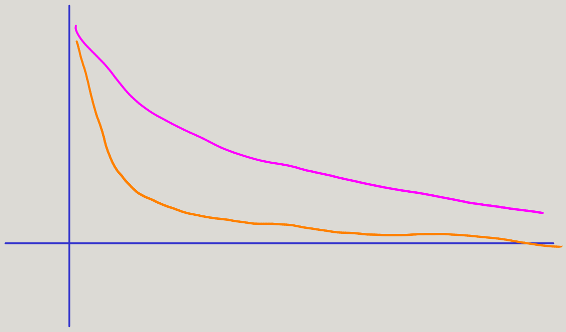
Markov's Bound

$$\Pr[X \geq a] \leq \frac{E[X]}{a}$$

Chebyshev's Bound

$$\Pr[\,|X - E[X]| \geq k\,] \leq \frac{Var(X)}{k^2}$$



Generalized Markov Bounds
Chernoff Bounds
Hoeffding

CS 70 Summer 2020 — Discrete Mathematics and Probability Theory Course Notes — DIS 07A

1 Probabilistic Bounds

A random variable X has variance $\mathrm{Var}(X) = 9$ and expectation $\mathbb{E}[X] = 2$. Furthermore, the value of X is never greater than 10. Given this information, provide either a proof or a counterexample for the following statements.

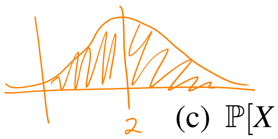
(a) $\mathbb{E}[X^2] = 13$. TRUE

$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}^2(X) \implies \mathbb{E}(X^2) = \mathrm{Var}(X) + \mathbb{E}^2(X)$
$= 13$

(b) $\mathbb{P}[X = 2] > 0$. FALSE

$X = \begin{cases} -1 & \text{w.p. } 1/2 \\ 5 & \text{w.p. } 1/2 \end{cases}$ $P[X=2] = 0$, $\mathbb{E}[X] = 2$, $\mathrm{Var}(X) = 9$

$\mathbb{E}(X) = 2 = \frac{a}{2} + \frac{b}{2} \Rightarrow a + b = 4$
$\mathbb{E}[X^2] = 13 = \frac{a^2}{2} + \frac{b^2}{2} \Rightarrow a^2 + b^2 = 26$
$-1 \quad 5$



(c) $\mathbb{P}[X \geq 2] = \mathbb{P}[X \leq 2]$. FALSE

$Y = \begin{cases} a = -7 & \text{w.p. } p = 1/10 \\ b = 3 & \text{w.p. } 1-p = 9/10 \end{cases}$

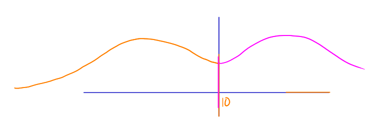
$p \neq \frac{1}{2}$
$\mathbb{E}[X] = ap + b(1-p) = 2$
$\mathbb{E}[X^2] = a^2 p + b^2(1-p) = 13$
$a, b < 10$

$b = 2 \pm \frac{3}{\sqrt{\frac{1-p}{p}}}$

(d) $\mathbb{P}[X \leq 1] \leq 8/9$.

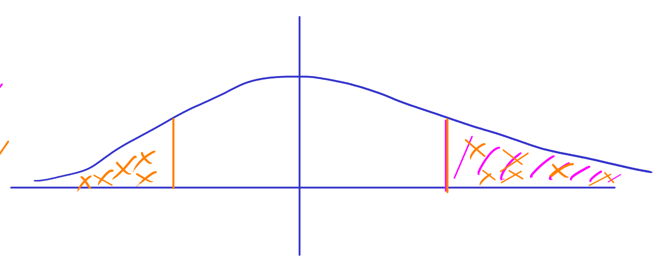
$-1(X - 10) = 10 - X = Y \quad \mathbb{E}[Y] = 10 - 2 = 8$

$\Pr[X \leq 1] = \Pr[X - 10 \leq -9]$
$= \Pr[10 - X \geq 9]$
$= \Pr[Y \geq 9] \leq \frac{8}{9}$



(e) $\mathbb{P}[X \geq 6] \leq 9/16$.

$\Pr[X \geq 6] = \Pr[X \geq 4 + 2]$
$= \Pr[X - 2 \geq 4]$
$= \Pr[X - \mathbb{E}(X) \geq k]$
$\leq \Pr[|X - \mathbb{E}(X)| \geq k]$
$\leq \frac{\mathrm{Var}(X)}{k^2} = \frac{9}{16}$



2 Easy A's

A friend tells you about a course called "Laziness in Modern Society" that requires almost no work. You hope to take this course next semester to give yourself a well-deserved break after working hard in CS 70. At the first lecture, the professor announces that grades will depend only on two homework assignments. Homework 1 will consist of three questions, each worth 10 points, and Homework 2 will consist of four questions, also each worth 10 points. He will give an A to any student who gets at least 60 of the possible 70 points.

However, speaking with the professor in office hours you hear some very disturbing news. He tells you that, in the spirit of the class, the GSIs are very lazy, and to save time the grading will be done as follows. For each student's Homework 1, the GSIs will choose an integer randomly from a distribution with mean $\mu = 5$ and variance $\sigma^2 = 1$. They'll mark each of the three questions with that score. To grade Homework 2, they'll again choose a random number from the same distribution, independently of the first number, and will mark all four questions with that score.

If you take the class, what will the mean and variance of your total class score be? Use Chebyshev's inequality to conclude that you have less than a 5% chance of getting an A when the grades are randomly chosen this way.

3 Working with the Law of Large Numbers

(a) A fair coin is tossed multiple times and you win a prize if there are more than 60% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

HHHHTTTHHH → 70% H ✓
HHHHTTTTTH → 50% H ✗

(b) A fair coin is tossed multiple times and you win a prize if there are more than 40% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

$$\le \frac{Var(X)}{k^2}$$

(c) A fair coin is tossed multiple times and you win a prize if there are between 40% and 60% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

(d) A fair coin is tossed multiple times and you win a prize if there are exactly 50% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

Let X = # of heads in $2n$ tosses.

$$\Pr[X=n] = \binom{2n}{n}\left(\frac{1}{2}\right)^n\left(1-\frac{1}{2}\right)^{2n-n} = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}$$

Let Y = # of heads in $2n+2$ tosses.

$$\binom{2n+2}{n+1} = \frac{(2n+2)(2n+1)2n!}{(n+1)(n+1)n!\,n!} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)}\binom{2n}{n}$$

$$\Pr[Y=n+1] = \binom{2n+2}{n+1}\left(\frac{1}{2}\right)^{2n+2}$$

$$= \underbrace{\frac{(2n+2)(2n+1)}{(n+1)(n+1)}\frac{1}{2^2}}_{<1}\binom{2n}{n}\left(\frac{1}{2}\right)^{2n} < \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}$$