COURSE: Concrete Mathematics
THEME: chap3 integer functions
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3.19 $\forall x\geq 1$ $\lfloor \log_b x\rfloor = \lfloor \log_b \lfloor x\rfloor\rfloor$ $(b>1)$ 的充要条件 观察

① b 为整数时 $b=1.8$

$\log_b x$ 为整数 → x 为整数 $x=b=1.8$ 时 $1\neq 0$

let $x=b^m+l$ $(0\leq l<b^{m-1})$ 即 $b^m\leq x<b^{m+1}$ b 取小数必然不成立

so $b^m\leq\lfloor x\rfloor = b^m+\lfloor l\rfloor < b^{m+1}$

$\therefore$ $m\leq \log_b x<m+1$

$m\leq \log_b\lfloor x\rfloor < m+1$

$\therefore$ $\lfloor\log_b x\rfloor = \lfloor\log_b\lfloor x\rfloor\rfloor = m$

② $\forall x\geq 1$ $\lfloor\log_b x\rfloor = \lfloor\log_b\lfloor x\rfloor\rfloor$ $(b>1)$ 时

$x=b$ 时 $\lfloor\log_b x\rfloor = 1$

$\lfloor\log_b\lfloor x\rfloor\rfloor\leq\lfloor\log_b x\rfloor = 1$. 当且仅当 $\lfloor x\rfloor = x$ 即 $\lfloor b\rfloor = b$ 时成立

$\therefore$ b 为整数

3.20 $$\sum_k kx[\alpha\leq kx\leq\beta] = x\sum_k k[\lceil\tfrac{\alpha}{x}\rceil\leq k\leq\lfloor\tfrac{\beta}{x}\rfloor]$$

$$= x\cdot\sum_{\lceil\frac{\alpha}{x}\rceil\leq k\leq\lceil\frac{\beta}{x}\rceil} k = x\cdot\frac{(\lceil\frac{\alpha}{x}\rceil+\lfloor\frac{\beta}{x}\rfloor)(\lfloor\frac{\beta}{x}\rfloor-\lceil\frac{\alpha}{x}\rceil+1)}{2}$$

$$= \frac{x}{2}(\lfloor\tfrac{\beta}{x}\rfloor^2-\lceil\tfrac{\alpha}{x}\rceil^2+\lfloor\tfrac{\beta}{x}\rfloor+\lceil\tfrac{\alpha}{x}\rceil)$$

$$= \frac{x}{2}(\lfloor\tfrac{\beta}{x}\rfloor(\lfloor\tfrac{\beta}{x}\rfloor+1)-\lceil\tfrac{\alpha}{x}\rceil(\lceil\tfrac{\alpha}{x}\rceil-1))$$

$$= \frac{x}{2}(\lfloor\tfrac{\beta}{x}\rfloor\lfloor\tfrac{\beta}{x}+1\rfloor-\lceil\tfrac{\alpha}{x}\rceil\lceil\tfrac{\alpha}{x}-1\rceil)$$

3.21 2^m 的十进制表示, 首位为 1

$2^0=\underline{1}$, $2^1=2$, $2^2=4$, $2^3=8$, $2^4=\underline{16}$, $2^5=32$, $2^6=64$, $\underline{2^7}=128$, $2^8=256$, $2^9=512$, $2^{10}=1024$

$2^m=\underline{10^k}+l$ $(0\leq l<10^k)$

$\lfloor\log_{10}2^m\rfloor = k$, $0\leq 2^m-10^k<10^k$

$$\sum_{0\leq m\leq M,\ k}[0\leq 2^m-10^k<10^k]$$

$$=\sum_{k,m}[0\leq m\leq M][10^k\leq 2^m<2\cdot 10^k]$$

$[k\lg 10\leq m<k\lg 10+1]$

$$=\sum_{k,m}[0\leq m\leq M][m=\lceil k\lg 10\rceil]$$

$$=\sum_k[0\leq\lceil k\lg 10\rceil\leq M] = \sum_k[-1<k\lg 10\leq M] = \lfloor\tfrac{M}{\lg 10}\rfloor-\lfloor-\tfrac{1}{\lg 10}\rfloor$$

$$=\lfloor\tfrac{M}{\lg 10}\rfloor+1$$