COURSE: Concrete Mathematics
THEME: chap3 integer functions
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3.22 $S_n = \sum_{k\ge1} \lfloor \frac{n}{2^k} + \frac{1}{2} \rfloor = \sum_{k\ge1}$ "距 $\frac{n}{2^k}$ 最近的整数"

n	$(n)_2$	$\lfloor \frac{n}{2} + \frac{1}{2} \rfloor$
13	1101	$110.1 \to 111^{+}$
14	1110	$111.0 \to 111$
15	1111	$111.1 \to 1000^{+}$

$n=13$

$$\begin{array}{r|l} 1101 & \\ 110 & +1 \\ 11 & \\ 1 & +1 \\ & +1 \\ \hline 1010 & +3 = 1101 \end{array}$$

$\sum_{k\ge1} \lfloor \frac{n}{2^k} + \frac{1}{2} \rfloor = \sum_{k\ge1} n \gg k + \sum$ "n二进制的位和"

$= \sum_{k\ge1} n \gg k + \sum$ "因移位而损失的1" $= n$

仅存在唯一奇数 q, $n = q \cdot 2^{k-1}$, 此时 $\lfloor \frac{n}{2^k} + \frac{1}{2} \rfloor = \lfloor \frac{q+1}{2} \rfloor$

而 $n-1$ 有 $\lfloor \frac{n-1}{2^k} + \frac{1}{2} \rfloor = \lfloor \frac{q+1}{2} \rfloor - 1$

对其他项均相同. 故 $\begin{cases} S_0 = 0 \\ S_n = S_{n-1} + 1 \end{cases}$ $S_n = n$

$\begin{cases} T_0 = 0 \\ T_n = T_{n-1} + q \cdot 2^k = T_n + 2n \end{cases}$ $T_n = n(n+1)$

3.23 第一个 m 下标 $\frac{(1+m-1)(m-1)}{2} + 1 = \frac{m(m-1)}{2} + 1 = \frac{m^2-m+2}{2}$

最后一个 m 下标 $\frac{m(m-1)}{2} + m = \frac{m^2+m}{2}$

$\forall k \in [\frac{m^2-m+2}{2}, \frac{m^2+m}{2}]$, $a_k = m$

$\forall n\ \exists m$

$$\sum_{\frac{m^2-m+2}{2} \le n \le \frac{m^2+m}{2}} [\lfloor \sqrt{2n} + \tfrac{1}{2} \rfloor = m]$$

$$= \sum_{\sqrt{(m-\frac{1}{2})^2 + \frac{7}{4}} \le \sqrt{2n} \le \sqrt{(m+\frac{1}{2})^2 - \frac{1}{4}}} [\lfloor \sqrt{2n} + \tfrac{1}{2} \rfloor = m]$$

$$= \sum_n [\underline{m - \tfrac{1}{2} < \sqrt{(m-\tfrac{1}{2})^2 + \tfrac{7}{4}} \le \sqrt{2n} \le \sqrt{(m+\tfrac{1}{2})^2 - \tfrac{1}{4}} < m + \tfrac{1}{2}}][\lfloor \sqrt{2n} + \tfrac{1}{2} \rfloor = m]$$

$$= \sum_{\frac{m^2-m+2}{2} \le n \le \frac{m^2+m}{2}} [m < \sqrt{2n} + \tfrac{1}{2} < m+1][\lfloor \sqrt{2n} + \tfrac{1}{2} \rfloor = m]$$

$$= \frac{m^2+m}{2} - \frac{m^2+m+2}{2} + 1$$

$$= m$$