Chapter 1: Linear Algebra

↱ Rank-Nullity Theorem

(1) let $L: V \to W$. Show $\dim(\ker(L)) + \dim(\mathrm{Im}(L)) = \dim(V)$

let $\{V_1, \cdots, V_k\}$ be a basis for $\ker(L) \Rightarrow \dim(\ker(L)) = k$

Since $\ker(L)$ is a subspace of V, we extend the basis of $\ker(L)$ to a basis of the entire vector space V

$\{V_1, \cdots, V_k, U_1, \cdots, U_m\} \Rightarrow \dim(V) = k + m$

We now want to show $\{L(U_1), \cdots, L(U_m)\}$ forms a basis for $\mathrm{Im}(L)$

① Assume $c_1 L(U_1) + \cdots + c_m L(U_m) = 0$

$L(c_1 U_1 + \cdots + c_m U_m) = 0$

since $\{U_1, \cdots, U_m\}$ are l.i., $c_1 = c_2 = \cdots = c_m = 0$

$\therefore \{L(U_1), \cdots, L(U_m)\}$ are l.i.

② Any vector $W \in \mathrm{Im}(L)$ can be written as $w = L(V)$ for some $V \in V$.

let $V = a_1 V_1 + \cdots + a_k V_k + b_1 U_1 + \cdots + b_m U_m$

$w = L(V) = L(a_1 V_1 + \cdots + a_k V_k + b_1 U_1 + \cdots + b_m U_m)$

$= a_1 L(V_1) + \cdots + a_k L(V_k) + \cdots b_1 L(U_1) + \cdots + b_m L(U_m)$

$= 0 + \cdots + 0 + b_1 L(U_1) + \cdots + b_m L(U_m) = b_1 L(U_1) + \cdots + b_m L(U_m)$

$\Rightarrow \{L(U_1) \cdots, L(U_m)\}$ spans $\mathrm{Im}(L)$

Since $\{L(U_1), \cdots L(U_m)\}$ is a basis for $\mathrm{Im}(L)$, $\dim(\mathrm{Im}(L)) = m$

$\therefore \dim(\ker(L)) + \dim(\mathrm{Im}(L)) = k + m = \dim(V)$

(2) a) Show P_3 forms a vector space of dimension four

Close under addition: let $p(x)$ and $q(x)$ in P_3 where

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, $q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$

$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 \in P_3$

Close under multiplication

$c \cdot p(x) = c \cdot a_0 + c \cdot a_1 x + c \cdot a_2 x^2 + c \cdot a_3 x^3 \in P_3$

Zero: $0 \in P_3$

Additive Inverse: $\forall p(x) \in P_3$, $\exists$ a polynomial $-1 \cdot p(x)$ ${}_{\wedge}{-p(x)}$ s.t.

$p(x) + (-p(x)) = 0$

$\{1, x, x^2, x^3\}$ is a basis for P_3. $\Rightarrow \dim(P_3) = 4$

b) $\{1, x, x^2, x^3\}$

c) $D(p(x) + q(x)) = D((a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + (a_3+b_3)x^3\}$

$= (a_1+b_1) + 2(a_2+b_2)x + 3(a_3+b_3)x^2$

$= (a + 2a_2x + 3a_3x^2) + (b_1 + 2b_2x + 3b_3x^2)$

$= D(p(x)) + D(q(x))$

$D(c \cdot p(x)) = D(ca_0 + ca_1x + ca_2x^2 + ca_3x^3)$

$= c(a_1 + 2a_2x + 3a_3x^2) = c \cdot p(x)$

Thus, D is a linear transformation

d) $D(1) = 0$, $D(x) = 1$, $D(x^2) = 2x$, $D(x^3) = 3x^2$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3, i $T(0_V) = 0_W \in Im(T)$

ii. let $w_1, w_2 \in Im(T)$, $\exists$ $V_1, V_2 \in V$ s.t. $T(V_1) = W_1$, $T(V_2) = W_2$

$T(V_1 + V_2) = T(V_1) + T(V_2) = W_1 + W_2 \in Im(T)$

iii. let $c \in \mathbb{R}$, $T(cV_1) = c \cdot T(V_1) = c \cdot W_1 \in Im(T)$

4. Show $(AB)^{-1} = B^{-1}A^{-1}$

① $AB \cdot (B^{-1}A^{-1}) = A(BB^{-1}A^{-1}) = A(IA^{-1}) = A \cdot A^{-1} = I$

② $(B^{-1}A^{-1})AB = (B^{-1}A^{-1}A)B = (B^{-1} \cdot I)B = B^{-1} \cdot B = I$

$\therefore$ $B^{-1}A^{-1}$ is the inverse of AB

5. $\det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix}$

$= (2-\lambda)(5-\lambda) - 3\cdot 3 = \lambda^2 - 7\lambda + 1$

$\lambda = \dfrac{7 \pm \sqrt{49-4}}{2} = \dfrac{7 \pm 3\sqrt{5}}{2}$ $\Rightarrow$ solve for eigenvectors

6. a) Consider the set of functions $\{1, x, x^2, \dots\}$

NTS that this set is l.i.

Consider a finite linear combination: $a_0 + a_1 x + \dots + a_n x^n = 0$

$a_0 = a_1 = \dots = a_n = 0$ in order for this polynomial to be 0 $\forall x \in \mathbb{R}$

Since any finite linear combination of the set that equals 0 implies that all coefficients must be 0, the set is l.i.

$\therefore C^\infty(\mathbb{R})$ is infinite-dimensional

b) let $f, g \in C^\infty(\mathbb{R})$, $c \in \mathbb{R}$

1. $D(f+g) = \frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x) = D(f) + D(g)$
2. $D(c f) = \frac{d}{dx}(c\cdot f(x)) = c \cdot \frac{d}{dx} f(x) = c\cdot f'(x) = c \cdot D(f)$

c) $\frac{d}{dx} f(x) = \lambda \cdot f(x)$

$f(x) = C\cdot e^{\lambda x}$, $C \in \mathbb{R}$, is a general solution.

$f(x) = e^{\lambda x}$ is an eigenvector of the differentiation operator $\frac{d}{dx}$ w/ eigenvalue λ

7. $V^* = \{v^* : V \to \mathbb{R} \mid v^* \text{ is linear}\}$, let $\{v_1, \dots, v_n\}$ be a basis for V

let $v_i^*(v_j) = \delta_{ij}$, $v_i^* \in V^*$ $\Rightarrow$ NTS $\{v_1^*, \dots v_n^*\}$ is a basis V

① Assume $c_1 v_1^* + \dots + c_n v_n^* = 0$, $c_1 \dots c_n \in \mathbb{R}$

Apply this to each basis in V

$(c_1 v_1^* + \dots + c_n v_n^*)(v_j) = c_1 v_1^*(v_j) + \dots + c_n v_n^*(v_j)$

$= c_1\delta_{1j} + \dots + c_n \delta_{nj} = 0$, $c_j = 0$ $\forall j$ $\Rightarrow$ $\{v_1^*, \dots v_n^*\}$ are l.i.

② let $\phi \in V^*$ be any linear functional. $\phi(v) = \phi(\sum a_i v_i) = \sum a_i \phi(v_i)$

$= \sum \phi(v_i) v_i^*(v)$