Probabilities on a Finite or Countable Space

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we assume that Ω is finite or countable and we take the σ-algebra $\mathcal{A} = 2^{\Omega}$ (the class of all subsets of Ω).

Theorem 4.1

(a) (uniqueness) A probability on the finite or countable set Ω is characterized by its values on the atoms: $p_w = P(\{w\}), w \in \Omega$.
(b) (existence) Let $(P_w)_{w\in\Omega}$ be a set of real numbers indexed by the finite or countable set Ω. Then there exists a unique probability P such that $p_w \geq 0, \forall w \in \Omega$ and $\sum_{w\in\Omega} p_w = 1$.

proof of Thm 4.1

(a) Let $A \in \mathcal{A}$. Then $A = \cup_{w\in A}\{w\}$, a finite or countable union of pairwise disjoint singletons. If P is probability, countable additivity yields

$$P(A) = P(\cup_{w\in A}\{w\}) = \sum_{w\in A} P(\{w\}) = \sum_{w\in A} p_w$$

(b)

(1) existence

Define a set function P such that for a $A \in \mathcal{A}$, $P(A) = \sum_{w \in A} P(\{w\})$, then

1. $P(\Omega) = \sum_{w \in A} P(\{w\}) = \sum_{w \in A} p_w = 1$
2. Supp' $A_1, A_2, \ldots$ are pairwise disjoint, then

$$P\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{w \in \cup_{n=1}^{\infty} A_n} P(\{w\}) = \sum_{n=1}^{\infty} \sum_{w \in A_n} P(\{w\}) = \sum_{n=1}^{\infty} P(A_n)$$

Thus, $P(\cdot)$ is a prob measure.

(2) uniqueness

The uniqueness is given by (a).

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Definition 4.1

A probability measure P on the finite set Ω is called uniform if $P(\{w_1\}) = P(\{w_2\}), \forall w_1, w_2 \in \Omega$.

In this case, it is immediate that

$$P(A) = \frac{\#(A)}{\#(\Omega)}.$$

Then computing the probability of any event A amounts to counting the number of points in a. On a given finite set Ω there is one and only one uniform probability.

Examples for finite space

1. Hypergeometric distribution

$$P(X = x) = \frac{\binom{N}{x}\binom{M}{n-x}}{\binom{N+M}{n}}, \quad 0 \leq x \leq N \text{ and } 0 \leq n - x \leq M.$$

2. Binomial distribution

$$P(X = x) = \binom{n}{x}\left(\frac{N}{N+M}\right)^{x}\left(\frac{M}{N+M}\right)^{n-x}, \quad \text{for } x = 0, 1, \ldots, n.$$

Examples for countable space

1. Poisson distribution

$$P(X = x) = e^{-\lambda}\frac{\lambda^n}{n!}, \quad n = 0, 1, 2, \ldots.$$

2. Geometric distribution

$$P(X = x) = (1 - \alpha)\alpha^n, \quad n = 0, 1, 2, \ldots.$$

Q4.1 (Poisson Approximation to the Binomial) Let P be a Binomial probability with prob of success p and number of trials n. Let $\lambda = pn$. Show that

$$P(k \text{ successes}) = \frac{\lambda^k}{k!}\left(1-\frac{\lambda}{n}\right)^n \left\{\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\cdots\left(\frac{n-k+1}{n}\right)\right\}\left(1-\frac{\lambda}{n}\right)^{-k}.$$

Let $n \to \infty$ and let p change so that λ remains constant. Conclude that for small p and large n.

$$P(k \text{ successes}) \approx e^{-\lambda}\frac{\lambda^k}{k!}, \text{ where } \lambda = pn.$$

proof of Q4.1

$P(k \text{ successes})$는 n번의 시행 중 k번의 성공, $n-k$번의 실패가 발생하는 확률이다. n번 중 k번을 선택하는 사건의 수는 $\binom{n}{k}$이므로 k번 성공할 확률은

$$P(k \text{ successes}) = \binom{n}{k} p^k (1-p)^{n-k}.$$

$\lambda = pn$라고 했으므로 다시 표현하면

$$
\begin{aligned}
\binom{n}{k} p^k (1-p)^{n-k} &= \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n \frac{n!}{n^k (n-k)!} \left(1 - \frac{\lambda}{n}\right)^{-k} \\
&= \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n \frac{n(n-1)\cdots(n-k+1)}{n^k} \left(1 - \frac{\lambda}{n}\right)^{-k} \\
&= \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n \left(\frac{n}{n}\right) \left(\frac{n-1}{n}\right) \cdots \left(\frac{n-k+1}{n}\right) \left(1 - \frac{\lambda}{n}\right)^{-k}
\end{aligned}
$$

이고, $n \to \infty$에 따라
$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}, \lim_{n\to\infty} \frac{n-k}{n} = 1, \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-k} = 1$이므로
$P(k \text{ successes}) \approx e^{-\lambda} \frac{\lambda^k}{k!}$이다. ■