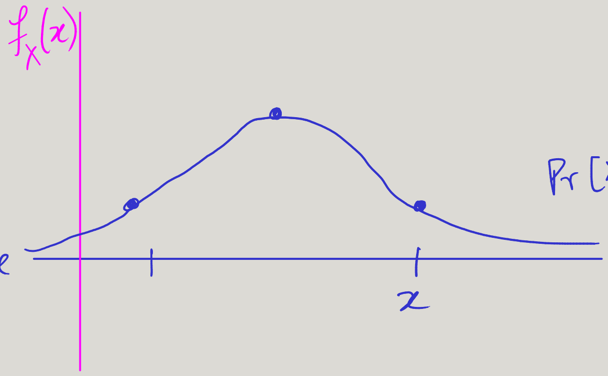
Probability density function

$X \sim \mathcal{D}$

$$f_X(x) \overset{\text{not exactly}}{=} Pr[X = x]$$

$$Pr[a \le X \le b] = \int_a^b f_X(x)\,dx$$



$Pr[X = x] = 0$

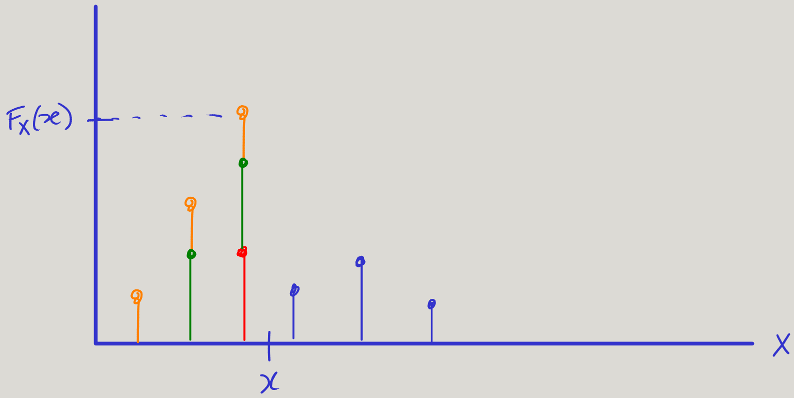
Cumulative Density func

$F_X(x)$

$$= \int_{-\infty}^{x} f_X(x)\,dx$$

$$= Pr[X < x]$$

$$= Pr[X \le x]$$



1 Condition on an Event

The random variable X has the PDF

$$f_X(x) = \begin{cases} cx^{-2}, & \text{if } 1 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases} \quad x \in [1, 2]$$

(a) Determine the value of c.

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_1^2 c x^{-2}\,dx$$

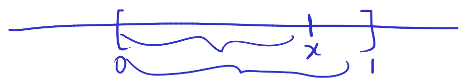
$$\int_1^2 c x^{-2}\,dx = \left.\frac{-c}{x}\right|_1^2 = \frac{-c}{2} + c = \underbrace{\frac{c}{2}}\overset{\text{set}}{=} 1 \Rightarrow c = 2$$

(b) Let A be the event $\{X > 1.5\}$. Calculate $\mathbb{P}(A)$ and the conditional PDF of X given that A has occurred.

$$\Pr[A] = \int_{1.5}^{\infty} f_X(x)\,dx = \int_{1.5}^{2} 2x^{-2}\,dx = \frac{1}{3}$$

$$f_{X|A}(x) = \frac{f_X(x)}{\Pr[A]} = \begin{cases} 6x^{-2}, & \underbrace{x \in [1.5, 2]} \\ 0, & \text{o/w} \end{cases}$$

2 Max of Uniforms



$\frac{x}{1} = x$

Let $X_1, \ldots X_n$ be independent $U[0,1]$ random variables, and let $X = \max(X_1, \ldots X_n)$. Compute each of the following in terms of n.

$U[a,b]\ f_U(x) = \frac{1}{b-a}$ $\quad U[0,1]\ f_{X_i}(x) = 1$ $\quad F_{X_i}(x) = \Pr[X_i \le x] = \int_0^x 1\,dx = x$

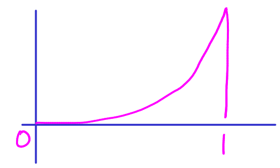
(a) What is the cdf of X? $F_X(x) = \Pr[X \le x]$

$= \Pr[\max(X_1, \ldots, X_n) \le x] = \overbrace{\Pr[X_1 \le x]}^{x} \cdots \overbrace{\Pr[X_n \le x]}^{x} = x^n$

$\Pr[(X_1 \le x) \wedge (X_2 \le x) \wedge \cdots \wedge (X_n \le x)]$

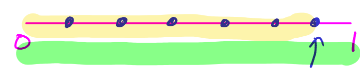
(b) What is the pdf of X?

$f_X(x) = \frac{d}{dx} F_X(x) = n x^{n-1}$



(c) What is $\mathbb{E}[X]$?

$$\int_{-\infty}^{\infty} x\, f_X(x)\,dx = \int_0^1 x \cdot n x^{n-1}\,dx = \int_0^1 n x^n\,dx = \frac{n}{n+1}$$



$\frac{n}{n+1}$

(d) What is $Var[X]$? $E[X^2]$

$$Var[X] = E[X^2] - E^2[X] = \boxed{\frac{n}{n+2} - \frac{n^2}{(n+1)^2}}$$

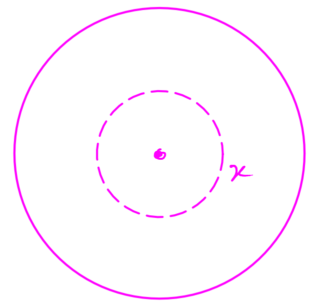
$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^1 x^2 \cdot n x^{n-1}\,dx$$

$$= \int_0^1 n x^{n+1}\,dx = \frac{n}{n+2}$$

3 Darts but with ML

Suppose Alice and Bob are playing darts on a circular board with radius 1. When Alice throws a dart, the distance of the dart from the center is uniform $[0,1]$. When Bob throws the dart, the location of the dart is uniform over the whole board. Let X the a random variable corresponding to the distance of the player's dart from center of the board.

(a) What is the pdf of X if Alice throws



$$f_{X|A}(x) = \frac{1}{1-0} = 1$$

(b) What is the pdf of X if Bob throws

$$F_{X|B}(x) = \frac{\pi x^2}{\pi 1^2} = x^2 = \Pr(X \le x)$$

$$f_{X|B}(x) = \frac{d}{dx} x^2 = 2x$$

Dist? cdf $\to$ pdf

(c) Suppose we let Alice throw the dart with probability p, and let Bob throw otherwise. What is the pdf of X (your answer should be in terms of p)?

$$F_X(x) = \Pr[X \le x] = \Pr[X \le x \cap A] + \Pr(X \le x \cap B] = \Pr[X \le x | A]\Pr[A] + \Pr[X \le x | B]\Pr[B]$$
$$= x \cdot p + x^2 \cdot (1-p)$$
$$= px + (1-p)x^2$$

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx}\left[px + (1-p)x^2\right] = \boxed{p + 2(1-p)x}$$

(d) Using the same premise as in part c, suppose you observe a dart on the board but don't know who threw it. Let x be the dart's distance from the center. We would like to come up with a decision rule to determine whether Alice or Bob is more likely to have thrown the dart given your observation, x. Specifically, if we let A be the event that Alice threw the dart and B be the event that Bob threw, we want to guess A if $\mathbb{P}[A|X \in [x, x+dx]] > \mathbb{P}[B|X \in [x, x+dx]]$ (what do these two probabilities have to sum up to?). For what values of x would we guess A? (your answer should be in terms of p)