CSE 404 HW 01

1.

a) $(2A)^T = 2\begin{bmatrix}1 & 2\\ 2 & 1\\ 3 & 4\end{bmatrix} = \begin{bmatrix}2 & 4\\ 4 & 2\\ 6 & 8\end{bmatrix}$
b) $(A - B)^T = it\ is\ not\ possible\ to\ compute\ since\ they\ dont\ have\ the\ same\ size$
c) $(3B^t - A)^T = \left(3\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 2\end{bmatrix} - \begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 4\end{bmatrix}\right)^T = \left(\begin{bmatrix}3 & 6 & 9\\ 0 & 3 & 6\end{bmatrix} - \begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 4\end{bmatrix}\right)^T =$
$\begin{bmatrix}2 & 4 & 6\\ -2 & 2 & 2\end{bmatrix}^T = \begin{bmatrix}2 & -2\\ 4 & 2\\ 6 & 2\end{bmatrix}$
d) $(-A)^T E = \begin{bmatrix}-1 & -2\\ -2 & -1\\ -3 & -3\end{bmatrix}\begin{bmatrix}3 & -2\\ 2 & 4\end{bmatrix} = \begin{bmatrix}-7 & -6\\ -8 & 0\\ -15 & -6\end{bmatrix}$
e) $(C + 2D^T + E)^T = it\ is\ not\ possible\ to\ compute\ since\ matrix\ E\ has\ different\ size$

2.

a) $AB = \begin{bmatrix}1 & 4\\ 3 & 2\end{bmatrix}\begin{bmatrix}2 & -1\\ -3 & 4\end{bmatrix} = \begin{bmatrix}-4 & 15\\ 0 & 5\end{bmatrix}$
b) $BA = \begin{bmatrix}2 & -1\\ -3 & 4\end{bmatrix}\begin{bmatrix}1 & 4\\ 3 & 2\end{bmatrix} = \begin{bmatrix}-1 & 6\\ 9 & -4\end{bmatrix}$
c) They are different.

3.

a) $v_1 \cdot v_2 = (-2, 0,1) \cdot (0, 1,0) = (0 + 0 + 0) = 0$
    a. $v_1 \cdot v_3 = (-2, 0,1) \cdot (2, 0,4) = (-4 + 0 + 4) = 0,$
    b. $v_2 \cdot v_3 = (0, 1,0) \cdot (-2, 0,1) = (0 + 0 + 0) = 0$
b) $v_1 = (-2,0,1) => \left(\frac{-2}{\sqrt{5}}, \frac{0}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)\ prove: \sqrt{\frac{-2}{\sqrt{5}}^2 + 0^2 + \frac{1}{\sqrt{5}}^2} = 1$
    a. $v_2 = (0,1,0)\ prove: \sqrt{0^2 + 1^2 + 0^2} = 1$
    b. $v_3 = (2,0,4) => \left(\frac{2}{\sqrt{20}}, \frac{0}{\sqrt{20}}, \frac{4}{\sqrt{20}}\right)\ prove: \sqrt{\frac{2}{\sqrt{20}}^2 + 0^2 + \frac{4}{\sqrt{20}}^2} = 1$

4.

a) $while\ m\ and\ n\ are\ bigger\ than\ 0\ (m > 0\ and\ n > 0), rank\ of\ the\ matrix\ xy^T\ is\ 1$
    a. $if\ m\ or\ n\ is\ 1, then\ the\ rank\ is\ 1\ since\ there\ is\ only\ one\ pivot\ point$
        i. $\begin{bmatrix}1\\ 2\end{bmatrix}\ or\ [1 \quad 2]$
    b. $if\ m\ and\ n\ are\ bigger\ than\ 1, still\ the\ rank\ is\ 1\ since\ every\ row\ can\ be\ divisible$ $by\ a\ row\ in\ the\ matrix$

i. $X = \begin{bmatrix}1\\2\\3\end{bmatrix}$ $Y = \begin{bmatrix}4 & 5 & 6 & 7\end{bmatrix}, XY^T =$

$\begin{bmatrix}4 & 5 & 6 & 7\\8 & 10 & 12 & 14\\12 & 15 & 18 & 21\end{bmatrix}, which\ row\ 2\ and\ row\ 3\ are\ multiple\ of\ row\ 1. So\ row\ 2\ and\ row3$

$would\ cancel\ out, making \begin{bmatrix}4 & 5 & 6 & 7\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix} that\ only\ has\ 1\ pivot\ point.$

c. $But\ if\ m\ or\ n\ is\ 0, then\ the\ rank\ is\ 0.$

i. $at\ this\ point, XY^T\ would\ be\ an\ empty\ matrix, which\ the\ rank\ is\ 0.$

5.

a) 'i' in X represents the row number, and 'i' in Y represents the column number.
By multiplying X and Y matrices, $XY = x_1 * y_1 + x_2 * y_2 + x_3 * y_3 + \cdots x_i * y_i\ which\ also\ can\ be\ written\ as\ \sum_{i=1}^{n} x_1 * y_1^T$

6.

a) $prove\ that\ X^TX\ is\ symmetric$

a. $say\ B = X^TX. B^T = (X^TX)^T = X^{T^T}X^T = X^TX = B$

b. $X^TX = (X^TX)^T, so\ it\ is\ symmetric$

b) $prove\ v^TAv \geq 0, for\ any\ v\ \in \mathbb{R}^{n*m}, which\ A = X^TX$

a. $prove\ v^TX^TXv \geq 0$

b. $v^T(X^TX)v = (Xv)^T(Xv)$

c. $by\ doing\ the\ multiplication\ of\ two\ matrix\ (Xv)^T(Xv), the\ pivot\ of\ the\ result\ has\ to\ be\ either\ 0\ or\ a\ positive\ number, which\ prove\ that\ all\ eigen\ values\ are\ non - negative.$

7.

a)

a. eigenvalues

i. $\det\left(\begin{bmatrix}2-\lambda & 1 & 3\\1 & 1-\lambda & 2\\3 & 2 & 5-\lambda\end{bmatrix}\right) = (2-\lambda) * \det\begin{bmatrix}1-\lambda & 2\\2 & 5-\lambda\end{bmatrix} - 1 * det\begin{bmatrix}1 & 2\\3 & 5-\lambda\end{bmatrix} + 3 * det\begin{bmatrix}1 & 1-\lambda\\3 & 2\end{bmatrix} = (2-\lambda) * \big((1-\lambda)(5-\lambda) - 4\big) - (5-\lambda-6) + 3\big(2-(1-\lambda)(3)\big) = (-\lambda^3 + 8\lambda^2 - 13\lambda + 2) - (5-\lambda-6) + (9\lambda - 3) = -\lambda^3 + 8\lambda^2 - 3\lambda$

ii. $\lambda = 0\ \&\ \lambda = 4 - \sqrt{13}\ \&\ \lambda = 4 + \sqrt{13}$

b. Eigenvector

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In [11]: import numpy as np
         import sympy as sym

In [17]: A = np.matrix("2, 1, 3;1, 1, 2; 3, 2, 5")
         A_sym = sym.Matrix(A)
         value, vec = np.linalg.eig(A)
         print("This is the Eigenvector of the matrix A, calculated using python numpy and sympy")
         vec

         This is the Eigenvector of the matrix A, calculated using python numpy and sympy

Out[17]: matrix([[-0.49079864, -0.65252078, -0.57735027],
                 [-0.31970025,  0.75130448, -0.57735027],
                 [-0.81049889,  0.0987837 ,  0.57735027]])
```

b) Eigen Decomposition of A

a. $A = U\Lambda U^T = \begin{bmatrix} -0.4908 & -0.6525 & -0.5773 \\ -0.3197 & 0.7513 & -0.5773 \\ -0.8105 & 0.0988 & 0.5773 \end{bmatrix} * \begin{bmatrix} 4+\sqrt{13} & 0 & 0 \\ 0 & 4-\sqrt{13} & 0 \\ 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} -0.4908 & -0.3197 & -0.8105 \\ -0.6525 & 0.7513 & 0.0988 \\ -0.05773 & -0.5773 & 0.5773 \end{bmatrix} =$

c) Rank of A

a. $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix} => divide\ R1\ by\ 2 = \begin{bmatrix} 1 & 0.5 & 1.5 \\ 1 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix} => R2 - R1 = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \\ 3 & 2 & 5 \end{bmatrix} => R3 - 3 * R1 = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} => divide\ R2\ by\ 0.5 = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0 & 1 & 1 \\ 0 & 0.5 & 0.5 \end{bmatrix} => R3 - 0.5 * R2 = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

b. So, the rank is 2

d) No, because not all eigen values are positive, 0 is one of them.

e) Yes, since all eigen values are non-negative.

f) Yes, since the determinant of A is 0.

a. $2 * det\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} - 1 * det\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} + 3 * det\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} = 2 - 1(-1) + 3(-1) = 0$