Exponential (λ) fixes # of arrivals to 1, X is waiting time between two Poisson arrivals

rate

Poisson (λ)

fixes interval

X is # of arrivals

$X \sim U[a,b]$

$$f_X(x) = \begin{cases} 0, & x \leq a \\ \frac{1}{2}, & a \leq x \leq b \\ 1, & o/w \end{cases}$$

$$\frac{d}{dx} F_Y = \frac{d}{dx} \_\_\_ F_X(\_\_\_)$$

$$= \_\_ \cdot \_\_\_ f_X(\_\_\_\_)$$

1 Exponential Distributions: Lightbulbs

λ #/ time

A brand new lightbulb has just been installed in our classroom, and you know the life span of a lightbulb is exponentially distributed with a mean of 50 days.

$X \sim \text{Expo}(1/50)$ $\quad \lambda \neq 50$ days $\quad \lambda = \frac{1}{50}$ lightbulbs/day

(a) Suppose an electrician is scheduled to check on the lightbulb in 30 days and replace it if it is broken. What is the probability that the electrician will find the bulb broken?

$$\Pr[X \le 30] = 1 - e^{-\lambda \cancel{x}^{30}} = 1 - e^{-\frac{1}{50}\cdot 30} \approx \boxed{0.451}$$

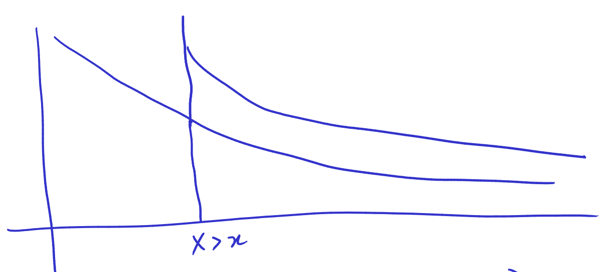
(b) Suppose the electrician finds the bulb broken and replaces it with a new one. What is the probability that the new bulb will last at least 30 days?

$$\Pr[Y > 30] = 1 - \Pr[Y \le 30] \approx \boxed{0.549}$$

(c) Suppose the electrician finds the bulb in working condition and leaves. What is the probability that the bulb will last at least another 30 days?

$$\Pr[X > 60 \mid X > 30] = \frac{\Pr[X > 60 \cap X > 30]}{\Pr[X > 30]} = \frac{e^{-\lambda \cdot 60}}{e^{-\lambda \cdot 30}} = e^{-\frac{1}{50}\cdot(60-30)} \approx \boxed{0.549}$$

$$\Pr[X > 60 \mid X > 30] = \Pr[X - 30 > 30 \mid X > 30] = \Pr[X > 30] \approx 0.549$$



$$\Pr[X > x+k \mid X > x] = \frac{e^{-\lambda(x+k)}}{e^{-\lambda x}} = e^{-\lambda(x+k-x)} = e^{-\lambda k}$$

2 Darts Again

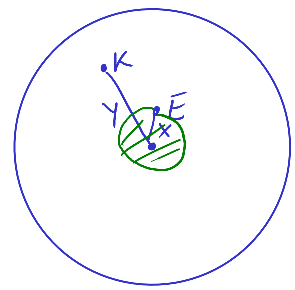
Edward and Khalil are playing darts.

Edward's throws are uniformily distributed over the entire dartboard, which has a radius of 10 inches. Khalil has good aim; the distance of his throws from the center of the dartboard follows an exponential distribution with parameter $1/2$ (Note this means that ~~Khalil~~ Khalil's dart can be arbitrarily far from the center).

Say that Edward and Khalil both throw one dart at the dartboard. Let X be the distance of Edward's dart from the center, and Y be the distance of Khalil's dart from the center of the dartboard. What is $\mathbb{P}(X < Y)$, the probability that Edward's throw is closer to the center of the board than Khalil's? Leave your answer in terms of an unevaluated integral.

[*Hint:* X is not uniform over $[0,10]$. Solve for the distribution of X by first computing the CDF of X, $\mathbb{P}(X < x)$.]

$$\Pr[X<Y] = \int_{x=0}^{x=10} \Pr[Y>x]\, \overbrace{f_X(x)\, dx}^{\Pr[X\in[x,x+dx]]}$$



$Y \sim Expo(1/2)$

$$\Pr[Y>y] = 1-F_Y(y) = 1-(1-e^{-\lambda y}) = e^{-\lambda y}$$

$$F_X(x) = \Pr[X \le x] = \frac{\pi x^2}{\pi (10)^2} = \frac{x^2}{100}$$

$$\therefore f_X(x) = \frac{d}{dx} F_X(x) = \frac{2x}{100} = \frac{x}{50}$$

$$= \boxed{\int_{x=0}^{x=10} e^{-\frac{1}{2}\cdot x} \cdot \frac{x}{50}\, dx}$$

≈ 0.0767

$Y=y \cap X<y$

$\Pr[X=x \cap Y>x]$

$\Pr[x-\varepsilon \le X \le x \cap Y>x]$

$\lim_{\varepsilon\to 0} \Pr[X\in[x+\varepsilon]]$

$= f_X(x)\cdot\varepsilon$

3 Why Is It Gaussian?

Let X be a normally distributed random variable with mean μ and variance σ^2. Let $Y = aX + b$, where $a > 0$ and b are non-zero real numbers. Show explicitly that Y is normally distributed with mean $a\mu + b$ and variance $a^2\sigma^2$. The PDF for the Gaussian Distribution is $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. One approach is to start with the cumulative distribution function of Y and use it to derive the probability density function of Y.

[1.You can use without proof that the pdf for any gaussian with mean and sd is given by the formula $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ where μ is the mean value for X and σ^2 is the variance. 2. The drivative of CDF gives PDF.]

$X \sim \mathcal{N}(\mu, \sigma^2) \qquad Y = aX + b \sim \mathcal{N}(a\mu+b, a^2\sigma^2)$, $a \geq 0$

$\text{cdf} \xrightarrow{d/dx} \text{pdf}$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{\sigma^2}\right)$$

$$\begin{aligned} F_Y(x) &= \Pr[Y \leq x] \\ &= \Pr[aX+b \leq x] \\ &= \Pr\left[X \leq \frac{x-b}{a}\right] \qquad \because a \geq 0 \\ &= F_X\left(\frac{x-b}{a}\right) \end{aligned}$$

$$\frac{d}{dx}F_Y(x) = F_X\left(\frac{x-b}{a}\right) \qquad \frac{d}{dx}\left(\frac{x-b}{a}\right) = \frac{1}{a}$$

$$= \frac{1}{a} \cdot f_X\left(\frac{x-b}{a}\right)$$

$$= \_\, f_X(\_\_\_\_)$$

$$\left(\frac{x-b}{a} - \mu\right)^2 = \frac{(x-b-a\mu)^2}{a^2}$$

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\left(\frac{x-b}{a} - \mu\right)^2 \Big/ \sigma^2\right)$$

$$= \frac{1}{\sqrt{2\pi\, a^2\sigma^2}} \exp\left(\frac{-(x-(a\mu+b))^2}{a^2\sigma^2}\right) \longrightarrow \text{pdf } \mathcal{N}(a\mu+b, a^2\sigma^2)$$

$$= \frac{1}{\sqrt{2\pi\_}} \exp\left(-\frac{(x - \_\_)^2}{\_^2}\right) \qquad e^x = \exp(x)$$