ECON 714. Quant Macro-Econ Theory

Homework II

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1 Steady State

Suppose $z_{ss} = 0$. I formulate the problem of the agent recursively.

$$
\begin{aligned}
V(k) = &\max_{c\geq 0, l\geq 0, k'\geq 0} \left\{ \log c - \frac{l^2}{2} + \beta V(k') \right\} \\
\text{s.t. } & c + k' = k^{\alpha} l^{1-\alpha} + (1-\delta)k
\end{aligned}
$$

where $\beta = 0.97$, $\alpha = 0.03$, and $\delta = 0.1$. I use the first order conditions and the envelope condition to derive the stochastic Euler equation.

$$
\begin{aligned}
\frac{1}{c} &= \lambda \\
l &= \lambda(1-\alpha)k^{\alpha} l^{-\alpha} \\
\beta V_1(k') &= \lambda \\
V_1(k) &= \lambda(\alpha k^{\alpha-1} l^{1-\alpha} + 1 - \delta)
\end{aligned}
$$

The intratemporal optimality condition and the Euler equation are

$$
\begin{aligned}
l &= \frac{1}{c}(1-\alpha)k^{\alpha} l^{-\alpha} \\
\frac{1}{c} &= \beta \frac{1}{c'}(\alpha k^{\alpha-1} l^{1-\alpha} + 1 - \delta)
\end{aligned}
$$

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The deterministic steady state variables c_{ss}, l_{ss}, and k_{ss} should satisfy the following equilibrium conditions derived above.

$$
\begin{aligned}
c_{ss} + k_{ss} &= k_{ss}^{\alpha} l_{ss}^{1-\alpha} + (1-\delta) k_{ss} \\
l_{ss} &= \frac{1}{c_{ss}} (1-\alpha) k_{ss}^{\alpha} l_{ss}^{-\alpha} \\
\frac{1}{c_{ss}} &= \beta \frac{1}{c_{ss}} (\alpha k_{ss}^{\alpha-1} l_{ss}^{1-\alpha} + 1 - \delta)
\end{aligned}
$$

To solve for c_{ss}, l_{ss}, and k_{ss}, I use the Euler equation and a budget constraint to get

$$
\begin{aligned}
l_{ss}(k_{ss}) &= \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha} \right)^{\frac{1}{1-\alpha}} k_{ss} \\
c_{ss}(k_{ss}) &= k_{ss}^{\alpha} l_{ss}(k_{ss})^{1-\alpha} -_{ss}
\end{aligned}
$$

The intratemporal optimality condition leads to a nonlinear equation of the form:

$$
l_{ss}(k_{ss}) = \frac{1}{c_{ss}(k_{ss})} (1-\alpha) k_{ss}^{\alpha} l_{ss}(k_{ss})^{-\alpha}.
$$

A solution for this equation is k_{ss}. The deterministic steady state variables when $z_{ss} = 0$ are as follows:

$$
\begin{aligned}
c_{ss} &= 1.1161 \\
l_{ss} &= 0.9465 \\
k_{ss} &= 3.7612 \\
y_{ss} &= k_{ss}^{\alpha} l_{ss}^{1-\alpha} = 1.4923 \\
w_{ss} &= (1-\alpha) k_{ss}^{\alpha} l_{ss}^{-\alpha} = 1.0564 \\
r_{ss} &= \alpha k_{ss}^{\alpha-1} w_{ss}^{1-\alpha} = 0.1309
\end{aligned}
$$

2 Value Function Iteration with a Fixed Grid

I formulate the problem of the agent recursively.

$$V(k,z) = \max_{c\geq 0, l\geq 0, k'\geq 0} \left\{ \log c - \frac{l^2}{2} + \sum_{z'} \pi(z'|z)\beta V(k',z') \right\}$$
$$\text{s.t. } c + k' = e^z k^\alpha l^{1-\alpha} + (1-\delta)k$$

where $\beta = 0.97$, $\alpha = 0.03$, and $\delta = 0.1$. I use the first order conditions and the envelope condition to derive the stochastic Euler equation.

$$\begin{aligned}
\frac{1}{c} &= \lambda \\
l &= \lambda(1-\alpha)e^z k^\alpha l^{-\alpha} \\
\beta \sum_{z'} \pi(z'|z) V_1(k',z') &= \lambda \\
V_1(k,z) &= \lambda(\alpha e^z k^{\alpha-1} l^{1-\alpha} + 1 - \delta)
\end{aligned}$$

The intratemporal optimality condition and the Euler equation are

$$\begin{aligned}
l &= \frac{1}{c}(1-\alpha)e^z k^\alpha l^{-\alpha} \\
\frac{1}{c} &= \beta \sum_{z'} \pi(z'|z) \frac{1}{c'}(\alpha e^{z'} k'^{\alpha-1} l'^{1-\alpha} + 1 - \delta)
\end{aligned}$$

I fix a grid of 250 points of capital, centered around k_{ss} with a coverage of $\pm 30\%$ of k_{ss} and equally spaced. I iterate on the value function of the household until the change in the sup norm between two iterations is less than 10^{-7}. A representative household works more and accumulates more capital under a positive technology shock because a positive technology shock increases the marginal productivity of labor and capital. It enables a representative household to consume more.

Figure 1: Fixed Grid: Consumption Policy Functions

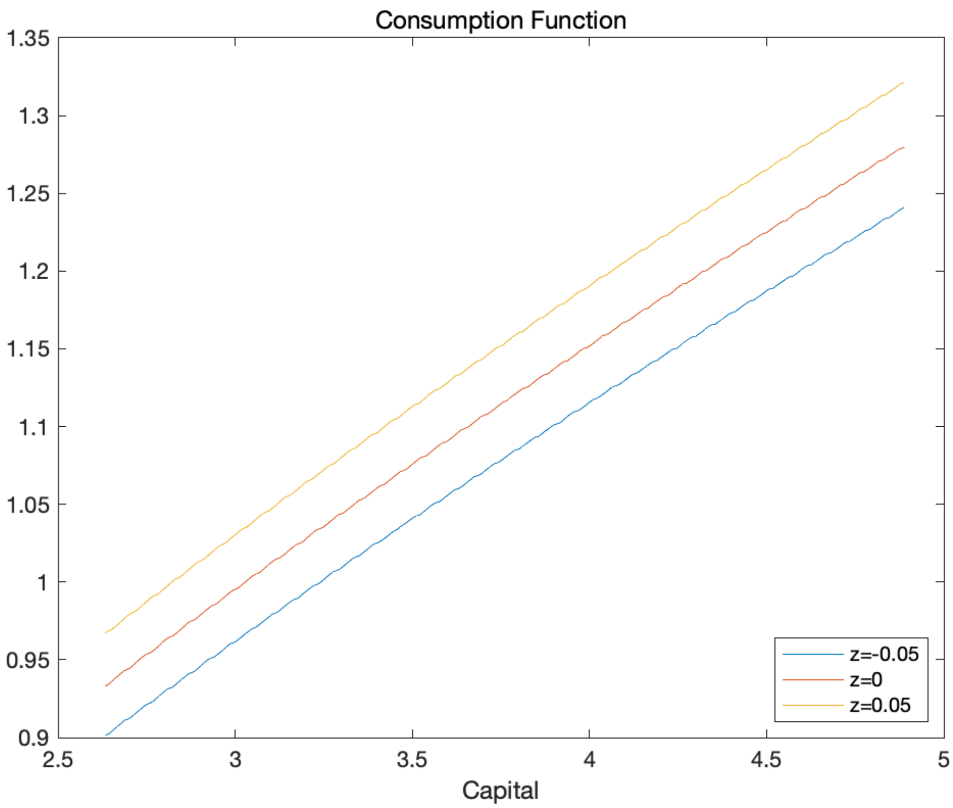


Figure 2: Fixed Grid: Labor Policy Functions

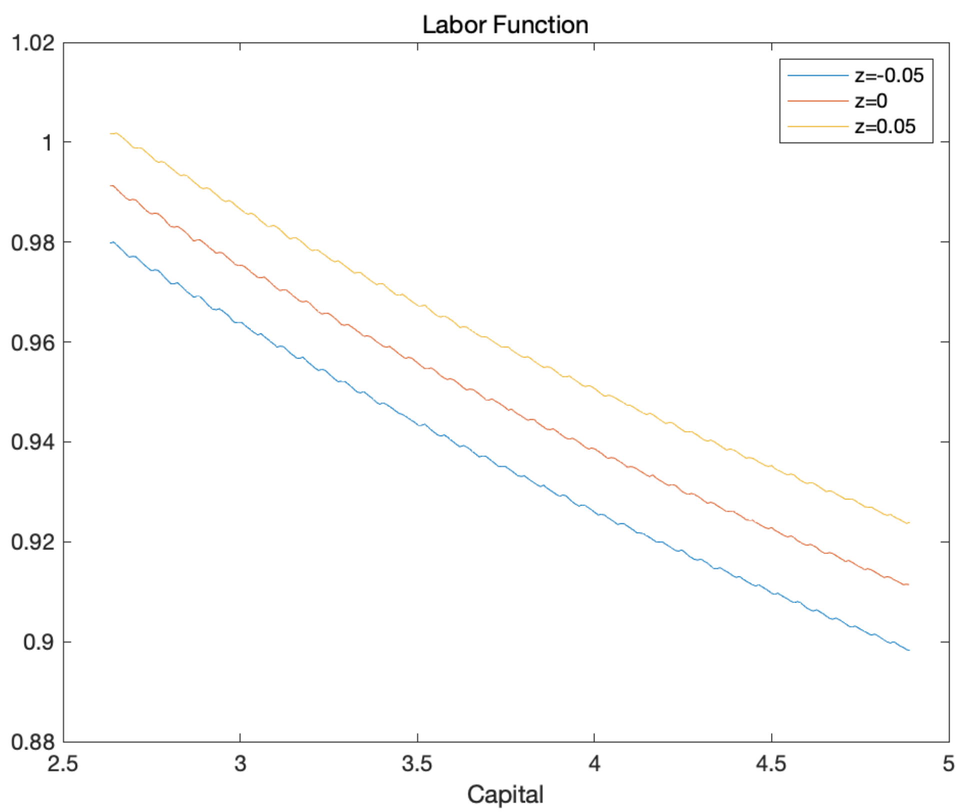
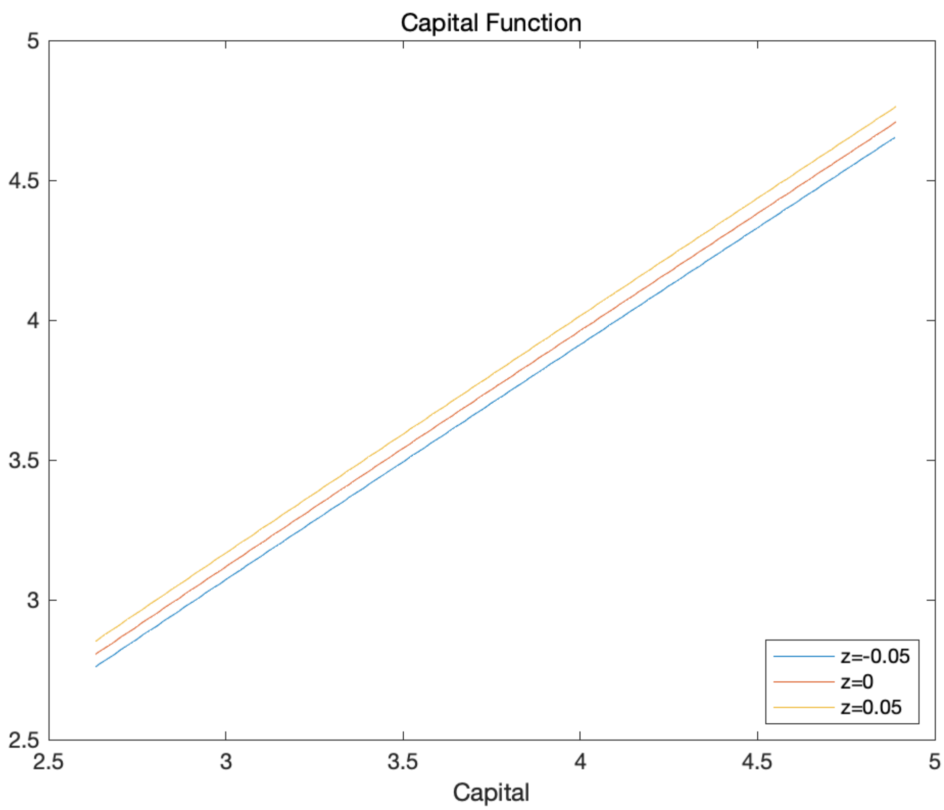


Figure 3: Fixed Grid: Capital Policy Functions



3 Accelerator

I use an accelerator to recompute my solution. I fix a grid of 250 points of capital, centered around k_{ss} with a coverage of $\pm 30\%$ of k_{ss} and equally spaced. I iterate on the value function of the household until the change in the sup norm between two iterations is less than 10^{-7}. The results are similar to the above results. A representative household consume more, works more, and accumulates more capital under a positive technology shock.

Figure 4: Accelerator: Consumption Policy Functions

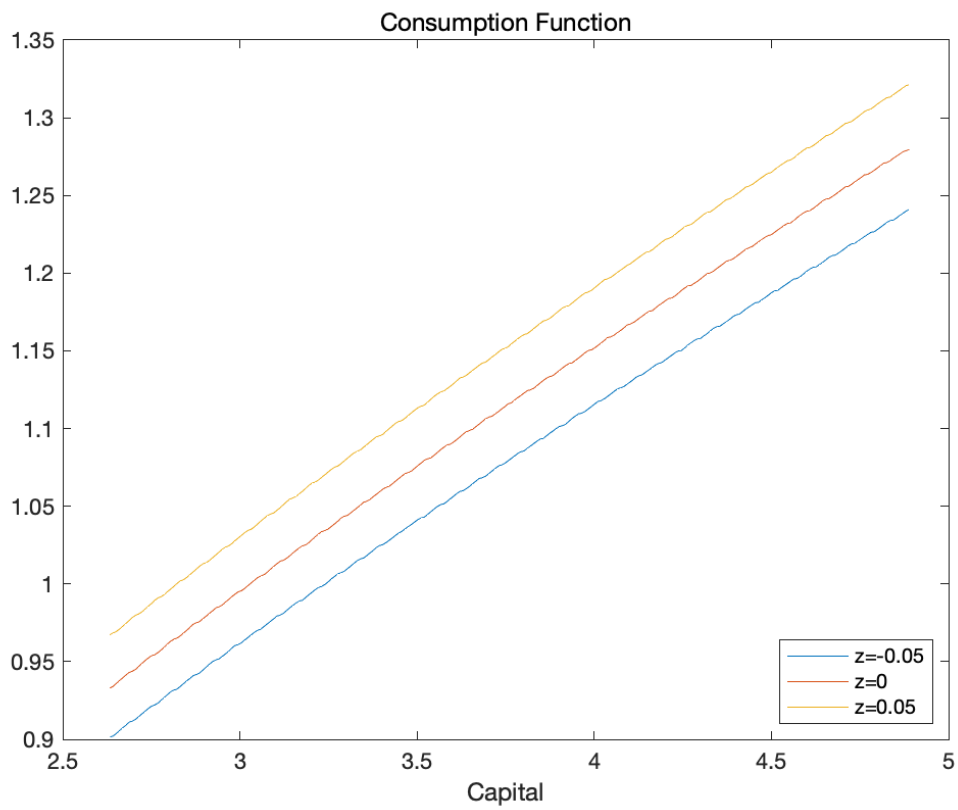


Figure 5: Accelerator: Labor Policy Functions

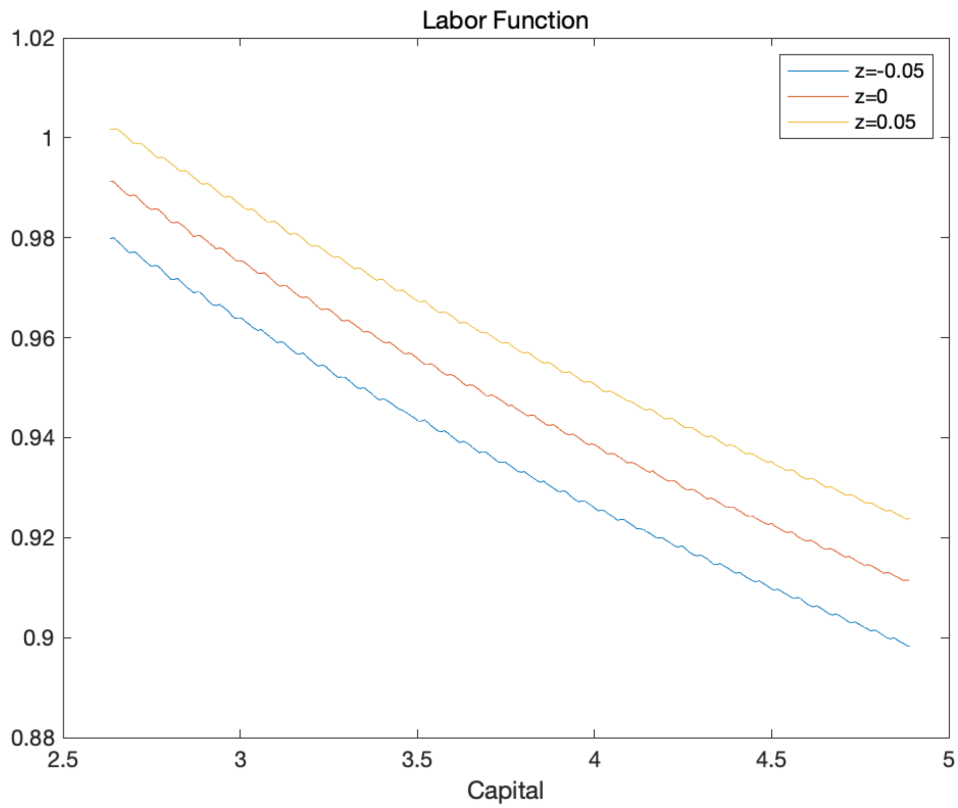
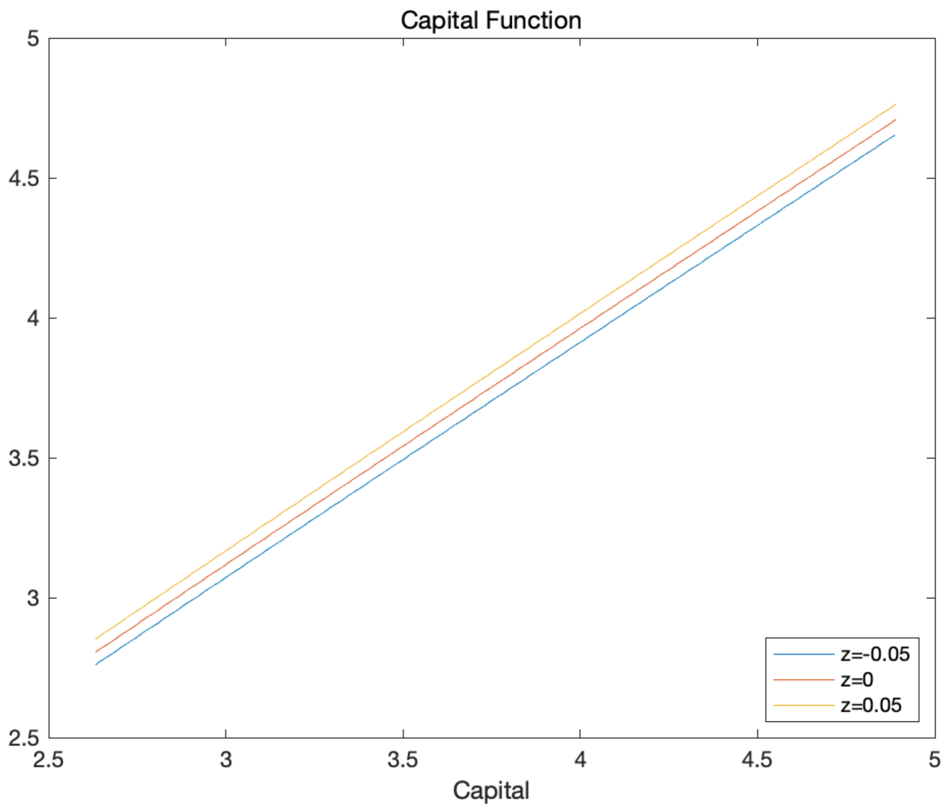


Figure 6: Accelerator: Capital Policy Functions



4 Multigrid

I implement a multigrid scheme for a value function iteration, which the grid of 10,000 points of capital, centered around k_{ss} with a coverage of $\pm 30\%$ of k_{ss} and equally spaced. I have 100 capital grid points in the first grid, 1,000 capital grid points in the second, and 10,000 capital grid points in the third. I iterate on the value function of the household until the change in the sup norm between two iterations is less than 10^{-7}. The results are similar to the previous ones. A representative household consume more, works more, and accumulates more capital under a positive technology shock.

Figure 7: Multigrid: Consumption Policy Functions

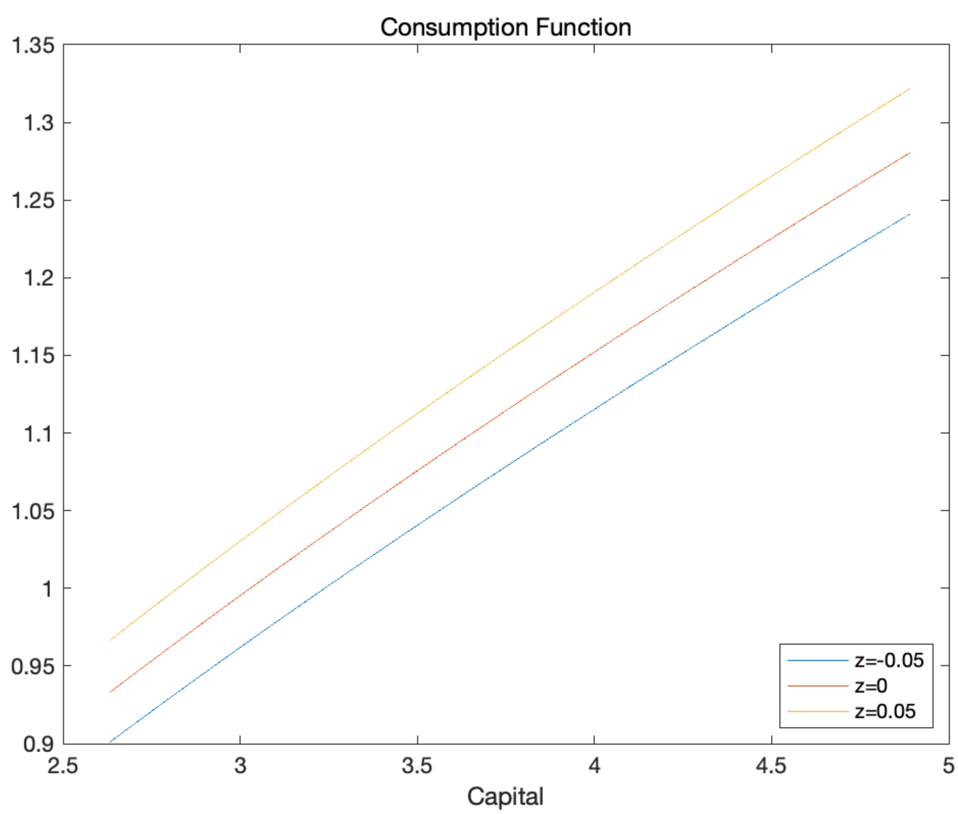


Figure 8: Multigrid: Labor Policy Functions

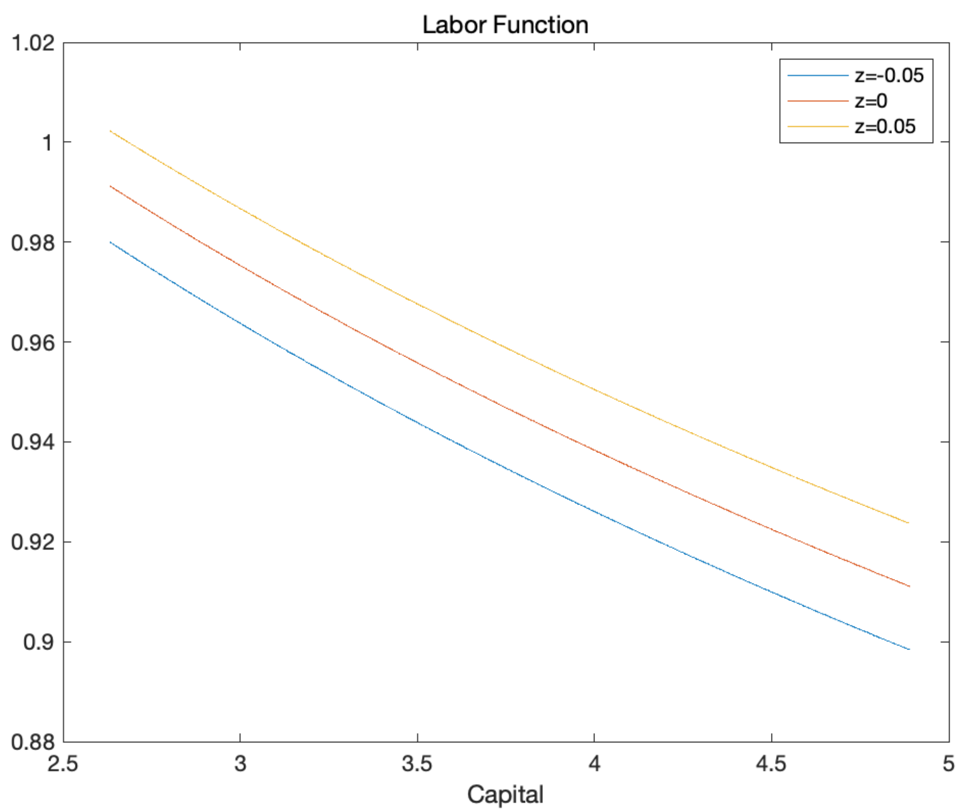


Figure 9: Multigrid: Capital Policy Functions

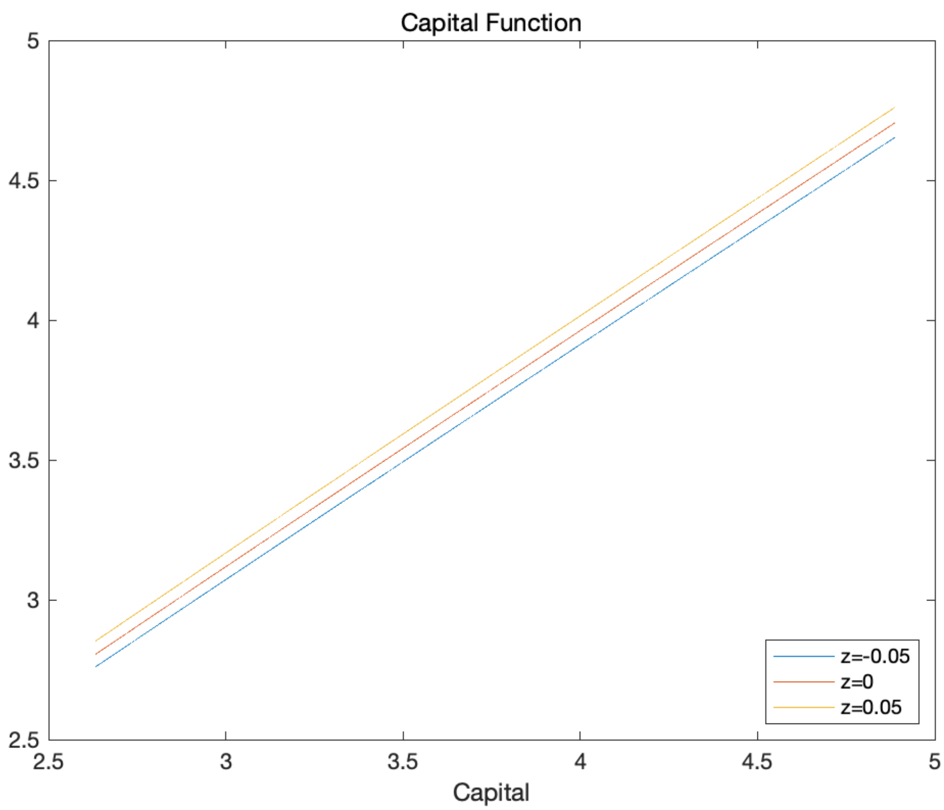


Table 1: Accuracy and Computing Time

	Fixed Grid	Accelerator	Multigrid
Computing time (sec.)	223.9936	63.7355	41.7036
Maximum of the Euler equation error	-2.7516	-2.7516	-3.7028
Mean of the Euler equation error	-3.5291	-3.5283	-4.4381

Although I implement a multigrid scheme with the grid of 10,000 points of capital, multigrid has the fastest running time and fixed grid has the slowest running time. Multigrid is the most accurate, while fixed grid and accelerator achieve similar levels of accuracy.

5 Chebyshev

I compute the solution to the model using 5 Chebyshev polynomials on capital for every level of productivity. Cheyshev Interpolation Theorem is useful to write code. The responses of the economy to a technology shock are similar to the previous ones. However the values of the solutions are somewhat different from the values of previous solutions. The values of the previous consumption functions range from 0.90 to 1.32, while the values of the consumption function I get using Chebyshev polynomials range from 0.87 to 1.20. The values of the previous labor functions range from 0.90 to 1.00, while the values of the labor function I get using Chebyshev polynomials range from 0.99 to 1.01. The previous capital functions range from 2.76 to 4.76, while the capital functions I get using Chebyshev polynomials range from 2.81 to 4.97.

Figure 10: Chebyshev: Consumption Policy Functions

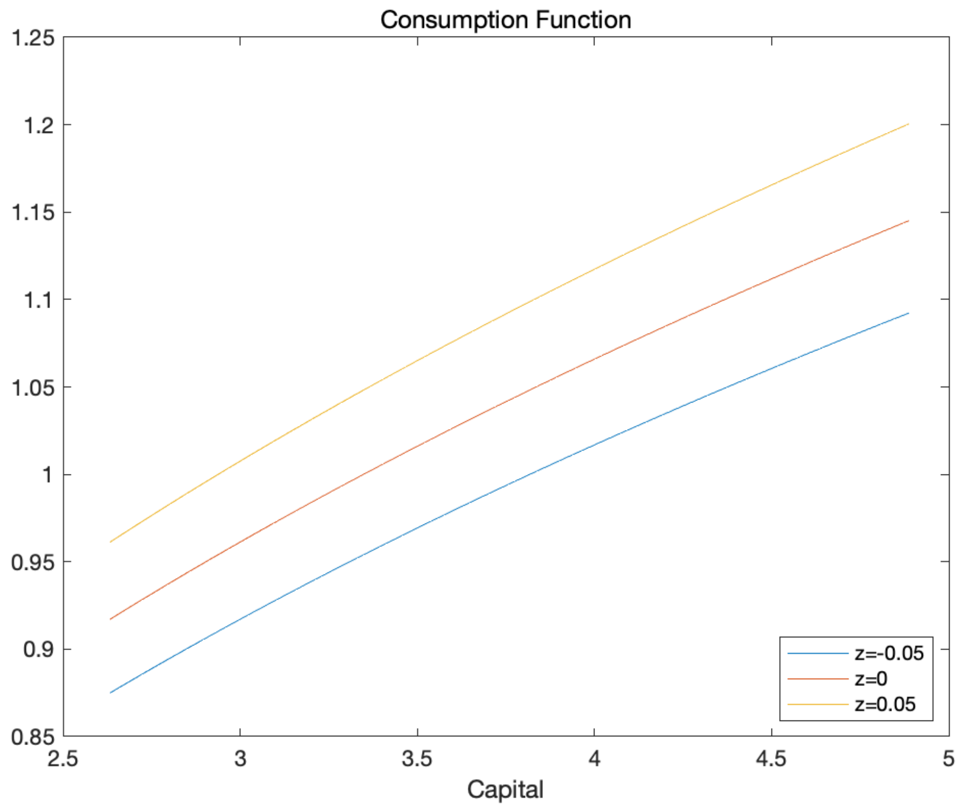


Figure 11: Chebyshev: Labor Policy Functions

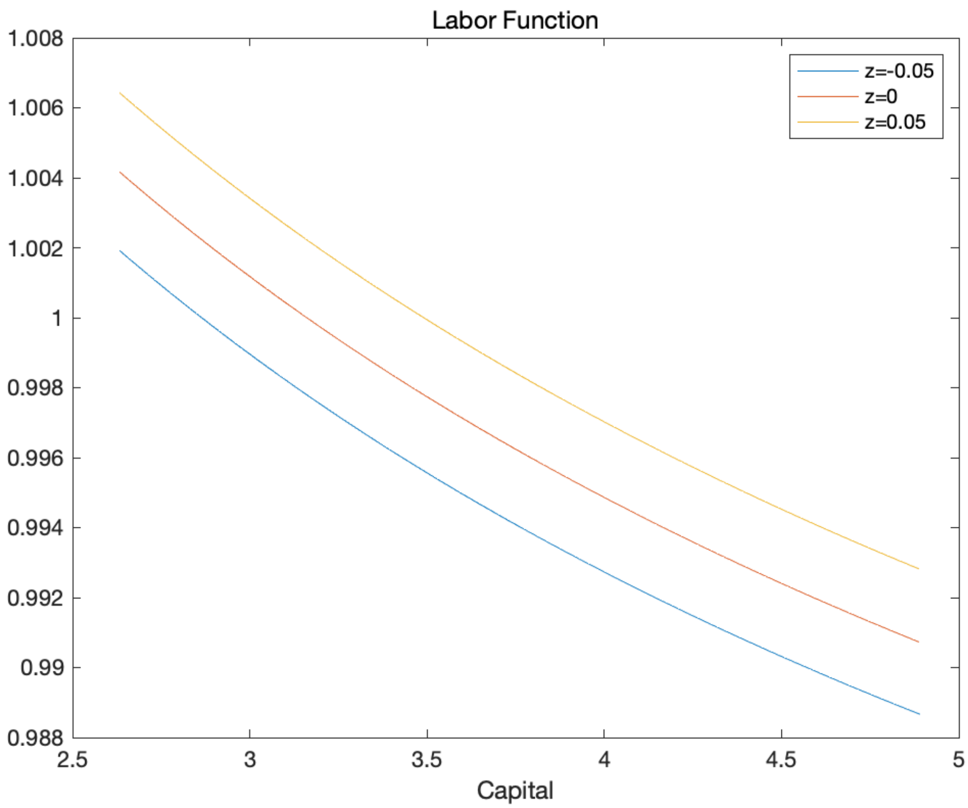
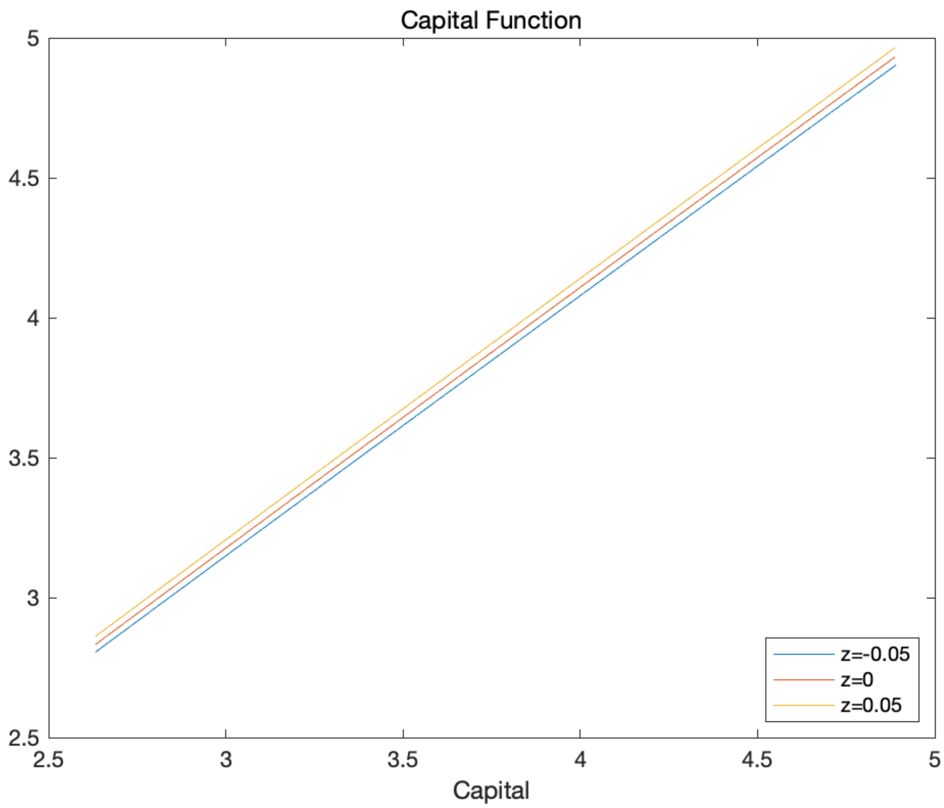


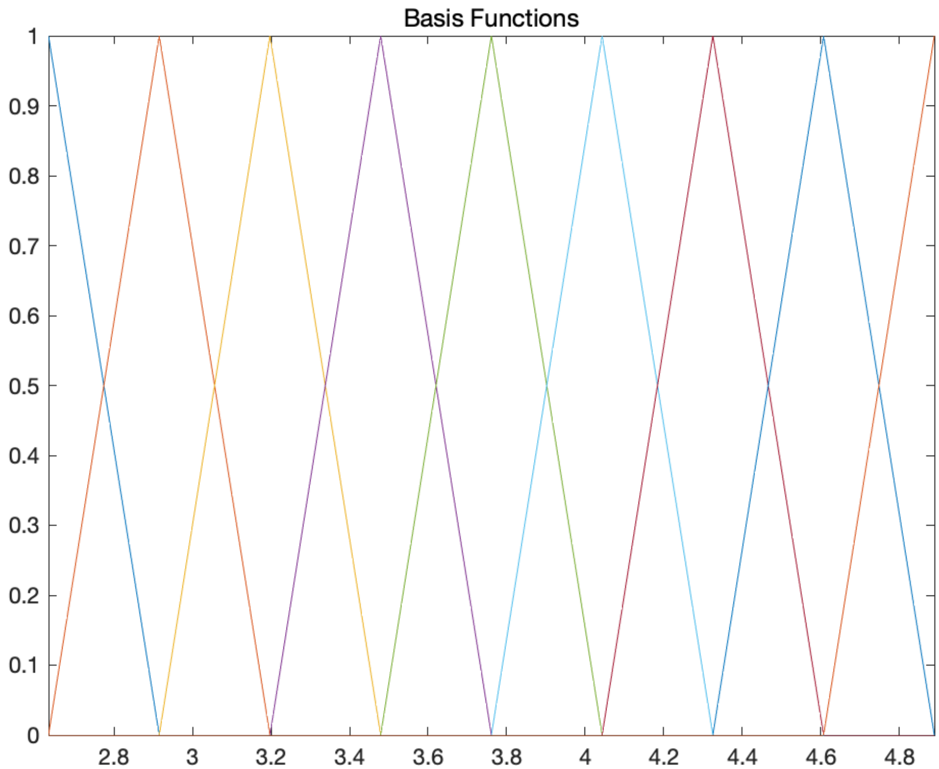
Figure 12: Chebyshev: Capital Policy Functions



6 Finite Elements

I compute the solution to the model using 8 finite elements on capital for every level of productivity. I use a Galerkin weighting scheme. Figure 13 shows the basis functions I use.

Figure 13: Finite Elements: Basis Functions



The responses of the economy to a technology shock are similar to the previous ones. However the values of the solutions are somewhat different from the values of previous solutions. The values of the previous consumption functions I get using Chebyshev polynomials range from 0.87 to 1.20, while the values of the consumption functions I get using finite elements range from 0.90 to 1.32. The values of the previous labor functions range from 0.99 to 1.01, while the values of the labor function I get using finite elements range from 0.90 to 1.00. The previous capital functions range from 2.81 to 4.97, while the capital functions I get using finite elements range from 2.76 to 4.76.

Figure 14: Finite Elements: Consumption Policy Functions

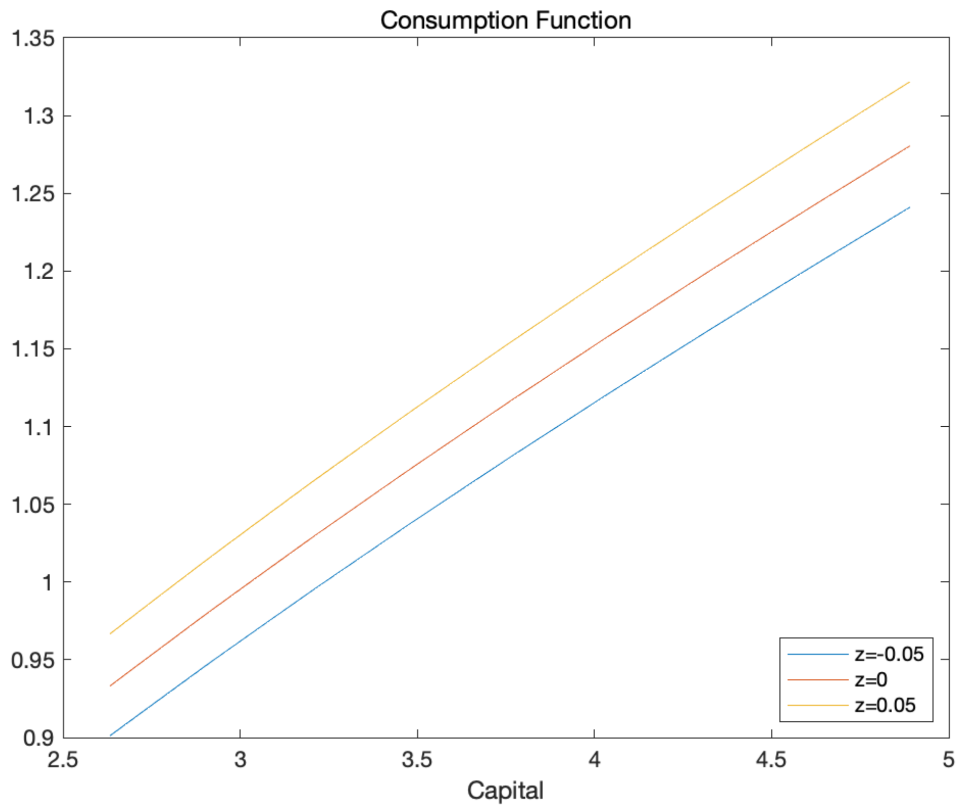


Figure 15: Finite Elements: Labor Policy Functions

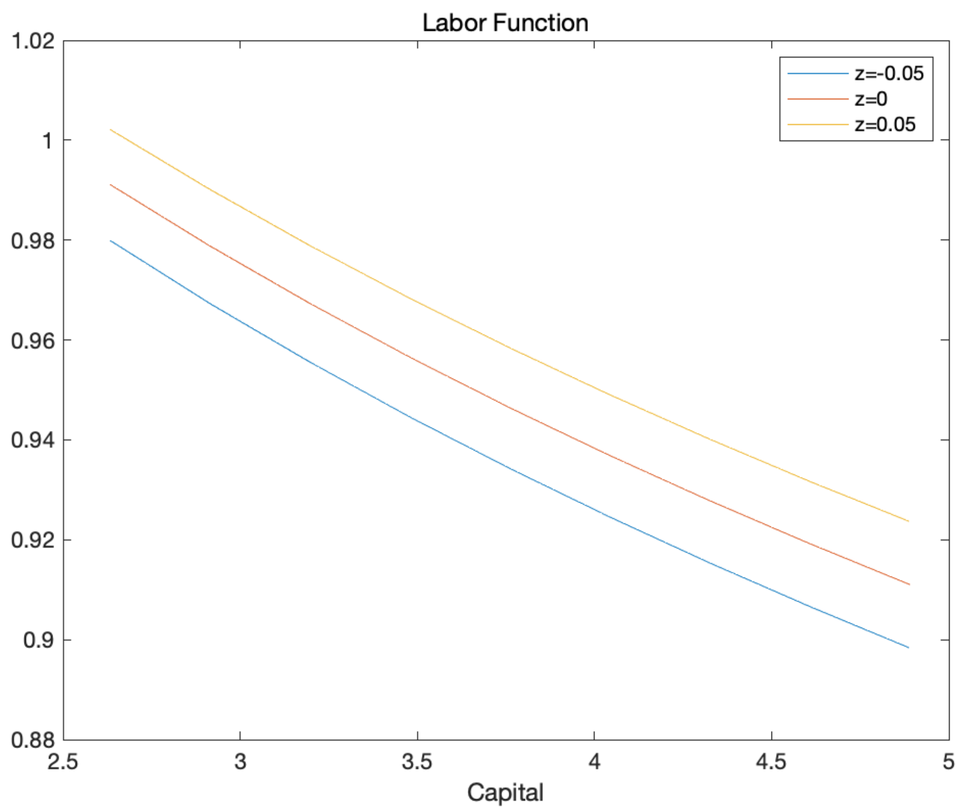


Figure 16: Finite Elements: Capital Policy Functions