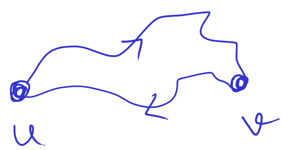
1 True or False Also, prove/give counterexample

(a) Any pair of vertices in a tree are connected by exactly one path. True.

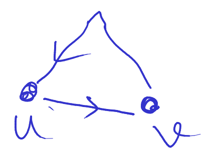


(1) connected

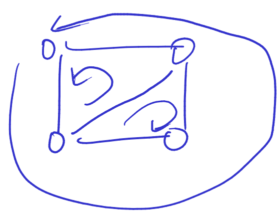
(2) no cycles

min # edges to be conn => must be only 1 path

(b) Adding an edge between two vertices of a tree creates a new cycle. True



(c) Adding an edge in a connected graph creates exactly one new cycle. False



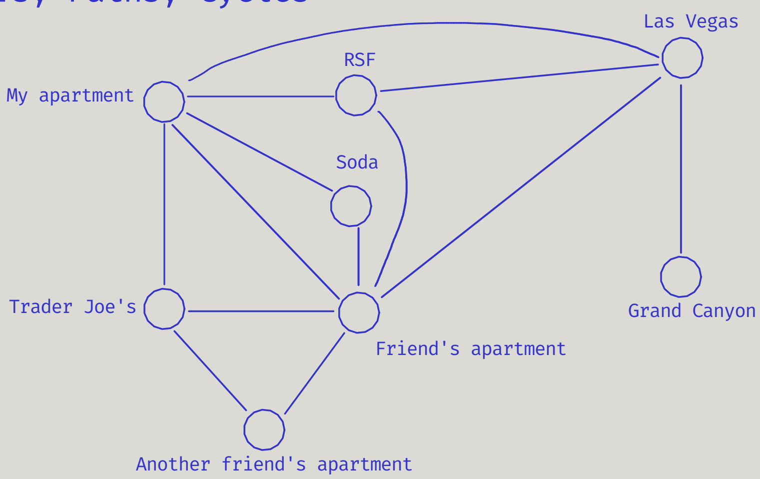
2 Bipartite Graph

A bipartite graph consists of 2 disjoint sets of vertices (say L and R), such that no 2 vertices in the same set have an edge between them. For example, here is a bipartite graph (with $L = \{$green vertices$\}$ and $R = \{$red vertices$\}$), and a non-bipartite graph.

Figure 1: A bipartite graph (left) and a non-bipartite graph (right).

Prove that a graph has no tours of odd length if it is a bipartite (This is equivalent to proving that, a graph G being a bipartite implies that G has no tours of odd length).

Walks, Tours, Paths, Cycles



Walk — pretty much anything

Tour — distinct edges, return to start

Path — no rep. vert
" " edges

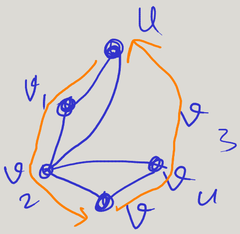
Cycle — no rep. vert
" " edges
start == end

(a) Any pair of vertices in a tree are connected by exactly one path.

Assume $\exists u, v \in V$ s.t. there are two distinct paths P_1, P_2 between u & v, where $G = (V, E)$ is a tree.

$[P_1 = (u, v_1, v_2, v), P_2 = (u, v_3, v_4, v)]$

Consider the path which starts at u, takes P_1 to v then takes the reverse of P_2 from v to U.



But this starts & ends at u, doesn't rep verts. (1)

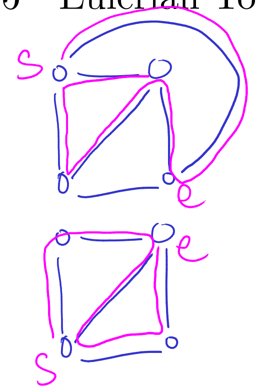
$\therefore$ there is a cycle!

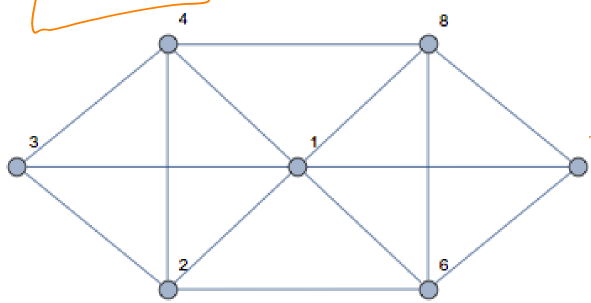
$\Rightarrow$ G is NOT a tree. This is contra!

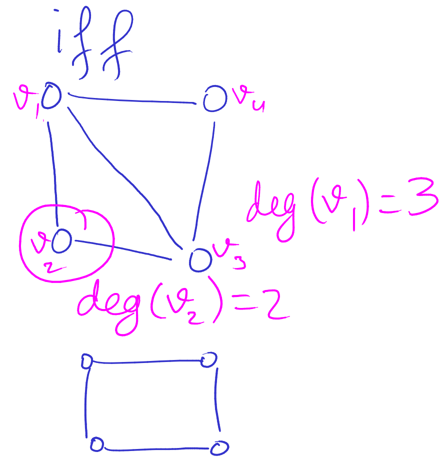
(2) P_1 & P_2 share some vert v'. Go from $v' \to v$ using P_1, $v \to v'$ using P_2.

Euler dge

3 Eulerian Tour and Eulerian Walk



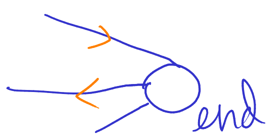


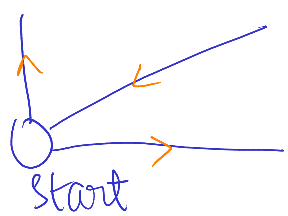


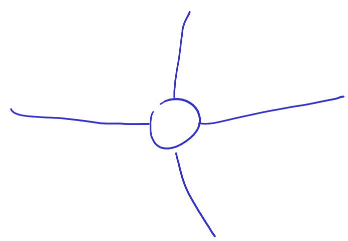
(a) Is there an Eulerian tour in the graph above? If no, give justification. If yes, provide an example. No, see thm 6.3

(b) Is there an Eulerian walk in the graph above? An Eulerian walk is a walk that uses each edge exactly once. If no, give justification. If yes, provide an example. Yes

(c) What is the condition that there is an Eulerian walk in an undirected graph? Briefly justify your answer.







4 Odd Degree Vertices

Claim: Let $G = (V,E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

Prove the claim above using:

(i) Direct proof (e.g., counting the number of edges in G). *Hint: in lecture, we proved that* $\sum_{v \in V} \deg v = 2|E|$.

(ii) Induction on $m = |E|$ (number of edges)

(iii) Induction on $n = |V|$ (number of vertices)