2}} - e^{-\frac{2\mu a}{\sigma^2}}}$$

□

8.6

Proof. The major difficulty is to make legitimate using Feymann-Kac formula while $(x - K)^+ \notin C_0^2$. For the conditions under which we can indeed apply Feymann-Kac formula to $(x - K)^+ \notin C_0^2$, c f. the book of Karatzas & Shreve, page 366. □

8.16 a)

Proof. Let $L_t = -\int_0^t \sum_{i=1}^n \frac{\partial h}{\partial x_i}(X_s)dB_s^i$. Then L is a square-integrable martingale. Furthermore, $\langle L\rangle_T = \int_0^T |\bigtriangledown h(X_s)|^2 ds$ is bounded, since $h \in C_0^1(\mathbb{R}^n)$. By Novikov's condition, $M_t = \exp\{L_t - \frac{1}{2}\langle L\rangle_t\}$ is a martingale. We define $\bar{P}$ on $\mathcal{F}_T$ by $d\bar{P} = M_T dP$. Then

$$dX_t = \bigtriangledown h(X_t)dt + dB_t$$

defines a BM under $\bar{P}$.

$$\begin{aligned}
& E^x[f(X_t)] \\
= \; & \bar{E}^x[M_t^{-1} f(X_t)] \\
= \; & \bar{E}^x[e^{\int_0^t \sum_{i=1}^n \frac{\partial h}{\partial x_i}(X_s)dX_s^i - \frac{1}{2}\int_0^t |\bigtriangledown h(X_s)|^2 ds} f(X_t)] \\
= \; & E^x[e^{\int_0^t \sum_{i=1}^n \frac{\partial h}{\partial x_i}(B_s)dB_s^i - \frac{1}{2}\int_0^t |\bigtriangledown h(B_s)|^2 ds} f(B_t)]
\end{aligned}$$

Apply Ito's formula to $Z_t = h(B_t)$, we get

$$h(B_t) - h(B_0) = \int_0^t \sum_{i=1}^n \frac{\partial h}{\partial x_i}(B_s)dB_s^i + \frac{1}{2}\int_0^t \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2}(B_s)ds$$

So

$$E^x[f(X_t)] = E^x[e^{h(B_t)-h(B_0)} e^{-\int_0^t V(B_s)ds} f(B_t)]$$

□

b)

Proof. If Y is the process obtained by killing B_t at a certain rate V, then it has transition operator

$$T_t^Y(g, x) = E^x[e^{-\int_0^t V(B_s)ds} g(B_t)]$$

So the equality in part a) can be written as

$$T_t^X(f, x) = e^{-h(x)} T_t^Y(fe^h, x)$$

□

9.11 a)

Proof. First assume F is closed. Let $\{\phi_n\}_{n\geq 1}$ be a sequence of bounded continuous functions defined on ∂D such that $\phi_n \to 1_F$ boundedly. This is possible due to Tietze extension theorem. Let $h_n(x) = E^x[\phi_n(B_\tau)]$. Then by Theorem 9.2.14, $h_n \in C(\bar{D})$ and $\Delta h_n(x) = 0$ in D. So by Poisson formula, for $z = re^{i\theta} \in D$,

$$h_n(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(t-\theta) h_n(e^{it})dt$$

Let $n \to \infty$, $h_n(z) \to E^x[1_F(B_\tau)] = P^x(B_\tau \in F)$ by bounded convergence theorem, and $RHS \to \frac{1}{2\pi}\int_0^{2\pi} P_r(t-\theta)1_F(e^{it})dt$ by dominated convergence theorem. Hence

$$P^z(B_\tau \in F) = \frac{1}{2\pi}\int_0^{2\pi} P_r(t-\theta)1_F(e^{it})dt$$

Then by $\pi - \lambda$ theorem and the fact Borel σ-field is generated by closed sets, we conclude

$$P^z(B_\tau \in F) = \frac{1}{2\pi}\int_0^{2\pi} P_r(t-\theta)1_F(e^{it})dt$$

for any Borel subset of ∂D. □

b)

Proof. Let B be a BM starting at 0. By example 8.5.9, $\phi(B_t)$ is, after a change of time scale $\alpha(t)$ and under the original probability measure P, a BM in the plane. $\forall F \in \mathcal{B}(\mathbb{R})$,

$$\begin{aligned}
& P(B \text{ exits } D \text{ from } \psi(F)) \\
= \ & P(\phi(B) \text{ exits upper half plane from } F) \\
= \ & P(\phi(B)_{\alpha(t)} \text{ exits upper half plane from } F) \\
= \ & \text{Probability of BM starting at i that exits from } F \\
= \ & \mu(F)
\end{aligned}$$

So by part a), $\mu(F) = \frac{1}{2\pi}\int_0^{2\pi} 1_{\psi(F)}(e^{it})dt = \frac{1}{2\pi}\int_0^{2\pi} 1_F(\phi(e^{it}))dt$. This implies

$$\int_R f(\xi)d\mu(\xi) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi(e^{it}))dt = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\phi(z))}{z}dz$$

□

c)

Proof. By change-of-variable formula,

$$\int_R f(\xi)d\mu(\xi) = \frac{1}{\pi}\int_{\partial H} f(\omega)\frac{d\omega}{|\omega - i|^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\frac{dx}{x^2+1}$$

□

d)

Proof. Let $g(z) = u + vz$, then g is a conformal mapping that maps i to $u + vi$ and keeps upper half plane invariant. Use the harmonic measure on x-axis of a BM starting from i, and argue as above in part a)-c), we can get the harmonic measure on x-axis of a BM starting from $u + iv$. □

12.1 a)

Proof. Let θ be an arbitrage for the market $\{X_t\}_{t\in[0,T]}$. Then for the market $\{\bar{X}_t\}_{t\in[0,T]}$:
(1) θ is self-financing, i.e. $d\bar{V}_t^\theta = \theta_t d\bar{X}_t$. This is (12.1.14).
(2) θ is admissible. This is clear by the fact $\bar{V}_t^\theta = e^{-\int_0^t \rho_s ds} V_t^\theta$ and ρ being bounded.
(3) θ is an arbitrage. This is clear by the fact $V_t^\theta > 0$ if and only if $\bar{V}_t^\theta > 0$.

So $\{\bar{X}_t\}_{t\in[0,T]}$ has an arbitrage if $\{X_t\}_{t\in[0,T]}$ has an arbitrage. Conversely, if we replace ρ with $-\rho$, we can calculate X has an arbitrage from the assumption that $\bar{X}$ has an aribitrage. □

12.2

Proof. By $V_t = \sum_{i=0}^n \theta_i X_i(t)$, we have $dV_t = \theta \cdot dX_t$. So θ is self-financing. □

12.6 (e)

Proof. Arbitrage exists, and one hedging strategy could be $\theta = (0, B_1 + B_2, B_1 - B_2 + \frac{1-3B_1+B_2}{5}, \frac{1-3B_1+B_2}{5})$. The final value would then become $B_1(T)^2 + B_2(T)^2$. □

12.10

Proof. Becasue we want to represent the contingent claim in terms of original BM B, the measure Q is the same as P. Solving SDE $dX_t = \alpha X_t dt + \beta X_t dB_t$ gives us $X_t = X_0 e^{(\alpha - \frac{\beta^2}{2})t + \beta B_t}$. So

$$\begin{aligned} & E^y[h(X_{T-t})] \\ = \;& E^y[X_{T-t}] \\ = \;& y e^{(\alpha-\frac{\beta^2}{2})(T-t)} e^{\frac{\beta^2}{2}(T-t)} \\ = \;& y e^{\alpha(T-t)} \end{aligned}$$

Hence $\phi = e^{\alpha(T-t)}\beta X_t = \beta X_0 e^{\alpha T - \frac{\beta^2}{2}t + \beta B_t}$. □

12.11 a)

Proof. According to (12.2.12), $\sigma(t,\omega) = \sigma$, $\mu(t,\omega) = m - X_1(t)$. So $u(t,\omega) = \frac{1}{\sigma}(m - X_1(t) - \rho X_1(t))$. By (12.2.2), we should define Q by setting

$$dQ|_{\mathcal{F}_t} = e^{-\int_0^t u_s dB_s - \frac{1}{2}\int_0^t u_s^2 ds} dP$$

Under Q, $\tilde{B}_t = B_t + \frac{1}{\sigma}\int_0^t (m - X_1(s) - \rho X_1(s))ds$ is a BM. Then under Q,

$$dX_1(t) = \sigma d\tilde{B}_t + \rho X_1(t) dt$$

So $X_1(T)e^{-\rho T} = X_1(0) + \int_0^T \sigma e^{-\rho t} d\tilde{B}_t$ and $E_Q[\xi(T)F] = E_Q[e^{-\rho T}X_1(T)] = x_1$. □

b)

Proof. We use Theorem 12.3.5. From part a), $\phi(t,\omega) = e^{-\rho t}\sigma$. We therefore should choose $\theta_1(t)$ such that $\theta_1(t)e^{-\rho t}\sigma = \sigma e^{-\rho t}$. So $\theta_1 = 1$ and θ_0 can then be chosen as 0. □

2 Extra Problems

EP1-1.

Proof. According to Borel-Cantelli lemma, the problem is reduced to proving $\forall\epsilon$,

$$\sum_{n=1}^{\infty} P(|S_n| > \epsilon) < \infty$$

where $S_n := \sum_{j=1}^{n}(B_{j/n} - B_{(j-1)/n})^2 - 1$. Set

$$X_j = (B_{j/n} - B_{(j-1)/n})^2 - 1/n$$

By the hint, if we consider the i.i.d. sequence $\{X_j\}_{j=1}^n$ normalized by its 4-th moment, we have

$$P(|S_n| > \epsilon) < \epsilon^{-4}E[S_n^4] \le \epsilon^{-4}CE[X_1^4]n^2$$

By integration-by-parts formula, we can easily calculate the 2k-th moment of $N(0,\sigma)$ is of order σ^k. So the order of $E[X_1^4]$ is n^{-4}. This suffices for the Borel-Cantelli lemma to apply. □

EP1-2.

Proof. We first see the second part of the problem is not hard, since $\int_0^t Y_s dB_s$ is a martingale with mean 0. For the first part, we do the following construction. We define $Y_t = 1$ for $t \in (0, 1/n]$, and for $t \in (j/n, (j+1)/n]$ $(1 \le j \le n-1)$

$$Y_t := C_j 1_{\{B_{(i+1)/n} - B_{i/n} \le 0, \quad 0 \le i \le j-1\}}$$

where each C_j is a constant to be determined.

Regarding this as a betting strategy, the intuition of Y is the following: We start with one dollar, if $B_{1/n} - B_0 > 0$, we stop the game and gain $(B_{1/n} - B_0)$ dollars. Otherwise, we bet C_1 dollars for the second run. If $B_{2/n} - B_{1/n} > 0$, we then stop the game and gain $C_1(B_{2/n} - B_{1/n}) - (B_{1/n} - B_0)$ dollars (if the difference is negative, it means we actually lose money, although we win the second bet). Otherwise, we bet C_2 dollar for the third run, etc. So in the end our total gain/loss of this betting is

$$\begin{aligned}\int_0^t Y_s dB_s \quad &= (B_{1/n} - B_0) + 1_{\{B_{1/n} - B_0 \le 0\}} C_1(B_{2/n} - B_{1/n}) + \cdots \\ &+ 1_{\{B_{1/n} - B_0 \le 0, \cdots, B_{(n-1)/n} - B_{(n-2)/n} \le 0\}} C_{n-1}(B_1 - B_{(n-1)/n})\end{aligned}$$

We now look at the conditions unde which $\int_0^1 Y_s dB_s \le 0$. There are several possibilities:
(1) $(B_{1/n} - B_0) \le 0$, $(B_{2/n} - B_{1/n}) > 0$, but $C_1(B_{2/n} - B_{1/n}) < |B_{1/n} - B_0|$;
(2) $(B_{1/n} - B_0) \le 0$, $(B_{2/n} - B_{1/n}) \le 0$, $(B_{3/n} - B_{2/n}) > 0$, but $C_2(B_{3/n} - B_{2/n}) < |B_{1/n} - B_0| + C_1|B_{2/n} - B_{1/n}|$;
$\cdots\cdots$;
(n) $(B_{1/n} - B_0) \le 0$, $(B_{2/n} - B_{1/n}) \le 0$, $\cdots$, $(B_1 - B_{(n-1)/n}) \le 0$.

The last event has the probability of $(1/2)^n$. The first event has the probability of

$$P(X \leq 0, Y > 0, 0 < Y < X/C_1) \leq P(0 < Y < X/C_1)$$

where X and Y are i.i.d. $N(0, 1/n)$ random variables. We can choose C_1 large enough so that this probability is smaller than $1/2^n$. The second event has the probability smaller than $P(0 < X < Y/C_2)$, where X and Y are independent Gaussian random variables with 0 mean and variances $1/n$ and $(C_1^2 + 1)/n$, respectively, we can choose C_2 large enough, so that this probability is smaller than $1/2^n$. We continue this process untill we get all the C_j's. Then the probability of $\int_0^1 Y_t dB_t \leq 0$ is at most $n/2^n$. For n large enough, we can have $P(\int_0^1 Y_t dB_t > 0) > 1 - \epsilon$ for given ϵ. The process Y is obviously bounded. $\square$

Comments: Different from flipping a coin, where the gain/loss is one dollar, we have now random gain/loss $(B_{j/n} - B_{(j-1)/n})$. So there is no sense checking our loss and making new strategy constantly. Put it into real-world experience, when times are tough and the outcome of life is uncertain, don't regret your loss and estimate how much more you should invest to recover that loss. Just keep trying as hard as you can. When the opportunity comes, you may just get back everything you deserve.

EP2-1.

Proof. This is another application of the fact hinted in Problem EP1-1. $E[Y_n] = 0$ is obvious. And

$$\begin{aligned} & E[(B^1_{j/n} - B^1_{(j-1)/n})^4 (B^2_{j/n} - B^2_{(j-1)/n})^4] \\ = \ & (3E[(B^1_{j/n} - B^1_{(j-1)/n})^2]^2)^2 \\ = \ & \frac{9}{n^4} \\ := \ & a_n \end{aligned}$$

We set $X_j = [B^1_{j/n} - B^1_{(j-1)/n}][B^2_{j/n} - B^2_{(j-1)/n}]/a_n^{\frac{1}{4}}$, and apply the hint in EP1-1,

$$E[Y_n^4] = a_n E(X_1 + \cdots + X_n)^4 \leq \frac{9}{n^4} cn^2 = \frac{9c}{n^2}$$

for some constant c. This implies $Y_n \to 0$ with probability one, by Borel-Cantelli lemma. $\square$

Comments: This following simple proposition is often useful in calculation. If X is a centered Gaussian random variable, then $E[X^4] = 3E[X^2]^2$. Furthermore, we can show $E[X^{2k}] = C_k E[X^{2k-2}]^2$ for some constant C_k. These results can be easily proved by integration-by-part formula. As a consequence, $E[B_t^{2k}] = Ct^k$ for some constant C.

EP3-1.

Proof. A short proof: For part (a), it suffices to set

$$Y_{n+1} = E[R_{n+1} - R_n | X_1, \cdots, X_{n+1} = 1]$$

(*What does this really mean, rigorously?*). For part (b), the answer is NO, and $R_n = \sum_{j=1}^n X_j^3$ gives the counter example.

A long proof:

We show the analysis behind the above proof and point out if $\{X_n\}_n$ is i.i.d. and symmetrically distributed, then Bernoulli type random variables are the only ones that have martingale representation property.

By adaptedness, $R_{n+1} - R_n$ can be represented as $f_{n+1}(X_1, \cdots, X_{n+1})$ for some Borel function $f_{n+1} \in \mathcal{B}(\mathbb{R}^{n+1})$. Martingale property and $\{X_n\}_n$ being i.i.d. Bernoulli random variables imply

$$f_{n+1}(X_1, \cdots, X_n, -1) = -f_{n+1}(X_1, \cdots, X_n, 1)$$

This inspires us set Y_{n+1} as

$$f_{n+1}(X_1, \cdots, X_n, 1) = E[R_{n+1} - R_n | X_1, \cdots, X_{n+1} = 1].$$

For part b), we just assume $\{X_n\}_n$ is i.i.d. and symmetrically distributed. If $(R_n)_n$ has martingale representation property, then

$$f_{n+1}(X_1, \cdots, X_{n+1})/X_{n+1}$$

must be a function of $X_1, \cdots, X_n$. In particular, for $n = 0$ and $f_1(x) = x^3$, we have X_1^2 =constant. So Bernoulli type random variables are the only ones that have martingale representation theorem.

□

EP5-1.

Proof. $\mathcal{A} = \frac{r}{x}\frac{d}{dx} + \frac{1}{2}\frac{d^2}{dx^2}$, so we can choose $f(x) = x^{1-2r}$ for $r \neq \frac{1}{2}$ and $f(x) = \log x$ for $r = \frac{1}{2}$. □

EP6-1. (a)

Proof. Assume the claim is false, then there exists $t_0 > 0$, $\epsilon > 0$ and a sequence $\{t_k\}_{k\geq 1}$ such that $t_k \uparrow t_0$, and

$$\left|\frac{f(t_k) - f(t_0)}{t_k - t_0} - f'_+(t_0)\right| > \epsilon$$

WLOG, we assume $f'_+(t_0) = 0$, otherwise we consider $f(t) - tf'_+(t_0)$. Because f'_+ is continuous, there exists $\delta > 0$, such that $\forall t \in (t_0 - \delta, t_0 + \delta)$,

$$|f'_+(t) - f'_+(t_0)| = |f'_+(t)| < \frac{\epsilon}{2}$$

Meanwhile, there exists infinitely many t_k's such that

$$\frac{f(t_k) - f(t_0)}{t_k - t_0} > \epsilon \text{ or } \frac{f(t_k) - f(t_0)}{t_k - t_0} < -\epsilon$$

By considering f or $-f$ and taking a subsequence, we can WLOG assume for all the t_k's, $t_k \in (t - \delta, t + \delta)$, and

$$\frac{f(t_k) - f(t_0)}{t_k - t_0} - f'_+(t_0) > \epsilon$$

Consider $h(t) = \epsilon(t - t_0) - [f(t) - f(t_0)] = (t - t_0)\left[\epsilon - \frac{f(t)-f(t_0)}{t-t_0}\right]$. Then $h(t_0) = 0$, $h'_+(t) = \epsilon - f'_+(t) > \epsilon/2$ for $t \in (t_0 - \delta, t_0 + \delta)$, and $h(t_k) > 0$. On one hand,

$$\int_{t_k}^{t_0} h'_+(t)dt > \frac{\epsilon}{2}(t_0 - t_k) > 0$$

On the other hand, if h is monotone increasing, then

$$\int_{t_k}^{t_0} h'_+(t)dt \leq h(t_0) - h(t_k) = 0 - h(t_k) < 0$$

Contradiction.

So it suffices to show h is monotone increasing on $(t_0 - \delta, t_0 + \delta)$. This is easily proved by showing h cannot obtain local maximum in the interior of $(t_0 - \delta, t_0 + \delta)$. □

(b)

Proof. $f(t) = |t - 1|$. □

(c)

Proof. $f(t) = 1_{\{t \geq 0\}}$. □

EP6-2. (a)

Proof. Since A is bounded, $\tau < \infty$ a.s..

$$\begin{aligned} E^x[M_{n+1} - M_n|\mathcal{F}_n] &= E^x[f(S_{n+1}) - f(S_n)|\mathcal{F}_n]1_{\{\tau \geq n+1\}} \\ &= (E^{S_n}[f(S_1)] - f(S_n))1_{\{\tau \geq n+1\}} \\ &= \Delta f(S_n)1_{\{\tau \geq n+1\}} \end{aligned}$$

Because $S_n \in A$ on $\{\tau \geq n+1\}$ and f is harmonic on $\bar{A}$, $\Delta f(S_n)1_{\{\tau \geq n+1\}} = 0$. So M is a martingale. □

(b)

Proof. For existence, set $f(x) = E^x[F(S_\tau)]$ $(x \in \bar{A})$, where $\tau = \inf\{n \geq 0 : S_n \notin A\}$. Clearly $f(x) = F(x)$ for $x \in \partial A$. For $x \in A$, $\tau \geq 1$ under P^x, and we have

$$\begin{aligned} \Delta f(x) &= E^x[f(S_1)] - f(x) \\ &= E^x[E^{S_1}[F(S_\tau)]] - f(x) \\ &= E^x[E^x[F(S_\tau) \circ \theta_1|S_1]] - f(x) \\ &= E^x[F(S_\tau) \circ \theta_1] - f(x) \\ &= E^x[F(S_\tau)] - f(x) \\ &= 0 \end{aligned}$$

For the 5th equality, we used the fact under P^x, $\tau \geq 1$ and hence $S_\tau \circ \theta_1 = S_\tau$.

For uniqueness, by part a), $f(S_{n\wedge\tau})$ is a martingale, so use optimal stopping time, we have

$$f(x) = E^x[f(S_0)] = E^x[f(S_{n\wedge\tau})]$$

Becasue f is bounded, we can use bounded convergence theorem and let $n \uparrow \infty$,

$$f(x) = E^x[f(S_\tau)] = E^x[F(S_\tau)]$$

□

(c)

Proof. Since $d \leq 2$, the random walk is recurrent. So $\tau < \infty$ a.s. even if A is bounded. The existence argument is exactly the same as part b). For uniqueness, we still have $f(x) = E^x[f(S_{n\wedge\tau})]$. Since f is bounded, we can let $n \uparrow \infty$, and get $f(x) = E^x[F(S_\tau)]$. □

(d)

Proof. Let $d = 1$ and $A = \{1, 2, 3, ...\}$. Then $\partial A = \{0\}$. If $F(0) = 0$, then both $f(x) = 0$ and $f(x) = x$ are solutions of the discrete Dirichlet problem. We don't have uniqueness. □

(e)

Proof. $A = \mathbb{Z}^3 - \{0\}$, $\partial A = \{0\}$, and $F(0) = 0$. $T_0 = \inf\{n \geq 0 : S_n \geq 0\}$. Let $c \in \mathbb{R}$ and $f(x) = cP^x(T_0 = \infty)$. Then $f(0) = 0$ since $T_0 = 0$ under P^0. f is clearly bounded. To see f is harmonic, the key is to show $P^x(T_0 = \infty | S_1 = y) = P^y(T_0 = \infty)$. This is due to Markov property: note $T_0 = 1 + T_0 \circ \theta_1$. Since c is arbitrary, we have more than one bounded solution.

□

EP6-3.

Proof.

$$\begin{aligned}
E^x[K_n - K_{n-1}|\mathcal{F}_{n-1}] &= E^x[f(S_n) - f(S_{n-1})|\mathcal{F}_{n-1}] - \Delta f(S_{n-1}) \\
&= E^{S_{n-1}}[f(S_1)] - f(S_{n-1}) - \Delta f(S_{n-1}) \\
&= \Delta f(S_{n-1}) - \Delta f(S_{n-1}) \\
&= 0
\end{aligned}$$

Applying Dynkin's formula is straightforward. □

EP6-4. (a)

Proof. By induction, it suffices to show if $|y - x| = 1$, then $E^y[T_A] < \infty$. We note $T_A = 1 + T_A \circ \theta_1$ for any sample path starting in A. So

$$E^x[T_A 1_{\{S_1\}}] = E^x[T_A | S_1 = y] P^x(S_1 = y) = E^y[T_A - 1] P^x(S_1 = y)$$

Since $E^x[T_A 1_{\{S_1\}}] \leq E^x[T_A] < \infty$ and $P^x(S_1 = y) > 0$, $E^y[T_A] < \infty$. □

(b)

Proof. If $y \in \partial A$, then under P^y, $T_A = 0$. So $f(y) = 0$. If $y \in A$,

$$\begin{aligned}\Delta f(y) &= E^y[f(S_1)] - f(y) \\ &= E^y[E^y[T_A \circ \theta_1 | S_1]] - f(y) \\ &= E^y[E^y[T_A - 1 | S_1]] - f(y) \\ &= E^y[T_A] - 1 - f(y) \\ &= -1\end{aligned}$$

To see uniqueness, use the martingale in EP6-3 for any solution f, we get

$$E^x[f(S_{T_A \wedge K})] = f(x) + E^x[\sum_{j=0}^{T_A - 1} \Delta f(S_j)] = f(x) - E^x[T_A]$$

Let $K \uparrow \infty$, we get $0 = f(x) - E^x[T_A]$.

□

EP7-1. a)

Proof. Since D is bounded, there exists $R > 0$, such that $D \subset\subset B(0, R)$. Let $\tau_R := \inf\{t > 0 : |B_t - B_0| \geq R\}$, then $\tau \leq \tau_R$. If $q \geq -\epsilon$

$$e(x) = E^x[e^{\epsilon\tau}] \leq E^x[e^{\epsilon\tau_R}] = E^x[\int_0^{\tau_R} \epsilon e^{\epsilon t} dt + 1] = 1 + \int_0^{\infty} P^x(\tau_R > t)\epsilon e^{\epsilon t} dt$$

For any $n \in \mathbb{N}$, $P^x(\tau_R > n) \leq P^x(\cap_{i=1}^n \{|B_k - B_{k-1}| < 2R\}) = a^n$, where $a = P^x(|B_1 - B_0| < 2R) < 1$. So $e(x) \leq 1 + \epsilon e^{\epsilon} \sum_{n=1}^{\infty} (ae^{\epsilon})^{n-1}$. For ϵ small enough, $ae^{\epsilon} < 1$, and hence $e(x) < \infty$. Obviously, ϵ is only dependent on D. □

c)

Proof. Since q is continuous and $\bar{D}$ is compact, q attains its minimum M. If $M \geq 0$, then we have nothing to prove. So WLOG, we assume $M < 0$. Then similar to part a),

$$\tilde{e}(x) \leq E^x[e^{-M(\tau \wedge \sigma_\epsilon)}] \leq E^x[e^{-M\sigma_\epsilon}] = 1 + \int_0^{\infty} P^x(\sigma_\epsilon > t)(-M)e^{-Mt} dt$$

Note $P^x(\sigma_\epsilon > t) = P^x(\sup_{s \leq t} |B_s - B_0| < \epsilon) = P^0(\sup_{s \leq t} |\epsilon B_{s/\epsilon^2}| < \epsilon) = P^x(\sigma_1 > t/\epsilon^2)$. So $\tilde{e}(x) = 1 + \int_0^{\infty} P^x(\sigma_1 > u)(-M\epsilon^2)e^{-M\epsilon^2 u} du = E^x[e^{-M\epsilon^2\sigma_1}]$. For ϵ small enough, $-M\epsilon^2$ will be so small that, by what we showed in the proof of part a), $E^x[e^{-M\epsilon^2\sigma_1}]$ will be finite. Obviously, ϵ is dependent on M and D only, hence q and D only. □

d)

Proof. Cf. Rick Durrett's book, *Stochastic Calculus: A Practical Introduction*, page 158-160. □

b)

Proof. From part d), it suffices to show for a give x, there is a $K = K(D, x) < \infty$, such that if $q = -K$, then $e(x) = \infty$. Since D is open, there exists $r > 0$, such that $B(x, r) \subset\subset D$.

Now we assume $q = -K < 0$, where K is to be determined. We have

$$e(x) = E^x[e^{K\tau}] \geq E^x[e^{K\tau_r}].$$

Here $\tau_r := \inf\{t > 0 : |B_t - B_0| \geq r\}$. Similar to part a), we have

$$E^x[e^{K\tau_r}] \geq 1 + \sum_{n=1}^{\infty} P^x(\tau_r \geq n)e^{kn}(1 - e^{-k})$$

So it suffices to show there exists $\delta > 0$, such that $P^x(\tau_r \geq n) \geq \delta^n$.

Note

$$P^x(\tau_r > n) = P^x(\max_{t\leq n} |B_t - B_0| < r) \geq P^x(\max_{t\leq n} |B_t^i - B_0^i| < C(d)r, i \leq d),$$

where B^i is the i-th coordinate of B and $C(d)$ is a constant dependent on d. Set $a = C(d)r$, then by independence

$$P^x(\tau_r > n) \geq P^0(\max_{t\leq n} |W_t| < a)^d$$

Here W is a standard one-dimensional BM. Let

$$\delta = \inf_{-\frac{a}{2}<x<\frac{a}{2}} P^x(\max_{t\leq 1} |W_t| < a, |W_0| < a/2, |W_1| < a/2)(> 0)$$

then we have

$$\begin{aligned}
& P^0(\max_{t\leq n} |W_t| < a) \\
\geq \; & P^0(\cap_{k=1}^{n}\{\max_{k-1\leq t\leq k} |W_t| < a, |W_{k-1}| < \frac{a}{2}, |W_k| < \frac{a}{2}\}) \\
= \; & P^0(\{\max_{n-1\leq t\leq n} |W_t| < a, |W_{n-1}| < \frac{a}{2}, |W_n| < \frac{a}{2}\} | \cap_{k=1}^{n-1} \\
& \{\max_{k-1\leq t\leq k} |W_t| < a, |W_{k-1}| < \frac{a}{2}, |W_k| < \frac{a}{2}\}) \\
& \times P^0(\cap_{k=1}^{n-1}\{\max_{k-1\leq t\leq k} |W_t| < a, |W_{k-1}| < \frac{a}{2}, |W_k| < \frac{a}{2}\}) \\
\geq \; & \delta P^0(\cap_{k=1}^{n-1}\{\max_{k-1\leq t\leq k} |W_t| < a, |W_{k-1}| < \frac{a}{2}, |W_k| < \frac{a}{2}\})
\end{aligned}$$

The last line is due to Markov property. By induction we have

$$P^0(\max_{t\leq n} |W_t| < a) > \delta^n,$$

and we are done. $\square$

EP7-2.

Proof. Consider the case of dimension 1. $D = \{x : x > 0\}$. Then for any $x > 0$, $P^x(\tau < \infty) = 1$. But by $P^x(\tau \in dt) = \frac{x}{2\pi t^3} e^{-\frac{x^2}{2t}} dt$, we can calculate that $E^x[\tau] = \infty$. So for every $\epsilon > 0$, $E^x[e^{\epsilon\tau}] \geq e^{\epsilon E[\tau]} = \infty$. □

EP8-1. a)

Proof.
$$E[e^{aX_1}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}+ax} dx = e^{\frac{a^2}{2}}$$
So $E[X_1 e^{aX_1}] = a e^{\frac{a^2}{2}}$. □

b)

Proof. We note $Z_n \in \mathcal{F}_n$ and X_{n+1} is independent of $\mathcal{F}_n$, so we have
$$\begin{aligned} & E[\frac{M_{n+1}}{M_n}|\mathcal{F}_n] \\ = & E[e^{-f(Z_n)X_{n+1}-\frac{1}{2}f^2(Z_n)}|\mathcal{F}_n] \\ = & E[e^{-f(z)X_{n+1}-\frac{1}{2}f^2(z)}]|_{z=Z_n} = e^{\frac{1}{2}f^2(Z_n)-\frac{1}{2}f^2(Z_n)} = 1 \end{aligned}$$
So $(M_n)_{n\geq 0}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$. □

c)

Proof.
$$\begin{aligned} & E[M_{n+1}Z_{n+1} - M_n Z_n|\mathcal{F}_n \\ = & M_n E[\frac{M_{n+1}}{M_n} Z_{n+1} - Z_n|\mathcal{F}_n] \\ = & M_n E[\frac{M_{n+1}}{M_n}(Z_n + f(Z_n) + X_{n+1}) - Z_n|\mathcal{F}_n] \\ = & M_n E[Z_n + f(Z_n) - Z_n + E[\frac{M_{n+1}}{M_n} X_{n+1}|\mathcal{F}_n]] \\ = & M_n[f(Z_n) + E[X_{n+1}e^{-f(Z_n)X_{n+1}-\frac{1}{2}f^2(Z_n)}|\mathcal{F}_n]] \\ = & M_n[f(Z_n) - f(Z_n)] \\ = & 0 \end{aligned}$$
So $(M_n Z_n)_{n\geq 0}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n\geq 0}$. □

d)

Proof. $\forall A \in \mathcal{F}_n$, $E^Q[Z_{n+1}; A] = E^P[M_{n+1}Z_{n+1}; A] = E^P[M_n Z_n; A] = E^Q[Z_n; A]$. So $E^Q[Z_{n+1}|\mathcal{F}_n] = Z_n$, that is, Z_n is a Q-martingale. □

EP8-2. a)

Proof. Let $Z_t = \exp\{\int_0^{t\wedge T_\epsilon} \frac{\alpha(\alpha-1)}{2B_s^2} ds\}$. Note $B^\alpha_{t\wedge T_\epsilon} = (\int_0^t 1_{\{s\le T_\epsilon\}} dB_s)^\alpha$, we have

$$dB^\alpha_{t\wedge T_\epsilon} = \alpha B^{\alpha-1}_{t\wedge T_\epsilon} 1_{\{t\le T_\epsilon\}} dB_t + \frac{\alpha(\alpha-1)}{2} B^{\alpha-2}_{t\wedge T_\epsilon} 1_{\{t\le T_\epsilon\}} dt$$

So $M_t = B^\alpha_{t\wedge T_\epsilon} Z_t$ satisfies

$$dM_t = B^\alpha_{t\wedge T_\epsilon} dZ_t + Z_t \alpha B_t^{\alpha-1} 1_{\{t\le T_\epsilon\}} dB_t + Z_t \frac{\alpha(\alpha-1)}{2} B_t^{\alpha-2} 1_{\{t\le T_\epsilon\}} dt$$

Meanwhile, $dZ_t = \frac{\alpha(\alpha-1)}{2B_t^2} 1_{\{t\le T_\epsilon\}} e^{\int_0^t \frac{\alpha(1-\alpha)}{2B_s^2} ds} dt$. So

$$B^\alpha_{t\wedge T_\epsilon} dZ_t + \frac{\alpha(\alpha-1)}{2} 1_{\{t\le T_\epsilon\}} B_t^{\alpha-2} Z_t dt = 0$$

Hence $dM_t = Z_t \alpha B_t^{\alpha-1} 1_{\{t\le T_\epsilon\}} dB_t$. To check M is a martingale, we note we actually have

$$E[\int_0^T Z_t^2 \alpha^2 B_t^{2\alpha-2}] 1_{\{t\le T_\epsilon\}} dt < \infty.$$

Indeed, $Z_t^2 1_{\{t\le T_\epsilon\}} \le e^{\frac{\alpha|1-\alpha|}{2\epsilon^2}T}$. If $\alpha \le t$, $B_t^{2\alpha-2} 1_{\{t\le T_\epsilon\}} \le \epsilon^{2\alpha-2}$; if $\alpha > 1$, $E[B_t^{2\alpha-2} 1_{\{t\le T_\epsilon\}}] \le t^{\alpha-1}$. Hence M is martingale. □

b)

Proof. Under Q, $Y_t = B_t - \int_0^t \frac{1}{M_s} d\langle M, B\rangle_s$ is a BM. We take $A_t = -\frac{\alpha}{B_t} 1_{\{t\le T_\epsilon\}}$. The SDE for B in terms of Y_t is

$$dB_t = dY_t + \frac{\alpha}{B_t} 1_{\{t\le T_\epsilon\}} dt$$

□

c)

Proof. Under Q, B satisfies the Bessel diffusion process before it hits $\frac{1}{2}$. That is, up to the time $T_{\frac{1}{2}}$, B satisfies the equation

$$dB_t = dY_t + \frac{\alpha}{B_t} dt$$

This line may sound fishy as we haven't defined what it means by an SDE defined up to a random time. Actually, a rigorous theory can be built for this notion. But we shall avoid this theoretical issue at this moment.

We choose $b > 1$, and define $\tau_b = \inf\{t > 0 : B_t \notin (\frac{1}{2}, b)\}$. Then $Q^1(T_{\frac{1}{2}} = \infty) = \lim_{b\to\infty} Q^1(B_{\tau_b} = b)$. By the results in EP5-1 and Problem 7.18 in Oksendal's book, we have

(i) If $\alpha > 1/2$, $\lim_{b\to\infty} Q^1(B_{\tau_b} = b) = \lim_{b\to\infty} \frac{1-(\frac{1}{2})^{1-2\alpha}}{b^{1-2\alpha}-(\frac{1}{2})^{1-2\alpha}} = 1 - (\frac{1}{2})^{2\alpha-1} > 0$. So in this case, $Q^1(T_{\frac{1}{2}} = \infty) > 0$.

(ii) If $\alpha < 1/2$, $\lim_{b\to\infty} Q^1(B_{\tau_b} = b) = \lim_{b\to\infty} \frac{1-(\frac{1}{2})^{1-2\alpha}}{b^{1-2\alpha}-(\frac{1}{2})^{1-2\alpha}} = 0$. So in this case, $Q^1(T_{\frac{1}{2}} = \infty) = 0$.

(iii) If $\alpha = 1/2$, $\lim_{b\to\infty} Q^1(B_{\tau_b} = b) = \lim_{b\to\infty} \frac{0-\log\frac{1}{2}}{\log b - \log\frac{1}{2}} = 0$. So in this case, $Q^1(T_{\frac{1}{2}} = \infty) = 0$.

$\square$

EP9-1. a)

Proof. Fix $z \in D$, consider $A = \{\omega \in D : \rho_D(z,\omega) < \infty\}$. Then A is clearly open. We show A is also closed. Indeed, if $\omega_k \in A$ and $\omega_k \to \omega_* \in D$, then for k sufficiently large, $|\omega_k - \omega_*| < \frac{1}{2}\mathrm{dist}(\omega_*, \partial D)$. So ω_k and ω_* are adjacent. By definition, $\rho_D(\omega_*, z) < \infty$, i.e. $\omega_* \in A$.

Since D is connected, and A is both closed and open, we conclude $A = D$. By the arbitrariness of z, $\rho_D(z,\omega) < \infty$ for any z, $\omega \in D$.

To see ρ_D is a metric on D, note $\rho_D(z,z) = 0$ by definition and $\rho(z,\omega) \geq 1$ for $z \neq \omega$. So $\rho_D(z,\omega) = 0$ iff $z = \omega$. If $\{x_k\}$ is a finite adjacent sequence connecting z_1 and z_2, and $\{y_l\}$ is a finite adjacent sequence connecting z_2 and z_3, then $\{x_k, z_2, y_l\}_{k,l}$ is a finite adjacent sequence connecting z_1 and z_3. So $\rho_D(z_1,z_3) \leq \rho_D(z_1,z_2) + \rho_D(z_2,z_3)$. Meanwhile, it's clear that $\rho_D(z,\omega) \geq$ and $\rho_D(z,\omega) = \rho_D(\omega, z)$. So ρ_D is a metric. $\square$

b)

Proof. $\forall z \in U_k$, then $\rho_D(z_0, z) \leq k$. Assume $z_0 = x_0$, x_1, $\cdots$, $x_k = z$ is a finite adjacent sequence. Then $|z - x_{k-1}| < \frac{1}{2}\max\{\mathrm{dist}(z,\partial D), \mathrm{dist}(x_{k-1},\partial D)\}$. For ω close to z,

$$|\omega - x_{k-1}| \leq |z-\omega| + |z - x_{k-1}| < \frac{1}{2}\max\{\mathrm{dist}(\omega,\partial D), \mathrm{dist}(x_{k-1},\partial D)\}.$$

Indeed, if $\mathrm{dist}(x_{k-1}, D) > \mathrm{dist}(z,\partial D)$, then for ω close to z, $\mathrm{dist}(\omega,\partial D)$ is also close to $\mathrm{dist}(z,\partial D)$, and hence $< \mathrm{dist}(x_{k-1},\partial D)$. Choose ω such that $|z-\omega| < \frac{1}{2}\mathrm{dist}(x_{k-1},\partial D) - |z - x_{k-1}|$, we then have

$$\begin{aligned} & |\omega - x_{k-1}| \\ \leq\ & |z-\omega| + |z-x_{k-1}| \\ <\ & \frac{1}{2}\mathrm{dist}(x_{k-1},\partial D) \\ =\ & \frac{1}{2}\max(\mathrm{dist}(x_{k-1},\partial D), \mathrm{dist}(\omega,\partial D)) \end{aligned}$$

If $\mathrm{dist}(x_{k-1},\partial D) \leq \mathrm{dist}(z,\partial D)$, then for ω close to z, $\frac{1}{2}\max\{\mathrm{dist}(\omega,\partial D), \mathrm{dist}(x_{k-1},\partial D)\}$ is very close to $\frac{1}{2}\max\{\mathrm{dist}(z,\partial D), \mathrm{dist}(x_{k-1},\partial D)\} = \frac{1}{2}\mathrm{dist}(z,\partial D)$. Hence, for ω close to z,

$$|\omega - x_{k-1}| \leq |z-\omega| + |z - x_{k-1}| < \frac{1}{2}\max(\mathrm{dist}(x_{k-1},\partial D), \mathrm{dist}(\omega,\partial D))$$

Therefore ω and x_{k-1} are adjacent. This shows $\rho_D(z_0,\omega) \leq k$, i.e. $\omega \in U_k$. $\square$

c)

Proof. By induction, it suffices to show there exists a constant $c > 0$, such that for adjacent z, $\omega \in D$, $h(z) \le ch(\omega)$. Indeed, let $r = \frac{1}{4}\min\{\mathrm{dist}(z, \partial D), \mathrm{dist}(\omega, \partial D)\}$, then by mean-value property, $\forall y \in B(\omega, r)$, we have $B(y, r) \subset B(\omega, 2r)$, so

$$h(\omega) = \frac{\int_{B(\omega,2r)} h(x)dx}{V(B(\omega, 2r))} \ge \frac{\int_{B(y,r)} h(x)dx}{V(B(\omega, 2r))} = \frac{V(B(y, r))}{V(B(\omega, 2r))} h(y) = \frac{h(y)}{2^d}$$

By using a sequence of small balls connecting ω and z, we are done. □

d)

Proof. Since K is compact and $\{U_1(x)\}_{x\in U}$ is an open covering of K, we can find a finite sub-covering $\{U_{n_i}(x)\}_{i=1}^N$ of K. This implies $\forall z, \omega \in K$, $\rho_D(z, \omega) \le N$. By the result in part c), we're done.

□

EP9-2. a)

Proof. We first have the following observation. Consider circles centered at 0, with radius r and $2r$, respectively. Let B be a BM on the plane and $\sigma_{2r} = \inf\{t > 0 : |B_t| = 2r\}$.

$\forall x \in \partial B(0, r)$, $P^x([B_0, B_{\sigma_{2r}}]$ doesn't loop around 0) is invariant for different x's on $\partial B(0, r)$, by the rotational invariance of BM. $\forall \theta > 0$, we define $\bar{B}_t = B_{\theta t}$, and $\bar{\sigma}_{2r} = \inf\{t > 0 : |\bar{B}_t| = 2r\}$. Since $\bar{B}$ and B have the same trajectories,

$$\begin{aligned} & P^x([B_0, B_{\sigma_{2r}}] \text{ doesn't loop around } 0) \\ = \; & P([B_0, B_{\sigma_{2r}}] + x \text{ doesn't loop around } 0) \\ = \; & P([\bar{B}_0, \bar{B}_{\bar{\sigma}_{2r}}] + x \text{ doesn't loop around } 0) \\ = \; & P(\tfrac{1}{\sqrt{\theta}}[\bar{B}_0, \bar{B}_{\bar{\sigma}_{2r}}] + \tfrac{x}{\sqrt{\theta}} \text{ doesn't loop around } 0) \end{aligned}$$

Define $W_t = \frac{\bar{B}_t}{\sqrt{\theta}} = \frac{B_{\theta t}}{\sqrt{\theta}}$, then W is a BM under P. If we set $\tau = \inf\{t > 0 : |W_t| = \frac{2r}{\sqrt{\theta}}\}$, then $\tau = \bar{\sigma}_{2r}$. So

$$\begin{aligned} & P(\tfrac{1}{\sqrt{\theta}}[\bar{B}_0, \bar{B}_{\bar{\sigma}_{2r}}] + \tfrac{x}{\sqrt{\theta}} \text{ doesn't loop around } 0) \\ = \; & P([W_0, W_\tau] + \tfrac{x}{\sqrt{\theta}} \text{ doesn't loop around } 0) \\ = \; & P^{\frac{x}{\sqrt{\theta}}}([W_0, W_\tau] \text{ doesn't loop around } 0) \end{aligned}$$

Note $\frac{x}{\sqrt{\theta}} \in \partial B(0, \frac{r}{\sqrt{\theta}})$, we conclude for different r's, the probability that BM starting from $\partial B(0, r)$ exits $B(0, 2r)$ without looping around 0 is the same.

Now we assume $2^{-n-1} \le |x| < 2^{-n}$ and $\sigma_n = \inf\{t > 0 : |B_t| = 2^{-n}\}$. Then for $E_j = \{[B_{\sigma_j}, B_{\sigma_{j-1}}]$ doesn't loop around $0\}$, $E \subset \cap_{j=1}^n E_j$. From what we observe above, $P^{B_{\sigma_j}}([B_0, B_{\sigma_{j-1}}]$ doesn't loop around 0) is a constant, say β. Use strong Markov property and induction, we have

$$P^x(\cap_{j=1}^n E_j) = P^x(\cap_{j=2}^n E_j; P^x(E_1|\mathcal{F}_{\sigma_1})) = \beta P^x(\cap_{j=2}^n E_j) = \beta^n = 2^{n\log\beta}$$

Set $-\log \beta = \alpha$, we have $P^x(E) \le 2^{-\alpha n} = 2^\alpha (2^{-n-1})^\alpha \le 2^\alpha |x|^\alpha$. Clearly $\beta \in (0,1)$. So $\alpha \in (0, \infty)$.

The above discussion relies on the assumtion $|x| < 1/2$. However, when $1/2 \le |x| < 1$, the desired inequality is trivial. Indeed, in this case $2^\alpha |x|^\alpha \ge 1$.

□

b)

Proof. $\forall x \in \partial D$, WLOG, we assume $x = 0$. $\forall \epsilon > 0$, let $\bar{B}_t = \epsilon B_{t/\epsilon^2}$, $\sigma = \inf\{t > 0 : |B_t| = 1\}$ and $\bar{\sigma} := \bar{\sigma}_\epsilon = \inf\{t > 0 : |\bar{B}_t| = \epsilon\}$, then $\bar{\sigma} = \epsilon^2 \sigma$. Hence $P^0\{[\bar{B}_0, \bar{B}_{\bar{\sigma}}] \text{ loops around } 0\} = P^0\{[B_0, B_\sigma] \text{ loops around } 0\}$. By part a), $P\{[B_0, B_\sigma] \text{ loops around } 0\} = 1$. So,

$$P^0(\bar{B} \text{ loops around } 0 \text{ before exiting } B(0,\epsilon)) = 1.$$

This means $P(\tau_D < \bar{\sigma}_\epsilon) = 1$, $\forall \epsilon > 0$. This is equivalent to x being regular. □

EP9-3. a)

Proof. We first establish a derivative estimate for harmonic functions. Let h be harmonic in D. Then $\frac{\partial h}{\partial z_i}$ is also harmonic. By mean-value property and integration-by-parts formula, $\forall z_0 \in D$ and $\forall r > 0$ such that $B(z_0, r) \subset U$, we have

$$\left| \frac{\partial h}{\partial z_i}(z_0) \right| = \left| \frac{\int_{B(z_0,r/2)} \frac{\partial h}{\partial z_i} dz}{V(B(z_0, r/2))} \right| = \left| \frac{\int_{\partial B(z_0,r/2)} h v_i dz}{V(B(z_0, r/2))} \right| \le \frac{2d}{r} \|h\|_{L^\infty(\partial B(z_0, r/2))}$$

Now fix K. There exists $\eta > 0$, such that when K is enlarged by a distance of η, the enlarged set is contained in the interior of a compact subset K' of U. Furthermore, if η is small enough, $\forall z, \omega \in K$ with $|z - \omega| < \eta$, we have $\cup_{\xi \in [z,\omega]} B(\xi, \eta) \subset K'$. Denote $\sup_n \sup_{z \in K'} |h_n(z)|$ by C, then by the above derivative estimate, for $z, \omega \in K$ with $|z - \omega| < \eta$,

$$|h_n(z) - h_n(\omega)| \le \frac{2d}{\eta} C |z - \omega|$$

This clearly shows the desired δ exists.

□

b)

Proof. Let K be a compact subset of D, then by part a) and Arzela-Ascoli theorem, $\{h_n\}_n$ is relatively compact in $C(K)$. So there is a subsequence $\{h_{n_j}\}$ such that $h_{n_j} \to h$ uniformly on K. Furthermore, by mean-value property, h must be also harmonic in the interior of K. By choosing a sequence of compact subsets $\{K_n\}$ increasing to D, and choosing diagonally subsequences, we can find a subsequence of $\{h_n\}$ such that it converges uniformly on any compact subset of D. This will consistently define a function h in D. Since harmonicity is a local property, h is harmonic in D.

□

EP10-1. a)

Proof. First, we note that

$$P^x(B_1 \geq 1; B_t > 0, \forall t \in [0,1]) = P^x(B_1 \geq 1) - P^x(\inf_{0 \leq s \leq 1} B_s \leq 0, B_1 \geq 1)$$

Let τ_0 be the first passage time of BM hitting 0, then by strong Markov property

$$\begin{aligned}
& P^x(\inf_{s \leq 1} B_s \leq 0, B_1 \geq 1) \\
= \quad & P^x(\tau_0 \leq 1, P^x(B_1 \geq 1 | \mathcal{F}_{\tau_0})) \\
= \quad & P^x(\tau_0 \leq 1, P^{B_{\tau_0}}(B_u \geq 1)|_{u=1-\tau_0}) \\
= \quad & P^x(\tau_0 \leq 1, P^0(B_u \geq 1)|_{u=1-\tau_0}) \\
= \quad & P^x(\tau_0 \leq 1, P^0(B_u \leq -1)|_{u=1-\tau_0}) \\
= \quad & P^x(\tau_0 \leq 1, B_1 \leq -1) \\
= \quad & P^x(B_1 \leq -1)
\end{aligned}$$

So

$$\begin{aligned}
& P^x(B_1 \geq 1; B_t > 0, \forall t \in [0,1]) \\
= \quad & P^x(B_1 \geq 1) - P^x(B_1 \leq -1) \\
= \quad & \int_{1-x}^{1+x} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \\
\geq \quad & 2x \frac{e^{-2}}{\sqrt{2\pi}}
\end{aligned}$$

where the last inequality is due to $x < 1$. $\square$

EP10-2.

Proof. Let $F(n) = P(E_{2^n})$ and let DLA be the shorthand for "doesn't loop around" then

$$\begin{aligned}
F(n+m) &= P(E_{2^{n+m}}) \\
&= P([B_0, B_{T_{2^{n+m}}}] \text{ DLA } 0) \\
&\leq P([B_0, B_{T_{2^n}}] \text{ DLA } 0; P([B_{T_{2^n}}, B_{T_{2^{n+m}}}] \text{ DLA } 0 | \mathcal{F}_{T^{2^n}})) \\
&= P([B_0, B_{T_{2^n}}] \text{ DLA } 0; P^{B_{T_{2^n}}}([B_0, B_{T_{2^{n+m}}}] \text{ DLA } 0))
\end{aligned}$$

By rotational invariance of BM $P^x([B_0, B_{T_{2^{n+m}}}] \text{ DLA } 0)$ is a constant for any $x \in \partial B(0, 2^n)$. By scaling, we have

$$P^x([B_0, B_{T_{2^{n+m}}}] \text{ DLA } 0) = P^{\frac{x}{2^n}}([B_0, B_{T_{2^m}}] \text{ DLA } 0) = P(E_{2^m}) = F(m)$$

So $F(n+m) \leq F(n)F(m)$. By the properties of submultiplicative functions, $\lim_{n\to\infty} \frac{\log F(n)}{n}$ exists. We set this limit by $-\alpha$. $\forall m \in \mathbb{N}$, for m large enough, we can find n, such that $2^n \leq m < 2^{n+1}$, then $P(E_{2^n}) \geq P(E_m) \geq P(E_{2^{n+1}})$. So

$$\frac{\log P(E_{2^n})}{\log 2^n} \frac{\log 2^n}{\log m} \geq \frac{\log P(E_m)}{\log m} \geq \frac{\log P(E_{2^{n+1}})}{\log 2^{n+1}} \frac{\log 2^{n+1}}{\log m}$$

Let $m \to \infty$, then $\log 2^n / \log m \to 1$ as seen by $\log 2^n \leq \log m < \log 2 + \log 2^n$. So $\lim_m \frac{\log P(E_m)}{\log m}$ exists and equals to $-\alpha$. To see $\alpha \in (0,1]$, note $F(1) < 1$ and $F(n) \leq F(1)^n$. So $\alpha > 0$. Furthermore, we note

$$\begin{aligned} & P^x([B_0, B_{T_n}] \text{ DLA } 0) \\ \geq \; & P^x(B^1 \text{ exits } (0,n) \text{ by hitting n}) \\ = \; & \frac{1}{n} \end{aligned}$$

So $\log P(E_n)/\log n \geq -1$. Hence $\alpha \leq 1$. □

EP10-3. a)

Proof. We assume $f_0(k) = 1$, $\forall k$ and $j, k = 1, \cdots, N$. We let P be the $N \times N$ matrix with $P_{jk} = p_{j,k}$. Then if we regard f_n as a row vector, we have $f_n = f_{n-1}P$. Define $M_n = \max_{k \leq N} f_n(k)$, then

$$f_{n+m} = f_0 P^{n+m} = f_0 P^m P^n = f_m P^n \leq M_m f_0 P^n = M_m f_n \leq M_m M_n f_0$$

So $M_{n+m} \leq M_n M_m$. By properties of submultiplicative functions, $\lim_n \frac{\log M_n}{n}$ exists and equals $\inf_n \frac{\log M_n}{n}$. Meanwhile, $\delta := \min_{j,k \leq N} p_{j,k} > 0$. So

$$M_n \geq f_n(k) \geq \delta \sum_{j=1}^{N} f_{n-1}(j) \geq \delta M_{n-1}$$

By induction, $M_n \geq \delta^n$. Hence $\inf_n \frac{\log M_n}{n} \geq \log \delta > -\infty$. Let $\beta = \inf_n \frac{\log M_n}{n}$, then $M_n \geq e^{\beta n}$. We set $\alpha = e^{\beta}$. Then $M_n \geq \alpha^n$. Meanwhile, there exists constant $C \in (0, \infty)$, such that for $m_n = \min_{k \leq N} f_n(k)$, $M_n \leq C m_n$. Indeed, for $n = 1$, $M_1 = m_1$, and for $n > 1$, $f_n(k) = \sum_j p_{j,k} f_{n-1}(j) \leq K \sum_j f_{n-1}(j)$ and $f_n(k) \geq \delta \sum_j f_{n-1}(j)$. So $M_n \leq \frac{K}{\delta} m_n$. Let $C = \frac{K}{\delta} \vee 1$, then

$$f_n(k) \geq m_n \geq \frac{M_n}{C} \geq \frac{\alpha^n}{C}$$

Similarly, we can show m_n is supermultiplicative and similar argument gives us the upper bound. □