Matrices

1.1

What are the entries a_{21}, and a_{23} of the matrix $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 7 & 8 \\ 0 & 9 & 4 \end{bmatrix}$.

Solution. a_{21} is the element in the second row and first column. $a_{21} = 2$.
a_{23} is the element in the second row and third column. $a_{23} = 8$. ■

1.2

Determine the products AB and BA for the following values of A and B: (I have not replicated the matrices here)

Solution. For the first set, $AB = \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}$

$BA = \begin{bmatrix} -20 & -20 & -12 \\ 24 & 33 & 32 \\ -9 & -12 & -11 \end{bmatrix}$

For the second set, $AB = \begin{bmatrix} 18 & 4 \\ 12 & 0 \end{bmatrix}$

$BA = \begin{bmatrix} 2 & 16 \\ 5 & 16 \end{bmatrix}$

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1.3

Let $A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$ be a row vector, and let $B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ be a column vector. Compute the products AB and BA.

Solution. AB is a 1×1 matrix. $AB = \begin{bmatrix} a_1b_1 + a_2b_2 + \cdots a_nb_n \end{bmatrix}$.

BA is an $n \times n$ matrix. $BA = \begin{bmatrix} a_1b_1 & a_2b_1 & \cdots & a_nb_1 \\ a_1b_2 & a_2b_2 & \cdots & a_nb_2 \\ \vdots & \vdots & \cdots & \vdots \\ a_1b_n & a_2b_n & \cdots & a_nb_n \end{bmatrix}$. ■

1.4

I skipped this because it was a self-checking problem.

1.5

Let A, B, and C be matrices of sizes $l \times m$, $m \times n$, and $n \times p$. How many multiplications are required to compute the product AB? In which order should the triple product ABC be computed, to minimize the number of multiplications required?

Solution. When an $l \times m$ matrix is multiplied with an $m \times n$ matrix, lmn matrix operations are required since one row multiplied with one column requires m multiplications, and we repeat it n times for each column and l times for each row.
There are two possible ways to multiply ABC, either as $(AB)C$ or $A(BC)$. When calculating $(AB)C$ we perform lmn multiplications to calculate AB. AB has a dimension of $l \times n$. Thus, we must multiply lnp times to calculate $(AB)C$. This leads to a total of $lmn + lnp$ multiplications.
When calculating $A(BC)$ we perform mnp multiplications to calculate BC. BC has a dimension of $m \times p$. Thus, we must multiply lmp times to calculate $A(BC)$. This leads to a total of $lmp + mnp$ multiplications. ■

1.6

Compute $\begin{bmatrix} 1 & a \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & a \\ & 1 \end{bmatrix}^n$

Solution. The product is $\begin{bmatrix} 1 & a+b \\ & 1 \end{bmatrix}$.

The second product is $\begin{bmatrix} 1 & na \\ & 1 \end{bmatrix}$. ■

1.7

Find a formula for $\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^n$ and prove it by induction.

Solution. When I try to multiply these matrices, I observe that $\begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix}$ is the resultant matrix. This happens because of how the terms add up (the element in the first row and the third column is just the sum of the first n natural numbers).

Proof by induction:
1. For base case ($n = 1$): The resultant matrix is just the matrix given above, since $n = 1$ and $\frac{n(n+1)}{2} = \frac{1*(2)}{2} = 1$.
2. We assume the formula holds for $n = k$. Thus, $\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & k & \frac{k(k+1)}{2} \\ & 1 & k \\ & & 1 \end{bmatrix}$. 3. For $n = k+1$, $\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^k \begin{bmatrix} 1 & k & \frac{k(k+1)}{2} \\ & 1 & k \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & k+1 & \frac{k(k+1)}{2} + k + 1 \\ & 1 & k+1 \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & k+1 & \frac{(k+2)(k+1)}{2} \\ & 1 & k+1 \\ & & 1 \end{bmatrix}$.
This completes the proof above. ■

1.8

I skipped this because I thought it was repetitive.

1.9

(a) When is $(A + B)(A - B) = A^2 - B^2$?
(b) Expand $(A + B)^3$.

Solution. (a) $(A + B)(A - B) = A^2 + AB - BA + B^2$. When $AB = BA$, or the matrices are commutative, $(A + B)(A - B) = A^2 - B^2$. (This is not true in general since matrix multiplication is not commutative).
(b) $(A + B)^2 = A^2 + AB + BA + B^2$.
$(A+B)^3 = (A+B)^2(A+B) = (A^2+AB+BA+B^2)(A+B) = A^3+A^2B+ABA+AB^2+BA^2 + BAB + B^2A + B^3$.

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1.10

Let D be the diagonal matrix with diagonal entries $d_1, \cdots, d_n$ and let $A = a_{ij}$ be an arbitrary $n \times n$ matrix. Compute the products DA and AD.

Solution. DA will be an $n \times n$ matrix. $DA = \begin{bmatrix} d_1 a_{11} & d_1 a_{12} & \cdots & d_1 a_{1n} \\ d_2 a_{21} & d_2 a_{22} & \cdots & d_2 c_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ d_n a_{n1} & d_n a_{n2} & \cdots & d_n c_{nn} \end{bmatrix}$.

AD will be an $n \times n$ matrix. $AD = \begin{bmatrix} d_1a_{11} & d_2a_{12} & \cdots & d_na_{1n} \\ d_1a_{21} & d_2a_{22} & \cdots & d_nc_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ d_1a_{n1} & d_2a_{n2} & \cdots & d_nc_{nn} \end{bmatrix}$.

■

1.11

Prove that the product of upper triangular matrices is upper triangular.

Solution. For a lower-triangular matrix, $u_{ij} = 0$ if $i > j$.
Let A be an $m \times n$ upper-triangular matrix and B be an $n \times p$ upper-triangular matrix. Let $C = AB$. Then, $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. For both a_{ik} and b_{kj} to be non-zero $i \leq k \leq j$. When $i > j$, then either or both a_{ik} and b_{kj} will be 0. So $c_{ij} = 0$ when $i > j$. Thus, C is an upper-triangular matrix. ■

1.12

I am not replicating the question here.

Solution. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$.
$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$.
For these matrices to be equal, $b = 0$ and $c = 0$. Thus A should be a diagonal matrix.
(Th remaining problems can be solved similarly).

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