空间直线和圆环面的交点

空间直线用参数t表示：

$$l: (\mathrm{x_0}, \mathrm{y_0}, \mathrm{z_0}) + (\mathrm{d_x}, \mathrm{d_y}, \mathrm{d_z})t$$

圆环面用两个参数θ, α表示：

$$\mathrm{S}: (\mathrm{R}\cos\theta + \mathrm{r}\cos\alpha\cos\theta, \mathrm{R}\sin\theta + \mathrm{r}\cos\alpha\sin\theta, \mathrm{r}\sin\alpha)$$

交点处，上面两式相等：

$$\begin{cases} \mathrm{x_0} + d_x t = \mathrm{R}\cos\theta + \mathrm{r}\cos\alpha\cos\theta \\ y_0 + d_y t = \mathrm{R}\sin\theta + \mathrm{r}\cos\alpha\sin\theta \\ \mathrm{z_0} + d_z t = \mathrm{r}\sin\alpha \end{cases}$$

依次得到：

$$(\mathrm{x_0} + d_x t)^2 + \left(y_0 + d_y t\right)^2 = (R + r\cos\alpha)^2$$

$$(\mathrm{x_0} + d_x t)^2 + \left(y_0 + d_y t\right)^2 = R^2 + r^2\cos^2\alpha + 2Rr\cos\alpha$$

$$(\mathrm{x_0} + d_x t)^2 + \left(y_0 + d_y t\right)^2 = R^2 + r^2(1 - \sin^2\alpha) + 2Rr\cos\alpha$$

$$(\mathrm{x_0} + d_x t)^2 + \left(y_0 + d_y t\right)^2 = R^2 + r^2 - (\mathrm{z_0} + d_z t)^2 + 2Rr\cos\alpha$$

$$(\mathrm{x_0} + d_x t)^2 + \left(y_0 + d_y t\right)^2 + (\mathrm{z_0} + d_z t)^2 - (R^2 + r^2) = 2Rr\cos\alpha$$

$$\left(d_x^2 + d_y^2 + d_z^2\right)t^2 + 2\left(x_0 d_x + y_0 d_y + z_0 d_z\right)t + x_0^2 + y_0^2 + z_0^2 - R^2 - r^2 = 2Rr\cos\alpha$$

记做：

$$At^2 + Bt + C = 2Rr\cos\alpha$$

$$(At^2 + Bt + C)^2 = 4R^2r^2(1 - \sin^2\alpha)$$

$$(At^2 + Bt + C)^2 = 4R^2r^2 - 4R^2r^2\left(\frac{z_0^2}{r^2} + 2\frac{z_0 d_z}{r^2}t + \frac{d_z^2}{r^2}t^2\right)$$

$$A^2t^4 + 2ABt^3 + (B^2 + 2AC)t^2 + 2BCt + C^2 = 4R^2r^2 - 4R^2r^2\left(\frac{z_0^2}{r^2} + 2\frac{z_0 d_z}{r^2}t + \frac{d_z^2}{r^2}t^2\right)$$

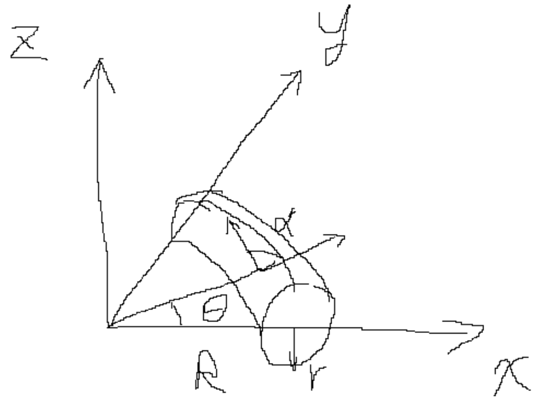
$$A^2t^4 + 2ABt^3 + (B^2 + 2AC + 4R^2d_z^2)t^2 + 2(BC + 4R^2z_0d_z)t + C^2 + 4R^2z_0^2 - 4R^2r^2 = 0$$

求解一元四次方程的根。

复数域内有根四个，实数域内有根 0~4 个，和实际情况复合。

补充：

圆环面的坐标系为



$$ax^4 + bx^3 + cx^2 + dx + e = 0 \quad (a \neq 0)$$

$$\Delta_1 = c^2 - 3bd + 12ae$$

$$\Delta_2 = 2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace$$

$$m_1 = \sqrt[3]{\Delta_2 + \sqrt{-4\Delta_1^3 + \Delta_2^2}}$$

$$\Delta_3 = \frac{\sqrt[3]{2}\Delta_1}{3am_1} + \frac{m_1}{3\sqrt[3]{2}a}$$

$$m_2 = \frac{b^2}{4a^2} - \frac{2c}{3a}$$

$$m_3 = -\frac{b^3}{a^3} + \frac{4bc}{a^2} - \frac{8d}{a}$$

$$m_4 = -\frac{b}{4a}$$

$$m_5 = \frac{1}{2}\sqrt{m_2 + \Delta_3}$$

$$m_{6a} = \frac{1}{2}\sqrt{2m_2 - \Delta_3 - \frac{m_3}{8m_5}}$$

$$m_{6b} = \frac{1}{2}\sqrt{2m_2 - \Delta_3 + \frac{m_3}{8m_5}}$$

$$\begin{cases} x_1 = m_4 - m_5 - m_{6a} \\ x_2 = m_4 - m_5 + m_{6a} \\ x_3 = m_4 + m_5 - m_{6b} \\ x_4 = m_4 + m_5 + m_{6b} \end{cases}$$