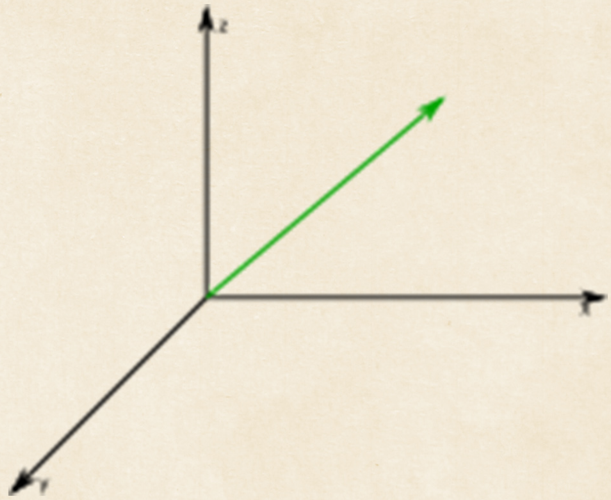
Student Guide to MATH 1410
Linear Algebra I



The student's guide to the basics of Linear Albebra

Linear systems. Vectors and matrices. Determinants. Orthogonality and applications. Vector geometry. Eigenvalues, eigenvectors, and applications. Complex numbers.

By slaying his assignment minions, defeating his midterm champion(s) and ending the final themselves. The heroic legacy of your achievement will be remembered on your academic transcript.

Disclaimer: A green dragon or a chromatic dragon of any kind, does not apologize for a failing grade.

Contents

Chapter 1: The Complex Numbers

Complex Numbers

Question(s)

1. Can we solve an equation $ax + b = 0$ where $a, b \in \mathbb{R}, a \neq 0$ for example $4x - \sqrt{7} = 0$?
2. Can we determine whether the equation $ax^2+bx+c = 0$ has any solution?

Answer(s)

1. Yes, using the quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

However, these are only real solutions if the value in the square root $b^2 - 4ac < 0$ then we have a number that doesn't exist in the real number space. We need a new space to "expand" on. This is where complex numbers come into place.

The discriminant is the equation in the square root.

This new set of numbers is call the **Complex Numbers**, they go beyond real numbers and have an *additional* dimension regarding their existence.

Definition 1.0.1. The imaginary unit

The imaginary unit i satisfies the two following properties:

1. $i^2 = -1$

2. if c is a real number with $c \geq 0$ then $\sqrt{-c} = i\sqrt{c}$

Definition 1.0.2. Complex Numbers

*A **complex number** is a number of the form $a + bi$, where a and b are real numbers and i is the imaginary unit. The set of complex number is denoted $\mathbb{C}$.*

This means all real numbers are complex numbers where the imaginary unit is zero.

Definition 1.0.3. Equality of complex numbers

Let $z = a+ib$ and $w-c+id$ be two complex numbers. We say that $z = w$ if and only if $a = c, b = d$.

Definition 1.0.4. Addition of complex numbers

*Given two complex numbers $z = a+ib$ and $w = c+id$, we define their **sum** to be complex number given by*

$$z + w = (a + c) + i(b + d)$$

.

Definition 1.0.5. Multiplication of complex numbers

*Given two complex numbers $z = a+ib$ and $w = c+id$, we define their **product** to be the complex number*

$$zw = (ac - bd) + i(ad + bc)$$

Note

This form $a + bi$ is called the *rectangular form* where $a, b \in \mathbb{R}$. a is the real part and b is the imaginary part.

Theorem 1.0.1. Properties of the Complex Conjugate

Let z and w be complex numbers.

- $z = a + bi$ then $\bar{z} = a - bi$
- $\bar{\bar{z}} = z$
- $\overline{z + w} = \bar{z} + \bar{w}$
- $\overline{zw} = \bar{z}\bar{w}$
- $\overline{z^n} = (\bar{z})^n$, *for any natural number n*
- *z is a real number if and only if* $\bar{z} = z$

Theorem 1.0.2. Properties of Complex Arithmetic

So complex numbers have the same properties as real numbers, you can distribute them, foil them, add them, subtract them, and what not and you will still get a number that is in the complex plane. In fact you can do more with complex numbers, you can represent a number where a square root has a negative number in it.

Polar Coordinates

We can use two number real numbers to represent a complex number. One for the real part, another for the imaginary part. However, we can use angles and distances to represent the same thing.

Definition 1.0.6. Polar Coordinates

Let z be a complex number that is not zero

1. *Modulus of z.*

   $$|z| = \text{is the distance between 0 and z.}$$

2. *Principle argument*

   $$\arg(z)$$

   is the angle (in radians) between the real axis and the line between 0 and z. The range of the principle argument is $-\pi \leq \arg(x) \leq \pi$.

3. *arguments of z Because there are multiple angles that represent the same principle angle or argument the arguments of z are all the angles that can represent the principle argument of z.*

   i.e., $\arg(z) = \{\arg(z) + 2k\pi | k \in \mathbb{R}\}$.

4. *Euler's formula*

   $$e^{i\theta} = \cos(\theta) + (\sin(\theta))i$$

5. *Polar form of z*

   $$z = |z|e^{i\theta}$$

Note

It is helpful to know the unit circle at this point.

Theorem 1.0.3. Conversion Between Rectangular and Polar Coordinates

Suppose P is represented in rectangular coordinates as (x, y) and in polar coordinates as (r, θ). Then

- $x = r\cos(\theta)$ *and* $y = r\sin(\theta)$
- $x^2 + y^2 = r^2$ *and* $\tan\theta = \frac{y}{x}$ *provided that* $x \neq 0$

The benefit of polar form is that it is easier to multiply complex numbers than in rectangular form. You must multiply the bases and add the powers for multiplication, or divide the bases and subtract the powers for division. Basically, the power rules from real numbers apply.

Definition 1.0.7. Complex n^{th} roots

Let z and w be complex numbers. If there is a natural number m such that $w^n = z$, then w is an n^{th} ***root*** *of z.*

Theorem 1.0.4. L

et $z \neq 0$ be a complex number with polar form $z = re^{i\theta}$. For each natural number n, z has n distinct n^{th} roots, which we denote by $w_0, w_1, \ldots, W_{n-1}$, and they are given by this formula

$$w_k = r^{\frac{1}{n}} e^{\left(\frac{\theta + 2\pi k}{n}\right)}$$

where $k = 1, 2, \ldots, n$

Chapter 2: Vectors

Introduction to Vectors

Vectors are basically maths way of describing direction and magnitude of something. It can be used to describe winds within a system. Aircraft flight directions, etc.

You can think of it as directed line with arrows

Definition 2.0.1. Vector

*A **Vectors** is a directed line segment.*

*Given point P and Q (either in the plane or in space), we denote with $\vec{PQ}$ the vector from P to Q. The point P is said to be the **initial point** of the vector, and the point Q is the **terminal point**.*

*The **magnitude, length, or norm** of $\vec{PQ}$ is the length of the line segment denoted $\bar{PQ} : ||\vec{PQ}|| = ||\bar{PQ}||$.*

*Two vectors are **equal** if they have the same magnitude and direction.*

A vector has a length and a direction, it can also be denoted as this $\vec{V}$

Definition 2.0.2. Component Form of a Vector

1. *The **component from** of a vector $\vec{v}$ in $\mathbb{R}^2$, whose terminal point is (a, b) when its initial point is $(0, 0)$, is $< a, b >$.*
2. *The **component form** of a vector $\vec{v}$ in $\mathbb{R}^3$, whose terminal point is (a, b, c) when its initial point is $(0, 0, 0)$, is $< a, b, c >$.*

*The numbers a, b (and c, respectively) are **components** of $\vec{v}$*

So for example in $\mathbb{R}^2$ if points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ form the vector $\vec{PQ} =< x_2 - x_1, y_2 - y_1 >$. $\mathbb{R}^3$ is the same but with a third component instead of two.

Definition 2.0.3. Magnitude of a vector

$$\bar{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + \cdots + (z_2 - z_1)^2}$$

Its basically the euclidean distance of the components, you find the square root of the sum of the difference squared of the individual components of the vector.

Definition 2.0.4. Algebra of vectors

1. *Addition and Subtraction: Given two vectors $\vec{u} =< u_1, u_2, \ldots, u_n >$ and $\vec{v} =< v_1, v_2, \ldots, v_n >$.*

$$v + u =< u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n >$$

$$v - u =< u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n >$$

You add and subtract component-wise.

2. *Scalar multiplication: If you have a vector in the form $c\vec{v}$ where $c \in \mathbb{R}$ then*

$$c\vec{v} =< cv_1, cv_2, \ldots, cv_n >$$

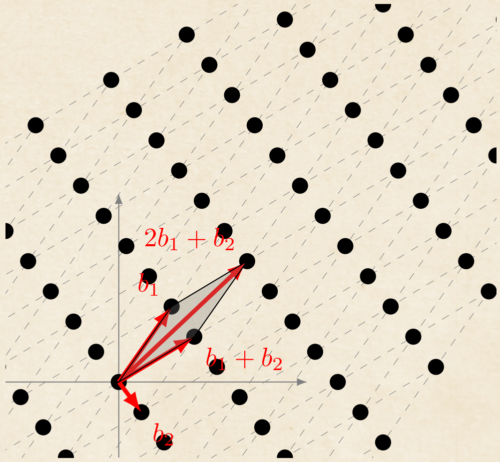
The scalar multiplication also multiplies the magnitude of the vector by the same amount as the scalar

The magnitude of a vector is 0 if and only if $\vec{v} =< 0, 0 >= \vec{0}$

Proof. $\Rightarrow$: If $\vec{v} = 0$, then the magnitude of the vector is zero. Its a point. $\Leftarrow$: If $\vec{v} =< v_1, v_2, \ldots, v_n$ and the magnitude of the vector is zero. Then.

$$\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = 0$$
$$v_1^2 + v_2^2 + \cdots + v_n^2 = 0$$
$$\vec{v} = 0$$

□



Definition 2.0.5. Unit vector

Is a vector whose magnitude is 1.

$$||\vec{v}|| = 1$$

The standard unit vector for $\mathbb{R}^3$ is $\vec{i} =< 1, 0, 0 >$, $\vec{j} =< 0, 1, 0 >$, $\vec{k} =< 0, 0, 1 >$. $\mathbb{R}^2$ uses the same as $\mathbb{R}^3$ just without $\vec{k}$ and one less component.

Theorem 2.0.1. Unit vector of any vector

To get the unit vector of any vector can be achieve using this formula

$$\frac{1}{||\vec{v}||}\vec{v}$$

Dot Product

The previous section introduced vectors and described how to add them together and how to multiply them by scalars. This section introduces a multiplication on vectors called **dot product**

Definition 2.0.6. Dot Product

Let $\vec{u} =< u_1, u_2, \ldots, u_n >$ *and* $\vec{v} =< v_1, v_2, \ldots, v_n >$.

$$\vec{u} \cdot \vec{v} =< u_1 \cdot v_1, u_2 \cdot v_2, \ldots, u_n \cdot v_n >$$

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \cdot ||\vec{v}|| \cos\theta$$

where $\theta,\ 0 \leq \theta < \pi$, *is the angle between* $\vec{u}$ *and* $\vec{v}$

Definition 2.0.7. Orthoognal

Vectors $\vec{v}$ *and* $\vec{u}$ *are* ***orthogonal*** *if their dot product is 0.*

Definition 2.0.8. Orhtogonal Projection

Let $\vec{u}$ *and* $\vec{v}$ *be given. The* ***orthogonal projection of*** $\vec{u}$ ***and*** $\vec{v}$, *denoted* $\text{proj}_{\vec{v}}\vec{u}$, *is*

$$\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right)\vec{v}$$

The projection is like a shadow that is cast upon one vector to another. If you have the projection of two vectors than you can figure the vector that is perpendicular to the projected vector and whose sum with the projected vector will give you the original vector that provided the projection

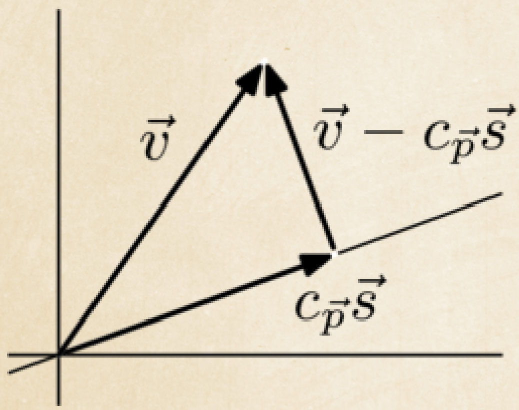


Figure 2.1: Vector Projection

Orthogonal Decomposition of Vectors

Let $\vec{u}$ and $\vec{v}$ be given. Then $\vec{u}$ can be written as the sum of two vectors, one of which is parallel to $\vec{v}$, and one of which is orthogonal to $\vec{v}$.

$$\vec{u} = \text{proj}_{\vec{v}}\vec{u} + (\vec{u} - \text{proj}_{\vec{v}}\vec{u})$$

- $\text{proj}_{\vec{v}}\vec{u}$ is the vector that is parallel with $\vec{v}$.
- $\vec{u} - \text{proj}_{\vec{v}}\vec{u})$ is the vector that is perpendicular with $\vec{v}$.

This is important when we are trying to find distances between planes and lines.

Cross Product

If we want to find the vector that is perpendicular or orthogonal to two vectors then we can use an operation that is called the **Cross Product** to create such a vector that is orthogonal to both of those vectors.

Definition 2.0.9. Cross Product

Let $\vec{u} =< u_1, u_2, u_3 >$ *and* $\vec{v} =< v_1, v_2, v_3 >$ *be vectors in* $\mathbb{R}^3$. *The* ***Cross Product*** *of* $\vec{u}$ *and* $\vec{v}$*m, denoted* $\vec{u} \times \vec{v}$, *is the vector*

$$\vec{u} \times \vec{v} =< u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 >$$

$$||\vec{u} \times \vec{v}|| = ||\vec{u}|| \cdot ||\vec{v}|| \cdot \sin\theta$$

$$0 \leq \theta < \pi$$

Application

The cross product is useful in a few areas.

Area of a parallelogram

Given two vectors $\vec{u}$ and $\vec{v}$ where they have the same origin point. You can calculate the are of a parallelogram by using the cross product.

$$A = ||\vec{u} \times \vec{v}||$$

$$A = ||\vec{u}|| \cdot ||\vec{v}|| \cdot \sin\theta$$

The magnitude of the resulting vector is the area of the parallelogram.

This idea is important when we want to figure out the volume of a parallelepiped

Area of a triangle

Because a parallelogram is basically two triangle of the same area together, we can use the same formula to calculate the area of a triangle

$$A = \frac{||\vec{u} \times \vec{v}||}{2}$$

$$A = \frac{||\vec{u}|| \cdot ||\vec{v}|| \cdot \sin\theta}{2}$$

Volume of a Parallelepiped

A Parallelepiped is a parallelogram that is 3-D. The volume of a 3-D object can be generalized as this $V = A \cdot H$ essentially. Well we know what the area of our base is. We just need the height. However, the height is just the projection of our area of the parallelogram (the vector that is perpendicular to the two vectors used to describe the parallelogram) with the third vector used to turn our parallelogram into a parallelepiped (give it depth/height).

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$$

Lines

So a vector is basically a line that is limited. Really a line is a scalar factor of a vector whose factor is infinity.

Definition 2.0.10. Equations of Lines in a Space

Consider the line in space that passes through $\vec{p} =< x_0, x_1, x_2$ *in the direction of* $\vec{d} =< a, b, c >$.

1. *The* ***vector equation*** *of the line is*

$$\ell(t) = \vec{p} + t\vec{d}$$

2. *The* ***parametric equations*** *of the line are:*

   $x = x_0 + at,\ \ y = y_0 + bt,\ \ z = z_0 + ct.$

3. *The* ***symmetric equations*** *of the line are:*

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Comparison of Lines

There are 4 different outcomes when we are comparing lines

Equal

Every point from one line is shared with the point from the other

Parallel

The direction vectors of the two lines are the same but they don't share any points

Intersection

They share only one common point.

Skew Lines (only exist in $\mathbb{R}^3$)

The direction vectors of the two lines are different and they don't share any points.

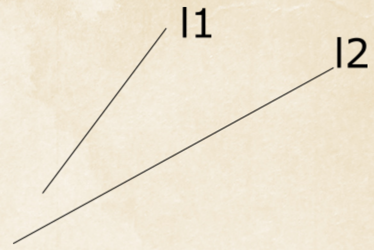


Figure 2.2: Skew Lines

Distance to a line

1. Distance from a point to a line The distance is the magnitude of the vector that is orthogonal to the line. Therefore, we need to find the projection of the vector of point PQ onto the direction vector of the line. Then subtract the vector PQ with the projected vector to get the orthogonal vector from the point to the line. The magnitude of that vector is the distance.

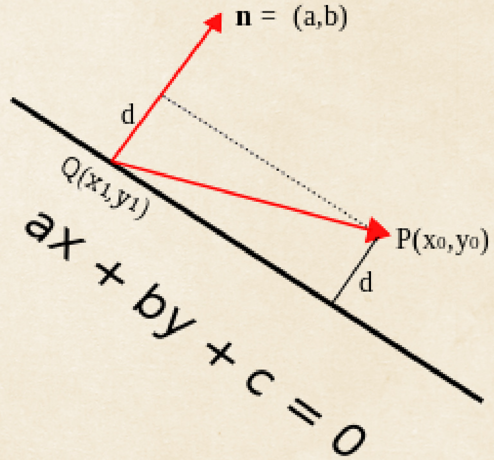


Figure 2.3: Distance from a line to a point

2. Distance between two parallel lines Is basically the same as the distance from a point to a line, just use the initial point of one of the lines.

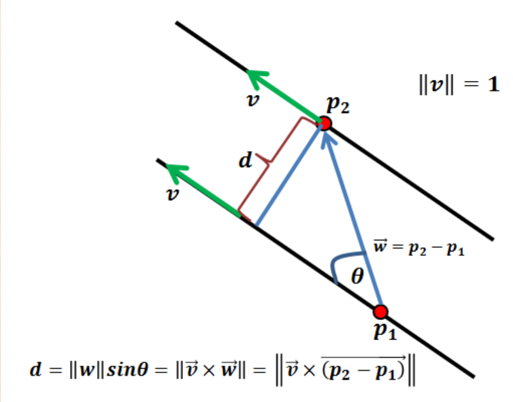


Figure 2.4: Distance from a line to a line

3. Distance between skew lines We need to find a vector that is orthogonal to both direction vectors. We can find this by using the cross product operation. Once we find the orthogonal vector, we use two initial point from each line create a vector where we can project onto the orthogonal vector to get vector where the termination point and the initial point lies on the two lines and its magnitude tells use the distance

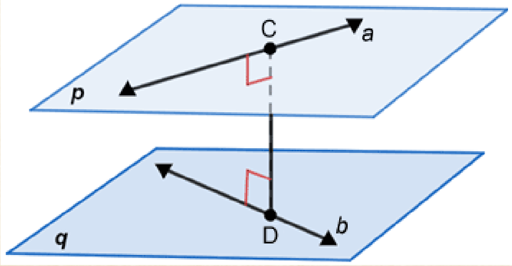


Figure 2.5: Distance from a line to a line skewed

Note

Skew lines always lie in parallel planes. In the next section we'll see that a plane can be determined by two non-parallel direction vector and a point on the plane. The distance between two skew lines is then equal ot the distance between the two parallel planes, which is given by the length of a line segment perpendicular to both planes, and therefore, to both lines.

Planes

Any flat surface can be considered a plane. But how do we describe such a thing in mathematics? Well, and two lines that lie on the plane can have a common vector that is perpendicular to both of them using the cross product. In fact, this "common" vector is perpendicular to all lines that lie on the plane. This "common" vector is called the **normal vector**. It is the vector that shows the direction of the face of the plane.

Definition 2.0.11. Equation of a Plane in Standard and General Forms

The plane passing through the point $P = (x_0, y_0, z_0)$ with normal vector $\vec{n} =< a, b, c >$ can be describe by an equation with ***standard form***

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0;$$

the equation's ***general form*** *is $ax + by + cz = d$.*

To find the normal vector of a plane with multiple points choose a point, get the vector from that point to all other points, then use the cross product.

Comparing planes

- intersection with a plane and a line
  - either the line and the plane is parallel in which case there is no solution
  - or it intersect at a point in which case there is only one solution.
- intersection with a plane and a plane
  - either the two planes do not intersect so they are parallel, i.e., there is no solution when solving.
  - or it intersect and creates a line along the intersection. i.e., you get a line.

Distances

Show an illustration

Chapter 3: Systems of Linear Equation

Introduction to linear equations

Definition 3.0.1. Linear Equation

A linear equations is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$$

where the x_i are variables (the unknowns), the a_i are the coefficients, and c is the constant.

A system of linear equations is a set of linear equations that involve the same variables.

$$\begin{cases} x_1 + x_2 = -1 \\ 2x_1 - 3x_2 = 8 \end{cases}$$

A solution to a system of linear equations is a set of values for the variables x_i such that each equation is the system is satisfied.

e.g.

$$\begin{cases} x_1 = 1 \\ x_2 = -2 \end{cases}$$

Using Matrices to Solve System of Linear Eqs

We convert the system of linear equations into a table called the **Augmented Matrix**.

Idea

Instead of writing equations we "play or do operations" directly on rows of the augmented matrix.

For example we have this system of linear equation:

$$\begin{cases} -2x + 3y = 2 \\ -x + y = 1 \end{cases}$$

$$\begin{gathered} y = 1 + x \\ -2x + 3(1 + x) = 2 \\ -2x + 3 + 3x = 2 \\ x + 3 = 2 \\ x = -1 \\ y = 0 \end{gathered}$$

In this case we used the substitution method to solve this system of linear equation. However, this can get unwieldy.

The augmented matrix form of this system of linear equation is such

$$\left[\begin{array}{cc|c} -2 & 3 & 2 \\ -1 & 1 & 1 \end{array}\right]$$

Goal:

1. row operations allowed
2. to obtain a matrix where each row has one 1 and everything else is a zero, basically we want an identity matrix.
3. Gaussian Elimination our method of getting 2.
4. Solution

Elementary row operations and Gaussian Elimination

Definition 3.0.2. Elementary row Operations

1. *Multiplication of a scalar*
2. *Swap rows*
3. *Add a row to a multiple of another one i.e., $cR_1 + R_2 \to R_2$*

Definition 3.0.3. Reduced Row Echelon Form Matrix

Satisfiers the following condition

1. *The 1st non zero entry in each row is a 1.*
2. *Each leading 1 comes in a column on the right of all the leaning 1's above it.*
3. *Rows of all zeros (when they exist) it has to be one the bottom of the matrix.*
4. *If the column has a leading 1 then all other entries are zeros.*

To put a matrix into RREF (Reduced Row Echelon Form) we

1. Create a leading 1
2. Use the leading 1 to put zeros underneath the leading one.

3. repeat until all possible rows have a leading 1, with the exception of all zeros being at the bottom.

4. Put zeros above leading ones.

Existence and uniqueness of solutions

Theorem 3.0.1. Existence and Uniqueness

Every system of linear equations has exactly zero, one, or infinitely many solutions. If a solution has 1 or infinitely many solution(s) then the system is consistent else it is inconsistent.

When the number of leading 1's less than the number of variables, then this leads to an equation with infinitely many solutions. We let one of these variables to be a *free variable* which means we can choose any value.

Here is an example of a system of equations that has one solution:

$$\begin{cases} x_1 + x_2 = 3 \\ 2x_1 + x_2 = 4 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 3 \end{array}\right]$$

$$R_2 - R_1 \to R_2$$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 0 & 1 \end{array}\right]$$

$$R_1 - R_2 \to R_1$$

$$\left[\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & 1 \end{array}\right]$$

$$R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

$$x_1 = 1$$

$$x_2 = 2$$

Geometrically these lines intersect at a point. They are also on the same plane. Here is an example with infinitely many solutions:

$$\begin{cases} x_2 - x_3 = 3 \\ x_1 + 2x_3 = 2 \\ -3x_2 + 3x_3 = -9 \end{cases}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 3 \\ 1 & 0 & 2 & 2 \\ 0 & -3 & 3 & -9 \end{array}\right]$$

$$RREF$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 + 2x_3 = 2 \\ x_2 - x_3 = 3 \end{cases}$$

x_3 is a "free variable

$$x_1 = 2 - 2t$$

$$x_2 = 3 + t$$

$$x_3 = t$$

So for any given $t \in \mathbb{R}$ will product three variables that will solve this system of linear equations.

Geometrically these planes intersect into a line.

Here is an example with no solution:

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + 2x_2 + x_3 = 2 \\ 2x_1 + 3x_2 + 2x_3 = 0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 0 \end{array}\right]$$

$$RREF$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 = 0 \\ 0 = 1 \end{cases}$$

$$0 \neq 1$$

Therefore, no solution exist for this system of linear equations.

Geometrically these three planes do not intersect.

Chapter 4: Matrix Algebra

Matrix Addition and Scalar Multiplication

A matrix is a rectangular array of numbers like this:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Definition 4.0.1. Matrix

A matrix is a rectangular array of numbers.

The horizontal lines of numbers form rows and the vertical lines of numbers form columns. A matrix with m rows and n columns is said to be an $m \times n$ matrix ("an m by n matrix, or a matrix of size $m \times n$).

The entries of an $m \times n$matrix are indexed as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

That is, a_{32} means "the number in the third row and second column." To save space, we will use the short hand $A = [a_{ij}]$ to denote a matrix A with entries a_{ij}. If we need to specify the size, we can also write $A = [a_{ij}]_{m \times n}$.

Definition 4.0.2. Matrix Equality

*Two $m \times n$ matrices A and B are **equal** if their corresponding entires are equal.*

Definition 4.0.3. Matrix Addition

*Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be an $m \times n$matrices. The **sum** of A and B, denoted $A + B$, is*

$$\begin{Bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{Bmatrix}$$

Definition 4.0.4. Scalar Multiplication

*Let $A = [a_{ij}]$ be an $m \times n$matrices and let k be a scalar. The **scalar multiplication** of A by k, denoted kA, is defined by*

$$\begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

Definition 4.0.5. Linear Combination

*Given $m \times n$ matrices $A_1, A_2, \dots, A_k$, a **linear combination** of these matrices is any expression of the form.*

$$B = c_1A_1 + c_2A_2 + \cdots + c_kA_k,$$

where $c_1, c_2, \dots, c_k$ are scalars.

Matrix Multiplication

Definition 4.0.6. Matrix Multiplication

*Let A be an $m \times r$ matrix, and let B be an $r \times n$ matrix. The **matrix product of A and B**, denoted $A \cdot B$, or simply AB, is the $m \times n$ matrix M whose entry in the i^{th} row and j^{th} column is the product of the i^{th} row of A and j^{th} column of B.*

$$[m_{ij}]_{m \times n} = \sum_{k=1}^{r} [a_{ik}][b_{kj}]$$

Note

This also means that the matrix multiplication is not commutative.

Theorem 4.0.1. Properties of Matrix Multiplication

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $k(AB) = (kA)B$
4. $IA = AI = A$

Definition 4.0.7. Identity Matrix

Is a $n \times n$ matrix whose entries are 1's on the diagonal and zero everywhere else.

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solving Matrix Equation $AX = B$

To solve a matrix equation $AX = B$ where A and B are given matrices.

1. Write the augmented matrix $[A|B]$
2. Use Gaussian Elimination to put the matrix into RREF

3.

(a) if the RREF is $[I|C]$. Then C is the unique solution of the equation $AX = B$

(b) If RREF is of the form $[R|C]$ where R is not the the identity and R has at least one row of zeros, then $AX = B$ has no solution or has infinitely many solutions.

The Matrix Inverse

Definition 4.0.8. Invertible Matrices and the Inverse of A

*We say that an $n \times n$ matrix A is **invertible** if thee exists a matrix X such that*

$$AX = XA = I_n$$

*When this is the case, we call the matrix X the **inverse** of A and write $X = A^{-1}$.*

This leads to the interesting conclusion where if we can solve $AX = I$ then there is a unique solution and it also solves $XA = I$. Then X is the inverse matrix of A so A^{-1}

Finding A^{-1}

Let A be an $n \times n$ matrix. To find A^{-1}, put the augmented matrix

$$[A|I_n]$$

into RREF. If the result is the form

$$[I_n|X],$$

then $A^{-1} = X$. If not, (that is, if the first n columns of the RREF are not I_n), then A is ot invertible.

Theorem 4.0.2. The Inverse of a 2×2 Matrix

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

A is invertible if and only if $ad - bc \neq 0$.

if $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Properties of the Matrix Inverse

Theorem 4.0.3. Invertible Matrix Theorem

Let A be an $n \times n$ matrix. The following statements are equivalent.

1. *A is invertible*
2. *The equation $A\vec{x} = \vec{0}$ has exactly one solution (namely, $\vec{x} = \vec{0}$).*
3. *The RREF of A is I.*
4. *The equation $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ vector $\vec{b}$.*
5. *There exist a matrix C such that $AC = I$.*
6. *There exist a matrix B such that $BA = I$.*

Theorem 4.0.4. Properties of Invertible Matrices

Let A and B be an $n \times n$ invertible matrices. Then:

1. *AB is invertible; $(AB)^{-1} = B^{-1}A^{-1}$.*
2. *A^{-1} is invertible; $(A^{-1})^{-1} = A$.*
3. *nA is invertible for any nonzero scalar n; $(nA)^{-1} = \frac{1}{n}A^{-1}$.*
4. *If A is a diagonal matrix, with diagonal entries $d_1, d_2, \ldots, d_n$, where none of the diagonal entries are 0, then A^{-1} exists and is a diagonal matrix. Furthermore, the diagonal entires of A^{-1} are $1/d_1, 1/d_2, \ldots, 1/d_n$.*

Furthermore,

1. *If a product AB is not invertible, then A or B is not invertible.*
2. *If A or B are not invertible,m then AB is not invertible.*

Elementary Matrices

Definition 4.0.9. Elementary Matrix

*An **elementary matrix** is an $n \times n$ matrix E that can be obtained from the identity matrix using a single row operation.*

Examples

$$E_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad R_1 \to R_1 + 2R_2 \tag{4.1}$$

$$E_2 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \quad 4R_1 \to R_1 \tag{4.2}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3 \tag{4.3}$$

Theorem 4.0.5. Effect of Multiplication by an Elementary Matrix

Let A be an $n \times k$ matrix, and suppose that the matrix B is obtained from A using an elementary row operation. Then $B = EA$, where E is the elementary matrix obtained from the identity matrix using the same row operation used to obtain B from A.s

Theorem 4.0.6. The inverse is a product of elementary matrices

Let A be an invertible $n \times n$ matrix, and let $E_1, E_2, \ldots, E_k$ be the elementary matrices corresponding (in order) to the elementary row operations used to reduce A to the identity matrix. Then

$$A^{-1} = E_K \ldots E_2 E_1.$$

Chapter 5: Operations on Matrices

The Matrix Transpose

Definition 5.0.1. Transpose

Let a be an $m \times n$ matrix. The transpose of A, denoted A^T, is the $n \times m$ matrix whose columns are the respective rows of A.

$$a_{ij} = a_{ji}$$

Definition 5.0.2. Symmetric and Skew Symmetric Matrices

A matrix A os symmetric if $A^T = A$

A matrix A is skew symmetric if $A^T = -A$

Theorem 5.0.1. Symmetric and Skew Symmetric Matrices

1. *Given any matrix A, the matrices AA^T and A^TA are symmetric.*
2. *Let A be a square matrix. The matrix $A + A^T$ is symmetric.*
3. *Let A be a square matrix. The matrix $A - A^T$ is skew symmetric.*

The Matrix Trace

Definition 5.0.3. The Trace

Let A be an $n \times n$ matrix. The trace of A, denoted $tr(A)$, is the sum of the diagonal elements of A. That is,

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

Theorem 5.0.2. Properties of the Matrix Trace

Let A and B be $n \times n$ matrices. Then:

1. $tr(A + B) = tr(A) + tr(B)$
2. $tr(A - B) = tr(A) - tr(B)$
3. $tr(kA) = k \cdot tr(A)$
4. $tr(AB) = tr(BA)$
5. $tr(A^T) = tr(A)$

The Determinant

Definition 5.0.4. Determinant

- *If A is a 1×1 matrix $A = [a]$, then $\det(A) = a$.*
- *If*
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$
The determinant of A. denoted by
$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$
- *If A is an $n \times n$ matrix, where $n \geq 2$, then $\det(A)$ is the number found by taking the cofactor expansion using the equation 5.0.6 along the first row of A. That is,*

$$\det(A) = a_{1,1}C_{1,1} + a_{1,2}C_{1,2} + \cdots + a_{1,n}C_{1,n}$$

Definition 5.0.5. Matrix Minor, Cofactor

Let A be an $n \times n$ matrix. the (i,j)-minor of A, denoted A_{ij}, is the determinant of the $(n-1) \times (n-1)$ matrix formed by deleting i^{th} row and j^{th} column of A.

The (i,j)-cofactor of A is the number

$$C_{ij} = (-1)^{i+j} A_{ij}.$$

Definition 5.0.6. Cofactor Expansion

Let $A = [a_{ij}]$ be an $n \times n$ matrix.

The cofactor expansion of A along the ith row is the sum

$$a_{i,1}C_{i,1} + a_{i,2}C_{i,2} + \cdots + a_{i,n}C_{i,n}$$

The cofactor expansion of A down the jth column is the sum

$$a_{1,j}C_{1,j} + a_{2,j}C_{2,j} + \cdots + a_{n,j}C_{n,j}$$

Properties of Determinants

Theorem 5.0.3. Cofactor Expansion Along Any Row or Column

Let A be an $n \times n$ matrix. The determinant of A can be computed using cofactor expansion along any row or column of A.

Key Idea: The Determinant of Triangular Matrices

The determinant of a triangular matrix is the product of its diagonal elements.

Let A be a $n \times n$ triangular matrix:

$$\det(A) = \prod_{i=1}^{n} a_{i,i}$$

Theorem 5.0.4. The Determinant and elementary Row Operations

Let A be an $n \times n$ matrix and let B be formed by performing one elementary row operation on A.

1. *If B is formed from A by adding a scalar multiple of one row to another, than $\det(B) = \det(A)$.*
2. *If B is formed from A by multiplying one row of A by a scalar k, then $\det(B) = k \cdot \det(A)$.*
3. *If B is formed from A by interchanging two rows of A, then $\det(B) = -\det(A)$.*

Theorem 5.0.5. The determinant of a non-invertible matrix

*If an $n \times n$ matrix A is **not** invertible, then $\det(A) = 0$.*

Theorem 5.0.6. Determinant Properties

Let A and B be $n \times n$ matrices and let k be a scalar. The following are true:

1. $\det(kA) = k^n \cdot \det(A)$
2. $\det(A^T) = \det(A)$
3. $\det(AB) = \det(A)\det(B)$

Theorem 5.0.7. The determinant of an inverse

If A is an invertible matrix, then $\det(A) \neq 0$, and

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Theorem 5.0.8. Invertible Matrix Theorem

Let A be an $n \times n$ matrix. The following statement are equivalent.

1. *A is invertible.*
2. $\det(A) \neq 0$

Application of the Determinant

Theorem 5.0.9. Cramer's Rule

Let A be an $n \times n$ matrix with $\det(A) \neq 0$ and let $\vec{b}$ be an $n \times 1$ column vector. Then the linear system

$$A\vec{x} = \vec{b}$$

has solution

$$x_i = \frac{\det(A_i(\vec{b}))}{\det(A)},$$

where $A_i(\vec{b})$ is the matrix formed by replacing the ith column of A with $\vec{b}$.

Definition 5.0.7. The adjugate of a matrix

Let A be an $n \times n$ matrix.

- *The **matrix of cofactors** of A is the $n \times n$ matrix*

  $$cof(A) = [C_{ij}]$$

  whose (i,j)-entry is given by the (i,j)-cofactor of A.
- *The **adjugate** of A is the $n \times n$ matrix*

  $$adj(A) = (cof(A))^T = [C_{ij}]^T.$$

Theorem 5.0.10. The adjugate formula for the inverse

Let A be an $n \times n$ matrix. If $\det(A) \neq 0$, then A is invertible, and

$$A^{-1} = \frac{1}{\det(A)} adj(A).$$

Chapter 6: Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

We have often explored new ideas in Linear Algebra by making connections to our previous algebraic experience. Adding two numbers, $x + y$, led us to adding vectors $\vec{x} + \vec{y}$ and adding matrices $A + B$. We explored multiplication, which then led us to solving the matrix equation $A\vec{x} = \vec{b}$, which was reminiscent of solving the algebra equation $ax = b$.

This chapter is motivated by another analogy. Consider: when we multiply an unknown number x by another number such as 5, what do we know about the result? Unless, $x = 0$, we know that in some sense $5x$ will be 5 times bigger than x. Now we apply this idea to vectors, we would readily agree that $5\vec{x}$ gives a vectors that is 5 times bigger than $\vec{x}$.

Definition 6.0.1. Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix, $\vec{x}$ a nonzero $n \times n$ column vector and λ a scalar. If

$$A\vec{x} = \lambda\vec{x},$$

then $\vec{x}$ is an eigenvector of A and λ is an eigenvalue of A.

Solving equations of 2 unknowns, in this case λ-a number and $\vec{x}$-a vector:

$$A\vec{x} = \lambda\vec{x}$$
$$= A\vec{x} - \lambda\vec{x} = \vec{\emptyset}$$
$$= A\vec{x} - (\lambda I)\vec{x} = \vec{\emptyset}$$
$$= (A - \lambda\vec{x})\vec{x} = \vec{\emptyset}$$
$$= B\vec{x} = \vec{0}$$

So we don't want B to be invertible. So we need the $\det(B) \neq 0$, so $\det(A - \lambda I) \neq 0$. This matrix B is called the **Characteristic Polynomial**

Definition 6.0.2. Characteristic Polynomial

Let A be an $n \times n$ matrix. The characteristic polynomial of A is the n-th degree polynomial $p(\lambda) = \det(A - \lambda I)$.

Key Idea: Finding the Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.

1. To find the eigenvalues of A, compute $p(\lambda)$, the characteristic polynomial of A, set it equal to 0, then solve for λ
2. To find the eigenvectors of A, for each eigenvalue solve the homogenous system $(A - \lambda I)\vec{x} = \vec{0}$. Do Gaussian Elimination to fond solution for x.

Properties of Eigenvalues and Eigenvectors

Theorem 6.0.1. Properties of Eigenvalues and Eigenvectors

Let A be an $n \times n$ invertible matrix. The following are true:

1. *If A is triangular, then the diagonal elements of A are the eigenvalues of A*
2. *If λ is an eigenvalue of A with eigenvectors $\vec{x}$, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with eigenvectors $\vec{x}$.*
3. *If λ is an eigenvalue of A then λ is an eigenvalue of A^T.*
4. *The sum of the eigenvalues of A is equal to the trace of A.*
5. *The product of the eigenvalues of A is equal to the determinant of A.*

Theorem 6.0.2. Invertible Matrix Theorem

Let A be an $n \times n$ matrix. The following statements are equivalent.

1. *A is invertible.*
2. *A does not have an eigenvalue of 0.*

Eigenvalues and Diagonalization

Definition 6.0.3. Similar Matrices

Let A and B be $n \times n$ matrices. We say that A is similar to B, and write A ~ B, if there exists an invertible $n \times n$ matrix P such that

$$A = P^{-1}BP.$$

Theorem 6.0.3. Shared properties of similar matrices

Let A and B be $n \times n$ matrices. If A ~ B, then:

1. $tr(A) = tr(B)$
2. $\det(A) = det(B)$
3. *A and B have the same eigenvalues.*

Chapter 7: Credits

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