1 Short Answers - Graphs

(a) Bob removed a degree 3 node from an n-vertex tree. How many connected components are there in the resulting graph?

(b) Given an n-vertex tree, Bob added 10 edges to it and then Alice removed 5 edges. If the resulting graph has 3 connected components, how many edges must be removed in order to remove all cycles from the resulting graph?

2 Planarity n6, pp. 11-13

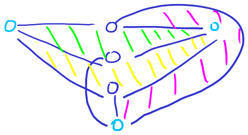
planar $\Rightarrow$ Euler, $\neg$Euler $\Rightarrow$ $\neg$Planar

K_5, $K_{3,3}$ are nonplanar

(a) Prove that $K_{3,3}$ is nonplanar.

(b) Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G, there are at least two edges among them. Use a proof by contradiction to show that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

(a)



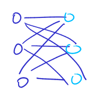
$4f \leq 2e$

$4 \times 5 = 20 \leq 2 \times 9 = 18$

$v + f = 2 + e$

$6 + f = 2 + 9$

$\Rightarrow f = 2 + 9 - 6 = 5$

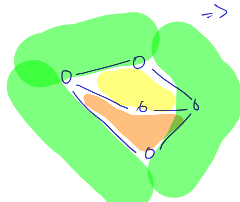


Planar $\Rightarrow$

$e \leq 2v - 4$

$\neg\ 9 \leq 2 \times 6 - 4 = 8$

$\Rightarrow$ nonplanar



face-edge adj $= 2e$

face-edge adj $\geq 4f$

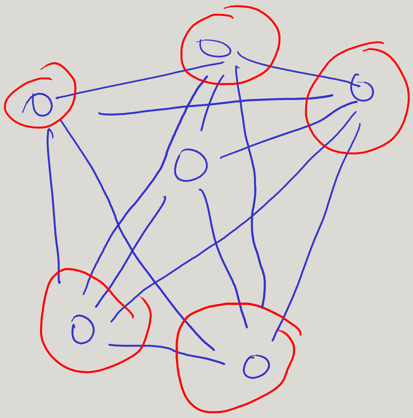
$4f \leq 2e$

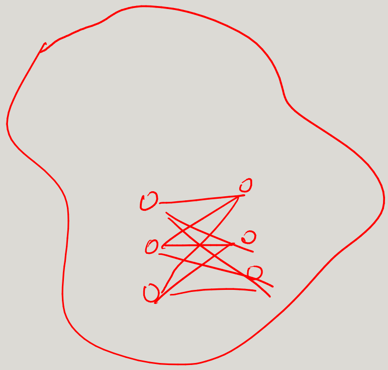
$v + f = 2 + e$

3 Graph Coloring

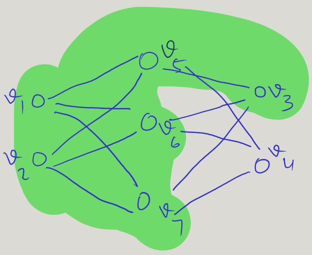
Prove that a graph with maximum degree at most k is $(k+1)$-colorable.

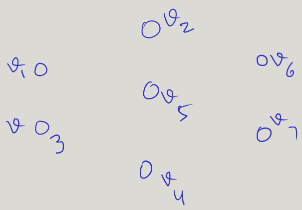
if K_5 or $K_{3,3}$ are a subgraph of G, then G is nonplanar





2.(b)





Assume $G=(V,E)$ is planar.

K_5 must NOT be a subgraph.

Consider any arbitrary set of 5 verts V_1

$\exists v_1, v_2 \in V_1 : \{v_1, v_2\} \notin E$

Consider the remaining verts $V_2 = V \setminus \{v_1, v_2\}$

$\exists v_3, v_4 \in V_2 : \{v_3, v_4\} \notin E$.

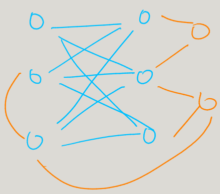
Let S be $(V \setminus V_1) \setminus V_2$

$(v_1, v_2, v) : v \in S = \{v_5, v_6, v_7\}$

$(v_3, v_4, v) : v \in S = \{v_5, v_6, v_7\}$

$\{v_1, v_5\}, \{v_1, v_6\}, \{v_1, v_7\}$ $\{v_3, v_5\}, \{v_3, v_6\}, \{v_3, v_7\}$

$\{v_2, v_5\}, \{v_2, v_6\}, \{v_2, v_7\}$



Verts v_1, v_2, v_3 don't share edges.

v_5, v_6, v_7 " " "

But v_1 is conn to v_5, v_6, v_7, same for v_2 & v_3 using the edges above, forming $K_{3,3}$ as a subgraph of G.

∴ G must be nonplanar.

4 Hypercubes

The vertex set of the n-dimensional hypercube $G = (V,E)$ is given by $V = \{0,1\}^n$ (recall that $\{0,1\}^n$ denotes the set of all n-bit strings). There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position. These problems will help you understand hypercubes.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that for any $n \geq 1$, the n-dimensional hypercube is bipartite.