# 1 Confidence Interval Introduction

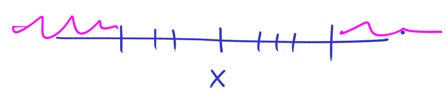
We observe a random variable $X$ which has mean $\mu$ and standard deviation $\sigma \in (0, \infty)$. Assume that the mean $\mu$ is unknown, but $\sigma$ is known.

We would like to give a 95% confidence interval for the unknown mean $\mu$. In other words, we want to give a random interval $(a, b)$ (it is random because it depends on the random observation $X$) such that the probability that $\mu$ lies in $(a, b)$ is at least 95%.

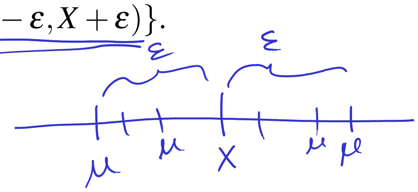
We will use a confidence interval of the form $(X - \varepsilon, X + \varepsilon)$, where $\varepsilon > 0$ is the width of the confidence interval. When $\varepsilon$ is smaller, it means that the confidence interval is narrower, i.e., we are giving a more *precise* estimate of $\mu$.

(a) Using Chebyshev's Inequality, calculate an upper bound on $\mathbb{P}\{|X - \mu| \geq \varepsilon\}$.

$$Pr[|X-\mu| \geq \varepsilon] \leq \frac{Var}{\varepsilon^2} = \frac{\sigma^2}{\varepsilon^2}$$




(b) Explain why $\mathbb{P}\{|X - \mu| < \varepsilon\}$ is the same as $\mathbb{P}\{\mu \in (X - \varepsilon, X + \varepsilon)\}$.




$|X-\mu| < \varepsilon$

$\Rightarrow -\varepsilon < X-\mu < \varepsilon$

$\Rightarrow -\varepsilon < X-\mu \ \wedge \ X-\mu < \varepsilon \Rightarrow \mu < X+\varepsilon \ \wedge \ \mu > X-\varepsilon$

(c) Using the previous two parts, choose the width of the confidence interval $\varepsilon$ to be large enough so that $\mathbb{P}\{\mu \in (X - \varepsilon, X + \varepsilon)\}$ is guaranteed to exceed 95%. [Note: Your confidence interval is allowed to depend on $X$, which is observed, and $\sigma$, which is known. Your confidence interval is not allowed to depend on $\mu$, which is unknown.]

$Pr[\mu \in (X-\varepsilon, X+\varepsilon)] > 0.95$

$Pr[\mu \notin (X-\varepsilon, X+\varepsilon)] \leq 0.05$

$Pr[\mu \notin (X-\varepsilon, X+\varepsilon)] \leq \frac{\sigma^2}{\varepsilon^2} \leq 0.05$

$$\frac{\sigma^2}{\varepsilon^2} < 0.05$$

$$\Rightarrow \varepsilon > \frac{\sigma}{\sqrt{0.05}} \approx 4.47\sigma$$

estimate of val

estimate of the std of

$(X - 4.47\sigma, X + 4.47\sigma)$

## 2 Poisson Confidence Interval

You collect $n$ samples ($n$ is a positive integer) $X_1, \ldots, X_n$, which are i.i.d. and known to be drawn from a Poisson distribution (with unknown mean). However, you have a bound on the mean: from a confidential source, you know that $\lambda \leq 2$. Find a $1-\delta$ confidence interval ($\delta \in (0,1)$) for $\lambda$ using Chebyshev's Inequality. (Hint: a good estimator for $\lambda$ is the *sample mean* $\bar{X} := n^{-1} \sum_{i=1}^{n} X_i$)

## 3 Hypothesis testing

We would like to test the null hypothesis claiming that a coin is fair, i.e. $P(H) = P(T) = 0.5$. To do this, we flip the coin $n = 100$ times. Let $Y$ be the number of heads in $n = 100$ flips of the coin. We decide to reject the hypothesis if we observe that the number of heads is less than $50 - c$ or larger than $50 + c$. However, we would like to avoid rejecting the hypothesis if it is true; we want to keep the probability of doing so less than 0.05. Please determine $c$. (*Hints: use the central limit theorem to estimate the probability of rejecting the hypothesis given it is actually true. Table is provided in the appendix.*)

$p = 0.5$

① Find dist of Y.

$Y \sim Binom(100, 0.5)$

$E[Y] = 50$

$Var(Y) = n(p)(1-p) = 25$

$\Pr[|Y - 50| > c] \leq 0.05$

$\Rightarrow \Pr[|Y - 50| \leq c] > 0.95$

② Apply CLT

$\frac{Y-50}{\sqrt{Var(Y)}} \sim N(0,1)$

$\frac{\sum_{i=1}^{100} X_i - 100 \cdot 0.5}{\sqrt{100 \cdot 0.25}} = \frac{Y-50}{5}$

$Y = \sum_{i=1}^{100} X_i$

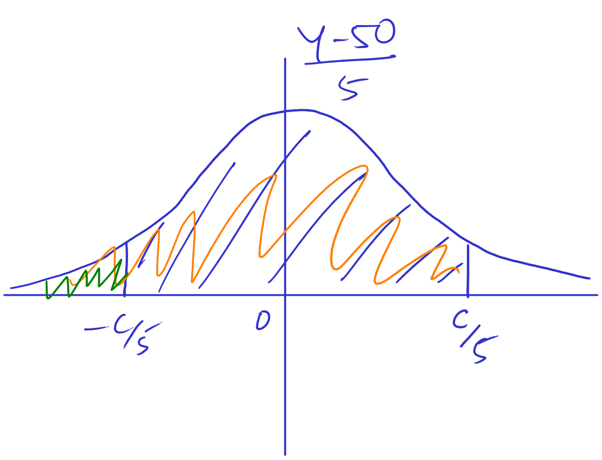
$E[X_i] = 0.5$

$Var(X_i) = 0.25$

$Var(Y) = \sum_{i=1}^{100} var(X_i)$

③ Use properties of $N(0,1)$

$\Pr[|Y - 50| \leq c] = \Pr\left[\frac{|Y-50|}{5} \leq \frac{c}{5}\right]$




$= \Pr\left[-\frac{c}{5} \leq \frac{Y-50}{5} \leq \frac{c}{5}\right]$

$= \Phi\left(\frac{c}{5}\right) - \Phi\left(\frac{-c}{5}\right)$

$N(0,1)$ symm about y-axis

$= \Phi\left(\frac{c}{5}\right) - \left(1 - \Phi\left(\frac{c}{5}\right)\right)$

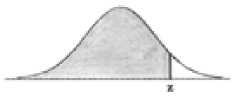
$= 2\Phi\left(\frac{c}{5}\right) - 1 \geq 0.95$

$\Rightarrow \Phi\left(\frac{c}{5}\right) \geq 0.975$

$\Rightarrow \frac{c}{5} \geq 1.96 \Rightarrow c \geq 9.8 \Rightarrow \boxed{c = 10}$

# 4 Appendix

$P[Z < 1.66] = 0.9515$




**Probability Content from -oo to Z**

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.0 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |

Table 1: Table of the Normal Distribution