CS 70 Spring 2020 — Discrete Mathematics and Probability Theory Course Notes — DIS 1A

1 Implication

Which of the following implications are always true, regardless of P? Give a counterexample for each false assertion (i.e. come up with a statement $P(x,y)$ that would make the implication false).

(a) $\forall x \forall y P(x,y) \implies \forall y \forall x P(x,y)$. True,

(x,y) P is true

(b) $\exists x \exists y P(x,y) \implies \exists y \exists x P(x,y)$. True,

(c) $\forall x \exists y P(x,y) \implies \exists y \forall x P(x,y)$. False

$P(x,y)$: $x = y$

(d) $\exists x \forall y P(x,y) \implies \forall y \exists x P(x,y)$. True

$x=5$ $x=5$

$\forall x \exists y \exists z P(x,y,z)$

$\not\Rightarrow \exists y \forall x \exists z P(x,y,z)$

2 XOR

The truth table of XOR (denoted by $\oplus$) is as follows.

A	B	A $\oplus$ B	A $\vee$ B
F	F	F	F
F	T	T	T
T	F	T	T
T	T	F	T

FF	F	F	T	F
FT	T	T	T	T
TF	T	T	T	T
FF	F	T	F	F

1. Express XOR using only $(\wedge, \vee, \neg)$ and parentheses.

$(A \vee B) \wedge \neg(A \wedge B)$

$(A \wedge \neg B) \vee (\neg A \wedge B)$

If unicorns exist then I'm a billionaire.

2. Does $(A \oplus B)$ imply $(A \vee B)$? Explain briefly.

$(A \oplus B) \Rightarrow (A \vee B)$ True

$(A \oplus B) \Leftarrow (A \vee B)$ False

3. Does $(A \vee B)$ imply $(A \oplus B)$? Explain briefly.

3 Truth Tables

Determine whether the following equivalences hold, by writing out truth tables. Clearly state whether or not each pair is equivalent.

(a) $P \wedge (Q \vee P) \equiv P \wedge Q$

(b) $(P \Rightarrow Q) \Rightarrow R \equiv P \Rightarrow (Q \Rightarrow R)$

(c) $(P \vee Q) \wedge R \equiv (P \wedge R) \vee (Q \wedge R)$

(d) $(P \wedge \neg Q) \Leftrightarrow (\neg P \vee Q) \equiv (Q \wedge \neg P) \Leftrightarrow (\neg Q \vee P)$

4 Converse and Contrapositive

Consider the statement "if a natural number is divisible by 4, it is divisible by 2".

(a) Write the statement in propositional logic. Prove that it is true or give a counterexample.

(b) Write the inverse of the implication in English and in propositional logic. Prove that it is true or give a counterexample. (The inverse of an implication $P \implies Q$ is $\neg P \implies \neg Q$.)

(c) Write the converse of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

(d) Write the contraposative of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

(a) $\forall x \in \mathbb{N}, 4|x \Rightarrow 2|x$, always true:

Proof: $4x = 2 \cdot \underbrace{(2x)}_{\text{an int}}$ $\quad 4|x \Rightarrow \exists k \in \mathbb{Z} : 4k = x$

$\Rightarrow 2(2k) = x$

$\Rightarrow 2\ell = x$

$\ell \in \mathbb{Z}$

There exists . . .

This implies . . $\quad \Rightarrow 2|x$ ∎

(b) $\forall x \in \mathbb{N}, 4 \nmid x \Rightarrow 2 \nmid x$, False, $x = 2$

(c) If x is div by 2 then it is div by 4. $\quad Q \Rightarrow P$

$\forall x \in \mathbb{N}, 2|x \Rightarrow 4|x$, False

$x = 2$

(d) If x is not div by 2 then it is not div by 4.

$\forall x \in \mathbb{N}, 2 \nmid x \Rightarrow 4 \nmid x$. True.

$\frac{x}{2} \notin \mathbb{N}$

$\frac{x}{4} = \frac{\left(\frac{x}{2}\right) \sim \notin \mathbb{N}}{2 \sim \in \mathbb{N}} \notin \mathbb{N}$

$P \Rightarrow Q \equiv \neg P \vee Q$

$\underbrace{\neg P \Rightarrow \neg Q}_{\text{inv}} \equiv \neg(\neg P) \vee \neg Q$

$\equiv P \vee \neg Q$

$\equiv \neg Q \vee P$

$\equiv \underbrace{Q \Rightarrow P}_{\text{conv}}$

$\overbrace{P \Rightarrow Q}^{\text{orig}} \equiv \underbrace{\neg Q \Rightarrow \neg P}_{\text{contra}}$

$\neg P \vee Q$

$\equiv \neg P \vee \neg(\neg Q)$

$\equiv \neg(\neg Q) \vee \neg P$

$\equiv \underbrace{\neg Q \Rightarrow \neg P}$