Simply-Typed Lambda Calculus

Principles of Programming Languages
CAS CS 320
Lecture 19

Practice Problem

```
let x = 0 in
let g = fun y -> x + 1 in
let x = 1 in
let f = fun y -> g x in
let x = 2 in
f
```

What (closure) does the ~~following~~ expression above *evaluate to? You don't need to give the derivation.*

Answer

```
let x = 0 in
let g = fun y -> x + 1 in
let x = 1 in
let f = fun y -> g x in
let x = 2 in
f
```

$$\begin{tikzcd}
\llparenthesis \{x \mapsto 1, g \mapsto ?\}, \lambda y.\, g\, x \rrparenthesis \\
\llparenthesis \{x \mapsto 0\}, \lambda y.\, x + 1 \rrparenthesis \arrow[u]
\end{tikzcd}$$

$$\boxed{\llparenthesis \{x \mapsto 1, g \mapsto \llparenthesis \{x \mapsto 0\}, \lambda y.\, x + 1 \rrparenthesis\}, \lambda y.\, g\, x \rrparenthesis}$$

Outline

Have a high-level discussion of type theory in general

Introduce and analyze the simply-typed lambda calculus (STLC)

Demo an implementation of the STLC

Learning Objectives

Give a derivation of a typing judgment in the STLC, both with Curry-style typing and Church-style typing

Given an example of expression that cannot be typed the STLC, but which can still be evaluated to a value

Implement the STLC

Type Theory

What is a Type?

```
let f : int -> int = ...
```

Who knows...

A **type** is an syntactic object that we give to an expression which describes something about its behavior

as opposed to semantic.

This description can be used to *restrict* the use of the expression *within* a program

Types help us delineate "well-behaved" programs

let sort : int list →

int sorted list

dependent type theory

Trade-offs

$$(\lambda x\,.\,xx)(\lambda x\,.\,xx)$$

`lambda term called Ω`

Types are *restrictive*. They tells us what we *can't* do in our programs

Types are *safe*. They make sure we don't do dumb things in our program

<u>The goal is to balance:</u>

» Simplicity/Usability
» Expressivity
» Safety/Theoretical Guarantees

OCaml

```
# let big_omega =
    let little_omega x = x x in
    little_omega little_omega;;
Error: This expression has type 'a -> 'b
       but an expression was expected of type 'a
       The type variable 'a occurs inside 'a -> 'b
```

The type system of OCaml tells us when we're trying to define an ill-behaved program

But OCaml also has strong *type inference* and *polymorphism* to balance these benefits with better ergonomics

The more expressive, the more complex the the type system, designing programming languages is finding the balance that works for you

Typing Judgments

$$\underbrace{\Gamma}_{\substack{\text{context}\\ \text{static environment}}} \vdash \underset{\text{subject}}{\boxed{e}} : \tau$$

<u>This judgment reads:</u>

e has type τ in the context Γ

We say that e is **well-typed** if $\cdot \vdash e : \tau$ for some type τ

empty

Most of what type theorists do is come up with rules for deriving typing judgments

ex. bidirectional type checking

$$\Gamma \vdash e \Rightarrow \tau$$

$$\Gamma \vdash e \Leftarrow \tau$$

ex. type inf.

$$\Gamma \vdash e : \tau \dashv \underbrace{C}_{\text{constraints.}}$$

ex. System F:

$$\Delta, \underbrace{\Gamma}_{\text{multiple contexts}} \vdash e : \tau$$

context

What is a Context?

BNF grammar

$$\left\{\begin{array}{l} \Gamma ::= \overset{\text{empty}}{\cdot} \mid \Gamma, \overbrace{x : \tau}^{\text{binding /declaration}} \\ x ::= \text{vars} \\ \tau ::= \text{types} \end{array}\right.$$

This depends...

In Theory: A context is an inductively-defined syntactic object, just like a type or a expression

In Practice: A context is a set (or ordered list, in some cases) of **variable declarations** /bindings

(a variable declaration a variable together with a type)

$$\{x : int\} \vdash x + 1 : int$$

example of set-contexts being "strange"

$$\dfrac{\overline{\{x : int\} \vdash x : int}}{\{x : bool\} \vdash fun\ x \to x : int \to int}$$

Inference Rules

$$\frac{\overbrace{\Gamma \vdash e_1 : \tau_1}^{\text{premise}} \qquad \dots \qquad \overbrace{\Gamma \vdash e_k : \tau_k}^{\text{premise}}}{\Gamma \vdash e : \tau}$$

Inference rules then tell us when we derive a new typing judgment from old typing judgments

An inference rule with no premises is called an **axiom**

<u>The questions we need to answer:</u>

» How do we know what rules to include?
» How do we know if we've chosen *good* rules?

ex.

$$\frac{\Gamma \vdash e_1 : \text{int} \qquad \Gamma \vdash e_2 : \text{int}}{\Gamma \vdash e_1 + e_2 : \text{int}}$$

Simply-Typed Lambda Calculus

Syntax

unit expression

type annotations

```
<e>  ::= () | <v> | <e> <e>
       | fun ( <v> : <ty> ) -> <e>
<ty> ::= unit | <ty> -> <ty>
<v>  ::= a | ... | z
```

types

The syntax is the same as that of the lambda calculus except:

» we include a unit expression

» we have types, which annotate arguments

This is the first time that **types are a part of our syntax**

(later we'll add more things like numbers)

(fun (x : unit) -> x) ()

Syntax

$$e ::= \bullet \mid x \mid \lambda x^{\tau} . e \mid ee$$

$$\tau ::= \overset{\text{unit type}}{\boxed{\top}} \mid \tau \to \tau$$

$$x ::= \mathit{variables}$$

The syntax is the same as that of the lambda calculus except:

» we include a unit expression

» we have types, which annotate arguments

This is the first time that **types are a part of our syntax**

(later we'll add more things like numbers)

Typing

$$\frac{}{\Gamma \vdash \bullet : \top}$$

"unit expr has unit type"

variables

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash x : \tau}$$

Γ is well-formed

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x^{\tau} . e : \tau \to \tau'}$$

abstraction

$$\frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'}$$

application

not allowed : $\lambda x . x x$

These rules enforce that a function can only be applied if we *know* that it's a function

In theory: We need to be careful that are contexts are well-formed...

Γ is well-formed $\equiv$ no duplicate variables

In practice: We will think of our context as as set

Type Annotations?

```
<e>  ::= () | <v> | <e> <e>
       | fun <v> -> <e>
<ty> ::= unit | <ty> -> <ty>
<v>  ::= a | ... | z
```

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x . e : \tau \to \tau'}$$

no annotation

Do we have to include the type annotation on function arguments?

No, but it does change the way typing works

Roughly speaking, if we include annotations we're using **Church-style typing.** If we drop annotations, we're using **Curry-style typing**

Church vs. Curry Typing

```
fun x -> x

fun (x : unit) -> x
```

What is the type of the first expression? How about the second?

In Curry-style typing, the type of an expression is *extrinsic*, the expression is just an expression in the lambda calculus

In Church-style typing, it's *intrinsic*, built into the expression and the semantics

Using Curry-style typing is not the same as having polymorphism

Curry:

$\cdot \vdash \lambda x . x : T \to T$

$\cdot \vdash \lambda x . x : (T \to T) \to (T \to T)$

both are possible.

Church:

$\cdot \vdash \lambda x^{T} . x : T \to T$

$\cdot \vdash \lambda x^{T \to T} . x : (T \to T) \to (T \to T)$

$\cdot \vdash \lambda f . (f \bullet , f (\lambda x . x))$

NOT POSSIBLE

Uniqueness of Types

> **Lemma.** If $\Gamma \vdash e : \underline{\tau_1}$ and $\Gamma \vdash e : \underline{\tau_2}$ then $\underline{\tau_1 = \tau_2}$

Proof. The rough idea is to do induction *on the derivations themselves* (whoa)

In the simply typed lambda calculus with Church-style typing, every expression has a *unique type*

In particular, the function `type_of` is well-defined

Semantics (Review)

$$\frac{}{\langle \mathcal{E}, \lambda \boxed{x^{\tau}} \,.\, e\rangle \Downarrow (\mathcal{E}, \lambda x \,.\, e)} \qquad \frac{}{\langle \mathcal{E}, \bullet \rangle \Downarrow \bullet} \qquad \frac{}{\langle \mathcal{E}, x\rangle \Downarrow \mathcal{E}(x)}$$

annotation ↗ ↖ no annotation

$$\frac{\langle \mathcal{E}, e_1\rangle \Downarrow (\mathcal{E}', \lambda x \,.\, e) \qquad \langle \mathcal{E}, e_2\rangle \Downarrow v_2 \qquad \langle \mathcal{E}'[x \mapsto v_2], e\rangle \Downarrow v}{\langle \mathcal{E}, e_1 e_2\rangle \Downarrow v}$$

The semantics are basically identical

(we can also consider small-step, or big-step with substitution)

This is part of the point. Type-checking only determines *whether* we go on to evaluate the program (whether it makes sense to)

It doesn't determine **how** we evaluate the program

Example (Curry)

$$\lambda x\,.\,xx$$

Not possible to type

What happens if we try to give a type to the above expression?

$\tau'' \to \tau = \tau''$

CONTRADICTION

$\tau' = \tau'' \to \tau$ $\qquad\qquad$ $\tau' = \tau''$

$$\dfrac{\dfrac{\{x:\tau'\}\vdash x:\tau''\to\tau \qquad \{x:\tau'\}\vdash x:\tau''}{\{x:\tau\}\vdash x\;x:\tau}}{\cdot\vdash \lambda x.xx:\tau}$$

Example (Church)

$$\lambda x^{\boxed{\tau}}. xx$$

What happens if we try to give a type to the above expression? What should τ be?

$$\frac{}{\cdot \vdash \lambda x^{\tau}. x x : ?}$$

Not possible.

$$\frac{\vdots}{\cdot \vdash \lambda x^{\tau \to \tau}. x x : ?}$$

Practice Problem

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash x : \tau} \qquad \frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x^{\tau} . e : \tau \to \tau'}$$

$$\frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'}$$

$$\cdot \vdash \lambda f^{\top \to \top} . \lambda x^{\top} . fx : (\top \to \top) \to \top \to \top$$

Give a derivation for the above judgment.

Answer

$$\cdot \vdash \lambda f^{\top \to \top}.\, \lambda x^{\top}.\, f x : (\top \to \top) \to \top \to \top$$

How do we know if we've defined a "good" programming language?

Type Safety

Big step: for STLC

Theorem. If $\cdot \vdash e : \tau$ then there is a value v such that $\langle \varnothing, e \rangle \Downarrow v$ and $\cdot \vdash v : \tau$

With small-step semantics, we can give a finer-grained analysis:

Small step:

Theorem. If $\cdot \vdash e : \tau$, then

- » *(progress)* either e is a value or there is an e' such that $e \longrightarrow e'$ } no stuck terms, e.g. if 2 then 2 else 2.
- » *(preservation)* If $\cdot \vdash e : \tau$ and $e \longrightarrow e'$ then $\cdot \vdash e' : \tau$
- » *(normalization)* there is a value v such that $e \twoheadrightarrow v$

only for stlc

These results are *fundamental*. They tell us that our programming language is well-behaved (it's a "good" programming language)

We will eventually drop normalization *(why?)*

we want non-termination

$$\cdot \vdash (\lambda x^T. x)\,\bullet : \boxed{T} \qquad v = \bullet$$

$$\langle \varnothing, (\lambda x^T. x)\,\bullet \rangle \Downarrow \bullet$$

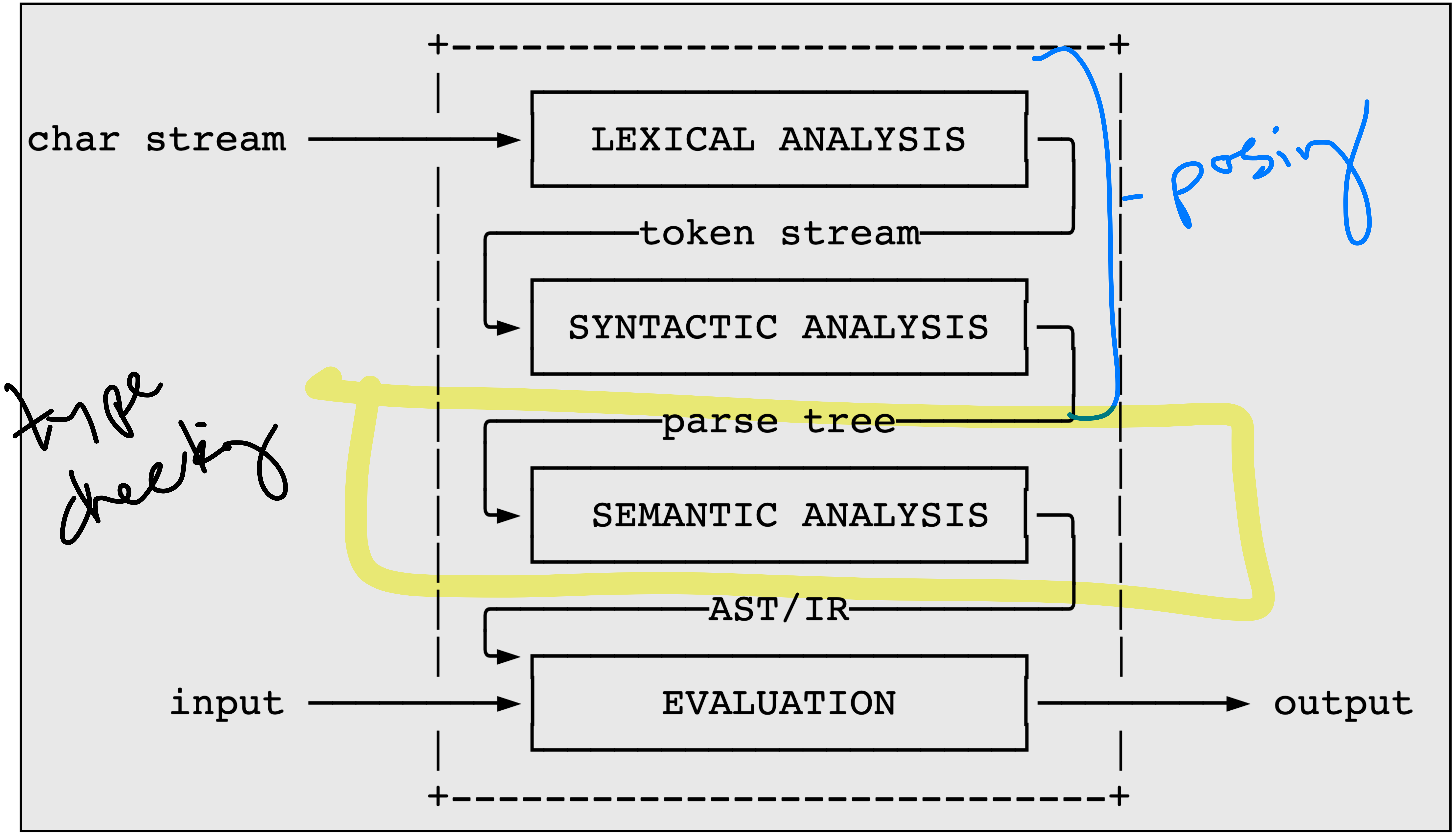
$$\cdot \vdash \bullet : \boxed{T}$$

if true then 2 else 3 ⟶ 2

: int   : int

Type Checking

The Picture



char stream

LEXICAL ANALYSIS

token stream

SYNTACTIC ANALYSIS

parse tree

SEMANTIC ANALYSIS

AST/IR

input

EVALUATION

output

parsing

type checking

Type Checking vs. Type Inference

```
type_check : expr -> ty -> bool

type_of : expr -> ty option
```

over-simplictions

Type checking the problem of determining whether a given expression is a given type

Type inference is the problem of *synthesizing* a type for a given expression, if possible

Theoretically, these two problems can be very different

For STLC, they are both easy

One Issue

$$\frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'}$$

How do we turn this into a type-checking procedure?

It seems like we need to do *some* amount of inference because it's not immediately clear what type we should check e_1 to be

Aside: If you're interested there is a way of *combining* checking and inference in what's called bidirectional type checking

Our solution: We'll just use type inference

```
⋮
| App (e1, e2) →
  let t1 = type_check e1   ?

                     ? → τ'

  let t2 = type_check e2   ?
```

General Recursion

```
let rec f x = f x
```

In the mini-projects, we will be implementing *unrestricted recursion*

If we have unrestricted recursion in our language, **it's no longer normalizing** *(why?)*

Again, it's a trade-off

Demo

Demo (Syntax)

```
<e>  ::= () | <v> | <e> <e> | fun ( <v> : <ty> ) -> <e>
       | let <v> : <ty> = <e> in <e>
       | let rec <v> ( <v> : <ty> ) : <ty> = <e> in <e>
       | if <e> then <e> else <e>
       | <e> + <e> | <e> - <e> | <e> * <e> | <e> = <e>
<ty> ::= unit | int | bool | <ty> -> <ty>
<v>  ::= ...
```

This is an extension of our demo from last lecture

(It would be good practice to write down the typing rules for this language)

Practice Problem

```
let rec f (x : t1) : t2 = e1 in e2
```

Write down (to the best of your ability) the typing rule for recursive let-expressions.

$$\frac{\Gamma, x : \tau_1, f : \tau_1 \to \tau_2 \vdash e_1 : \tau_2 \qquad \Gamma, f : \tau_1 \to \tau_2 \vdash e_2 : \tau}{\Gamma \vdash \text{let rec } f \ (x : \tau_1) : \tau_2 = e_1 \text{ in } e_2 : \tau}$$

Practice Problem

```
let rec f (x : t1) : t2 = e1 in e2
```

Write down (to the best of your ability) the typing rule for recursive let-expressions.