Combinatorial Proofs

1) Id what is being counted

↳ # of ways to choose k items out of n

2) LHS counts (1)

↳ by def^n

3) RHS counts (1)

↳ # of hats NOT chosen uniquely determines the chosen items

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$= \frac{n!}{(n-k)!\,(n-n+k)!}$$

$$= \frac{n!}{(n-k)!\,(n-(n-k))!}$$

$$= \binom{n}{n-k}$$

correct but bad

not chosen

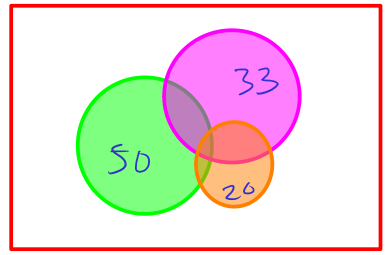
chosen

99

1 The Count

(a) How many of the first 100 positive integers are divisible by 2, 3, or 5?

$50 + 33 + 20 - 16 - 6 - 10 + 3$



99 8 7

☆☆|☆|☆|

(b) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have? 9 bars, 7 stars ☆☆|||||☆|||☆|☆☆

9941000

☆|☆|☆|☆

(c) Now instead of non-increasing, they must be strictly decreasing. So 9983220 is no longer valid, while 9753210 is valid. How many choices for a new phone number does he nave now?

98 76

2 CS70: The Musical

Edward, one of the previous head TA's, has been hard at work on his latest project, *CS70: The Musical*. It's now time for him to select a cast, crew, and directing team to help him make his dream a reality.

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity: $\binom{2n}{2} = 2\binom{n}{2} + n^2$

(b) Edward would now like to select a crew out of n people, Use this to provide a combinatorial argument that proves the following identity: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ (this is called pascal's identity)

(c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity: $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity: $\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}$.

(a) LHS counts # of ways to select 2 directors out of $2n$.

RHS: Divide the $2n$ dirs into 2 groups of size n.

$\binom{n}{2}$ ways to select 2 within each of 2 groups, yielding $2\binom{n}{2}$.

n^2 ways of choosing one dir from group 1 & another from group 2.

(b) LHS: Def^n

RHS:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

(c) What are we counting?

# of ways to make a team of size k out of n ppl and select a lead.

LHS: ① Select k.

② Select k ppl out of n to cast: $\binom{n}{k}$

③ Choose 1 out of k to be lead actor: $\binom{k}{1} = k$

$$\sum_{i=1}^{n} i\binom{n}{i}$$

RHS: out of n ppl, select 1 lead actor: $\binom{n}{1}$ ways.

2^{n-1} subsets of remaining ppl.

$\therefore$ Total # $= n2^{n-1}$

(d) LHS: ① select k.

② Select k ppl out of n to cast: $\binom{n}{k}$

③ Choose j out of k to be lead actors: $\binom{k}{j}$

$$\sum_{i=1}^{n} \binom{i}{j}\binom{n}{i}$$

RHS: out of n ppl, select j lead actor: $\binom{n}{j}$ ways.

$n-j$ remaining.

2^{n-j} subsets of remaining ppl.

$\therefore$ Total # $= \binom{n}{j} 2^{n-j}$

3 Bit String

0, 1

How many bit strings of length 10 contain at least five consecutive 0's?

$$\binom{6}{5} + \binom{5}{4} + \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \binom{1}{0}$$

$$6 + 5 + 4 + 3 + 2 + 1$$

$\underline{00000}\ \underbrace{\_\ \_\ \_\ \_\ \_}_{2^5}$

$\binom{6}{1} 2^5 = 192$

$\underline{1}\ \underline{00000}\ \overbrace{\_\ \_\ \_\ \_}^{2^4}$

$\_\ \underline{1}\ \underline{00000}\ \_\ \_\ \_$

$\_\ \_\ \underline{1}\ \underline{00000}\ \_\ \_$

$\_\ \_\ \_\ \underline{1}\ \underline{00000}\ \_$

$\_\ \_\ \_\ \_\ \underline{1}\ \underline{00000}$

$2^5 + 5 \cdot 2^4$

$= 112$

$\underline{1}\ \underline{1}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0\ 0\ 0\ 0\ 0}$

$\underline{1}\ \underline{1}\ \underline{00000}$