Matrices and Matrix Operations

D.1-1

Show that if A and B are symmetric $n \times n$ matrices, then so are $A + B$ and $A - B$.

Proof. Let an element of A be represented by a_{ij}. Since A is a symmetric matrix, $a_{ij} = a_{ji}$.
Similarly, let an element of B be represented by b_{ij}. Since B is a symmetric matrix, $b_{ij} = b_{ji}$.
Let $C = A + B$ and an element of C be denoted by c_{ij}. Then $c_{ij} = a_{ij} + b_{ij} = a_{ji} + b_{ji} = c_{ji}$.
So C is a symmetric matrix.
Let $D = A - B$ and an element of D be denoted by d_{ij}. Then $d_{ij} = a_{ij} - b_{ij} = a_{ji} - b_{ji} = d_{ji}$.
So D is a symmetric matrix.
Hence proved. □

D.1-2

Prove that $(AB)^T = B^T A^T$ and that $A^T A$ is always a symmetric matrix.

Proof. □

$(AB)^T_{ij} = (AB)_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki}$.
$(B^T A^T)_{ij} = \sum_{k=1}^{n} b^T_{ik} a^T_{kj}$.
$= \sum_{k=1}^{n} b_{ik} a_{jk}$.
Thus, $(AB)^T = B^T A^T$.

$(A^T A)^T = A^T (A^T)^T = A^T A$. So $A^T A$ is a symmetric matrix.

D.1-3

Prove that the product of two lower-triangular matrices is lower-triangular.

Proof. □

For a lower-triangular matrix, $l_{ij} = 0$ if $i < j$.
Let A be an $m \times n$ lower-triangular matrix and B be an $n \times p$ lower-triangular matrix. Let $C = AB$. Then,
$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. For both a_{ik} and b_{kj} to be non-zero $i \geq k \geq j$. When $i < j$, then either or both a_{ik} and b_{kj} will be 0. So $c_{ij} = 0$ when $i < j$.
Thus, C is a lower-triangular matrix.

D.1-4

Prove that if P is an $n \times n$ permutation matrix and A is an $n \times n$ matrix, then the matrix product PA is A with its rows permuted, and the matrix product AP is A with its columns permuted. Prove that the product of two permutation matrices is a permutation matrix.

Proof. □

A permutation matrix P has exactly one 1 in each row or column.

Let $X(i)$ be the column in row i for which $p_{ij} = 1$.
Then $(PA)_{i,j} = \sum_{k=1}^{n} p_{ik} a_{kj} = p_{i,X(i)} a_{X(i),j} = A_{X(i),j}$. (Since $p_{i,k} = 0$ when $k \neq X(i)$). This proves the first part.

Let $X(j)$ be the row in column j for which $p_{ij} = 1$.
Then $(AP)_{i,j} = \sum_{k=1}^{n} a_{ik} p_{kj} = a_{i,X(j)} p_{X(j),j} = A_{i,X(j)}$. (Since $p_{k,j} = 0$ when $k \neq X(j)$). This proves the second part.

Let PP' be the product of two permutation matrices. P will only change the ordering of the rows in P'. So P' will still have rows with only one 1 in every row and every column. Hence proved.