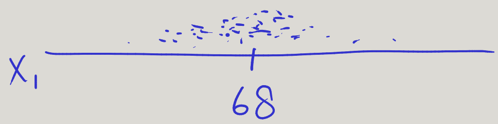
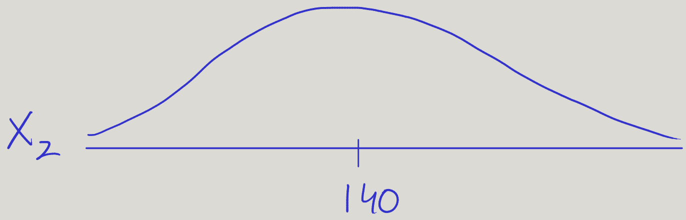
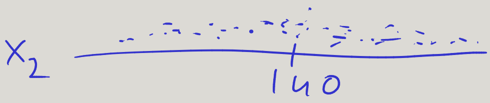
X_1 — height

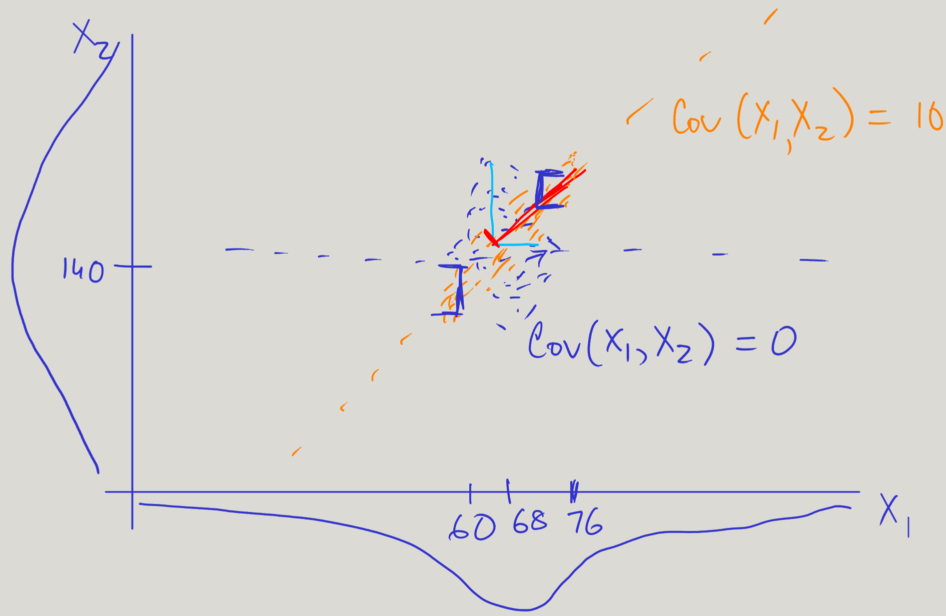
X_2 — weight







$X = (X_1, X_2)$



$$Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$$

$$(X_1 + X_2 + X_3)^2 = X_1^2 + X_2^2 + X_3^2 + X_1X_2 + X_1X_3 + X_2X_3$$
$$+ X_2X_1 + X_3X_1 + X_3X_2$$
$$+2(X_1X_2 + X_1X_3 + X_2X_3)$$

$$\left(\sum_{i=1}^{n} X_i\right)^2 = \sum_{i=1}^{n} X_i^2 + \sum_{i \neq j} X_i X_j$$

$$+2 \sum_{1 \le i < j \le n} X_i X_j$$

$X_i \sim \underline{\text{Bernoulli}}(p)$

$X_i^2 \sim \underline{\text{Bernoulli}}(p)$

$E[X_i^2] = P[X_i^2 = 1] = P[X_i = 1]$

What's dist of X_i^2?

① What vals does X_i^2 take on?

0, 1

② What are $P[X_i^2 = 0]$ & $\underbrace{P[X_i^2 = 1]}$?

$P[X_i^2 = 1] = p$

$X_i \sim Bern(p)$ $\quad$ $X_i X_j \sim Bern(\cancel{pq})$ only if $X_i \perp X_j$

$X_j \sim Bern(q)$ $\quad$ $X_i \perp X_j$

$E[X_i X_j] = P[X_i X_j = 1] = \cancel{pq}\ P[X_i = 1 \cap X_j = 1] = \left(\frac{n-2}{n}\right)^m$

what is dist of $X_i X_j$?

① Vals 0, 1

② Prob?

$X_j \backslash X_i$	0	1	
0	$(1-p)(1-q)$	$p(1-q)$	
1	$(1-p)q$	pq	$X_i X_j = 1$

$X \sim Bern(p)$

$E[X] = 0 \cdot (1-p) + 1 \cdot p$

$= p = Pr[X = 1]$

1 Ball in Bins

You are throwing k balls into n bins. Let X_i be the number of balls thrown into bin i.

(a) What is $\mathbb{E}[X_i]$?

(b) What is the expected number of empty bins?

(c) Define a collision to occur when two balls land in the same bin (if there are n balls in a bin, count that as $n-1$ collisions). What is the expected number of collisions?

2 Variance

If the random variables are independent, we could just sum up the variances individually. If not, we generally use this technique that we will show in this problem. This problem will give you practice to compute the variance of a sum of random variables that are not pairwise independent. Recall that $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

(a) A building has n floors numbered $1, 2, \ldots, n$, plus a ground floor G. At the ground floor, m people get on the elevator together, and each gets off at a uniformly random one of the n floors (independently of everybody else). What is the expected number of floors the elevator stops at (not counting the ground floor)?

$E[A_i] = P(A_i = 1)$

$A_i = \begin{cases} 1 \text{ if elevator stops at } i \\ 0 \text{ o/w} \end{cases}$

$P[A_i = 1] = 1 - \left(\frac{n-1}{n}\right)^m$

$= 1 - P[\text{no one gets out}]$

$A = \sum A_i$, $E[A] = \sum E[A_i] = \sum P(A_i = 1)$

$= n \cdot \left[1 - \left(\frac{n-1}{n}\right)^m\right]$

(b) What is the *variance* of the number of floors the elevator *does not* stop at? (In fact, the variance of the number of floors the elevator *does* stop at must be the same (make sure you understand why), but the former is a little easier to compute.)

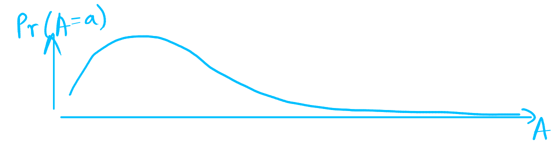
floors stop = $n -$ (# of floors where don't stop)

$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}^2[X]$

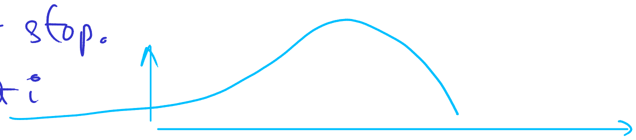
$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

$\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$

$\mathrm{Var}(A) = \mathrm{Var}((-1)A+n) = (-1)^2\,\mathrm{Var}(A)$



Let X be # of floors where don't stop.



$X_i = \begin{cases} 1 & \text{if elevator does NOT stop at } i \\ 0 & \text{o/w} \end{cases}$

$P(X_i = 1) = \mathbb{E}[X_i] = \left(\frac{n-1}{n}\right)^m$, $\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = n\left(\frac{n-1}{n}\right)^m$

$\mathbb{E}[X^2] = \mathbb{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] = \sum_{i=1}^{n}\mathbb{E}[X_i^2] + \sum_{1\le i\ne j\le n}\mathbb{E}[X_iX_j] = n\left(\frac{n-1}{n}\right)^m + 2\binom{n}{2}\left(\frac{n-2}{n}\right)^m$

$\mathrm{Var}(X) = n\left(\frac{n-1}{n}\right)^m + 2\binom{n}{2}\left(\frac{n-2}{n}\right)^m - n^2\left(\frac{n-1}{n}\right)^{2m}$

(c) A group of three friends has n books they would all like to read. Each friend (independently of the other two) picks a random permutation of the books and reads them in that order, one book per week (for n consecutive weeks). Let X be the number of weeks in which all three friends are reading the same book. Compute $\mathrm{Var}(X)$.

count

$X_i = \begin{cases} 1 & \text{if all frnds reading the same book on week } i \\ 0 & \text{o/w} \end{cases}$

$\mathbb{E}[X_i] = \Pr[X_i = 1] = 1\cdot\left(\frac{1}{n}\right)\left(\frac{1}{n}\right) = \frac{1}{n^2}$

$\mathbb{E}[X] = n\cdot\frac{1}{n^2} = \frac{1}{n} = \mu_1$

$\mathbb{E}[X_iX_j] = \Pr[X_iX_j = 1]$

$= \Pr[X_i = 1 \cap X_j = 1]$

$= 1\cdot\left(\frac{1}{n}\right)\cdot\left(\frac{1}{n}\right)\cdot 1\cdot\left(\frac{1}{n-1}\right)\left(\frac{1}{n-1}\right)$

$= \left(\frac{1}{n(n-1)}\right)^2$

$\mathbb{E}[X^2] = \sum_{i=1}^{n}\mathbb{E}[X_i^2] + \sum_{1\le i\ne j\le n}\mathbb{E}[X_iX_j]$

$= n\cdot\frac{1}{n^2} + 2\binom{n}{2}\left(\frac{1}{n(n-1)}\right)^2$

$2\binom{n}{2} = \frac{n(n-1)}{\cancel{2}\,1}$

$= \frac{1}{n} + \frac{1}{n(n-1)} = \mu_2$

$\mathrm{Var}(X) = \mu_2 - \mu_1^2 = \boxed{\frac{1}{n} + \frac{1}{n(n-1)} - \frac{1}{n^2}}$

$\frac{n}{n}\cdot\frac{1}{n}\cdot\frac{1}{n}\cdot\frac{n-1}{n-1}\cdot\frac{1}{n-1}\cdot\frac{1}{n-1}$

3 Covariance

We have a bag of 5 red and 5 blue balls. We take two balls uniformly at random from the bag without replacement. Let X_1 and X_2 be indicator random variables for the first and second ball being red. What is $\text{cov}(X_1, X_2)$? Recall that $\text{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$.