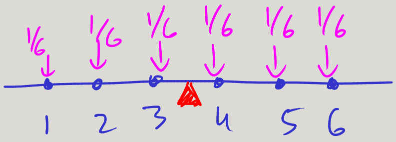
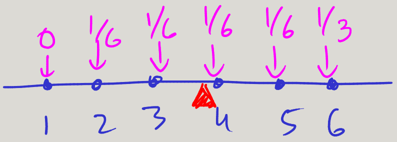
Expected value

X	$P(X)$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

$$E[X] = \sum_{x \in X} x P(X=x)$$



$$E[X] = 3.5$$



$$E[X] = 1 \cdot 0 + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{3}$$

$$= \frac{1}{3} + \frac{1}{2} + \frac{2}{3} + \frac{5}{6} + 2$$

$$= 3\frac{5}{6} = 3.833$$

Indicators

n letters   n envelopes

Expected $\overset{X}{\overset{\parallel}{\#}}$ of envelopes w/ correct letter?

$$X_i = \begin{cases} 1 & \text{if letter } i \text{ goes to the correct envelope} \\ 0 & \text{o/w} \end{cases}$$

$$X_i \sim \text{Bernoulli}(1/n)$$

$$X = \sum_{i=1}^{n} X_i$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} 1/n = n \cdot \frac{1}{n} = 1$$

$$\text{Var}(X) = E[X^2] - (E[X])^2$$

CS 70 Discrete Mathematics and Probability Theory
Summer 2020 Course Notes

Dis 5C

1 Head Count

Consider a coin with $\mathbb{P}(\text{Heads}) = 2/5$. Suppose you flip the coin 20 times, and define X to be the number of heads.

(a) Name the distribution of X and what its parameters are.

(b) What is $\mathbb{P}(X = 7)$?

(c) What is $\mathbb{P}(X \geq 1)$? Hint: You should be able to do this without a summation.

(d) What is $\mathbb{P}(12 \leq X \leq 14)$?

2 Family Planning

Mr. and Mrs. Brown decide to continue having children until they either have their first girl or until they have three children. Assume that each child is equally likely to be a boy or a girl, independent of all other children, and that there are no multiple births. Let G denote the numbers of girls that the Browns have. Let C be the total number of children they have.

(a) Determine the sample space, along with the probability of each sample point.

$\Omega = \{g, bg, bbg, bbb\}$ $\quad P(g) = 1/2,\ P(bg) = \frac{1}{4},\ P(bbg) = P(bbb) = \frac{1}{8}$

(b) Compute the joint distribution of G and C. Fill in the table below.

	$C = 1$	$C = 2$	$C = 3$	
$G = 0$	0	0	1/8	1/8
$G = 1$	1/2	1/4	1/8	7/8
	1/2	1/4	1/4	

$P(G = x \cap C = y)$

$P(G = x) = \sum_{y \in C} P(G = x \cap C = y)$

(c) Use the joint distribution to compute the marginal distributions of G and C and confirm that the values are as you'd expect. Fill in the tables below.

$\mathbb{P}(G=0)$	1/8
$\mathbb{P}(G=1)$	7/8

$\mathbb{P}(C=1)$	$\mathbb{P}(C=2)$	$\mathbb{P}(C=3)$
1/2	1/4	1/4

joint is product of marginals when indep

(d) Are G and C independent? $P(C=1, G=0) \neq P(C=1)P(G=0)$

$\overset{\shortparallel}{0}$ $\overset{\shortparallel}{1/16}$

NOT indep

(e) What is the expected number of girls the Browns will have? What is the expected number of children that the Browns will have?

$$\mathbb{E}[G] = 7/8$$

$$\mathbb{E}[C] = 7/4 = 1.75$$

3 How Many Queens?

You shuffle a standard 52-card deck, before drawing the first three cards from the top of the pile. Let X denote the number of queens you draw.

(a) What is $\mathbb{P}(X=0)$, $\mathbb{P}(X=1)$, $\mathbb{P}(X=2)$ and $\mathbb{P}(X=3)$?

$$P(X=0) = \frac{48}{52}\cdot\frac{47}{51}\cdot\frac{46}{50} = \frac{4324}{5525}$$

queens

not queens

$$P(X=1) = \frac{\#\text{ of fav. outcomes}}{\#\text{ of total outcomes}} = \frac{\binom{4}{1}\binom{48}{2}}{\binom{52}{3}} = \frac{1128}{5525}$$

$$P(X=2) = \frac{\binom{4}{2}\binom{48}{1}}{\binom{52}{3}} = \frac{72}{5525}$$

$$P(X=3) = \frac{\binom{4}{3}\binom{48}{0}}{\binom{52}{3}} = \frac{1}{5525}$$

4324
1128
5452
72
5524
1
1

(b) What do your answers you computed in part a add up to?

$$\sum_{x\in X} P(X=x) = \frac{5525}{5525} = 1$$

(c) Compute $\mathbb{E}(X)$ from the definition of expectation.

$$\mathbb{E}[X] = 1\cdot\frac{1128}{5525} + 2\cdot\frac{72}{5525} + 3\cdot\frac{1}{5525} = \frac{1275}{5525} = \frac{3}{13}$$

$$+\,0\cdot\frac{4324}{5525}$$

$$\frac{1}{13}+\frac{1}{13}+\frac{1}{13}$$

1128
144
1272
3
1275

(d) Suppose we define indicators X_i, $1 \le i \le 3$, where X_i is the indicator variable that equals 1 if the ith card is a queen and 0 otherwise. Compute $\mathbb{E}(X)$ using linearity of expectation.

$X = \#$ of queens $\quad X = X_1 + X_2 + X_3$

$\mathbb{E}[X_2 \mid X_1] \neq \mathbb{E}[X_2]$

$$\mathbb{E}[X] = \mathbb{E}[X_1+X_2+X_3] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] = \frac{1}{13}+\frac{1}{13}+\frac{1}{13} = \frac{3}{13}$$

(e) Are the X_i indicators independent? Does this affect your solution to part (d)?

$P(X_1) \neq P(X_1 \mid X_2) \Rightarrow$ NOT indep.

$4/52 = 1/13 \qquad 3/51 = 1/17$

Linearity of $\mathbb{E}$ does NOT care abt indep.

4 Linearity

Solve each of the following problems using linearity of expectation. Explain your methods clearly.

(a) In an arcade, you play game A 10 times and game B 20 times. Each time you play game A, you win with probability $1/3$ (independently of the other times), and if you win you get 3 tickets (redeemable for prizes), and if you lose you get 0 tickets. Game B is similar, but you win with probability $1/5$, and if you win you get 4 tickets. What is the expected total number of tickets you receive?

(b) A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence “book” appears?