CS 70 Discrete Mathematics and Probability Theory
Summer 2020 Course Notes

DIS 1D

1 Set Operations

- $\mathbb{R}$, the set of real numbers
- $\mathbb{Q}$, the set of rational numbers: $\{a/b : a, b \in \mathbb{Z} \wedge b \neq 0\}$
- $\mathbb{Z}$, the set of integers: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $\mathbb{N}$, the set of natural numbers: $\{0, 1, 2, 3, \ldots\}$

$\{1,2\} = \{2,1\}$

(a) Given a set $A = \{1,2,3,4\}$, what is $\mathscr{P}(A)$ (Power Set)?

$\{\{\}, \{1\}, \{2\}, \ldots, \{1,2\}, \{1,3\}, \{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}\}$

(b) Given a generic set B, how do you describe $\mathscr{P}(B)$ using set comprehension notation? (Set Comprehension is $\{x \mid x \in A\}$.)

$\mathscr{P}(B) = \{x \mid x \subseteq B\}$

(c) What is $\mathbb{R} \cap \mathscr{P}(A)$?

$\emptyset$ ℝ : Set<f64> 𝒫(A) : Set<Set<f64>>

(d) What is $\mathbb{R} \cap \mathbb{Z}$?

$\mathbb{Z}$, $\because \mathbb{Z} \subseteq \mathbb{R}$

(e) What is $\mathbb{N} \cup \mathbb{Q}$?

$\mathbb{Q}$, $\because \mathbb{N} \subseteq \mathbb{Q}$

(f) What kind of numbers are in $\mathbb{R} \setminus \mathbb{Q}$?

Irrational

(g) If $S \subseteq T$, what is $S \setminus T$?

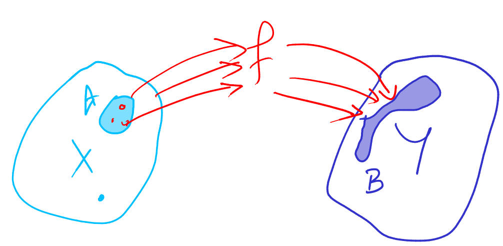
$S \cap T^c = \emptyset$

2 Image and Preimage

Let X and Y be sets, and $f : X \to Y$ be a function. For a subset, $A \subseteq X$, define it's image to be $f(A) = \{f(x) \mid x \in A\}$. For a subset $B \subseteq Y$, define it's preimage $f^{-1}(B) = \{x \mid f(x) \in B\}$. Note that in this context f^{-1} does not refer to an inverse function, as f may not have an inverse.

(a) Let $B \subseteq F(X)$. Prove that $f(f^{-1}(B)) = B$

(b) Let $A \subseteq X$. Prove that $A \subseteq f^{-1}(f(A))$



range
NOT im

$f(x) = x^2$ $f : \mathbb{R} \to \mathbb{R}$

$X = \mathbb{R}$, $Y = \mathbb{R}$

$A \subset X \quad A \subset \mathbb{R}$

$A = [1, 4] = \{x \in \mathbb{R} \mid x \geq 1 \wedge x \leq 4\}$

$f(A) = \{f(x) \mid x \in A\} = [1, 16]$

$B \subset Y, \quad B \subset \mathbb{R}$

$B = [4, 9]$

$f^{-1}(B) = [2, 3] \cup [-3, -2]$

$B = [-4, -1] = \{x \in \mathbb{R} \mid x \geq -4 \wedge x \leq -1\}$

$f^{-1}(B) = \phi$

(a) $B \subseteq f(X) \Rightarrow f^{-1}(B) \subseteq X$. Consider $x \in f^{-1}(B)$. $f(x) \in B$

$f(f^{-1}(x)) \in B \Rightarrow f(f^{-1}(B)) \subseteq B$ — 1

$B \subseteq f(f^{-1}(B))$ — 2

$B = f(f^{-1}(B))$

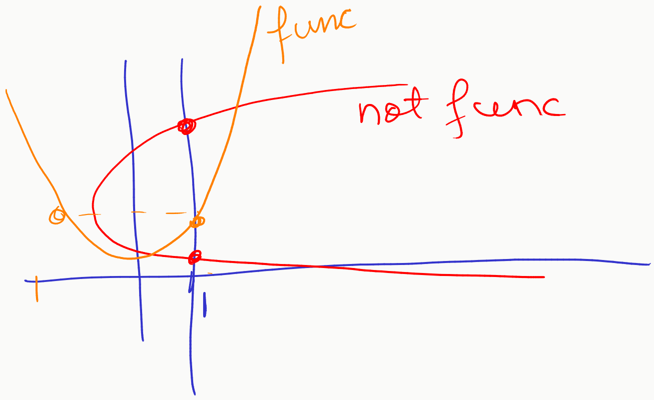
$y \in B$

$\exists x \in X : f(x) = y$

is also in $f^{-1}(B)$

$y \in f(f^{-1}(B))$

$B \subseteq f(f^{-1}(B))$



$X = Y$

$X \subseteq Y \wedge Y \subseteq X \Rightarrow X = Y$

(c) Give an example of when $A \neq f^{-1}(f(A))$

(d) Suppose f is injective. Is it true that $A = f^{-1}(f(A))$? Prove or provide a counter-example.

3 Bijections

Consider the function

$$f(x) = \begin{cases} x, & \text{if } x \geq 1; \\ x^2, & \text{if } -1 \leq x < 1; \\ 2x+3, & \text{if } x < -1. \end{cases}$$

(a) If the domain and range of f are $\mathbb{N}$, is f injective (one-to-one), surjective (onto), bijective?

(b) If the domain and range of f are $\mathbb{Z}$, is f injective (one-to-one), surjective (onto), bijective?

(c) If the domain and range of f are $\mathbb{R}$, is f injective (one-to-one), surjective (onto), bijective?